Chapter 2
Algorithmic Operators

In Chap. 5 we will present several methods for solving convex optimization problems. We will focus our study on *iterative methods* (we also call them *iterative procedures* or *algorithms*) which are given in the form of the following recurrence

$$x^{k+1} = T_k x^k \tag{2.1}$$

defined on a closed convex subset $X \subseteq \mathcal{H}$, where $T_k : X \to X$ is a sequence of operators. We suppose that the starting point x^0 is an element of a starting subset $X_0 \subseteq X$. Usually, one supposes that $X_0 = X$. A sequence $\{x^k\}_{k=0}^{\infty}$ generated by the iterative method (2.1) is called an *approximating sequence*. If $T_k = T$ for all $k \geq 0$, then this sequence is called an *orbit* of T. Any iterative method for solving a convex optimization problem is constructed in such a way that the approximating sequences $\{x^k\}_{k=0}^{\infty}$ generated by this method converge (at least weakly) to a solution of the optimization problem. As we will see, the solution is a fixed point of an operator $S : X \to \mathcal{H}$, which is usually a nonexpansive one. The form of this operator depends on the considered optimization problem. A sequence of operators T_k which defines the iterative method is usually constructed in such a way that $\operatorname{Fix} S \subseteq \bigcap_{k=0}^{\infty} \operatorname{Fix} T_k$.

In this chapter we deal with general properties of operators which define algorithms for solving convex optimization problems. In one iteration of the algorithm an appropriate operator $T : X \to X$ defines an *actualization*, also called an *update* x^+ of the current approximation x of a solution of the convex optimization problem. Usually, this actualization has the form $x^+ = Tx$. We call T an *algorithmic operator*. One can also consider algorithms, where the actualization has the form $x^+ \in Tx$ for a mapping (multifunction) $T : X \rightrightarrows X$. In this case, T is called an *algorithmic mapping*.

Operators defining iterations of an algorithm usually depend on some parameters which are constant or vary during the iteration process. The properties of approximating sequences depend on the properties of algorithmic operators defining the iterative method as well as on the choice of parameters defining these operators.

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2.1 Basic Definitions and Properties

Let $\mathcal{H}$ be a Hilbert space. In what follows, we consider operators which are defined on a nonempty closed convex subset $X \subseteq \mathcal{H}$.

Remark 2.1.1. Let $U_i : X \to X$, $i \in I := \{1, 2, \dots, m\}$. If (i) $U = \sum_{i \in I} \omega_i U_i$, where $w = (\omega_1, \omega_2, \dots, \omega_m) \in \Delta_m$, or (ii) $U = U_m U_{m-1} \dots U_1$, then the following obvious inclusion holds

$$\bigcap_{i \in I} \operatorname{Fix} U_i \subseteq \operatorname{Fix} U$$

The converse inclusion needs not to be true even if all U_i, $i \in I$, have a common fixed point (see Example 2.1.27).

Definition 2.1.2. Let $T : X \to \mathcal{H}$ and $\lambda \in [0, 2]$. The operator $T_\lambda : X \to \mathcal{H}$ defined by

$$T_\lambda := (1 - \lambda) \operatorname{Id} + \lambda T$$

is called a *λ-relaxation or, shortly, relaxation* of the operator T. Obviously, $T_\lambda = \operatorname{Id} + \lambda(T - \operatorname{Id})$. We call λ a *relaxation parameter*. *If* $\lambda \in (0, 1)$, then T_λ is called an *under-relaxation* of T. If $\lambda \in (1, 2)$, then T_λ is called an *over-relaxation* of T and if $\lambda = 2$, then T_λ is called the *reflection* of T. If $\lambda \in (0, 2)$, then T_λ is called a *strict relaxation* of T.

A relaxation T_λ of an operator T can be defined for any $\lambda \in \mathbb{R}$. However, if we do not extend explicitly the range of λ, we assume that $\lambda \in [0, 2]$.

Remark 2.1.3. Note that the equality $(T_\lambda)_\mu = T_{\lambda\mu}$ holds for all $\lambda, \mu \in \mathbb{R}$, consequently $(T_\lambda)_{\lambda^{-1}} = T$ for $\lambda \neq 0$.

Remark 2.1.4. It is clear that $\operatorname{Fix} T = \operatorname{Fix} T_\lambda$ whenever $\lambda \neq 0$.

Let $U_i : X \to X$, $i \in I := \{1, 2, \dots, m\}$, $U := U_m U_{m-1} \dots U_1$ and $Q_i := U_i U_{i-1} \dots U_1 U_m \dots U_{i+1}$, $i = 1, 2, \dots, m$. Denote $Q_0 := Q_m = U$ and $U_0 := U_m$. There exists a relationship among the subsets of fixed points of operators Q_i, which is expressed by the following theorem.

Theorem 2.1.5. *For $i = 1, 2, \dots, m$ there holds*

$$\operatorname{Fix} Q_i = U_i(\operatorname{Fix} Q_{i-1}). \tag{2.2}$$

Proof. Let $i \in I$. First we prove the inclusion

$$\operatorname{Fix} Q_i \supseteq U_i(\operatorname{Fix} Q_{i-1}). \tag{2.3}$$

Let $z^{i-1} \in \operatorname{Fix} Q_{i-1}$ and $z^i = U_i z^{i-1}$. Then we have

$$\begin{aligned} z^i &= U_i z^{i-1} = U_i Q_{i-1} z^{i-1} = U_i U_{i-1} \dots U_1 U_m \dots U_{i+1} U_i z^{i-1} \\ &= U_i U_{i-1} \dots U_1 U_m \dots U_{i+1} z^i = Q_i z^i, \end{aligned}$$

which proves that (2.3) holds for any $i \in I$. Consequently,

$$\begin{aligned}
\operatorname{Fix} Q_i &\supseteq U_i(\operatorname{Fix} Q_{i-1}) \supseteq U_i U_{i-1}(\operatorname{Fix} Q_{i-2}) \\
&\supseteq \ldots \supseteq U_i U_{i-1} \ldots U_1(\operatorname{Fix} Q_0) = U_i U_{i-1} \ldots U_1(\operatorname{Fix} Q_m) \\
&\supseteq U_i U_{i-1} \ldots U_1 U_m(\operatorname{Fix} Q_{m-1}) \\
&\supseteq \ldots \supseteq U_i U_{i-1} \ldots U_1 U_m U_{m-1} \ldots U_{i+1}(\operatorname{Fix} Q_i) \\
&= Q_i(\operatorname{Fix} Q_i) = (\operatorname{Fix} Q_i),
\end{aligned}$$

$i \in I$, and all inclusions are, actually, equations. In particular, $\operatorname{Fix} Q_i = U_i(\operatorname{Fix} Q_{i-1})$, i.e., (2.2) is satisfied for all $i \in I$. □

2.1.1 Nonexpansive Operators

Definition 2.1.6. We say that an operator $T : X \to \mathcal{H}$ is:

(i) *Nonexpansive* (NE), if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in X$,

(ii) *Strictly nonexpansive* if

$$\|Tx - Ty\| < \|x - y\| \text{ or } x - y = Tx - Ty$$

for all $x, y \in X$,

(iii) An α-*contraction*, where $\alpha \in (0, 1)$ or, shortly, a *contraction* if

$$\|Tx - Ty\| \leq \alpha \|x - y\|$$

for all $x, y \in X$.

