Chapter 2
Vector and Tensor Analysis in Euclidean Space

2.1 Vector- and Tensor-Valued Functions, Differential Calculus

In the following we consider a vector-valued function $\boldsymbol{x}(t)$ and a tensor-valued function $\mathbf{A}(t)$ of a real variable t. Henceforth, we assume that these functions are continuous such that

$$\lim_{t \to t_0} [\boldsymbol{x}(t) - \boldsymbol{x}(t_0)] = \boldsymbol{0}, \quad \lim_{t \to t_0} [\mathbf{A}(t) - \mathbf{A}(t_0)] = \mathbf{0} \tag{2.1}$$

for all t_0 within the definition domain. The functions $\boldsymbol{x}(t)$ and $\mathbf{A}(t)$ are called differentiable if the following limits

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \lim_{s \to 0} \frac{\boldsymbol{x}(t+s) - \boldsymbol{x}(t)}{s}, \quad \frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \lim_{s \to 0} \frac{\mathbf{A}(t+s) - \mathbf{A}(t)}{s} \tag{2.2}$$

exist and are finite. They are referred to as the derivatives of the vector- and tensor-valued functions $\boldsymbol{x}(t)$ and $\mathbf{A}(t)$, respectively.

For differentiable vector- and tensor-valued functions the usual rules of differentiation hold.

1. Product of a scalar function with a vector- or tensor-valued function:

$$\frac{\mathrm{d}}{\mathrm{d}t}[u(t)\,\boldsymbol{x}(t)] = \frac{\mathrm{d}u}{\mathrm{d}t}\boldsymbol{x}(t) + u(t)\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}, \tag{2.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[u(t)\,\mathbf{A}(t)] = \frac{\mathrm{d}u}{\mathrm{d}t}\mathbf{A}(t) + u(t)\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}. \tag{2.4}$$

2. Mapping of a vector-valued function by a tensor-valued function:

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{A}(t)\,\boldsymbol{x}(t)] = \frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\boldsymbol{x}(t) + \mathbf{A}(t)\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}. \tag{2.5}$$

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3. Scalar product of two vector- or tensor-valued functions:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\boldsymbol{x}(t)\cdot\boldsymbol{y}(t)\right]=\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\cdot\boldsymbol{y}(t)+\boldsymbol{x}(t)\cdot\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t}, \tag{2.6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{A}(t):\mathbf{B}(t)\right]=\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}:\mathbf{B}(t)+\mathbf{A}(t):\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}. \tag{2.7}$$

4. Tensor product of two vector-valued functions:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\boldsymbol{x}(t)\otimes\boldsymbol{y}(t)\right]=\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\otimes\boldsymbol{y}(t)+\boldsymbol{x}(t)\otimes\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t}. \tag{2.8}$$

5. Composition of two tensor-valued functions:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{A}(t)\,\mathbf{B}(t)\right]=\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\mathbf{B}(t)+\mathbf{A}(t)\,\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}. \tag{2.9}$$

6. Chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}\left[u(t)\right]=\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}t},\quad \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}\left[u(t)\right]=\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}t}. \tag{2.10}$$

7. Chain rule for functions of several arguments:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}\left[u(t),v(t)\right]=\frac{\partial\boldsymbol{x}}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial\boldsymbol{x}}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}t}, \tag{2.11}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}\left[u(t),v(t)\right]=\frac{\partial\mathbf{A}}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial\mathbf{A}}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}t}, \tag{2.12}$$

where $\partial/\partial u$ denotes the partial derivative. It is defined for vector and tensor valued functions in the standard manner by

$$\frac{\partial\boldsymbol{x}(u,v)}{\partial u}=\lim_{s\to 0}\frac{\boldsymbol{x}(u+s,v)-\boldsymbol{x}(u,v)}{s}, \tag{2.13}$$

$$\frac{\partial\mathbf{A}(u,v)}{\partial u}=\lim_{s\to 0}\frac{\mathbf{A}(u+s,v)-\mathbf{A}(u,v)}{s}. \tag{2.14}$$

The above differentiation rules can be verified with the aid of elementary differential calculus. For example, for the derivative of the composition of two second-order tensors (2.9) we proceed as follows. Let us define two tensor-valued functions by

$$\mathbf{O}_1(s)=\frac{\mathbf{A}(t+s)-\mathbf{A}(t)}{s}-\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t},\quad \mathbf{O}_2(s)=\frac{\mathbf{B}(t+s)-\mathbf{B}(t)}{s}-\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}. \tag{2.15}$$

Bearing the definition of the derivative (2.2) in mind we have

$$\lim_{s\to 0}\mathbf{O}_1(s)=\mathbf{0},\quad \lim_{s\to 0}\mathbf{O}_2(s)=\mathbf{0}.$$

Then,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{A}(t)\,\mathbf{B}(t)\right] &= \lim_{s\to 0}\frac{\mathbf{A}(t+s)\,\mathbf{B}(t+s)-\mathbf{A}(t)\,\mathbf{B}(t)}{s} \\
&= \lim_{s\to 0}\frac{1}{s}\left\{\left[\mathbf{A}(t)+s\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}+s\mathbf{O}_1(s)\right]\left[\mathbf{B}(t)+s\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}+s\mathbf{O}_2(s)\right]\right. \\
&\qquad \left. -\mathbf{A}(t)\,\mathbf{B}(t)\right\} \\
&= \lim_{s\to 0}\left\{\left[\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}+\mathbf{O}_1(s)\right]\mathbf{B}(t)+\mathbf{A}(t)\left[\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}+\mathbf{O}_2(s)\right]\right\} \\
&\quad +\lim_{s\to 0}s\left[\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}+\mathbf{O}_1(s)\right]\left[\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}+\mathbf{O}_2(s)\right]=\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\mathbf{B}(t)+\mathbf{A}(t)\,\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}.
\end{aligned}
$$

2.2 Coordinates in Euclidean Space, Tangent Vectors

Definition 2.1. A coordinate system is a one to one correspondence between vectors in the n-dimensional Euclidean space $\mathbb{E}^n$ and a set of n real numbers $(x^1, x^2, \ldots, x^n)$. These numbers are called coordinates of the corresponding vectors.

Thus, we can write

$$x^i = x^i(\boldsymbol{r}) \quad \Leftrightarrow \quad \boldsymbol{r} = \boldsymbol{r}\left(x^1, x^2, \ldots, x^n\right), \tag{2.16}$$

where $\boldsymbol{r} \in \mathbb{E}^n$ and $x^i \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$. Henceforth, we assume that the functions $x^i = x^i(\boldsymbol{r})$ and $\boldsymbol{r} = \boldsymbol{r}\left(x^1, x^2, \ldots, x^n\right)$ are sufficiently differentiable.

Example 2.1. Cylindrical coordinates in $\mathbb{E}^3$. The cylindrical coordinates (Fig. 2.1) are defined by

$$\boldsymbol{r} = \boldsymbol{r}(\varphi, z, r) = r\cos\varphi\,\boldsymbol{e}_1 + r\sin\varphi\,\boldsymbol{e}_2 + z\boldsymbol{e}_3 \tag{2.17}$$

and

$$
\begin{gathered}
r = \sqrt{(\boldsymbol{r}\cdot\boldsymbol{e}_1)^2 + (\boldsymbol{r}\cdot\boldsymbol{e}_2)^2}, \quad z = \boldsymbol{r}\cdot\boldsymbol{e}_3, \\
\varphi = \begin{cases} \arccos\dfrac{\boldsymbol{r}\cdot\boldsymbol{e}_1}{r} & \text{if } \boldsymbol{r}\cdot\boldsymbol{e}_2 \geq 0, \\ 2\pi - \arccos\dfrac{\boldsymbol{r}\cdot\boldsymbol{e}_1}{r} & \text{if } \boldsymbol{r}\cdot\boldsymbol{e}_2 < 0, \end{cases}
\end{gathered} \tag{2.18}
$$

where $\boldsymbol{e}_i$ $(i = 1, 2, 3)$ form an orthonormal basis in $\mathbb{E}^3$.

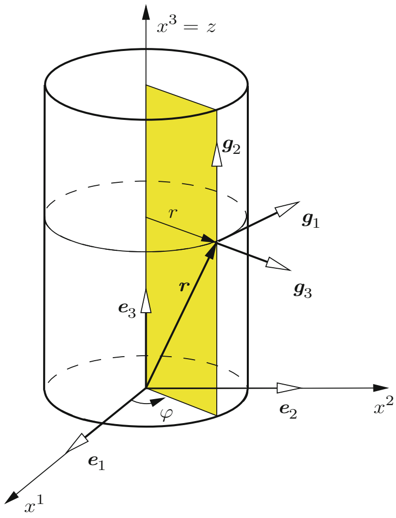

Fig. 2.1 Cylindrical coordinates in three-dimensional space

The vector components with respect to a fixed basis, say $\mathcal{H} = \{\boldsymbol{h}_1, \boldsymbol{h}_2, \ldots, \boldsymbol{h}_n\}$, obviously represent its coordinates. Indeed, according to Theorem 1.5 of the previous chapter the following correspondence is one to one

$$\boldsymbol{r} = x^i \boldsymbol{h}_i \quad \Leftrightarrow \quad x^i = \boldsymbol{r} \cdot \boldsymbol{h}^i, \quad i = 1, 2, \ldots, n, \tag{2.19}$$

where $\boldsymbol{r} \in \mathbb{E}^n$ and $\mathcal{H}' = \left\{\boldsymbol{h}^1, \boldsymbol{h}^2, \ldots, \boldsymbol{h}^n\right\}$ is the basis dual to $\mathcal{H}$. The components x^i $(2.19)_2$ are referred to as the linear coordinates of the vector $\boldsymbol{r}$.

The Cartesian coordinates result as a special case of the linear coordinates (2.19) where $\boldsymbol{h}_i = \boldsymbol{e}_i$ $(i = 1, 2, \ldots, n)$ so that

$$\boldsymbol{r} = x^i \boldsymbol{e}_i \quad \Leftrightarrow \quad x^i = \boldsymbol{r} \cdot \boldsymbol{e}_i, \quad i = 1, 2, \ldots, n. \tag{2.20}$$

Let $x^i = x^i(\boldsymbol{r})$ and $y^i = y^i(\boldsymbol{r})$ $(i = 1, 2, \ldots, n)$ be two arbitrary coordinate systems in $\mathbb{E}^n$. Since their correspondences are one to one, the functions

$$x^i = \hat{x}^i\left(y^1, y^2, \ldots, y^n\right) \;\Leftrightarrow\; y^i = \hat{y}^i\left(x^1, x^2, \ldots, x^n\right), \; i = 1, 2, \ldots, n \tag{2.21}$$

are invertible. These functions describe the transformation of the coordinate systems. Inserting one relation (2.21) into another one yields

$$\begin{aligned} y^i = \hat{y}^i \big(&\hat{x}^1\left(y^1, y^2, \ldots, y^n\right), \\ &\hat{x}^2\left(y^1, y^2, \ldots, y^n\right), \ldots, \hat{x}^n\left(y^1, y^2, \ldots, y^n\right)\big). \end{aligned} \tag{2.22}$$

The further differentiation with respect to y^j delivers with the aid of the chain rule

$$\frac{\partial y^i}{\partial y^j} = \delta_{ij} = \frac{\partial y^i}{\partial x^k}\frac{\partial x^k}{\partial y^j}, \quad i, j = 1, 2, \ldots, n. \tag{2.23}$$

The determinant of the matrix (2.23) takes the form

$$\left|\delta_{ij}\right| = 1 = \left|\frac{\partial y^i}{\partial x^k}\frac{\partial x^k}{\partial y^j}\right| = \left|\frac{\partial y^i}{\partial x^k}\right|\left|\frac{\partial x^k}{\partial y^j}\right|. \tag{2.24}$$

The determinant $\left|\partial y^i / \partial x^k\right|$ on the right hand side of (2.24) is referred to as Jacobian determinant of the coordinate transformation $y^i = \hat{y}^i\left(x^1, x^2, \ldots, x^n\right)$ $(i = 1, 2, \ldots, n)$. Thus, we have proved the following theorem.

