Preface

Nonlinear functional analysis is an important branch of contemporary mathematics; it has grown from geometry, fluid and elastic mechanics, physics, chemistry, biology, control theory and economics, etc. It is related to many areas of mathematics: topology, ordinary differential equations, partial differential equations, groups, dynamical systems, differential geometry, measure theory, etc.

We mainly present our new results on the three fundamental methods in nonlinear functional analysis: Variational, Topological and Partial Order Methods with their Applications. They have been used extensively to solve questions of the existence of solutions for elliptic equations, wave equations, Schrödinger equations, Hamiltonian systems, etc. Also they have been used to study the existence of multiple solutions and the properties of solutions.

Hilbert posed his famous 23 problems on the occasion of his speech at the centennial assembly of the International Congress 1900 in Paris. Three of these were related to the calculus of variations. Included are minimization methods, minimax methods, Morse theory, category, Ljusternik–Schnirelmann theory, etc. in the calculus of variations. We should mention that Ambrosetti and Rabinowitz's work [11] in the 1970s is the beginning of the minimax method, making it possible for people to deal with functionals that are unbounded from below, which come from the study of nonlinear elliptic equations, Hamiltonian systems, geometry, and mathematical physics. In the 1930s, Morse developed a theory which set up the relationship between critical points of a non-degenerate function and the topology of the underlying compact manifold. In the 1960s Palais [149] and Smale [164] et al. extended Morse theory to infinite-dimensional manifolds by using the Palais–Smale condition.

Topological methods and partial order methods are basic and important tools in nonlinear functional analysis too. The Brouwer degree is a powerful tool in algebraic topology; the Leray–Schauder degree is an extension of the Brouwer degree from finite-dimensional spaces to infinite-dimensional Banach spaces, which has been introduced by Leray and Schauder in the study of nonlinear partial differential equations in the 1930s. Rabinowitz's global bifurcation theorem is based on the computation of the Leray–Schauder degree. In many problems that arise in population biology, economics, and the study of infectious diseases, we need to discuss the existence of nonnegative solutions with some desired qualitative properties, so cones are used to develop partial order methods and fixed point index theory. Then one gets fixed point theorems and applications to many kinds of differential equation, etc.

In Chap. 1, we present preliminaries: some basic concepts, and useful famous theorems and results so that the reader may easily find information if need may be.

In Chap. 2, we introduce three kinds of operator: increasing operators, decreasing operators, and mixed monotone operators. Some fixed point theorems and applications to integral equations and differential equations are included. One equivalent condition of the normal cone is given.

In Chap. 3, we present the minimax methods including the Mountain Pass Theorem, linking methods, local linking methods, and critical groups; next, we treat some applications to elliptic boundary value problems.

In Chap. 4, we use bifurcation and critical point theory together to study the structure of the solutions of elliptic equations; also we have results on three sign-changing solutions.

In Chap. 5, we consider the boundary value problems for a class of Monge–Ampère equations. First we prove that any solution on the ball is radially symmetric by the moving plane argument. Then we show that there exists a critical radius such that, if the radius of a ball is smaller than this critical value, then there exists a solution, and vice versa. Using a comparison between domains we prove that this phenomenon occurs for every domain. By using the Lyapunov–Schmidt reduction method we get the local structure of the solutions near a degenerate point; by Leray–Schauder degree theory, a priori estimates, and using bifurcation theory we get the global structure.

In Chap. 6, on superlinear systems of Hammerstein integral equations and applications, we use the Leray–Schauder degree to obtain new results on the existence of solutions, and apply them to two-point boundary problems of systems of equations. We also are concerned with the existence of (component-wise) positive solutions for a semilinear elliptic system, where the nonlinear term is superlinear in one equation and sublinear in the other equation. By constructing a cone $K_1 \times K_2$, which is the Cartesian product of two cones in the space $C(\overline{\Omega})$, and computing the fixed point index in $K_1 \times K_2$, we establish the existence of positive solutions for the system.

In Chap. 7, we show some results on the Dancer–Fučik spectrum for bounded domains. We are concerned with the Fučik point spectrum for Schrödinger operators, $-\Delta + V$, in $L^2(\mathbb{R}^N)$ for certain types of potential, $V : \mathbb{R}^N \to \mathbb{R}$. We use the Dancer–Fučik spectrum to asymptotically linear elliptic problems to get one-sign solutions.

In Chap. 8, we introduce some results on sign-changing solutions of elliptic and p-Laplacian, including using Nehri manifold, invariant sets of descent flows, Morse theory, etc.

In Chap. 9, we show that if $u_0 \in W_0^{1,p}(\Omega)$ is a local minimizer of J in the C^1-topology, it is still a local minimizer of the functional J in $W_0^{1,p}(\Omega)$. This extends the famous results of Brezis–Nirenberg to $p > 2$. We thus obtain multiple solutions and structures of solutions for p-Laplacian equations. Finally, we also show uniqueness results of various kinds.

In Chap. 10, we obtain nontrivial solutions of a class of nonlocal quasilinear elliptic boundary value problems using the Yang index and critical groups, and we obtain sign-changing solutions of a class of nonlocal quasilinear elliptic boundary value problems using variational methods and invariant sets of descent flows. We also show a uniqueness result.

In Chap. 11, we study free boundary problems, Schrödinger systems from Bose–Einstein condensates, and competing systems with many species. We prove the existence and uniqueness result of the Dirichlet boundary value problem of elliptic competing systems. We show that, for the singular limit, species are spatially segregated; they satisfy a remarkable system of differential inequalities as $\kappa \to +\infty$. We also introduce optimal partition problems related to eigenvalues and nonlinear eigenvalues. Finally, some recent new results on Schrödinger systems from Bose–Einstein condensates are presented.

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I dedicate this book to my father Deren Zhang, my wife Jimin Fang and my son Fan Zhang.

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