The theorem below, called the *Banach fixed point theorem* or the *Banach theorem on contractions*, is widely applied in various areas of mathematics. The theorem holds for any complete metric space, and hence, in particular, for every closed subset of a Hilbert space.

Theorem 2.1.7 (Banach, 1922). *Let $\mathcal{X}$ be a complete metric space and $T : \mathcal{X} \to \mathcal{X}$ be a contraction. Then T has exactly one fixed point $x^* \in \mathcal{X}$. Furthermore, for any $x \in \mathcal{X}$, the orbit $\{T^k x\}_{k=0}^{\infty}$ converges to x^* with a rate of geometric progression.*

Proof. See, e.g., original paper of Banach [15], [185, Theorem 1.1], [267, Theorem 24.2], [184, Theorem 2.1], [183, Theorem 2.1] or [36, Theorem 2.1]. □

The Banach fixed point theorem is a widely applied tool for an iterative approximation of fixed points. Unfortunately, its application is restricted to contractions. We will need, however, appropriate tools for an iterative approximation of fixed points of nonexpansive operators T with $\operatorname{Fix} T \neq \emptyset$.

Below, we present several classical fixed points theorems.

Theorem 2.1.8 (Brouwer, 1912). *Let $X \subseteq \mathbb{R}^n$ be nonempty compact and convex and $T : X \to X$ be continuous. Then T has a fixed point.*

Proof. See, e.g., original paper of Brouwer [43] or [191, Chap. II, §5, Theorem 7.2] or [183, Theorem 7.6]. □

The Brouwer fixed point theorem was generalized by Juliusz Schauder.

Theorem 2.1.9 (Schauder, 1930). *Let X be a nonempty compact and convex subset of a Banach space and $T : X \to X$ be continuous. Then T has a fixed point.*

Proof. See, e.g., original paper of Schauder [302] or [191, Chap. II, §6, Theorem 3.2] or [183, Theorem 8.1]. □

For nonexpansive operators in a Hilbert space $\mathcal{H}$ the compactness of $X \subseteq \mathcal{H}$ in the Schauder theorem can be replaced by the boundedness of X. The following theorem was proved independently by Browder [45, Theorem 1], Göhde [188] and by Kirk [227]. The proof can also be found, e.g., in [185, Theorem 5.1], [191, Chap. I, §4, Theorem 1.3], [183, Theorem 4.1] or [36, Theorem 3.1].

Theorem 2.1.10 (Browder–Göhde–Kirk, 1965). *Let X be a nonempty closed, convex and bounded subset of a uniformly convex Banach space (e.g., of a Hilbert space $\mathcal{H}$) and $U : X \to X$ be nonexpansive. Then U has a fixed point.*

Contrary to the Banach fixed point theorem, the theorems of Brouwer, Schauder and of Browder–Göhde–Kirk are only of existential nature. In Chap. 3 we present theorems which can be applied to iterative methods for determining fixed points of nonexpansive operators.

Below, we present some properties of the subset of fixed points of a nonexpansive operator. The following result can be found in [185, Proposition 5.3].

Proposition 2.1.11. *The subset of fixed points of a nonexpansive operator $T : X \to \mathcal{H}$ is closed and convex.*

Proof. (cf. [185, Proposition 5.3]) Let $x^k \in \operatorname{Fix} T$ and $x^k \to x$. We have $x \in X$ because X is closed. By the continuity of T,

$$x = \lim x^k = \lim_k T x^k = T x,$$

i.e., $\operatorname{Fix} T$ is a closed subset. Now we show the convexity of $\operatorname{Fix} T$. Let $x, y \in \operatorname{Fix} T$, $x \neq y$ and $z = (1-\lambda)x + \lambda y$ for $\lambda \in (0,1)$. By the nonexpansivity of T and by the positive homogeneity of the norm we have

$$\|x - Tz\| = \|Tx - Tz\| \leq \|x - z\| = \lambda \|x - y\| \tag{2.4}$$

and

$$\|Tz - y\| = \|Tz - Ty\| \leq \|z - y\| = (1 - \lambda)\, \|x - y\| \,. \tag{2.5}$$

Now, the triangle inequality yields

$$\begin{aligned}\|x - y\| &\leq \|x - Tz\| + \|Tz - y\| \\ &\leq \lambda \, \|x - y\| + (1 - \lambda)\, \|x - y\| \\ &= \|x - y\| \,.\end{aligned}$$

Consequently,

$$\|x - y\| = \|x - Tz\| + \|Tz - y\| \,.$$

By the strict convexity of the norm, the vectors $x - Tz$ and $Tz - y$ are positive linearly dependent. Therefore, $\alpha(x - Tz) + \beta(y - Tz) = 0$ for some $\alpha, \beta \geq 0$. Since $x \neq y$, it follows that $\alpha + \beta > 0$, and hence, $Tz = \frac{\alpha}{\alpha+\beta}x + \frac{\beta}{\alpha+\beta}y$. Now, the nonexpansivity of T and inequalities (2.4) and (2.5) yield

$$\frac{\beta}{\alpha + \beta}\, \|x - y\| = \|x - Tz\| = \|Tx - Tz\| \leq \|x - z\| = \lambda \, \|x - y\| \tag{2.6}$$

and

$$\frac{\alpha}{\alpha + \beta}\, \|x - y\| = \|Tz - y\| = \|Tz - Ty\| \leq \|z - y\| = (1 - \lambda)\, \|x - y\| \,. \tag{2.7}$$

If at least one inequality in (2.6) and (2.7) is strict, then by summing up (2.6) and (2.7) we would obtain a contradiction. Therefore, $\frac{\beta}{\alpha+\beta} = \lambda$ and $\frac{\alpha}{\alpha+\beta} = (1-\lambda)$, consequently $Tz = (1 - \lambda)x + \lambda y = z$. □

The closedness and convexity of the subset of fixed points of a nonexpansive operator follows also from a property presented in Sect. 2.2 (see Corollary 2.2.48).

Lemma 2.1.12. *Let $S_i : X \to X$, $i \in I := \{1, 2, \ldots, m\}$, be nonexpansive. Then:*

(i) A convex combination $S := \sum_{i \in I} \omega_i S_i$, where $w = (\omega_1, \ldots, \omega_m) \in \Delta_m$, is nonexpansive. If, furthermore, at least one operator S_i is a contraction and the corresponding weight $\omega_i > 0$, then S is a contraction;

(ii) A composition $S := S_m S_{m-1} \ldots S_1$ is nonexpansive. If, furthermore, at least one operator S_i is a contraction, then S is a contraction.

Proof. Let $x, y \in X$ and S_i be nonexpansive, i.e., $\|S_i x - S_i y\| \leq \alpha_i \, \|x - y\|$, where $\alpha_i \in (0, 1]$, $i \in I$.

(i) Let $w \in \Delta_m$, $S := \sum_{i \in I} \omega_i S_i$ and $\alpha = \sum_{j \in I} \omega_j \alpha_j$. It is clear that $\alpha \in (0, 1]$. By the convexity of the norm and the nonexpansivity of S_i, $i \in I$, we have

$$\|Sx - Sy\| = \left\| \sum_{i \in I} \omega_i (S_i x - S_i y) \right\|$$
$$\leq \sum_{i \in I} \omega_i \|S_i x - S_i y\|$$
$$\leq \sum_{i \in I} \omega_i \alpha_i \|x - y\|$$
$$= \sum_{i \in I} \frac{\omega_i \alpha_i}{\sum_{j \in I} \omega_j \alpha_j} \alpha \|x - y\|$$
$$= \alpha \|x - y\|,$$

i.e., S is a nonexpansive operator. Now suppose that S_{i_0} is a contraction, i.e., $\alpha_{i_0} < 1$ and that $\omega_{i_0} > 0$, for some $i_0 \in I$. Then $\alpha \in (0, 1)$, i.e., S is a contraction.