Theorem 2.1. *If the transformation of the coordinates* $y^i = \hat{y}^i\left(x^1, x^2, \ldots, x^n\right)$ *admits an inverse form* $x^i = \hat{x}^i\left(y^1, y^2, \ldots, y^n\right)$ $(i = 1, 2, \ldots, n)$ *and if* J *and* K *are the Jacobians of these transformations then* $JK = 1$.

One of the important consequences of this theorem is that

$$J = \left|\frac{\partial y^i}{\partial x^k}\right| \neq 0. \tag{2.25}$$

Now, we consider an arbitrary curvilinear coordinate system

$$\theta^i = \theta^i(\boldsymbol{r}) \;\Leftrightarrow\; \boldsymbol{r} = \boldsymbol{r}\left(\theta^1, \theta^2, \ldots, \theta^n\right), \tag{2.26}$$

where $\boldsymbol{r} \in \mathbb{E}^n$ and $\theta^i \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$. The equations

$$\theta^i = const, \; i = 1, 2, \ldots, k-1, k+1, \ldots, n \tag{2.27}$$

define a curve in $\mathbb{E}^n$ called θ^k-coordinate line. The vectors (see Fig. 2.2)

$$\boldsymbol{g}_k = \frac{\partial \boldsymbol{r}}{\partial \theta^k}, \quad k = 1, 2, \ldots, n \tag{2.28}$$

are called the tangent vectors to the corresponding θ^k-coordinate lines (2.27).

One can verify that the tangent vectors are linearly independent and form thus a basis of $\mathbb{E}^n$. Conversely, let the vectors (2.28) be linearly dependent. Then, there are scalars $\alpha^i \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$, not all zero, such that $\alpha^i \boldsymbol{g}_i = \boldsymbol{0}$. Let further $x^i = x^i(\boldsymbol{r})$ $(i = 1, 2, \ldots, n)$ be linear coordinates in $\mathbb{E}^n$ with respect to a basis $\mathcal{H} = \{\boldsymbol{h}_1, \boldsymbol{h}_2, \ldots, \boldsymbol{h}_n\}$. Then,

$$\boldsymbol{0} = \alpha^i \boldsymbol{g}_i = \alpha^i \frac{\partial \boldsymbol{r}}{\partial \theta^i} = \alpha^i \frac{\partial \boldsymbol{r}}{\partial x^j}\frac{\partial x^j}{\partial \theta^i} = \alpha^i \frac{\partial x^j}{\partial \theta^i}\boldsymbol{h}_j.$$

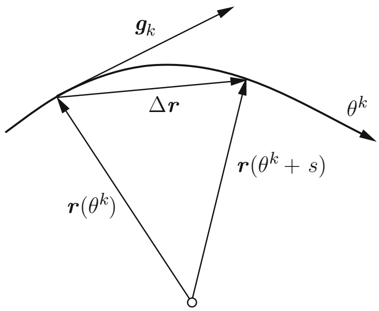

Fig. 2.2 Illustration of the tangent vectors

Since the basis vectors $\boldsymbol{h}_j$ $(j = 1, 2, \ldots, n)$ are linearly independent

$$\alpha^i \frac{\partial x^j}{\partial \theta^i} = 0, \quad j = 1, 2, \ldots, n.$$

This is a homogeneous linear equation system with a non-trivial solution α^i $(i = 1, 2, \ldots, n)$. Hence, $\left|\partial x^j / \partial \theta^i\right| = 0$, which obviously contradicts relation (2.25).

Example 2.2. Tangent vectors and metric coefficients of cylindrical coordinates in $\mathbb{E}^3$. By means of (2.17) and (2.28) we obtain

$$\boldsymbol{g}_1 = \frac{\partial \boldsymbol{r}}{\partial \varphi} = -r \sin \varphi \boldsymbol{e}_1 + r \cos \varphi \boldsymbol{e}_2,$$
$$\boldsymbol{g}_2 = \frac{\partial \boldsymbol{r}}{\partial z} = \boldsymbol{e}_3,$$
$$\boldsymbol{g}_3 = \frac{\partial \boldsymbol{r}}{\partial r} = \cos \varphi \boldsymbol{e}_1 + \sin \varphi \boldsymbol{e}_2. \tag{2.29}$$

The metric coefficients take by virtue of (1.24) and $(1.25)_2$ the form

$$\left[g_{ij}\right] = \left[\boldsymbol{g}_i \cdot \boldsymbol{g}_j\right] = \begin{bmatrix} r^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \left[g^{ij}\right] = \left[g_{ij}\right]^{-1} = \begin{bmatrix} r^{-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.30}$$

The dual basis results from $(1.21)_1$ by

$$\boldsymbol{g}^1 = \frac{1}{r^2} \boldsymbol{g}_1 = -\frac{1}{r} \sin \varphi \boldsymbol{e}_1 + \frac{1}{r} \cos \varphi \boldsymbol{e}_2,$$
$$\boldsymbol{g}^2 = \boldsymbol{g}_2 = \boldsymbol{e}_3,$$
$$\boldsymbol{g}^3 = \boldsymbol{g}_3 = \cos \varphi \boldsymbol{e}_1 + \sin \varphi \boldsymbol{e}_2. \tag{2.31}$$

2.3 Coordinate Transformation. Co-, Contra- and Mixed Variant Components

Let $\theta^i = \theta^i(\boldsymbol{r})$ and $\bar{\theta}^i = \bar{\theta}^i(\boldsymbol{r})$ $(i = 1, 2, \ldots, n)$ be two arbitrary coordinate systems in $\mathbb{E}^n$. It holds

$$\bar{\boldsymbol{g}}_i = \frac{\partial \boldsymbol{r}}{\partial \bar{\theta}^i} = \frac{\partial \boldsymbol{r}}{\partial \theta^j} \frac{\partial \theta^j}{\partial \bar{\theta}^i} = \boldsymbol{g}_j \frac{\partial \theta^j}{\partial \bar{\theta}^i}, \quad i = 1, 2, \ldots, n. \tag{2.32}$$

If $\boldsymbol{g}^i$ is the dual basis to $\boldsymbol{g}_i$ $(i = 1, 2, \ldots, n)$, then we can write

$$\bar{\boldsymbol{g}}^i = \boldsymbol{g}^j \frac{\partial \bar{\theta}^i}{\partial \theta^j}, \quad i = 1, 2, \ldots, n. \tag{2.33}$$

Indeed,

$$\bar{\boldsymbol{g}}^i \cdot \bar{\boldsymbol{g}}_j = \left(\boldsymbol{g}^k \frac{\partial \bar{\theta}^i}{\partial \theta^k} \right) \cdot \left(\boldsymbol{g}_l \frac{\partial \theta^l}{\partial \bar{\theta}^j} \right) = \boldsymbol{g}^k \cdot \boldsymbol{g}_l \left(\frac{\partial \bar{\theta}^i}{\partial \theta^k} \frac{\partial \theta^l}{\partial \bar{\theta}^j} \right)$$

$$= \delta^k_l \left(\frac{\partial \bar{\theta}^i}{\partial \theta^k} \frac{\partial \theta^l}{\partial \bar{\theta}^j} \right) = \frac{\partial \bar{\theta}^i}{\partial \theta^k} \frac{\partial \theta^k}{\partial \bar{\theta}^j} = \frac{\partial \bar{\theta}^i}{\partial \bar{\theta}^j} = \delta^i_j, \; i, j = 1, 2, \ldots, n. \tag{2.34}$$

One can observe the difference in the transformation of the dual vectors (2.32) and (2.33) which results from the change of the coordinate system. The transformation rules of the form (2.32) and (2.33) and the corresponding variables are referred to as covariant and contravariant, respectively. Covariant and contravariant variables are denoted by lower and upper indices, respectively.

The co- and contravariant rules can also be recognized in the transformation of the components of vectors and tensors if they are related to tangent vectors. Indeed, let

$$\boldsymbol{x} = x_i \boldsymbol{g}^i = x^i \boldsymbol{g}_i = \bar{x}_i \bar{\boldsymbol{g}}^i = \bar{x}^i \bar{\boldsymbol{g}}_i, \tag{2.35}$$

$$\mathbf{A} = \mathrm{A}_{ij} \boldsymbol{g}^i \otimes \boldsymbol{g}^j = \mathrm{A}^{ij} \boldsymbol{g}_i \otimes \boldsymbol{g}_j = \mathrm{A}^{i}_{\cdot j} \boldsymbol{g}_i \otimes \boldsymbol{g}^j$$

$$= \bar{\mathrm{A}}_{ij} \bar{\boldsymbol{g}}^i \otimes \bar{\boldsymbol{g}}^j = \bar{\mathrm{A}}^{ij} \bar{\boldsymbol{g}}_i \otimes \bar{\boldsymbol{g}}_j = \bar{\mathrm{A}}^{i}_{\cdot j} \bar{\boldsymbol{g}}_i \otimes \bar{\boldsymbol{g}}^j. \tag{2.36}$$

Then, by means of (1.28), (1.88), (2.32) and (2.33) we obtain

$$\bar{x}_i = \boldsymbol{x} \cdot \bar{\boldsymbol{g}}_i = \boldsymbol{x} \cdot \left(\boldsymbol{g}_j \frac{\partial \theta^j}{\partial \bar{\theta}^i} \right) = x_j \frac{\partial \theta^j}{\partial \bar{\theta}^i}, \tag{2.37}$$

$$\bar{x}^i = \boldsymbol{x} \cdot \bar{\boldsymbol{g}}^i = \boldsymbol{x} \cdot \left(\boldsymbol{g}^j \frac{\partial \bar{\theta}^i}{\partial \theta^j} \right) = x^j \frac{\partial \bar{\theta}^i}{\partial \theta^j}, \tag{2.38}$$

$$\bar{A}_{ij} = \bar{\boldsymbol{g}}_i \mathbf{A} \bar{\boldsymbol{g}}_j = \left(\boldsymbol{g}_k \frac{\partial \theta^k}{\partial \bar{\theta}^i} \right) \mathbf{A} \left(\boldsymbol{g}_l \frac{\partial \theta^l}{\partial \bar{\theta}^j} \right) = \frac{\partial \theta^k}{\partial \bar{\theta}^i} \frac{\partial \theta^l}{\partial \bar{\theta}^j} A_{kl}, \tag{2.39}$$

$$\bar{A}^{ij} = \bar{\boldsymbol{g}}^i \mathbf{A} \bar{\boldsymbol{g}}^j = \left(\boldsymbol{g}^k \frac{\partial \bar{\theta}^i}{\partial \theta^k} \right) \mathbf{A} \left(\boldsymbol{g}^l \frac{\partial \bar{\theta}^j}{\partial \theta^l} \right) = \frac{\partial \bar{\theta}^i}{\partial \theta^k} \frac{\partial \bar{\theta}^j}{\partial \theta^l} A^{kl}, \tag{2.40}$$

$$\bar{A}^{i}_{\cdot j} = \bar{\boldsymbol{g}}^i \mathbf{A} \bar{\boldsymbol{g}}_j = \left(\boldsymbol{g}^k \frac{\partial \bar{\theta}^i}{\partial \theta^k} \right) \mathbf{A} \left(\boldsymbol{g}_l \frac{\partial \theta^l}{\partial \bar{\theta}^j} \right) = \frac{\partial \bar{\theta}^i}{\partial \theta^k} \frac{\partial \theta^l}{\partial \bar{\theta}^j} A^{k}_{\cdot l}. \tag{2.41}$$

Accordingly, the vector and tensor components x_i, A_{ij} and x^i, A^{ij} are called covariant and contravariant, respectively. The tensor components $A^{i}_{\cdot j}$ are referred to as mixed variant. The transformation rules (2.37)–(2.41) can similarly be written for tensors of higher orders as well. For example, one obtains for third-order tensors

$$\bar{\mathsf{A}}_{ijk} = \frac{\partial \theta^r}{\partial \bar{\theta}^i} \frac{\partial \theta^s}{\partial \bar{\theta}^j} \frac{\partial \theta^t}{\partial \bar{\theta}^k} \mathsf{A}_{rst}, \quad \bar{\mathsf{A}}^{ijk} = \frac{\partial \bar{\theta}^i}{\partial \theta^r} \frac{\partial \bar{\theta}^j}{\partial \theta^s} \frac{\partial \bar{\theta}^k}{\partial \theta^t} \mathsf{A}^{rst}, \ldots \tag{2.42}$$

From the very beginning we have supplied coordinates with upper indices which imply the contravariant transformation rule. Indeed, let us consider the transformation of a coordinate system $\bar{\theta}^i = \bar{\theta}^i \left(\theta^1, \theta^2, \ldots, \theta^n \right)$ $(i = 1, 2, \ldots, n)$. It holds:

$$d\bar{\theta}^i = \frac{\partial \bar{\theta}^i}{\partial \theta^k} d\theta^k, \quad i = 1, 2, \ldots, n. \tag{2.43}$$

Thus, the differentials of the coordinates really transform according to the contravariant law (2.33).