(ii) We have

$$\|Sx - Sy\| = \|S_m S_{m-1} \ldots S_1 x - S_m S_{m-1} \ldots S_1 y\| \leq \alpha \|x - y\|,$$

where $\alpha = \alpha_m \alpha_{m-1} \ldots \alpha_1 \in (0, 1]$. If S_{i_0} is a contraction for some $i_0 \in I$, i.e., $\alpha_{i_0} \in (0, 1)$, then, of course, $\alpha \in (0, 1)$ and S is a contraction. $\square$

Theorem 2.1.13. *Let $U_i : X \to X$ be nonexpansive for all $i \in I := \{1, 2, \ldots, m\}$, and $U := U_m U_{m-1} \ldots U_1$. If $U_j(X)$ is bounded for at least one $j \in I$, then* $\operatorname{Fix} U \neq \emptyset$.

Proof. Let $U_j(X)$ be bounded for some $j \in I$. Since U_i are nonexpansive, $i \in I$, the boundedness of $U_j(X)$ yields the boundedness of $U(X)$. Therefore, $Y := \operatorname{cl}\operatorname{conv} U(X)$ is closed, convex and bounded. Since $U(X) \subseteq X$ and X is closed and convex, we have $Y \subseteq X$. The operator $U\mid_Y$ maps a closed, convex and bounded subset Y into itself. By the Browder–Göhde–Kirk Fixed Point Theorem, the operator $U\mid_Y$ has a fixed point $z \in Y$. Of course, $Uz = U\mid_Y(z) = z$. $\square$

Theorem 2.1.14. *Let $U_i : X \to \mathcal{H}$, $i \in I := \{1, 2, \ldots, m\}$, be nonexpansive operators with a common fixed point and $U := \sum_{i \in I} \omega_i U_i$ with $w \in \operatorname{ri} \Delta_m$. Then*

$$\operatorname{Fix} U = \bigcap_{i \in I} \operatorname{Fix} U_i.$$

Proof. The inclusion $\bigcap_{i \in I} \operatorname{Fix} U_i \subseteq \operatorname{Fix} U$ is always true (see Remark 2.1.1). Now we show that the converse inclusion also holds. Let $z \in \operatorname{Fix} U$ and $u \in \bigcap_{i \in I} \operatorname{Fix} U_i$. If $z = u$, then, of course, $z \in \bigcap_{i \in I} \operatorname{Fix} U_i$. Otherwise, for $z \neq u$, by the convexity of the norm and by the nonexpansivity of U_i, $i \in I$, we have

$$\begin{aligned}\|z-u\| &= \|Uz-u\| \\ &= \left\|\sum_{i\in I}\omega_i U_i z - u\right\| = \left\|\sum_{i\in I}\omega_i (U_i z - u)\right\| \\ &\le \sum_{i\in I}\omega_i \|U_i z - u\| = \sum_{i\in I}\omega_i \|U_i z - U_i u\| \\ &\le \sum_{i\in I}\omega_i \|z-u\| = \|z-u\|\,.\end{aligned}$$

Consequently,

$$\left\|\sum_{i\in I}\omega_i (U_i z - u)\right\| = \sum_{i\in I}\omega_i \|U_i z - u\| = \sum_{i\in I}\omega_i \|z-u\|\,. \tag{2.8}$$

Since $\omega_i > 0$ for all $i \in I$, the first equality in (2.8) yields a positive linear dependence of all pairs of vectors $U_i z - u$ and $U_j z - u$, $i, j \in I$, $i \neq j$, i.e.,

$$\|U_i z - u\|\,(U_j z - u) = \left\|U_j z - u\right\|\,(U_i z - u). \tag{2.9}$$

The second equality in (2.8), together with the inequality $\|U_i z - u\| \le \|z-u\|$, $i \in I$, and the assumption $\omega_i > 0$, $i \in I$, yield

$$\|U_i z - u\| = \|z-u\| \tag{2.10}$$

for all $i \in I$. Since $z \neq u$, we have $U_i z \neq u$, $i \in I$. Now, it follows from (2.9) and (2.10) that $U_i z = v$ for all $i \in I$ and for some $v \in \mathcal{H}$. Consequently,

$$z = Uz = \sum_{j\in I}\omega_j U_j z = \sum_{j\in I}\omega_j v = v = U_i z,$$

for all $i \in I$, i.e., $z \in \bigcap_{i\in I} \operatorname{Fix} U_i$. □

2.1.2 *Quasi-nonexpansive Operators*

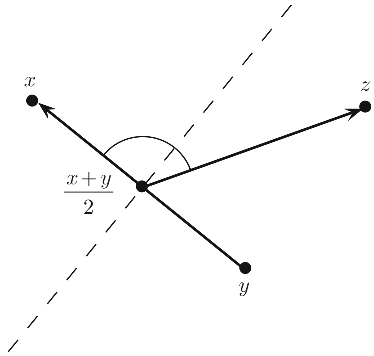
Definition 2.1.15. We say that an operator $T : X \to \mathcal{H}$ is:

(i) *Fejér monotone* (FM) with respect to a nonempty subset $C \subseteq X$ if

$$\|Tx - z\| \le \|x - z\|$$

for all $x \in X$ and $z \in C$,

Fig. 2.1 Equivalence (2.11)



(ii) *Strictly Fejér monotone* with respect to a nonempty subset $C \subseteq X$ if

$$\|Tx - z\| < \|x - z\|$$

for all $x \notin C$ and $z \in C$.

Remark 2.1.16. Because of the following obvious equivalence

$$\|z - y\| \leq \|z - x\| \Longleftrightarrow \left\langle z - \frac{y + x}{2}, y - x \right\rangle \geq 0 \tag{2.11}$$

for arbitrary $x, y, z \in \mathcal{H}$ (see Fig. 2.1), an operator $T : X \to \mathcal{H}$ is Fejér monotone with respect to C if and only if

$$\left\langle z - \frac{Tx - x}{2}, Tx - x \right\rangle \geq 0. \tag{2.12}$$

Furthermore, T is strictly Fejér monotone if and only if inequality (2.12) is strict for all $x \notin C$. We have not supposed that C is closed convex in Definition 2.1.15. Inequality (2.12) yields, however, that if T is (strictly) Fejér monotone with respect to C, then T is (strictly) Fejér monotone with respect to $\operatorname{conv} C$. Furthermore, the continuity of the norm yields that if T is Fejér monotone with respect to C, then T is Fejér monotone with respect to $\operatorname{cl} C$. Therefore, we can suppose, without loss of generality, that C is closed convex in Definition 2.1.15 (i) and that C is convex in Definition 2.1.15 (ii).

There exists the largest subset, with respect to which an operator T is Fejér monotone. This subset is closed and convex, as follows from the following lemma.

Lemma 2.1.17. *Let $T : X \to \mathcal{H}$. If the subset*

$$\operatorname{Fej} T := \bigcap_{x \in X} \left\{ z \in X : \left\langle z - \frac{Tx + x}{2}, Tx - x \right\rangle \geq 0 \right\} \tag{2.13}$$

is nonempty, then $\operatorname{Fej} T$ *is the largest subset, with respect to which T is Fejér monotone.*

Proof. The assertion follows directly from the equivalence (2.11). □

Remark 2.1.18. Because of frequent use we state some obvious properties of Fejér monotone operators:

(i) If T is (strictly) Fejér monotone with respect to a nonempty subset $C \subseteq \mathcal{H}$, then for an arbitrary $\lambda \in (0,1)$ its relaxation T_λ is also (strictly) Fejér monotone with respect to C.
(ii) If T is (strictly) Fejér monotone with respect to a nonempty subset $C \subseteq \mathcal{H}$, then T is (strictly) Fejér monotone with respect to any nonempty subset $D \subseteq C$.
(iii) Every composition and every convex combination of operators which are Fejér monotone with respect to a nonempty subset $C \subseteq \mathcal{H}$ is Fejér monotone with respect C.

Definition 2.1.19. We say that an operator $T : X \to \mathcal{H}$ having a fixed point is:

(i) *Quasi-nonexpansive* (QNE) if T is Fejér monotone with respect to $\operatorname{Fix} T$, i.e.,

$$\|Tx - z\| \leq \|x - z\|$$

for all $x \in X$ and $z \in \operatorname{Fix} T$,
(ii) *Strictly quasi-nonexpansive* (sQNE) if T is strictly Fejér monotone with respect to $\operatorname{Fix} T$, i.e.,

$$\|Tx - z\| < \|x - z\|$$

for all $x \notin \operatorname{Fix} T$ and $z \in \operatorname{Fix} T$,
(iii) C*-strictly quasi-nonexpansive* (C-sQNE), where $C \neq \emptyset$ and $C \subseteq \operatorname{Fix} T$, if T is quasi-nonexpansive and

$$\|Tx - z\| < \|x - z\|$$

for all $x \notin \operatorname{Fix} T$ and $z \in C$.

For an operator T having a fixed point the following relation is clear:

$$T \text{ is sQNE} \Longrightarrow T \text{ is } C\text{-sQNE}$$

where $C \subseteq \operatorname{Fix} T$. Furthermore, by definition,

$$T \text{ is } \operatorname{Fix} T\text{-sQNE} \Longrightarrow T \text{ is sQNE.}$$

The metric projection onto a closed convex subset is a typical example of a strictly quasi-nonexpansive operator.

A nonexpansive and strictly Fejér monotone operator is also called *attracting* (see [22, Definition 2.1]). Yamada and Ogura use the name an *attracting quasi-nonexpansive* operator for a strictly quasi-nonexpansive one (see [346, page 623]). Vasin and Ageev call these operators *strongly Q-quasi-nonexpansive* (see [333, Definition 2.2]). Reich and Zaslavski define a more general operator than the strictly quasi-nonexpansive one and call it an *F-attracting mapping*, where $F = \operatorname{Fix} T$ (see [297, Sect. 1]). A continuous strictly quasi-nonexpansive operator is also called a *paracontraction* (see, [164, Definition 1]). The class of quasi-nonexpansive operators is denoted in [126, page 161] by $\mathcal{F}^0$. Properties of quasi-nonexpansive operators in metric spaces have been intensively studied since 1969 (see, e.g., [145, 148, 283], [50, Sect. 1], [113]), but the name quasi-nonexpansive was introduced by Dotson [147].

Lemma 2.1.20. *A nonexpansive operator $U : X \to \mathcal{H}$ with a fixed point is quasi-nonexpansive.*

Proof. Let U be nonexpansive and $z \in \operatorname{Fix} U$. Then

$$\|Ux - z\| = \|Ux - Uz\| \leq \|x - z\|,$$

i.e., U is quasi-nonexpansive. □

It is clear that the class of nonexpansive operators having a fixed point is an essential subclass of quasi-nonexpansive operators, because a quasi-nonexpansive operator needs not to be continuous. Moreover, a quasi-nonexpansive operator needs not to be nonexpansive even if it is continuous (see Exercise 2.5.2). In this section we present properties of the family of quasi-nonexpansive operators. In further parts of the book we show that these operators play an important role in iterative methods for fixed point problems.

The following lemma gives a relation between the subset $\operatorname{Fej} T$ and the subset $\operatorname{Fix} T$ for an operator $T : X \to \mathcal{H}$ (cf. [24, Proposition 2.6 (ii)]).

Lemma 2.1.21. *For any operator $T : X \to \mathcal{H}$ the inclusion* $\operatorname{Fej} T \subseteq \operatorname{Fix} T$ *holds. If* $\operatorname{Fix} T \neq \emptyset$ *and T is quasi-nonexpansive, then the converse inclusion also holds. Consequently, the subset of fixed points of a quasi-nonexpansive operator is closed and convex.*

Proof. If $\operatorname{Fej} T = \emptyset$, then the first part of the assertion is obvious. Now let $\operatorname{Fej} T \neq \emptyset$ and $w \in \operatorname{Fej} T$. Then, for $z = x = w$ in (2.13), we obtain

$$\begin{aligned} 0 &\leq \langle w - \frac{Tw + w}{2}, Tw - w\rangle \\ &= -\frac{1}{2}\|Tw - w\|^2 \leq 0, \end{aligned}$$

i.e., $Tw = w$. Therefore, $\operatorname{Fej} T \subseteq \operatorname{Fix} T$. Now suppose that $\operatorname{Fix} T \neq \emptyset$ and that T is quasi-nonexpansive, i.e., T is Fejér monotone with respect to $\operatorname{Fix} T$. Then, Lemma 2.1.17 yields the inclusion $\operatorname{Fix} T \subseteq \operatorname{Fej} T$, which together with the first part of the lemma gives $\operatorname{Fix} T = \operatorname{Fej} T$. The convexity and the closedness of $\operatorname{Fix} T$ follows now from Lemma 2.1.17 and from the fact that the intersection of closed half-spaces is closed and convex. □

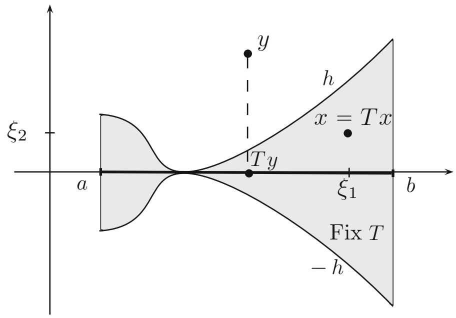


Fig. 2.2 Nonconvex Fix T for a Fejér monotone operator T

Remark 2.1.22. It follows from Remark 2.1.18 (ii), Lemmas 2.1.17 and 2.1.21 that a quasi-nonexpansive operator $T : X \to X$ is Fejér monotone with respect to any nonempty subset of Fix T. Therefore, we will restrict our further consideration of Fejér monotone operators to quasi-nonexpansive ones. Note, however, that without the quasi nonexpansivity of T the equality Fix T = Fej T needs not to be true. In this case, Fix T needs not to be convex, even if T is Fejér monotone.

Example 2.1.23. Let $\mathcal{H} = \mathbb{R}^2$, $X := [a, b] \times \mathbb{R}$ for $-\infty \le a \le b \le +\infty$ and $h : X \to \mathbb{R}_+$ be a function with $\inf_{x\in[a,b]} h(x) = 0$. Define the operator $T : X \to \mathbb{R}^2$ by

$$Tx := \begin{cases} x & \text{if } |\xi_2| \le h(\xi_1) \\ (\xi_1, 0) & \text{if } |\xi_2| > h(\xi_1), \end{cases}$$

where $x = (\xi_1, \xi_2)$ (see Fig. 2.2).
The reader may check that Fej $T = [a, b] \times \{0\}$ and that Fix $T = \{x \in X : |\xi_2| \le h(\xi_1)\}$. If h is positive in at least one point, then Fej $T \ne$ Fix T. If, moreover, h is not concave, then Fix T is not convex.