Example 2.3. Transformation of linear coordinates into cylindrical ones (2.17). Let $x^i = x^i(\boldsymbol{r})$ be linear coordinates with respect to an orthonormal basis $\boldsymbol{e}_i$ $(i = 1, 2, 3)$ in $\mathbb{E}^3$:

$$x^i = \boldsymbol{r} \cdot \boldsymbol{e}_i \quad \Leftrightarrow \quad \boldsymbol{r} = x^i \boldsymbol{e}_i. \tag{2.44}$$

By means of (2.17) one can write

$$x^1 = r \cos\varphi, \quad x^2 = r \sin\varphi, \quad x^3 = z \tag{2.45}$$

and consequently

$$\begin{aligned}
&\frac{\partial x^1}{\partial \varphi} = -r\sin\varphi = -x^2, && \frac{\partial x^1}{\partial z} = 0, && \frac{\partial x^1}{\partial r} = \cos\varphi = \frac{x^1}{r}, \\
&\frac{\partial x^2}{\partial \varphi} = r\cos\varphi = x^1, && \frac{\partial x^2}{\partial z} = 0, && \frac{\partial x^2}{\partial r} = \sin\varphi = \frac{x^2}{r}, \\
&\frac{\partial x^3}{\partial \varphi} = 0, && \frac{\partial x^3}{\partial z} = 1, && \frac{\partial x^3}{\partial r} = 0.
\end{aligned} \tag{2.46}$$

The reciprocal derivatives can easily be obtained from (2.23) by inverting the matrix $\left[\frac{\partial x^i}{\partial \varphi} \; \frac{\partial x^i}{\partial z} \; \frac{\partial x^i}{\partial r}\right]$. This yields:

$$\begin{aligned}
&\frac{\partial \varphi}{\partial x^1} = -\frac{1}{r}\sin\varphi = -\frac{x^2}{r^2}, && \frac{\partial \varphi}{\partial x^2} = \frac{1}{r}\cos\varphi = \frac{x^1}{r^2}, && \frac{\partial \varphi}{\partial x^3} = 0,\\
&\frac{\partial z}{\partial x^1} = 0, && \frac{\partial z}{\partial x^2} = 0, && \frac{\partial z}{\partial x^3} = 1,\\
&\frac{\partial r}{\partial x^1} = \cos\varphi = \frac{x^1}{r}, && \frac{\partial r}{\partial x^2} = \sin\varphi = \frac{x^2}{r}, && \frac{\partial r}{\partial x^3} = 0.
\end{aligned} \tag{2.47}$$

2.4 Gradient, Covariant and Contravariant Derivatives

Let $\Phi = \Phi\left(\theta^1, \theta^2, \ldots, \theta^n\right)$, $\boldsymbol{x} = \boldsymbol{x}\left(\theta^1, \theta^2, \ldots, \theta^n\right)$ and $\mathbf{A} = \mathbf{A}\left(\theta^1, \theta^2, \ldots, \theta^n\right)$ be, respectively, a scalar-, a vector- and a tensor-valued differentiable function of the coordinates $\theta^i \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$. Such functions of coordinates are generally referred to as fields, as for example, the scalar field, the vector field or the tensor field. Due to the one to one correspondence (2.26) these fields can alternatively be represented by

$$\Phi = \Phi(\boldsymbol{r}), \quad \boldsymbol{x} = \boldsymbol{x}(\boldsymbol{r}), \quad \mathbf{A} = \mathbf{A}(\boldsymbol{r}). \tag{2.48}$$

In the following we assume that the so-called directional derivatives of the functions (2.48)

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}s}\Phi(\boldsymbol{r} + s\boldsymbol{a})\right|_{s=0} &= \lim_{s\to 0}\frac{\Phi(\boldsymbol{r} + s\boldsymbol{a}) - \Phi(\boldsymbol{r})}{s},\\
\left.\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{x}(\boldsymbol{r} + s\boldsymbol{a})\right|_{s=0} &= \lim_{s\to 0}\frac{\boldsymbol{x}(\boldsymbol{r} + s\boldsymbol{a}) - \boldsymbol{x}(\boldsymbol{r})}{s},\\
\left.\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{A}(\boldsymbol{r} + s\boldsymbol{a})\right|_{s=0} &= \lim_{s\to 0}\frac{\mathbf{A}(\boldsymbol{r} + s\boldsymbol{a}) - \mathbf{A}(\boldsymbol{r})}{s}
\end{aligned} \tag{2.49}$$

exist for all $\boldsymbol{a} \in \mathbb{E}^n$. Further, one can show that the mappings $\boldsymbol{a} \to \left.\frac{\mathrm{d}}{\mathrm{d}s}\Phi(\boldsymbol{r} + s\boldsymbol{a})\right|_{s=0}$, $\boldsymbol{a} \to \left.\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{x}(\boldsymbol{r} + s\boldsymbol{a})\right|_{s=0}$ and $\boldsymbol{a} \to \left.\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{A}(\boldsymbol{r} + s\boldsymbol{a})\right|_{s=0}$ are linear with respect to the vector $\boldsymbol{a}$. For example, we can write for the directional derivative of the scalar function $\Phi = \Phi(\boldsymbol{r})$

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\Phi[\boldsymbol{r} + s(\boldsymbol{a} + \boldsymbol{b})]\right|_{s=0} = \left.\frac{\mathrm{d}}{\mathrm{d}s}\Phi[\boldsymbol{r} + s_1\boldsymbol{a} + s_2\boldsymbol{b}]\right|_{s=0}, \tag{2.50}$$

where s_1 and s_2 are assumed to be functions of s such that $s_1 = s$ and $s_2 = s$. With the aid of the chain rule this delivers

$$\begin{aligned}
&\left.\frac{\mathrm{d}}{\mathrm{d}s}\Phi\left[\boldsymbol{r} + s_1\boldsymbol{a} + s_2\boldsymbol{b}\right]\right|_{s=0} \\
&\quad = \left.\left\{\frac{\partial}{\partial s_1}\Phi\left[\boldsymbol{r} + s_1\boldsymbol{a} + s_2\boldsymbol{b}\right]\frac{\mathrm{d}s_1}{\mathrm{d}s} + \frac{\partial}{\partial s_2}\Phi\left[\boldsymbol{r} + s_1\boldsymbol{a} + s_2\boldsymbol{b}\right]\frac{\mathrm{d}s_2}{\mathrm{d}s}\right\}\right|_{s=0} \\
&\quad = \left.\frac{\partial}{\partial s_1}\Phi\left(\boldsymbol{r} + s_1\boldsymbol{a} + s_2\boldsymbol{b}\right)\right|_{s_1=0,s_2=0} + \left.\frac{\partial}{\partial s_2}\Phi\left(\boldsymbol{r} + s_1\boldsymbol{a} + s_2\boldsymbol{b}\right)\right|_{s_1=0,s_2=0} \\
&\quad = \left.\frac{\mathrm{d}}{\mathrm{d}s}\Phi\left(\boldsymbol{r} + s\boldsymbol{a}\right)\right|_{s=0} + \left.\frac{\mathrm{d}}{\mathrm{d}s}\Phi\left(\boldsymbol{r} + s\boldsymbol{b}\right)\right|_{s=0}
\end{aligned}$$

and finally

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\Phi\left[\boldsymbol{r} + s\left(\boldsymbol{a} + \boldsymbol{b}\right)\right]\right|_{s=0} = \left.\frac{\mathrm{d}}{\mathrm{d}s}\Phi\left(\boldsymbol{r} + s\boldsymbol{a}\right)\right|_{s=0} + \left.\frac{\mathrm{d}}{\mathrm{d}s}\Phi\left(\boldsymbol{r} + s\boldsymbol{b}\right)\right|_{s=0} \tag{2.51}$$

for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{E}^n$. In a similar fashion we can write

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}s}\Phi\left(\boldsymbol{r} + s\alpha\boldsymbol{a}\right)\right|_{s=0} &= \left.\frac{\mathrm{d}}{\mathrm{d}\left(\alpha s\right)}\Phi\left(\boldsymbol{r} + s\alpha\boldsymbol{a}\right)\frac{\mathrm{d}\left(\alpha s\right)}{\mathrm{d}s}\right|_{s=0} \\
&= \alpha\left.\frac{\mathrm{d}}{\mathrm{d}s}\Phi\left(\boldsymbol{r} + s\boldsymbol{a}\right)\right|_{s=0}, \quad \forall\boldsymbol{a} \in \mathbb{E}^n, \ \forall\alpha \in \mathbb{R}.
\end{aligned} \tag{2.52}$$

Representing $\boldsymbol{a}$ with respect to a basis as $\boldsymbol{a} = a^i\boldsymbol{g}_i$ we thus obtain

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}s}\Phi\left(\boldsymbol{r} + s\boldsymbol{a}\right)\right|_{s=0} &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\Phi\left(\boldsymbol{r} + sa^i\boldsymbol{g}_i\right)\right|_{s=0} = a^i\left.\frac{\mathrm{d}}{\mathrm{d}s}\Phi\left(\boldsymbol{r} + s\boldsymbol{g}_i\right)\right|_{s=0} \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}s}\Phi\left(\boldsymbol{r} + s\boldsymbol{g}_i\right)\right|_{s=0}\boldsymbol{g}^i \cdot \left(a^j\boldsymbol{g}_j\right),
\end{aligned} \tag{2.53}$$

where $\boldsymbol{g}^i$ form the basis dual to $\boldsymbol{g}_i$ $(i = 1, 2, \ldots, n)$. This result can finally be expressed by

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\Phi\left(\boldsymbol{r} + s\boldsymbol{a}\right)\right|_{s=0} = \mathrm{grad}\Phi \cdot \boldsymbol{a}, \quad \forall\boldsymbol{a} \in \mathbb{E}^n, \tag{2.54}$$

where the vector denoted by $\mathrm{grad}\Phi \in \mathbb{E}^n$ is referred to as gradient of the function $\Phi = \Phi\left(\boldsymbol{r}\right)$. According to (2.53) and (2.54) it can be represented by

$$\mathrm{grad}\Phi = \left.\frac{\mathrm{d}}{\mathrm{d}s}\Phi\left(\boldsymbol{r} + s\boldsymbol{g}_i\right)\right|_{s=0}\boldsymbol{g}^i. \tag{2.55}$$

Example 2.4. Gradient of the scalar function $\|\boldsymbol{r}\|$. Using the definition of the directional derivative (2.49) we can write

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\|\boldsymbol{r}+s\boldsymbol{a}\|\Big|_{s=0} &= \frac{\mathrm{d}}{\mathrm{d}s}\sqrt{(\boldsymbol{r}+s\boldsymbol{a})\cdot(\boldsymbol{r}+s\boldsymbol{a})}\Big|_{s=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\sqrt{\boldsymbol{r}\cdot\boldsymbol{r}+2s\,(\boldsymbol{r}\cdot\boldsymbol{a})+s^2\,(\boldsymbol{a}\cdot\boldsymbol{a})}\Big|_{s=0} \\ &= \frac{1}{2}\frac{2\,(\boldsymbol{r}\cdot\boldsymbol{a})+2s\,(\boldsymbol{a}\cdot\boldsymbol{a})}{\sqrt{\boldsymbol{r}\cdot\boldsymbol{r}+2s\,(\boldsymbol{r}\cdot\boldsymbol{a})+s^2\,(\boldsymbol{a}\cdot\boldsymbol{a})}}\Bigg|_{s=0} = \frac{\boldsymbol{r}\cdot\boldsymbol{a}}{\|\boldsymbol{r}\|}.\end{aligned}$$

Comparing this result with (2.54) delivers

$$\operatorname{grad}\|\boldsymbol{r}\| = \frac{\boldsymbol{r}}{\|\boldsymbol{r}\|}. \tag{2.56}$$

Similarly to (2.54) one defines the gradient of the vector function $\boldsymbol{x}=\boldsymbol{x}(\boldsymbol{r})$ and the gradient of the tensor function $\mathbf{A}=\mathbf{A}(\boldsymbol{r})$:

$$\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{x}(\boldsymbol{r}+s\boldsymbol{a})\Big|_{s=0} = (\operatorname{grad}\boldsymbol{x})\,\boldsymbol{a}, \quad \forall\boldsymbol{a}\in\mathbb{E}^n, \tag{2.57}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{A}(\boldsymbol{r}+s\boldsymbol{a})\Big|_{s=0} = (\operatorname{grad}\mathbf{A})\,\boldsymbol{a}, \quad \forall\boldsymbol{a}\in\mathbb{E}^n. \tag{2.58}$$

Herein, $\operatorname{grad}\boldsymbol{x}$ and $\operatorname{grad}\mathbf{A}$ represent tensors of second and third order, respectively.