Let $U_i : X \to X$, $i \in I := \{1, 2, \dots, m\}$, and:

(i) $U := \sum_{i\in I} \omega_i U_i$, where $w = (\omega_1, \omega_2, \dots, \omega_m) \in \Delta_m$ or
(ii) $U := U_m U_{m-1} \dots U_1$.

As we observed before, the following inclusion holds

$$\bigcap_{i\in I} \operatorname{Fix} U_i \subseteq \operatorname{Fix} U \tag{2.14}$$

(see Remark 2.1.1) and the converse inclusion holds in case (i) when all U_i, $i \in I$, are nonexpansive operators with a common fixed point and $w \in \operatorname{ri} \Delta_m$ (see Theorem 2.1.14). It turns out that, in both cases (i) and (ii), the inclusion converse to (2.14)) is true for strictly quasi-nonexpansive operators (see [22, Proposition 2.12], where the property was formulated for attracting operators). In case (i) this property is also true for a more general form of the operator $U = \sum_{i\in I} \omega_i U$, where the weights ω_i, $i \in I$, may depend on x.

Definition 2.1.24. A function $w : X \to \Delta_m$, with $w(x) = (\omega_1(x), \dots, \omega_m(x))$ is called a *weight function*.

Definition 2.1.25. Let $U_i : X \to \mathcal{H}$, $i \in I$. We say that the weight function $w : X \to \Delta_m$ is *appropriate with respect to the family* $\{U_i\}_{i\in I}$ or, shortly, *appropriate*, if for any $x \notin \bigcap_{i\in I} \operatorname{Fix} U_i$ there exists an index $j \in I$ such that

$$\omega_j(x) \parallel U_j x - x \parallel \neq 0. \tag{2.15}$$

Denote

$$I(x) := \{i \in I : x \notin \operatorname{Fix} U_i\} \tag{2.16}$$

for a family of operators $U_i : X \to \mathcal{H}$, $i \in I$. The subset $I(x)$ is called a subset of *violated constraints*. Note that w is appropriate if and only if

$$w_j(x) > 0 \text{ for some } j \in I(x) \tag{2.17}$$

and for any $x \notin \bigcap_{i\in I} \operatorname{Fix} U_i$ (or, equivalently, for any $x \in \mathcal{H}$ such that $I(x) \neq \emptyset$).

A weight function $w : X \to \operatorname{ri} \Delta_m$ is appropriate with respect to any family of operators $\{U_i\}_{i\in I}$ if:

(i) $w \in \operatorname{ri} \Delta_m$ is a vector of constant weights (this case was considered in [22, Proposition 2.12]), or if
(ii) $w_i(x) > 0$ for all $x \notin \operatorname{Fix} U_i$ and for all $i \in I$.

It is clear that property (2.15) is weaker than conditions (i) and (ii) above.

The following theorem extends important results of [22, Proposition 2.12], where $C = \bigcap_{i\in I} \operatorname{Fix} U_i$ and only constant weights are considered (see also [25, Proposition 2.5] for a related result). These extended results will be applied in further parts of the book.

Theorem 2.1.26. *Let the operators $U_i : X \to X$, $i \in I$, with $\bigcap_{i\in I} \operatorname{Fix} U_i \neq \emptyset$, be C-strictly quasi-nonexpansive, where $C \subseteq \bigcap_{i\in I} \operatorname{Fix} U_i$, $C \neq \emptyset$. If U has one of the following forms:*

(i) $U := \sum_{i\in I} \omega_i U_i$ and the weight function $w : X \to \Delta_m$ is appropriate,
(ii) $U := U_m U_{m-1} \dots U_1$,

then

$$\operatorname{Fix} U = \bigcap_{i\in I} \operatorname{Fix} U_i \tag{2.18}$$

and U is C-strictly quasi-nonexpansive.

Proof. The inclusion $\bigcap_{i\in I} \operatorname{Fix} U_i \subseteq \operatorname{Fix} U$ is obvious. Now we show that $\operatorname{Fix} U \subseteq \bigcap_{i\in I} \operatorname{Fix} U_i$. This inclusion is clear if $\bigcap_{i\in I} \operatorname{Fix} U_i = X$. Now suppose that $x \notin \bigcap_{i\in I} \operatorname{Fix} U_i$. Let $z \in \bigcap_{i\in I} \operatorname{Fix} U_i$. If $z \in C$, then the C-strict quasi nonexpansivity of U_i, $i \in I$, yields

$$\|U_i x - z\| < \|x - z\| \text{ for any } i \in I(x). \tag{2.19}$$

(i) Let $Ux = \sum_{i \in I} \omega_i(x) U_i x$, where the weight function $w : X \to \Delta_m$ is appropriate. Then the convexity of the norm, (2.19) and (2.15) yield

$$\begin{aligned} \|Ux - z\| &= \left\| \sum_{i \in I} \omega_i(x)(U_i x - z) \right\| \\ &\le \sum_{i \in I} \omega_i(x) \|U_i x - z\| \le \sum_{i \in I} \omega_i(x) \|x - z\| = \|x - z\|, \end{aligned}$$

where the second inequality is strict if $z \in C$.

(ii) Let $j := \min\{i \in I : x \notin \operatorname{Fix} U_i\}$. Then we have $U_j U_{j-1} \dots U_1 x = U_j x$ and (2.19) yields

$$\begin{aligned} \|Ux - z\| &= \|U_m \dots U_1 x - z\| \\ &= \left\| U_m \dots U_j x - z \right\| \\ &\le \left\| U_{m-1} \dots U_j x - z \right\| \\ &\le \dots \le \left\| U_j x - z \right\| \le \|x - z\|, \end{aligned}$$

where the latter inequality is strict if $z \in C$.

Now it is clear that $x \notin \operatorname{Fix} U$ because, otherwise, for $z \in C$, in both cases (i) and (ii) we would obtain

$$\|x - z\| = \|Ux - z\| < \|x - z\|,$$

a contradiction. We have proved that $\operatorname{Fix} U \subseteq \bigcap_{i \in I} \operatorname{Fix} U_i$. Hence, (2.18) holds and, in both cases (i) and (ii), U is C-strictly quasi-nonexpansive. □

Note that equality (2.18) needs not to be true for nonexpansive operators, even if they have a common fixed point.

Example 2.1.27. (cf. [22, Remark 2.11]) Let $X \subseteq \mathcal{H}$ be a subspace with $\dim X > 0$. Let $U_i : X \to X$, $U_i := -\operatorname{Id}$, $i = 1, 2$. We have $U_2 U_1 = \operatorname{Id}$, consequently, $\operatorname{Fix}(U_2 U_1) = X$, but $\operatorname{Fix} U_1 \cap \operatorname{Fix} U_2 = \{0\}$.

The assumption on the C-strict quasi nonexpansivity in Theorem 2.1.26 (i) can be weakened. In this case it suffices to suppose that all U_i are quasi-nonexpansive, $i \in I$, and at least one of them is C-strictly quasi-nonexpansive. The assumption that the weight function w is appropriate should be replaced in this case by a stronger one, namely: $w_j(x) > 0$ for all x such that $I(x) \neq \emptyset$ and for all $j \in I(x)$. We leave the proof of this fact to the reader.

A stronger version of the first part of Theorem 2.1.26 (ii) for two operators is stated below (cf. [346, Proposition 1(d) (i)]).

Theorem 2.1.28. *Let* $S : X \to X$ *be quasi-nonexpansive,* $T : X \to X$ *be strictly quasi-nonexpansive and* $\operatorname{Fix} S \cap \operatorname{Fix} T \neq \emptyset$. *Then* $\operatorname{Fix} ST = \operatorname{Fix} TS = \operatorname{Fix} S \cap \operatorname{Fix} T$. *Furthermore,* ST *is quasi-nonexpansive and* TS *is strictly quasi-nonexpansive.*

Proof. The inclusions $\operatorname{Fix} S \cap \operatorname{Fix} T \subseteq \operatorname{Fix} ST$ and $\operatorname{Fix} S \cap \operatorname{Fix} T \subseteq \operatorname{Fix} TS$ are clear.

(i) We prove that $\operatorname{Fix} ST \subseteq \operatorname{Fix} S \cap \operatorname{Fix} T$. The inclusion is obvious if $\operatorname{Fix} ST = \emptyset$. Suppose that $\operatorname{Fix} ST \neq \emptyset$ and let $x \in \operatorname{Fix} ST$ be such that $x \notin \operatorname{Fix} S \cap \operatorname{Fix} T$. We consider two cases:

(a) $x \in \operatorname{Fix} T$. Then $x = STx = Sx$, i.e., $x \in \operatorname{Fix} S$. Therefore, $x \in \operatorname{Fix} S \cap \operatorname{Fix} T$.
(b) $x \notin \operatorname{Fix} T$. Let $z \in \operatorname{Fix} S \cap \operatorname{Fix} T$. By the quasi nonexpansivity of S and by the strict quasi nonexpansivity of T, we have

$$\|x - z\| = \|STx - z\| \leq \|Tx - z\| < \|x - z\| .$$

In both cases we obtain a contradiction, which proves that $\operatorname{Fix} ST \subseteq \operatorname{Fix} S \cap \operatorname{Fix} T$.

(ii) We prove that $\operatorname{Fix} TS \subseteq \operatorname{Fix} T \cap \operatorname{Fix} S$. The inclusion is obvious if $\operatorname{Fix} TS = \emptyset$. Suppose that $\operatorname{Fix} TS \neq \emptyset$ and let $x \in \operatorname{Fix} TS$ be such that $x \notin \operatorname{Fix} T \cap \operatorname{Fix} S$. Consider two cases:

(a) $Sx \in \operatorname{Fix} T$. Then $x = TSx = Sx$, consequently, $x \in \operatorname{Fix} S$. Now we have $x = Sx \in \operatorname{Fix} T$, i.e., $x \in \operatorname{Fix} T \cap \operatorname{Fix} S$.
(b) $Sx \notin \operatorname{Fix} T$. Let $z \in \operatorname{Fix} T \cap \operatorname{Fix} S$. By the strict quasi nonexpansivity of T and by the quasi nonexpansivity of S, we have

$$\|x - z\| = \|TSx - z\| < \|Sx - z\| \leq \|x - z\| .$$

In both cases we obtain a contradiction, which proves that $\operatorname{Fix} TS \subseteq \operatorname{Fix} T \cap \operatorname{Fix} S$.

Let now $z \in \operatorname{Fix} TS = \operatorname{Fix} T \cap \operatorname{Fix} S$ and $x \in X$. We have

$$\|STx - z\| \leq \|Tx - z\| \leq \|x - z\| ,$$

i.e., ST is quasi-nonexpansive. Furthermore,

$$\|TSx - z\| \leq \|Sx - z\| \leq \|x - z\| ,$$

where the second inequality is strict if $x \notin \operatorname{Fix} S$ and the first one is strict if $x \in \operatorname{Fix} S$ and $x \notin \operatorname{Fix} T$. Hence, TS is strictly quasi-nonexpansive. $\square$

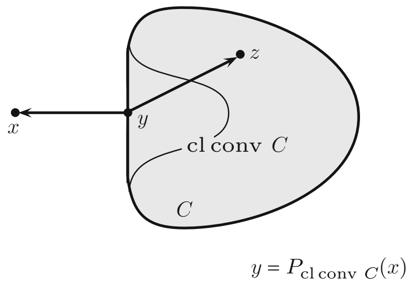


Fig. 2.3 $P_{\mathrm{cl\,conv}\,C}$ is a separator of C

Corollary 2.1.29. *Let* $U := U_m U_{m-1} \ldots U_1$*, where* $U_1, U_2, \ldots, U_{m-1} : X \to X$ *are quasi-nonexpansive,* $U_m : X \to X$ *is strictly quasi-nonexpansive and* $\bigcap_{i \in I} \operatorname{Fix} U_i \neq \emptyset$*. Then* $\operatorname{Fix} U = \bigcap_{i \in I} \operatorname{Fix} U_i$ *and* U *is strictly quasi-nonexpansive.*

Proof. The corollary follows from Theorem 2.1.28. We leave to the reader an easy proof by induction with respect to m. □

The assumption of Theorem 2.1.28 that T is strictly quasi-nonexpansive is essential. Note that the composition of quasi-nonexpansive operators needs not to be quasi-nonexpansive (see Example 2.1.52). Furthermore, the assumptions of Theorem 2.1.28 do not yield the strict quasi nonexpansivity of the operator ST (see Example 2.1.54).

2.1.3 Cutters and Strongly Quasi-nonexpansive Operators

Definition 2.1.30. Let $x \in \mathcal{H}$. We say that $y \in \mathcal{H}$ *separates a subset* $C \subseteq \mathcal{H}$ from x if

$$\langle x - y, z - y \rangle \le 0$$

for all $z \in C$. We say that an operator $T : X \to \mathcal{H}$ is a *separator of a subset* $C \subseteq X$ or T *separates a subset* C, if $y := Tx$ separates C from x for all $x \in \mathcal{H}$. We say that T is an α*-relaxed separator* of C, where $\alpha \in [0, 2]$, if T is an α-relaxation of a separator of C. Let T have a fixed point. We say that T is a *cutter* if T is a separator of $\operatorname{Fix} T$, i.e.,

$$\langle x - Tx, z - Tx \rangle \le 0 \tag{2.20}$$

for all $x \in X$ and all $z \in \operatorname{Fix} T$. We say that T is an α*-relaxed cutter*, where $\alpha \in [0, 2]$, if T is an α-relaxed separator of $\operatorname{Fix} T$.

For any nonempty $C \subseteq \mathcal{H}$ the projection $P_{\mathrm{cl\,conv}\,C}$ is a separator of C (see Fig. 2.3). In general, a separator of C is not uniquely determined.