In order to evaluate the above gradients (2.54), (2.57) and (2.58) we represent the vectors $\boldsymbol{r}$ and $\boldsymbol{a}$ with respect to the linear coordinates (2.19) as

$$\boldsymbol{r}=x^i\boldsymbol{h}_i, \quad \boldsymbol{a}=a^i\boldsymbol{h}_i. \tag{2.59}$$

With the aid of the chain rule we can further write for the directional derivative of the function $\Phi=\Phi(\boldsymbol{r})$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\Phi(\boldsymbol{r}+s\boldsymbol{a})\Big|_{s=0} &= \frac{\mathrm{d}}{\mathrm{d}s}\Phi\left[\left(x^i+sa^i\right)\boldsymbol{h}_i\right]\Big|_{s=0} \\ &= \frac{\partial\Phi}{\partial\left(x^i+sa^i\right)}\frac{\mathrm{d}\left(x^i+sa^i\right)}{\mathrm{d}s}\Bigg|_{s=0} = \frac{\partial\Phi}{\partial x^i}a^i \\ &= \left(\frac{\partial\Phi}{\partial x^i}\boldsymbol{h}^i\right)\cdot\left(a^j\boldsymbol{h}_j\right) = \left(\frac{\partial\Phi}{\partial x^i}\boldsymbol{h}^i\right)\cdot\boldsymbol{a}, \quad \forall\boldsymbol{a}\in\mathbb{E}^n.\end{aligned}$$

Comparing this result with (2.54) and bearing in mind that it holds for all vectors $\boldsymbol{a}$ we obtain

$$\operatorname{grad}\Phi = \frac{\partial \Phi}{\partial x^i}\boldsymbol{h}^i. \tag{2.60}$$

The representation (2.60) can be rewritten in terms of arbitrary curvilinear coordinates $\boldsymbol{r} = \boldsymbol{r}\left(\theta^1, \theta^2, \ldots, \theta^n\right)$ and the corresponding tangent vectors (2.28). Indeed, in view of (2.33) and (2.60)

$$\operatorname{grad}\Phi = \frac{\partial \Phi}{\partial x^i}\boldsymbol{h}^i = \frac{\partial \Phi}{\partial \theta^k}\frac{\partial \theta^k}{\partial x^i}\boldsymbol{h}^i = \frac{\partial \Phi}{\partial \theta^i}\boldsymbol{g}^i. \tag{2.61}$$

Comparison of the last result with (2.55) yields

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\Phi\left(\boldsymbol{r} + s\boldsymbol{g}_i\right)\right|_{s=0} = \frac{\partial \Phi}{\partial \theta^i}, \quad i = 1, 2, \ldots, n. \tag{2.62}$$

According to the definition (2.54) the gradient is independent of the choice of the coordinate system. This can also be seen from relation (2.61). Indeed, taking (2.33) into account we can write for an arbitrary coordinate system $\bar{\theta}^i = \bar{\theta}^i\left(\theta^1, \theta^2, \ldots, \theta^n\right)$ $(i = 1, 2, \ldots, n)$:

$$\operatorname{grad}\Phi = \frac{\partial \Phi}{\partial \theta^i}\boldsymbol{g}^i = \frac{\partial \Phi}{\partial \bar{\theta}^j}\frac{\partial \bar{\theta}^j}{\partial \theta^i}\boldsymbol{g}^i = \frac{\partial \Phi}{\partial \bar{\theta}^j}\bar{\boldsymbol{g}}^j. \tag{2.63}$$

Similarly to relation (2.61) one can express the gradients of the vector-valued function $\boldsymbol{x} = \boldsymbol{x}\left(\boldsymbol{r}\right)$ and the tensor-valued function $\mathbf{A} = \mathbf{A}\left(\boldsymbol{r}\right)$ by

$$\operatorname{grad}\boldsymbol{x} = \frac{\partial \boldsymbol{x}}{\partial \theta^i} \otimes \boldsymbol{g}^i, \quad \operatorname{grad}\mathbf{A} = \frac{\partial \mathbf{A}}{\partial \theta^i} \otimes \boldsymbol{g}^i. \tag{2.64}$$

Example 2.5. Deformation gradient and its representation in the case of simple shear. Let $\boldsymbol{x}$ and $\boldsymbol{X}$ be the position vectors of a material point in the current and reference configuration, respectively. The deformation gradient $\mathbf{F} \in \mathbf{Lin}^3$ is defined as the gradient of the function $\boldsymbol{x}\left(\boldsymbol{X}\right)$ as

$$\mathbf{F} = \operatorname{grad}\boldsymbol{x}. \tag{2.65}$$

For the Cartesian coordinates in $\mathbb{E}^3$ where $\boldsymbol{x} = x^i\boldsymbol{e}_i$ and $\boldsymbol{X} = X^i\boldsymbol{e}_i$ we can write by using $(2.64)_1$

$$\mathbf{F} = \frac{\partial \boldsymbol{x}}{\partial X^j} \otimes \boldsymbol{e}^j = \frac{\partial x^i}{\partial X^j}\boldsymbol{e}_i \otimes \boldsymbol{e}^j = \mathrm{F}^i_{\cdot j}\boldsymbol{e}_i \otimes \boldsymbol{e}^j, \tag{2.66}$$

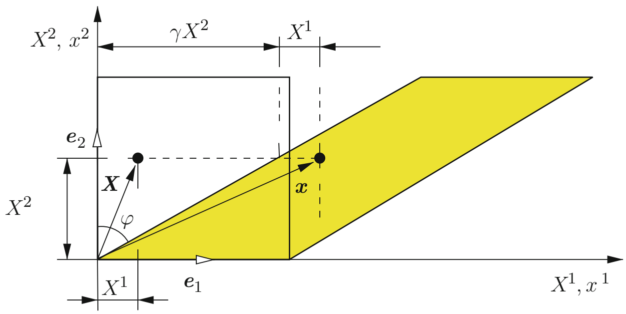

Fig. 2.3 Simple shear of a rectangular sheet

where the matrix $\left[F^i_{\cdot j}\right]$ is given by

$$\left[F^i_{\cdot j}\right] = \begin{bmatrix} \dfrac{\partial x^1}{\partial X^1} & \dfrac{\partial x^1}{\partial X^2} & \dfrac{\partial x^1}{\partial X^3} \\ \dfrac{\partial x^2}{\partial X^1} & \dfrac{\partial x^2}{\partial X^2} & \dfrac{\partial x^2}{\partial X^3} \\ \dfrac{\partial x^3}{\partial X^1} & \dfrac{\partial x^3}{\partial X^2} & \dfrac{\partial x^3}{\partial X^3} \end{bmatrix}. \tag{2.67}$$

In the case of simple shear it holds (see Fig. 2.3)

$$x^1 = X^1 + \gamma X^2, \quad x^2 = X^2, \quad x^3 = X^3, \tag{2.68}$$

where γ denotes the amount of shear. Insertion into (2.67) yields

$$\left[F^i_{\cdot j}\right] = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.69}$$

Henceforth, the derivatives of the functions $\Phi = \Phi\left(\theta^1, \theta^2, \ldots, \theta^n\right)$, $\boldsymbol{x} = \boldsymbol{x}\left(\theta^1, \theta^2, \ldots, \theta^n\right)$ and $\mathbf{A} = \mathbf{A}\left(\theta^1, \theta^2, \ldots, \theta^n\right)$ with respect to curvilinear coordinates θ^i will be denoted shortly by

$$\Phi_{,i} = \frac{\partial \Phi}{\partial \theta^i}, \quad \boldsymbol{x}_{,i} = \frac{\partial \boldsymbol{x}}{\partial \theta^i}, \quad \mathbf{A}_{,i} = \frac{\partial \mathbf{A}}{\partial \theta^i}. \tag{2.70}$$

They obey the covariant transformation rule (2.32) with respect to the index i since

$$\frac{\partial \Phi}{\partial \theta^i} = \frac{\partial \Phi}{\partial \bar{\theta}^k} \frac{\partial \bar{\theta}^k}{\partial \theta^i}, \quad \frac{\partial \boldsymbol{x}}{\partial \theta^i} = \frac{\partial \boldsymbol{x}}{\partial \bar{\theta}^k} \frac{\partial \bar{\theta}^k}{\partial \theta^i}, \quad \frac{\partial \mathbf{A}}{\partial \theta^i} = \frac{\partial \mathbf{A}}{\partial \bar{\theta}^k} \frac{\partial \bar{\theta}^k}{\partial \theta^i} \tag{2.71}$$

and represent again a scalar, a vector and a second-order tensor, respectively. The latter ones can be represented with respect to a basis as

$$\boldsymbol{x}_{,i} = x^j|_i \, \boldsymbol{g}_j = x_j|_i \, \boldsymbol{g}^j ,$$

$$\mathbf{A}_{,i} = \mathrm{A}^{kl}|_i \, \boldsymbol{g}_k \otimes \boldsymbol{g}_l = \mathrm{A}_{kl}|_i \, \boldsymbol{g}^k \otimes \boldsymbol{g}^l = \mathrm{A}^k_{\cdot l}|_i \, \boldsymbol{g}_k \otimes \boldsymbol{g}^l , \tag{2.72}$$

where $(\bullet)|_i$ denotes some differential operator on the components of the vector $\boldsymbol{x}$ or the tensor $\mathbf{A}$. In view of (2.71) and (2.72) this operator transforms with respect to the index i according to the covariant rule and is called covariant derivative. The covariant type of the derivative is accentuated by the lower position of the coordinate index.

On the basis of the covariant derivative we can also define the contravariant one. To this end, we formally apply the rule of component transformation $(1.95)_1$ as $(\bullet)|^i = g^{ij} \, (\bullet)|_j$. Accordingly,

$$x^j|^i = g^{ik} x^j|_k , \qquad x_j|^i = g^{ik} x_j|_k ,$$

$$\mathrm{A}^{kl}|^i = g^{im} \mathrm{A}^{kl}|_m , \qquad \mathrm{A}_{kl}|^i = g^{im} \mathrm{A}_{kl}|_m , \qquad \mathrm{A}^k_{\cdot l}|^i = g^{im} \mathrm{A}^k_{\cdot l}|_m . \tag{2.73}$$

For scalar functions the covariant and the contravariant derivative are defined to be equal to the partial one so that:

$$\Phi|_i = \Phi|^i = \Phi_{,i} . \tag{2.74}$$

In view of (2.63)–(2.70), (2.72) and (2.74) the gradients of the functions $\Phi = \Phi\left(\theta^1, \theta^2, \ldots, \theta^n\right)$, $\boldsymbol{x} = \boldsymbol{x}\left(\theta^1, \theta^2, \ldots, \theta^n\right)$ and $\mathbf{A} = \mathbf{A}\left(\theta^1, \theta^2, \ldots, \theta^n\right)$ take the form

$$\mathrm{grad}\Phi = \Phi|_i \, \boldsymbol{g}^i = \Phi|^i \, \boldsymbol{g}_i ,$$

$$\mathrm{grad}\boldsymbol{x} = x^j|_i \, \boldsymbol{g}_j \otimes \boldsymbol{g}^i = x_j|_i \, \boldsymbol{g}^j \otimes \boldsymbol{g}^i = x^j|^i \, \boldsymbol{g}_j \otimes \boldsymbol{g}_i = x_j|^i \, \boldsymbol{g}^j \otimes \boldsymbol{g}_i ,$$

$$\begin{aligned} \mathrm{grad}\mathbf{A} &= \mathrm{A}^{kl}|_i \, \boldsymbol{g}_k \otimes \boldsymbol{g}_l \otimes \boldsymbol{g}^i = \mathrm{A}_{kl}|_i \, \boldsymbol{g}^k \otimes \boldsymbol{g}^l \otimes \boldsymbol{g}^i = \mathrm{A}^k_{\cdot l}|_i \, \boldsymbol{g}_k \otimes \boldsymbol{g}^l \otimes \boldsymbol{g}^i \\ &= \mathrm{A}^{kl}|^i \, \boldsymbol{g}_k \otimes \boldsymbol{g}_l \otimes \boldsymbol{g}_i = \mathrm{A}_{kl}|^i \, \boldsymbol{g}^k \otimes \boldsymbol{g}^l \otimes \boldsymbol{g}_i = \mathrm{A}^k_{\cdot l}|^i \, \boldsymbol{g}_k \otimes \boldsymbol{g}^l \otimes \boldsymbol{g}_i . \end{aligned} \tag{2.75}$$

2.5 Christoffel Symbols, Representation of the Covariant Derivative

In the previous section we have introduced the notion of the covariant derivative but have not so far discussed how it can be taken. Now, we are going to formulate a procedure constructing the differential operator of the covariant derivative. In other words, we would like to express the covariant derivative in terms of the vector or tensor components. To this end, the partial derivatives of the tangent vectors (2.28) with respect to the coordinates are first needed. Since these derivatives again represent vectors in $\mathbb{E}^n$, they can be expressed in terms of the tangent vectors $\boldsymbol{g}_i$ or dual vectors $\boldsymbol{g}^i$ $(i = 1, 2, \ldots, n)$ both forming bases of $\mathbb{E}^n$. Thus, one can write

$$\boldsymbol{g}_i,_j = \Gamma_{ijk}\boldsymbol{g}^k = \Gamma^k_{ij}\boldsymbol{g}_k, \quad i, j = 1, 2, \ldots, n, \tag{2.76}$$

where the components Γ_{ijk} and Γ^k_{ij} $(i, j, k = 1, 2, \ldots, n)$ are referred to as the Christoffel symbols of the first and second kind, respectively. In view of the relation $\boldsymbol{g}^k = g^{kl}\boldsymbol{g}_l$ $(k = 1, 2, \ldots, n)$ (1.21) these symbols are connected with each other by

$$\Gamma^k_{ij} = g^{kl}\Gamma_{ijl}, \quad i, j, k = 1, 2, \ldots, n. \tag{2.77}$$

Keeping the definition of tangent vectors (2.28) in mind we further obtain

$$\boldsymbol{g}_i,_j = \boldsymbol{r},_{ij} = \boldsymbol{r},_{ji} = \boldsymbol{g}_j,_i, \quad i, j = 1, 2, \ldots, n. \tag{2.78}$$

With the aid of (1.28) the Christoffel symbols can thus be expressed by

$$\Gamma_{ijk} = \Gamma_{jik} = \boldsymbol{g}_i,_j \cdot \boldsymbol{g}_k = \boldsymbol{g}_j,_i \cdot \boldsymbol{g}_k, \tag{2.79}$$

$$\Gamma^k_{ij} = \Gamma^k_{ji} = \boldsymbol{g}_i,_j \cdot \boldsymbol{g}^k = \boldsymbol{g}_j,_i \cdot \boldsymbol{g}^k, \quad i, j, k = 1, 2, \ldots, n. \tag{2.80}$$

For the dual basis $\boldsymbol{g}^i$ $(i = 1, 2, \ldots, n)$ one further gets by differentiating the identities $\boldsymbol{g}^i \cdot \boldsymbol{g}_j = \delta^i_j$ (1.15):

$$\begin{aligned} 0 = \left(\delta^i_j\right),_k &= \left(\boldsymbol{g}^i \cdot \boldsymbol{g}_j\right),_k = \boldsymbol{g}^i,_k \cdot \boldsymbol{g}_j + \boldsymbol{g}^i \cdot \boldsymbol{g}_j,_k \\ &= \boldsymbol{g}^i,_k \cdot \boldsymbol{g}_j + \boldsymbol{g}^i \cdot \left(\Gamma^l_{jk}\boldsymbol{g}_l\right) = \boldsymbol{g}^i,_k \cdot \boldsymbol{g}_j + \Gamma^i_{jk}, \quad i, j, k = 1, 2, \ldots, n. \end{aligned}$$

Hence,

$$\Gamma^i_{jk} = \Gamma^i_{kj} = -\boldsymbol{g}^i,_k \cdot \boldsymbol{g}_j = -\boldsymbol{g}^i,_j \cdot \boldsymbol{g}_k, \quad i, j, k = 1, 2, \ldots, n \tag{2.81}$$

and consequently

$$\boldsymbol{g}^i,_k = -\Gamma^i_{jk}\boldsymbol{g}^j = -\Gamma^i_{kj}\boldsymbol{g}^j, \quad i, k = 1, 2, \ldots, n. \tag{2.82}$$

By means of the identities following from (2.79)

$$g_{ij},_k = \left(\boldsymbol{g}_i \cdot \boldsymbol{g}_j\right),_k = \boldsymbol{g}_i,_k \cdot \boldsymbol{g}_j + \boldsymbol{g}_i \cdot \boldsymbol{g}_j,_k = \Gamma_{ikj} + \Gamma_{jki}, \tag{2.83}$$

where $i, j, k = 1, 2, \ldots, n$ and in view of (2.77) we finally obtain

$$\Gamma_{ijk} = \frac{1}{2} \left(g_{ki},_j + g_{kj},_i - g_{ij},_k \right), \tag{2.84}$$

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(g_{li},_j + g_{lj},_i - g_{ij},_l \right), \quad i, j, k = 1, 2, \ldots, n. \tag{2.85}$$

It is seen from (2.84) and (2.85) that all Christoffel symbols identically vanish in the Cartesian coordinates (2.20). Indeed, in this case

$$g_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}, \quad i, j = 1, 2, \ldots, n \tag{2.86}$$

and hence

$$\Gamma_{ijk} = \Gamma_{ij}^k = 0, \quad i, j, k = 1, 2, \ldots, n. \tag{2.87}$$

Example 2.6. Christoffel symbols for cylindrical coordinates in $\mathbb{E}^3$ (2.17). By virtue of relation $(2.30)_1$ we realize that $g_{11},_3 = 2r$, while all other derivatives $g_{ik},_j$ $(i, j, k = 1, 2, 3)$ (2.83) are zero. Thus, Eq. (2.84) delivers

$$\Gamma_{131} = \Gamma_{311} = r, \quad \Gamma_{113} = -r, \tag{2.88}$$

while all other Christoffel symbols of the first kind Γ_{ijk} $(i, j, k = 1, 2, 3)$ are likewise zero. With the aid of (2.77) and $(2.30)_2$ we further obtain

$$\Gamma_{ij}^1 = g^{11}\Gamma_{ij1} = r^{-2}\Gamma_{ij1}, \quad \Gamma_{ij}^2 = g^{22}\Gamma_{ij2} = \Gamma_{ij2},$$

$$\Gamma_{ij}^3 = g^{33}\Gamma_{ij3} = \Gamma_{ij3}, \quad i, j = 1, 2, 3. \tag{2.89}$$

By virtue of (2.88) we can further write

$$\Gamma_{13}^1 = \Gamma_{31}^1 = \frac{1}{r}, \quad \Gamma_{11}^3 = -r, \tag{2.90}$$

while all remaining Christoffel symbols of the second kind Γ_{ij}^k $(i, j, k = 1, 2, 3)$ (2.85) vanish.

Now, we are in a position to express the covariant derivative in terms of the vector or tensor components by means of the Christoffel symbols. For the vector-valued function $\boldsymbol{x} = \boldsymbol{x}\left(\theta^1, \theta^2, \ldots, \theta^n\right)$ we can write using (2.76)

$$\begin{aligned} \boldsymbol{x},_j &= \left(x^i \boldsymbol{g}_i\right),_j = x^i,_j \boldsymbol{g}_i + x^i \boldsymbol{g}_i,_j \\ &= x^i,_j \boldsymbol{g}_i + x^i \Gamma_{ij}^k \boldsymbol{g}_k = \left(x^i,_j + x^k \Gamma_{kj}^i\right) \boldsymbol{g}_i, \end{aligned} \tag{2.91}$$

or alternatively using (2.82)

$$\boldsymbol{x},_j = \left(x_i \boldsymbol{g}^i\right),_j = x_i,_j \boldsymbol{g}^i + x_i \boldsymbol{g}^i,_j$$

$$= x_i,_j \boldsymbol{g}^i - x_i \Gamma^i_{kj} \boldsymbol{g}^k = \left(x_i,_j - x_k \Gamma^k_{ij}\right) \boldsymbol{g}^i. \tag{2.92}$$

Comparing these results with (2.72) yields

$$x^i|_j = x^i,_j + x^k \Gamma^i_{kj}, \quad x_i|_j = x_i,_j - x_k \Gamma^k_{ij}, \quad i, j = 1, 2, \ldots, n. \tag{2.93}$$

Similarly, we treat the tensor-valued function $\mathbf{A} = \mathbf{A}\left(\theta^1, \theta^2, \ldots, \theta^n\right)$:

$$\mathbf{A},_k = \left(\mathrm{A}^{ij} \boldsymbol{g}_i \otimes \boldsymbol{g}_j\right),_k$$

$$= \mathrm{A}^{ij},_k \boldsymbol{g}_i \otimes \boldsymbol{g}_j + \mathrm{A}^{ij} \boldsymbol{g}_i,_k \otimes \boldsymbol{g}_j + \mathrm{A}^{ij} \boldsymbol{g}_i \otimes \boldsymbol{g}_j,_k$$

$$= \mathrm{A}^{ij},_k \boldsymbol{g}_i \otimes \boldsymbol{g}_j + \mathrm{A}^{ij} \left(\Gamma^l_{ik} \boldsymbol{g}_l\right) \otimes \boldsymbol{g}_j + \mathrm{A}^{ij} \boldsymbol{g}_i \otimes \left(\Gamma^l_{jk} \boldsymbol{g}_l\right)$$

$$= \left(\mathrm{A}^{ij},_k + \mathrm{A}^{lj} \Gamma^i_{lk} + \mathrm{A}^{il} \Gamma^j_{lk}\right) \boldsymbol{g}_i \otimes \boldsymbol{g}_j. \tag{2.94}$$

Thus,

$$\mathrm{A}^{ij}|_k = \mathrm{A}^{ij},_k + \mathrm{A}^{lj} \Gamma^i_{lk} + \mathrm{A}^{il} \Gamma^j_{lk}, \quad i, j, k = 1, 2, \ldots, n. \tag{2.95}$$

By analogy, we further obtain

$$\mathrm{A}_{ij}|_k = \mathrm{A}_{ij},_k - \mathrm{A}_{lj} \Gamma^l_{ik} - \mathrm{A}_{il} \Gamma^l_{jk},$$

$$\mathrm{A}^i_{\cdot j}|_k = \mathrm{A}^i_{\cdot j},_k + \mathrm{A}^l_{\cdot j} \Gamma^i_{lk} - \mathrm{A}^i_{\cdot l} \Gamma^l_{jk}, \quad i, j, k = 1, 2, \ldots, n. \tag{2.96}$$

Similar expressions for the covariant derivative can also be formulated for tensors of higher orders.