The name *cutter* expresses the fact that, for any $x \notin \operatorname{Fix} T$, the hyperplane $H(x - Tx, \langle Tx, x - Tx \rangle)$ cuts the space into two half-spaces, one of which contains the point x while the other one contains the subset $\operatorname{Fix} T$ (see Fig. 2.4). In the literature one can find different names for cutters. Bauschke and Combettes call the class of

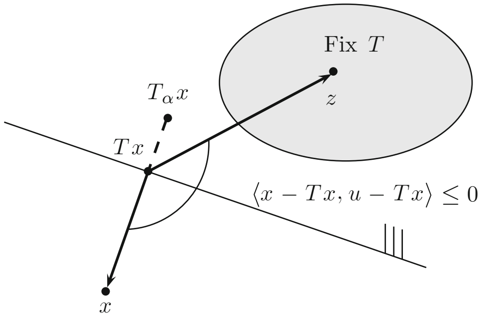


Fig. 2.4 A cutter and an α-relaxed cutter

cutters a $\mathcal{T}$*-class* (see [24, Definition 2.2] and [121, Definition 2.1]). Yamada and Ogura (see [346, Sect. B]) and Măruşter (see [254]) call the operators *firmly quasi-nonexpansive*. Zaknoon, Segal and Censor denoted cutters as *directed operators* (see [104–106, 307, 356]). In [69] these operators were called *separating operators*. The name *cutter* was proposed by Cegielski and Censor in [70].

Note that a separator and, in particular, a cutter need not to be continuous operators.

Remark 2.1.31. Let $T : X \to \mathcal{H}$ have a fixed point. Then, by Lemma 1.2.5, the operator T is a cutter if and only if

$$\langle Tx - x, z - x\rangle \geq \|Tx - x\|^2 \tag{2.21}$$

holds for all $x \in X$ and for all $z \in \operatorname{Fix} T$ (cf. [121, Proposition 2.3 (ii)]), and T is an α-relaxed cutter, where $\alpha \in [0, 2]$, if and only if

$$\alpha\langle Tx - x, z - x\rangle \geq \|Tx - x\|^2 \tag{2.22}$$

holds for all $x \in X$ and for all $z \in \operatorname{Fix} T$. Furthermore, if T is a cutter, then $T\,|_{\operatorname{Fix} T} = \operatorname{Id}$. Therefore, T is a cutter (respectively, an α-relaxed cutter) if and only if inequality (2.21) (respectively, (2.22)) is satisfied for all $x \notin \operatorname{Fix} T$ and for all $z \in \operatorname{Fix} T$. Relaxed cutters were also studied in [253, 255, 346] and in [249], where they were called *averaged quasi-nonexpansive* mappings.

Remark 2.1.32. Let $T : X \to \mathcal{H}$ be a separator of a subset $C \subseteq X$. Then the following obvious properties of T hold:

(i) T is a separator of the closed convex hull of C.
(ii) T is a separator of any subset $D \subseteq C$.
(iii) For an arbitrary $\lambda \in [0, 1]$, the relaxation T_λ of T is a separator of C.
(iv) $C \subseteq \operatorname{Fix} T$.

Corollary 2.1.33. *Let* $U : X \to \mathcal{H}$*,* $T := \frac{1}{2}(U + \operatorname{Id})$ *and* $C \subseteq X$*. Then:*

(i) U *is Fejér monotone with respect to* C *if and only if* T *is a separator of* C*.*

(ii) If Fix $U \neq \emptyset$*, then* U *is quasi-nonexpansive if and only if* T *is a cutter.*

Proof. The corollary follows easily from equivalence (2.11) (see [24, Proposition 2.3 (v)⇔(vi)] for a different proof). □

Corollary 2.1.34. *Let* $T : X \to \mathcal{H}$ *and* $C \subseteq X$*. If* T *is Fejér monotone with respect to* C*, then* T *is Fejér monotone with respect to the closed convex hull of* C*.*

Proof. The corollary follows directly from Corollary 2.1.33 (i) and from Remark 2.1.32 (i). □

By Remark 2.1.32 (iii), the right hand side of the equivalence in Corollary 2.1.33 (i) can be written in the form: U_λ *is a separator of* C *for all* $\lambda \in [0, \frac{1}{2}]$. Similarly, the right hand side of the equivalence in Corollary 2.1.33 (ii) can be written in the form: U_λ *is a cutter for all* $\lambda \in [0, \frac{1}{2}]$. Corollary 2.1.33 (ii) can also be written equivalently as follows:

U is quasi-nonexpansive if and only if there is a cutter $S : X \to H$ and $\mu \in [0, 2]$ such that $U = S_\mu$.

A subset $C \subseteq X$ for which the operator $T : X \to \mathcal{H}$ is a separator needs not to be convex. However there exists the largest subset for which T is a separator, which is closed and convex. This fact follows from the following lemma.

Lemma 2.1.35. *Let* $T : X \to \mathcal{H}$*. If the subset*

$$\operatorname{Sep} T := \bigcap_{x \in X} \{z \in X : \langle z - Tx, x - Tx \rangle \leq 0\}$$

is nonempty, then $\operatorname{Sep} T$ *is the largest subset for which* T *is a separator. Furthermore,* $\operatorname{Sep} T$ *is a closed convex subset.*

Proof. The first part of the lemma follows directly from Definition 2.1.30. The second part follows from the fact that an intersection of closed convex subsets is closed and convex. □

If T is nonexpansive, then Fix T is a closed convex subset (see Proposition 2.1.11). It turns out that cutters have the same property. The second part of the following lemma was proved in [24, Proposition 2.6 (i)–(ii)].

Lemma 2.1.36. *Let* $T : X \to \mathcal{H}$*. The following inclusion holds*

$$\operatorname{Sep} T \subseteq \operatorname{Fix} T. \tag{2.23}$$

If T *is a cutter, then a converse inclusion is also true. Hence, the subset of fixed points of a cutter is closed and convex.*

Proof. Let $y \in \operatorname{Sep} T$, i.e., $\langle x - Tx, y - Tx \rangle \leq 0$ for all $x \in X$. If we take $x = y$, we get $\|y - Ty\| \leq 0$, and hence, $Ty = y$, i.e., $y \in \operatorname{Fix} T$. Now suppose that T is a cutter and that $y \in \operatorname{Fix} T$. Then for any $x \in X$ we have $\langle y - Tx, x - Tx \rangle \leq 0\}$, i.e.,

$y \in \operatorname{Sep} T$. Therefore, we have $\operatorname{Sep} T = \operatorname{Fix} T$. The subset $\operatorname{Fix} T$ is closed convex as an intersection of closed convex subsets. □

It follows from Remark 2.1.32 (ii) and from Lemmas 2.1.35 and 2.1.36 that a cutter $T : X \to X$ is a separator of any nonempty subset of $\operatorname{Fix} T$. Therefore, we will restrict our further considerations of separators to cutters. Note, however, that the converse inclusion of (2.23) is not true in general (see Example 2.2.7). Hence, there is a separator with a fixed point which is not a cutter.

It is an immediate consequence of the characterization of the metric projection (see Theorem 1.2.4) that an operator $T : \mathcal{H} \to \mathcal{H}$ is a metric projection onto a closed convex subset if and only if $T^2 = T$ and T is a cutter (a more general fact will be presented in Theorem 2.2.5). In this case, we have $T = P_{\operatorname{Fix} T}$. Even if a cutter T is not idempotent, T is closely related to the metric projection. The following corollary was proved in [121, Proposition 2.3 (iii)].