From (2.87), (2.93), (2.95) and (2.96) it is seen that the covariant derivative taken in Cartesian coordinates (2.20) coincides with the partial derivative:

$$x^i|_j = x^i,_j, \quad x_i|_j = x_i,_j,$$

$$\mathrm{A}^{ij}|_k = \mathrm{A}^{ij},_k, \ \mathrm{A}_{ij}|_k = \mathrm{A}_{ij},_k, \ \mathrm{A}^i_{\cdot j}|_k = \mathrm{A}^i_{\cdot j},_k, \quad i, j, k = 1, 2, \ldots, n. \tag{2.97}$$

Formal application of the covariant derivative (2.93), (2.95) and (2.96) to the tangent vectors (2.28) and metric coefficients $(1.90)_{1,2}$ yields by virtue of (2.76), (2.77), (2.82) and (2.84) the following identities referred to as Ricci's Theorem:

$$\boldsymbol{g}_i|_j = \boldsymbol{g}_i,_j - \boldsymbol{g}_l \Gamma^l_{ij} = \boldsymbol{0}, \quad \boldsymbol{g}^i|_j = \boldsymbol{g}^i,_j + \boldsymbol{g}^l \Gamma^i_{lj} = \boldsymbol{0}, \tag{2.98}$$

$$g_{ij}|_k = g_{ij},_k - g_{lj}\Gamma^l_{ik} - g_{il}\Gamma^l_{jk} = g_{ij},_k - \Gamma_{ikj} - \Gamma_{jki} = 0, \tag{2.99}$$

$$g^{ij}|_k = g^{ij},_k + g^{lj}\Gamma^i_{lk} + g^{il}\Gamma^j_{lk} = g^{il}g^{jm}\left(-g_{lm},_k + \Gamma_{mkl} + \Gamma_{lkm}\right) = 0, \tag{2.100}$$

where $i, j, k = 1, 2, \ldots, n$. The latter two identities can alternatively be proved by taking (1.25) into account and using the product rules of differentiation for the covariant derivative which can be written as (Exercise 2.7)

$$\mathrm{A}_{ij}|_k = a_i|_k\, b_j + a_i b_j|_k \quad \text{for} \quad \mathrm{A}_{ij} = a_i b_j, \tag{2.101}$$

$$\mathrm{A}^{ij}|_k = a^i|_k\, b^j + a^i b^j|_k \quad \text{for} \quad \mathrm{A}^{ij} = a^i b^j, \tag{2.102}$$

$$\mathrm{A}^i_{j}|_k = a^i|_k\, b_j + a^i b_j|_k \quad \text{for} \quad \mathrm{A}^i_{j} = a^i b_j, \quad i, j, k = 1, 2, \ldots, n. \tag{2.103}$$

2.6 Applications in Three-Dimensional Space: Divergence and Curl

Divergence of a tensor field. One defines the divergence of a tensor field $\mathbf{S}(\boldsymbol{r})$ by

$$\mathrm{div}\mathbf{S} = \lim_{V \to 0} \frac{1}{V} \int_A \mathbf{S}\boldsymbol{n}\mathrm{d}A, \tag{2.104}$$

where the integration is carried out over a closed surface area A with the volume V and the outer unit normal vector $\boldsymbol{n}$ illustrated in Fig. 2.4.

For the integration we consider a curvilinear parallelepiped with the edges formed by the coordinate lines $\theta^1, \theta^2, \theta^3$ and $\theta^1 + \Delta\theta^1, \theta^2 + \Delta\theta^2, \theta^3 + \Delta\theta^3$ (Fig. 2.5). The infinitesimal surface elements of the parallelepiped can be defined in a vector form by

$$\mathrm{d}\boldsymbol{A}^{(i)} = \pm\left(d\theta^j \boldsymbol{g}_j\right) \times \left(d\theta^k \boldsymbol{g}_k\right) = \pm g\boldsymbol{g}^i d\theta^j d\theta^k, \quad i = 1, 2, 3, \tag{2.105}$$

where $g = [\boldsymbol{g}_1\boldsymbol{g}_2\boldsymbol{g}_3]$ (1.31) and i, j, k is an even permutation of 1,2,3. The corresponding infinitesimal volume element can thus be given by (no summation over i)

$$\mathrm{d}V = \mathrm{d}\boldsymbol{A}^{(i)} \cdot \left(d\theta^i \boldsymbol{g}_i\right) = \left[d\theta^1 \boldsymbol{g}_1\, d\theta^2 \boldsymbol{g}_2\, d\theta^3 \boldsymbol{g}_3\right]$$
$$= [\boldsymbol{g}_1\boldsymbol{g}_2\boldsymbol{g}_3]\, d\theta^1 d\theta^2 d\theta^3 = g d\theta^1 d\theta^2 d\theta^3. \tag{2.106}$$

We also need the identities

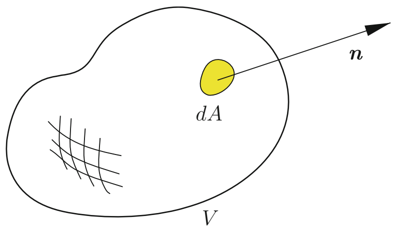

Fig. 2.4 Definition of the divergence: closed surface with the area A, volume V and the outer unit normal vector $\boldsymbol{n}$

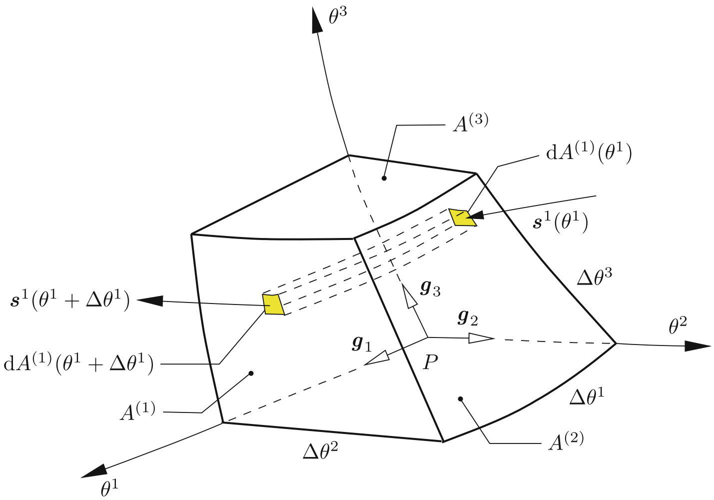

Fig. 2.5 Derivation of the divergence in three-dimensional space

$$
\begin{aligned}
g_{,k} = \left[\boldsymbol{g}_1\boldsymbol{g}_2\boldsymbol{g}_3\right]_{,k} &= \Gamma^l_{1k}\left[\boldsymbol{g}_l\boldsymbol{g}_2\boldsymbol{g}_3\right] + \Gamma^l_{2k}\left[\boldsymbol{g}_1\boldsymbol{g}_l\boldsymbol{g}_3\right] + \Gamma^l_{3k}\left[\boldsymbol{g}_1\boldsymbol{g}_2\boldsymbol{g}_l\right] \\
&= \Gamma^l_{lk}\left[\boldsymbol{g}_1\boldsymbol{g}_2\boldsymbol{g}_3\right] = \Gamma^l_{lk}\, g,
\end{aligned} \tag{2.107}
$$

$$
\left(g\boldsymbol{g}^i\right)_{,i} = g_{,i}\,\boldsymbol{g}^i + g\boldsymbol{g}^i{}_{,i} = \Gamma^l_{li}\, g\boldsymbol{g}^i - \Gamma^i_{li}\, g\boldsymbol{g}^l = \boldsymbol{0}, \tag{2.108}
$$

following from (1.39), (2.76) and (2.82). With these results in hand, one can express the divergence (2.104) as follows

$$
\begin{aligned}
\mathrm{div}\mathbf{S} &= \lim_{V\to 0}\frac{1}{V}\int_A \mathbf{S}\boldsymbol{n}\,\mathrm{d}A \\
&= \lim_{V\to 0}\frac{1}{V}\sum_{i=1}^{3}\int_{A^{(i)}}\left[\mathbf{S}\left(\theta^i + \Delta\theta^i\right)\mathrm{d}\boldsymbol{A}^{(i)}\left(\theta^i + \Delta\theta^i\right) + \mathbf{S}\left(\theta^i\right)\mathrm{d}\boldsymbol{A}^{(i)}\left(\theta^i\right)\right].
\end{aligned}
$$

Keeping (2.105) and (2.106) in mind and using the abbreviation

$$\boldsymbol{s}^i\left(\theta^i\right) = \mathbf{S}\left(\theta^i\right) g\left(\theta^i\right) \boldsymbol{g}^i\left(\theta^i\right), \quad i = 1, 2, 3 \tag{2.109}$$

we can thus write

$$\begin{aligned}
\operatorname{div}\mathbf{S} &= \lim_{V\to 0} \frac{1}{V} \sum_{i=1}^{3} \int_{\theta^k}^{\theta^k+\Delta\theta^k} \int_{\theta^j}^{\theta^j+\Delta\theta^j} \left[\boldsymbol{s}^i\left(\theta^i + \Delta\theta^i\right) - \boldsymbol{s}^i\left(\theta^i\right)\right] \mathrm{d}\theta^j \mathrm{d}\theta^k \\
&= \lim_{V\to 0} \frac{1}{V} \sum_{i=1}^{3} \int_{\theta^k}^{\theta^k+\Delta\theta^k} \int_{\theta^j}^{\theta^j+\Delta\theta^j} \int_{\theta^i}^{\theta^i+\Delta\theta^i} \frac{\partial \boldsymbol{s}^i}{\partial\theta^i} \mathrm{d}\theta^i \mathrm{d}\theta^j \mathrm{d}\theta^k \\
&= \lim_{V\to 0} \frac{1}{V} \sum_{i=1}^{3} \int_V \frac{\boldsymbol{s}^i{}_{,i}}{g} \mathrm{d}V,
\end{aligned} \tag{2.110}$$

where i, j, k is again an even permutation of 1,2,3. Assuming continuity of the integrand in (2.110) and applying (2.108) and (2.109) we obtain

$$\operatorname{div}\mathbf{S} = \frac{1}{g}\boldsymbol{s}^i{}_{,i} = \frac{1}{g}\left[\mathbf{S} g \boldsymbol{g}^i\right]_{,i} = \frac{1}{g}\left[\mathbf{S}_{,i}\, g \boldsymbol{g}^i + \mathbf{S}\left(g\boldsymbol{g}^i\right)_{,i}\right] = \mathbf{S}_{,i}\, \boldsymbol{g}^i, \tag{2.111}$$

which finally yields by virtue of $(2.72)_2$

$$\operatorname{div}\mathbf{S} = \mathbf{S}_{,i}\, \boldsymbol{g}^i = \mathrm{S}_{j\cdot}^{\ i}|_i\, \boldsymbol{g}^j = \mathrm{S}^{ji}|_i\, \boldsymbol{g}_j. \tag{2.112}$$

Example 2.7. The momentum balance in Cartesian and cylindrical coordinates. Let us consider a material body or a part of it with a mass m, volume V and outer surface A. According to the Euler law of motion the vector sum of external volume forces $\boldsymbol{f}\mathrm{d}V$ and surface tractions $\boldsymbol{t}\mathrm{d}A$ results in the vector sum of inertia forces $\ddot{\boldsymbol{x}}\mathrm{d}m$, where $\boldsymbol{x}$ stands for the position vector of a material element $\mathrm{d}m$ and the superposed dot denotes the material time derivative. Hence,

$$\int_m \ddot{\boldsymbol{x}}\mathrm{d}m = \int_A \boldsymbol{t}\mathrm{d}A + \int_V \boldsymbol{f}\mathrm{d}V. \tag{2.113}$$

Applying the Cauchy theorem (1.77) to the first integral on the right hand side and using the identity $\mathrm{d}m = \rho\mathrm{d}V$ it further delivers

$$\int_V \rho\ddot{\boldsymbol{x}}\mathrm{d}V = \int_A \boldsymbol{\sigma}\boldsymbol{n}\mathrm{d}A + \int_V \boldsymbol{f}\mathrm{d}V, \tag{2.114}$$

where ρ denotes the density of the material. Dividing this equation by V and considering the limit case $V \to 0$ we obtain by virtue of (2.104)

$$\rho \ddot{\boldsymbol{x}} = \operatorname{div} \boldsymbol{\sigma} + \boldsymbol{f}. \tag{2.115}$$

This vector equation is referred to as the momentum balance.

Representing vector and tensor variables with respect to the tangent vectors $\boldsymbol{g}_i$ $(i = 1, 2, 3)$ of an arbitrary curvilinear coordinate system as

$$\ddot{\boldsymbol{x}} = a^i \boldsymbol{g}_i, \quad \boldsymbol{\sigma} = \sigma^{ij} \boldsymbol{g}_i \otimes \boldsymbol{g}_j, \quad \boldsymbol{f} = f^i \boldsymbol{g}_i$$

and expressing the divergence of the Cauchy stress tensor by (2.112) we obtain the component form of the momentum balance (2.115) by

$$\rho a^i = \sigma^{ij}|_j + f^i, \quad i = 1, 2, 3. \tag{2.116}$$

With the aid of (2.95) the covariant derivative of the Cauchy stress tensor can further be written by

$$\sigma^{ij}|_k = \sigma^{ij},_k + \sigma^{lj} \Gamma^i_{lk} + \sigma^{il} \Gamma^j_{lk}, \quad i, j, k = 1, 2, 3 \tag{2.117}$$

and thus,

$$\sigma^{ij}|_j = \sigma^{ij},_j + \sigma^{lj} \Gamma^i_{lj} + \sigma^{il} \Gamma^j_{lj}, \quad i = 1, 2, 3. \tag{2.118}$$

By virtue of the expressions for the Christoffel symbols (2.90) and keeping in mind the symmetry of the Cauchy stress tensors $\sigma^{ij} = \sigma^{ji}$ $(i \neq j = 1, 2, 3)$ we thus obtain for cylindrical coordinates:

$$\begin{aligned}
\sigma^{1j}|_j &= \sigma^{11},_\varphi + \sigma^{12},_z + \sigma^{13},_r + \frac{3\sigma^{31}}{r}, \\
\sigma^{2j}|_j &= \sigma^{21},_\varphi + \sigma^{22},_z + \sigma^{23},_r + \frac{\sigma^{32}}{r}, \\
\sigma^{3j}|_j &= \sigma^{31},_\varphi + \sigma^{32},_z + \sigma^{33},_r - r\sigma^{11} + \frac{\sigma^{33}}{r}.
\end{aligned} \tag{2.119}$$

The balance equations finally take the form

$$\begin{aligned}
\rho a^1 &= \sigma^{11},_\varphi + \sigma^{12},_z + \sigma^{13},_r + \frac{3\sigma^{31}}{r} + f^1, \\
\rho a^2 &= \sigma^{21},_\varphi + \sigma^{22},_z + \sigma^{23},_r + \frac{\sigma^{32}}{r} + f^2, \\
\rho a^3 &= \sigma^{31},_\varphi + \sigma^{32},_z + \sigma^{33},_r - r\sigma^{11} + \frac{\sigma^{33}}{r} + f^3.
\end{aligned} \tag{2.120}$$

In Cartesian coordinates, where $\boldsymbol{g}_i = \boldsymbol{e}_i$ $(i = 1, 2, 3)$, the covariant derivative coincides with the partial one, so that

$$\sigma^{ij}|_j = \sigma^{ij},_j = \sigma_{ij},_j \,. \tag{2.121}$$

Thus, the balance equations reduce to

$$\begin{aligned}
\rho\ddot{x}_1 &= \sigma_{11},_1 + \sigma_{12},_2 + \sigma_{13},_3 + f_1, \\
\rho\ddot{x}_2 &= \sigma_{21},_1 + \sigma_{22},_2 + \sigma_{23},_3 + f_2, \\
\rho\ddot{x}_3 &= \sigma_{31},_1 + \sigma_{32},_2 + \sigma_{33},_3 + f_3,
\end{aligned} \tag{2.122}$$

where $\ddot{x}_i = a_i$ $(i = 1, 2, 3)$.

Divergence and curl of a vector field. Now, we consider a differentiable vector field $\boldsymbol{t}\left(\theta^1, \theta^2, \theta^3\right)$. One defines the divergence and curl of $\boldsymbol{t}\left(\theta^1, \theta^2, \theta^3\right)$ respectively by

$$\mathrm{div}\boldsymbol{t} = \lim_{V \to 0} \frac{1}{V} \int\limits_A (\boldsymbol{t} \cdot \boldsymbol{n})\, \mathrm{d}A, \tag{2.123}$$

$$\mathrm{curl}\boldsymbol{t} = \lim_{V \to 0} \frac{1}{V} \int\limits_A (\boldsymbol{n} \times \boldsymbol{t})\, \mathrm{d}A = -\lim_{V \to 0} \frac{1}{V} \int\limits_A (\boldsymbol{t} \times \boldsymbol{n})\, \mathrm{d}A, \tag{2.124}$$

where the integration is again carried out over a closed surface area A with the volume V and the outer unit normal vector $\boldsymbol{n}$ (see Fig. 2.4). Considering (1.66) and (2.104), the curl can also be represented by

$$\mathrm{curl}\boldsymbol{t} = -\lim_{V \to 0} \frac{1}{V} \int\limits_A \hat{\boldsymbol{t}}\boldsymbol{n} \mathrm{d}A = -\mathrm{div}\hat{\boldsymbol{t}}. \tag{2.125}$$

Treating the vector field in the same manner as the tensor field we can write

$$\mathrm{div}\boldsymbol{t} = \boldsymbol{t},_i \cdot \boldsymbol{g}^i = t^i|_i \tag{2.126}$$

and in view of $(2.75)_2$ (see also Exercise 1.44)

$$\mathrm{div}\boldsymbol{t} = \mathrm{tr}\,(\mathrm{grad}\boldsymbol{t})\,. \tag{2.127}$$

The same procedure applied to the curl (2.124) leads to

$$\mathrm{curl}\boldsymbol{t} = \boldsymbol{g}^i \times \boldsymbol{t},_i \,. \tag{2.128}$$

By virtue of $(2.72)_1$ and (1.44) we further obtain (see also Exercise 2.8)

$$\mathrm{curl}\boldsymbol{t} = t_i|_j\; \boldsymbol{g}^j \times \boldsymbol{g}^i = e^{jik}\frac{1}{g} t_i|_j\; \boldsymbol{g}_k. \tag{2.129}$$

With respect to the Cartesian coordinates (2.20) with $\boldsymbol{g}_i = \boldsymbol{e}_i$ $(i = 1, 2, 3)$ the divergence (2.126) and curl (2.129) simplify to

$$\mathrm{div}\boldsymbol{t} = t^i{}_{,i} = t^1{}_{,1} + t^2{}_{,2} + t^3{}_{,3} = t_{1,1} + t_{2,2} + t_{3,3}\,, \tag{2.130}$$

$$\begin{aligned}\mathrm{curl}\boldsymbol{t} &= e^{jik} t_{i,j}\,\boldsymbol{e}_k \\ &= (t_{3,2} - t_{2,3})\,\boldsymbol{e}_1 + (t_{1,3} - t_{3,1})\,\boldsymbol{e}_2 + (t_{2,1} - t_{1,2})\,\boldsymbol{e}_3.\end{aligned} \tag{2.131}$$

Now, we are going to discuss some combined operations with a gradient, divergence, curl, tensor mapping and products of various types (see also Exercise 2.12).

1. Curl of a gradient:
$$\mathrm{curl}\,\mathrm{grad}\Phi = \boldsymbol{0}. \tag{2.132}$$
2. Divergence of a curl:
$$\mathrm{div}\,\mathrm{curl}\boldsymbol{t} = 0. \tag{2.133}$$
3. Divergence of a vector product:
$$\mathrm{div}\,(\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot \mathrm{curl}\boldsymbol{u} - \boldsymbol{u} \cdot \mathrm{curl}\boldsymbol{v}. \tag{2.134}$$
4. Gradient of a divergence:
$$\mathrm{grad}\,\mathrm{div}\boldsymbol{t} = \mathrm{div}\,(\mathrm{grad}\boldsymbol{t})^{\mathrm{T}}\,, \tag{2.135}$$
$$\mathrm{grad}\,\mathrm{div}\boldsymbol{t} = \mathrm{curl}\,\mathrm{curl}\boldsymbol{t} + \mathrm{div}\,\mathrm{grad}\boldsymbol{t} = \mathrm{curl}\,\mathrm{curl}\boldsymbol{t} + \Delta\boldsymbol{t}\,, \tag{2.136}$$
where the combined operator $\Delta\boldsymbol{t} = \mathrm{div}\,\mathrm{grad}\boldsymbol{t}$ is known as the Laplacian.
5. Skew-symmetric part of a gradient
$$\mathrm{skew}\,(\mathrm{grad}\boldsymbol{t}) = \frac{1}{2}\widehat{\mathrm{curl}\boldsymbol{t}}. \tag{2.137}$$
6. Divergence of a (left) mapping
$$\mathrm{div}\,(\boldsymbol{t}\mathbf{A}) = \mathbf{A} : \mathrm{grad}\boldsymbol{t} + \boldsymbol{t} \cdot \mathrm{div}\mathbf{A}. \tag{2.138}$$
7. Divergence of a product of a scalar-valued function and a vector-valued function
$$\mathrm{div}\,(\Phi\boldsymbol{t}) = \boldsymbol{t} \cdot \mathrm{grad}\Phi + \Phi\mathrm{div}\boldsymbol{t}. \tag{2.139}$$
8. Divergence of a product of a scalar-valued function and a tensor-valued function
$$\mathrm{div}\,(\Phi\mathbf{A}) = \mathbf{A}\mathrm{grad}\Phi + \Phi\mathrm{div}\mathbf{A}. \tag{2.140}$$

We prove, for example, identity (2.132). To this end, we apply $(2.75)_1$, (2.82) and (2.128). Thus, we write

$$\begin{aligned}
\operatorname{curl}\operatorname{grad}\Phi &= \boldsymbol{g}^j \times \left(\Phi|_i\ \boldsymbol{g}^i\right)_{,j} = \Phi_{,ij}\ \boldsymbol{g}^j \times \boldsymbol{g}^i + \Phi_{,i}\ \boldsymbol{g}^j \times \boldsymbol{g}^i{}_{,j} \\
&= \Phi_{,ij}\ \boldsymbol{g}^j \times \boldsymbol{g}^i - \Phi_{,i}\ \Gamma^i_{kj} \boldsymbol{g}^j \times \boldsymbol{g}^k = \boldsymbol{0}
\end{aligned} \tag{2.141}$$

taking into account that $\Phi_{,ij} = \Phi_{,ji}$, $\Gamma^l_{ij} = \Gamma^l_{ji}$ and $\boldsymbol{g}^i \times \boldsymbol{g}^j = -\boldsymbol{g}^j \times \boldsymbol{g}^i$ $(i \neq j,\ i, j = 1, 2, 3)$.

Example 2.8. Balance of mechanical energy as an integral form of the momentum balance. Using the above identities we are now able to formulate the balance of mechanical energy on the basis of the momentum balance (2.115). To this end, we multiply this vector relation scalarly by the velocity vector $\boldsymbol{v} = \dot{\boldsymbol{x}}$

$$\boldsymbol{v} \cdot (\rho \ddot{\boldsymbol{x}}) = \boldsymbol{v} \cdot \operatorname{div}\boldsymbol{\sigma} + \boldsymbol{v} \cdot \boldsymbol{f}.$$

Using (2.138) we can further write

$$\boldsymbol{v} \cdot (\rho \ddot{\boldsymbol{x}}) + \boldsymbol{\sigma} : \operatorname{grad}\boldsymbol{v} = \operatorname{div}(\boldsymbol{v}\boldsymbol{\sigma}) + \boldsymbol{v} \cdot \boldsymbol{f}.$$

Integrating this relation over the volume of the body V yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_m \left(\frac{1}{2}\boldsymbol{v} \cdot \boldsymbol{v}\right) \mathrm{d}m + \int\limits_V \boldsymbol{\sigma} : \operatorname{grad}\boldsymbol{v}\,\mathrm{d}V = \int\limits_V \operatorname{div}(\boldsymbol{v}\boldsymbol{\sigma})\,\mathrm{d}V + \int\limits_V \boldsymbol{v} \cdot \boldsymbol{f}\,\mathrm{d}V,$$

where again $\mathrm{d}m = \rho \mathrm{d}V$ and m denotes the mass of the body. Keeping in mind the definition of the divergence (2.104) and applying the Cauchy theorem (1.77) according to which the Cauchy stress vector is given by $\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n}$, we finally obtain the relation

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_m \left(\frac{1}{2}\boldsymbol{v} \cdot \boldsymbol{v}\right) \mathrm{d}m + \int\limits_V \boldsymbol{\sigma} : \operatorname{grad}\boldsymbol{v}\,\mathrm{d}V = \int\limits_A \boldsymbol{v} \cdot \boldsymbol{t}\,\mathrm{d}A + \int\limits_V \boldsymbol{v} \cdot \boldsymbol{f}\,\mathrm{d}V \tag{2.142}$$

expressing the balance of mechanical energy. Indeed, the first and the second integrals on the left hand side of (2.142) represent the time rate of the kinetic energy and the stress power, respectively. The right hand side of (2.142) expresses the power of external forces i.e. external tractions $\boldsymbol{t}$ on the boundary of the body A and external volume forces $\boldsymbol{f}$ inside of it.

Example 2.9. Axial vector of the spin tensor. The spin tensor is defined as a skew-symmetric part of the velocity gradient by

$$\mathbf{w} = \operatorname{skew}(\operatorname{grad}\boldsymbol{v}). \tag{2.143}$$

By virtue of (1.158) we can represent it in terms of the axial vector

$$\mathbf{w} = \hat{\boldsymbol{w}}, \tag{2.144}$$

which in view of (2.137) takes the form:

$$\boldsymbol{w} = \frac{1}{2}\operatorname{curl}\boldsymbol{v}. \tag{2.145}$$

Example 2.10. Navier-Stokes equations for a linear-viscous fluid in Cartesian and cylindrical coordinates. A linear-viscous fluid (also called Newton fluid or Navier-Poisson fluid) is defined by a constitutive equation

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\eta\mathbf{d} + \lambda\left(\operatorname{tr}\mathbf{d}\right)\mathbf{I}, \tag{2.146}$$

where

$$\mathbf{d} = \operatorname{sym}\left(\operatorname{grad}\boldsymbol{v}\right) = \frac{1}{2}\left[\operatorname{grad}\boldsymbol{v} + \left(\operatorname{grad}\boldsymbol{v}\right)^{\mathrm{T}}\right] \tag{2.147}$$

denotes the rate of deformation tensor, p is the hydrostatic pressure while η and λ represent material constants referred to as shear viscosity and second viscosity coefficient, respectively. Inserting (2.147) into (2.146) and taking (2.127) into account delivers

$$\boldsymbol{\sigma} = -p\mathbf{I} + \eta\left[\operatorname{grad}\boldsymbol{v} + \left(\operatorname{grad}\boldsymbol{v}\right)^{\mathrm{T}}\right] + \lambda\left(\operatorname{div}\boldsymbol{v}\right)\mathbf{I}. \tag{2.148}$$

Substituting this expression into the momentum balance (2.115) and using (2.135) and (2.140) we obtain the relation

$$\rho\dot{\boldsymbol{v}} = -\operatorname{grad}p + \eta\operatorname{div}\operatorname{grad}\boldsymbol{v} + (\eta+\lambda)\operatorname{grad}\operatorname{div}\boldsymbol{v} + \boldsymbol{f} \tag{2.149}$$

referred to as the Navier-Stokes equation. By means of (2.136) it can be rewritten as

$$\rho\dot{\boldsymbol{v}} = -\operatorname{grad}p + (2\eta+\lambda)\operatorname{grad}\operatorname{div}\boldsymbol{v} - \eta\operatorname{curl}\operatorname{curl}\boldsymbol{v} + \boldsymbol{f}. \tag{2.150}$$

For an incompressible fluid characterized by the kinematic condition $\operatorname{tr}\mathbf{d} = \operatorname{div}\boldsymbol{v} = 0$, the latter two equations simplify to

$$\rho\dot{\boldsymbol{v}} = -\operatorname{grad}p + \eta\Delta\boldsymbol{v} + \boldsymbol{f}, \tag{2.151}$$

$$\rho\dot{\boldsymbol{v}} = -\operatorname{grad}p - \eta\operatorname{curl}\operatorname{curl}\boldsymbol{v} + \boldsymbol{f}. \tag{2.152}$$

With the aid of the identity $\Delta\boldsymbol{v} = \boldsymbol{v},_i|^i$ (see Exercise 2.14) we thus can write

$$\rho\dot{\boldsymbol{v}} = -\operatorname{grad}p + \eta\boldsymbol{v},_i|^i + \boldsymbol{f}. \tag{2.153}$$

In Cartesian coordinates this relation is thus written out as

$$\rho \dot{v}_i = -p_{,i} + \eta \left(v_{i,11} + v_{i,22} + v_{i,33} \right) + f_i, \qquad i = 1, 2, 3. \tag{2.154}$$

For arbitrary curvilinear coordinates we use the following representation for the vector Laplacian (see Exercise 2.16)

$$\Delta \boldsymbol{v} = g^{ij} \left(v^k{}_{,ij} + 2\Gamma^k_{li} v^l{}_{,j} - \Gamma^m_{ij} v^k{}_{,m} + \Gamma^k_{li,j} v^l + \Gamma^k_{mj} \Gamma^m_{li} v^l - \Gamma^m_{ij} \Gamma^k_{lm} v^l \right) \boldsymbol{g}_k. \tag{2.155}$$

For the cylindrical coordinates it takes by virtue of (2.30) and (2.90) the following form

$$\begin{aligned} \Delta \boldsymbol{v} = & \left(r^{-2} v^1{}_{,11} + v^1{}_{,22} + v^1{}_{,33} + 3r^{-1} v^1{}_{,3} + 2r^{-3} v^3{}_{,1} \right) \boldsymbol{g}_1 \\ & + \left(r^{-2} v^2{}_{,11} + v^2{}_{,22} + v^2{}_{,33} + r^{-1} v^2{}_{,3} \right) \boldsymbol{g}_2 \\ & + \left(r^{-2} v^3{}_{,11} + v^3{}_{,22} + v^3{}_{,33} - 2r^{-1} v^1{}_{,1} + r^{-1} v^3{}_{,3} - r^{-2} v^3 \right) \boldsymbol{g}_3. \end{aligned}$$

Inserting this result into (2.151) and using the representations $\dot{\boldsymbol{v}} = \dot{v}^i \boldsymbol{g}_i$ and $\boldsymbol{f} = f^i \boldsymbol{g}_i$ we finally obtain

$$\begin{aligned} \rho \dot{v}^1 &= f^1 - \frac{\partial p}{\partial \varphi} + \eta \left(\frac{1}{r^2} \frac{\partial^2 v^1}{\partial \varphi^2} + \frac{\partial^2 v^1}{\partial z^2} + \frac{\partial^2 v^1}{\partial r^2} + \frac{3}{r} \frac{\partial v^1}{\partial r} + \frac{2}{r^3} \frac{\partial v^3}{\partial \varphi} \right), \\ \rho \dot{v}^2 &= f^2 - \frac{\partial p}{\partial z} + \eta \left(\frac{1}{r^2} \frac{\partial^2 v^2}{\partial \varphi^2} + \frac{\partial^2 v^2}{\partial z^2} + \frac{\partial^2 v^2}{\partial r^2} + \frac{1}{r} \frac{\partial v^2}{\partial r} \right), \\ \rho \dot{v}^3 &= f^3 - \frac{\partial p}{\partial r} + \eta \left(\frac{1}{r^2} \frac{\partial^2 v^3}{\partial \varphi^2} + \frac{\partial^2 v^3}{\partial z^2} + \frac{\partial^2 v^3}{\partial r^2} - \frac{2}{r} \frac{\partial v^1}{\partial \varphi} + \frac{1}{r} \frac{\partial v^3}{\partial r} - \frac{v^3}{r^2} \right). \end{aligned} \tag{2.156}$$

Exercises

2.1. Evaluate tangent vectors, metric coefficients and the dual basis of spherical coordinates in $\mathbb{E}^3$ defined by (Fig. 2.6)

$$\boldsymbol{r}(\varphi, \phi, r) = r \sin\varphi \sin\phi \boldsymbol{e}_1 + r \cos\phi \boldsymbol{e}_2 + r \cos\varphi \sin\phi \boldsymbol{e}_3. \tag{2.157}$$

2.2. Evaluate the coefficients $\dfrac{\partial \bar{\theta}^i}{\partial \theta^k}$ (2.43) for the transformation of linear coordinates in the spherical ones and vice versa.

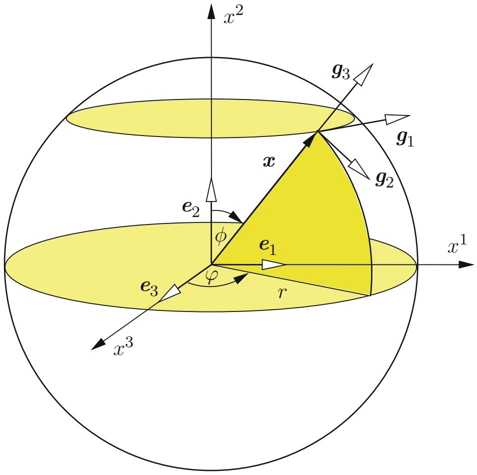

Fig. 2.6 Spherical coordinates in three-dimensional space

2.3. Evaluate gradients of the following functions of $\boldsymbol{r}$:

(a) $\dfrac{1}{\|\boldsymbol{r}\|}$, (b) $\boldsymbol{r}\cdot\boldsymbol{w}$, (c) $\boldsymbol{r}\mathbf{A}\boldsymbol{r}$, (d) $\mathbf{A}\boldsymbol{r}$, (e) $\boldsymbol{w}\times\boldsymbol{r}$,

where $\boldsymbol{w}$ and $\mathbf{A}$ are some vector and tensor, respectively.

2.4. Evaluate the Christoffel symbols of the first and second kind for spherical coordinates (2.157).

2.5. Verify relations (2.96).

2.6. Prove identities (2.99) and (2.100) by using (1.91).

2.7. Prove the product rules of differentiation for the covariant derivative (2.101)–(2.103).

2.8. Verify relation (2.129) applying (2.112), (2.125) and using the results of Exercise 1.23.

2.9. Write out the balance equations (2.116) in spherical coordinates (2.157).

2.10. Evaluate tangent vectors, metric coefficients, the dual basis and Christoffel symbols for cylindrical surface coordinates defined by

$$\boldsymbol{r}\,(r, s, z) = r\cos\frac{s}{r}\boldsymbol{e}_1 + r\sin\frac{s}{r}\boldsymbol{e}_2 + z\boldsymbol{e}_3. \tag{2.158}$$

2.11. Write out the balance equations (2.116) in cylindrical surface coordinates (2.158).

2.12. Prove identities (2.133)–(2.140).

2.13. Write out the gradient, divergence and curl of a vector field $\boldsymbol{t}(\boldsymbol{r})$ in cylindrical and spherical coordinates (2.17) and (2.157), respectively.

2.14. Prove that the Laplacian of a vector-valued function $\boldsymbol{t}(\boldsymbol{r})$ can be given by $\Delta\boldsymbol{t} = \boldsymbol{t},_i|^i$. Specify this identity for Cartesian coordinates.

2.15. Write out the Laplacian $\Delta\Phi$ of a scalar field $\Phi(\boldsymbol{r})$ in cylindrical and spherical coordinates (2.17) and (2.157), respectively.

2.16. Write out the Laplacian of a vector field $\boldsymbol{t}(\boldsymbol{r})$ in component form in an arbitrary curvilinear coordinate system. Specify the result for spherical coordinates (2.157).

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