Corollary 2.1.37. *Let $T : X \to \mathcal{H}$ be a cutter. Then, for any $x \in X$, it holds*

$$\|Tx - x\| \leq \|P_{\operatorname{Fix} T}x - x\| . \tag{2.24}$$

Proof. If $x \in \operatorname{Fix} T$, then inequality (2.24) is obvious. Now let $x \notin \operatorname{Fix} T$. Then it follows from inequality (2.21) for $z := P_{\operatorname{Fix} T}x$ together with the Cauchy–Schwarz inequality that

$$\|Tx - x\| \leq \frac{\langle Tx - x, P_{\operatorname{Fix} T}x - x\rangle}{\|Tx - x\|} \leq \|P_{\operatorname{Fix} T}x - x\|$$

which completes the proof. □

Definition 2.1.38. Let $\alpha \geq 0$ and assume that $T : X \to \mathcal{H}$ has a fixed point. We say that T is *α-strongly quasi-nonexpansive* (α-SQNE), if

$$\|Tx - z\|^2 \leq \|x - z\|^2 - \alpha \|Tx - x\|^2 \tag{2.25}$$

for all $x \in X$ and $z \in \operatorname{Fix} T$. If T satisfies (2.25) with $\alpha > 0$, then T is called *strongly quasi-nonexpansive* (*SQNE*).

A property which is more general than the strong nonexpansivity was introduced by Halperin [198, Sect. 2] and was called φ-property, where $\varphi : [0, \infty) \to [0, \infty)$ is a nondecreasing function. If $\varphi(t) = t^2$ for all $t \in [0, \infty)$, then φ-property is equivalent to the strong quasi-nonexpansivity. The notion strong quasi nonexpansivity was introduced by Bruck [50, Sect. 1] for operators defined on a metric space. Strongly quasi-nonexpansive operators are widely studied in the literature. Bauschke and Borwein use the name *strongly attracting operators* for operators which are NE and SQNE (see [22, Definition 2.1]). Reich and Zaslavski define a more general operator and call it a *uniformly F-attracting mapping*, where $F = \operatorname{Fix} T$ (see [297, Sect. 1]). Vasin and Ageev call the α-SQNE operators, where $\alpha \in (0, 1)$, *Q-pseudocontractive operators* (see [333, Definition 2.3]). Yamada and Ogura use the notation *α-attracting quasi-nonexpansive* for the α-SQNE operators [346]. Crombez denotes the class of α-SQNE operators by $\mathcal{F}^\alpha$ (see [126, pages 160–161]) and gives several equivalent conditions for $T \in \mathcal{F}^\alpha$ (see [126, Theorem 2.1]).

It follows easily from the equivalence (a)$\Leftrightarrow$(c) of Lemma 1.2.5 that an operator T which has a fixed point is a cutter if and only if it is 1-strongly quasi-nonexpansive. The following theorem extends this property to relaxations of T (cf. [121, Proposition 2.3 (ii)]).

Theorem 2.1.39. *Assume that $T : X \to \mathcal{H}$ has a fixed point and let $\lambda \in (0, 2]$. Then T is a cutter if and only if its relaxation T_λ is $\frac{2-\lambda}{\lambda}$-strongly quasi-nonexpansive, i.e.,*

$$\|T_\lambda x - z\|^2 \leq \|x - z\|^2 - \frac{2-\lambda}{\lambda} \|T_\lambda x - x\|^2 \tag{2.26}$$

for all $x \in X$ and for all $z \in \operatorname{Fix} T$.

Proof. Since

$$T_\lambda x - x = \lambda(Tx - x),$$

the properties of the inner product yield

$$\begin{aligned}
&\|T_\lambda x - z\|^2 - \|x - z\|^2 + \frac{2-\lambda}{\lambda} \|T_\lambda x - x\|^2 \\
&= \|x - z + \lambda(Tx - x)\|^2 - \|x - z\|^2 + \lambda(2-\lambda) \|Tx - x\|^2 \\
&= 2\lambda(\|Tx - x\|^2 - \langle z - x, Tx - x\rangle) \\
&= 2\lambda\langle z - Tx, x - Tx\rangle
\end{aligned}$$

for all $x \in X$ and for all $z \in C$. The assertion follows directly from the equalities above. $\square$

The following corollary is an equivalent formulation of Theorem 2.1.39.

Corollary 2.1.40. *Assume that $U : X \to \mathcal{H}$ has a fixed point and let $\alpha \in (0, 2]$. Then U is an α-relaxed cutter if and only if U is $\frac{2-\alpha}{\alpha}$-strongly quasi-nonexpansive.*

In general, a relaxation T_λ of a cutter T with $\lambda \geq 2$ needs not to be strongly quasi-nonexpansive. Nevertheless, the following proposition holds.

Proposition 2.1.41. *Let $T : X \to \mathcal{H}$ be a cutter with* $\operatorname{int}\operatorname{Fix} T \neq \emptyset$ *and $\lambda > 0$. Then for any $z \in \operatorname{int}\operatorname{Fix} T$ and $x \notin \operatorname{Fix} T$ it holds*

$$\|T_\lambda x - z\|^2 \leq \|x - z\|^2 - \lambda\left(2 + \frac{2\delta}{\|Tx - x\|} - \lambda\right) \|Tx - x\|^2, \tag{2.27}$$

where $\delta > 0$ is such that $B(z, \delta) \subseteq \operatorname{Fix} T$. If X is bounded, then T_λ is $\operatorname{int}\operatorname{Fix} T$-strictly quasi-nonexpansive for any $\lambda \in (0, 2]$.

Proof. Let $z \in \operatorname{int}\operatorname{Fix} T$ and $x \notin \operatorname{Fix} T$. Then $w := z - \delta \frac{Tx-x}{\|Tx-x\|} \in \operatorname{Fix} T \subseteq X$ and inequality (2.21) yields

$$\begin{aligned}
\|T_\lambda x - z\|^2 &= \|x + \lambda(Tx - x) - z\|^2 \\
&= \|x - z\|^2 + \lambda^2 \|Tx - x\|^2 - 2\lambda\langle z - x, Tx - x\rangle \\
&= \|x - z\|^2 + \lambda^2 \|Tx - x\|^2 \\
&\quad -2\lambda\langle z - w, Tx - x\rangle - 2\lambda\langle w - x, Tx - x\rangle \\
&\le \|x - z\|^2 + \lambda^2 \|Tx - x\|^2 \\
&\quad -2\lambda\delta \|Tx - x\| - 2\lambda \|Tx - x\|^2 \\
&= \|x - z\|^2 - \lambda(2 + \frac{2\delta}{\|Tx - x\|} - \lambda) \|Tx - x\|^2 .
\end{aligned}$$

Let X be bounded and $d > 0$ be such that $\|Tu - u\| \le d$ for any $u \in X$. The existence of such d follows from Corollary 2.1.37. Denote $\varepsilon := \frac{2\delta}{d}$. Then (2.27) yields

$$\|T_\lambda x - z\|^2 \le \|x - z\|^2 - \lambda(2 + \varepsilon - \lambda) \|Tx - x\|^2 .$$

Consequently, T_λ is $\operatorname{int}\operatorname{Fix} T$-strictly quasi-nonexpansive for any $\lambda \in (0, 2]$. □

The corollary below follows immediately from Proposition 2.1.41 and from Theorem 2.1.26.

Corollary 2.1.42. *Let $U_i : X \to \mathcal{H}$, $i \in I$ be quasi-nonexpansive with $C := \bigcap_{i\in I} \operatorname{Fix} U_i \neq \emptyset$ and let $U := U_m U_{m-1} \ldots U_1$. If $\operatorname{int} C \neq \emptyset$, then $\operatorname{Fix} U = \bigcap_{i\in I} \operatorname{Fix} U_i$ and U is int C-strictly quasi-nonexpansive.*

An equivalent formulation of the following result appeared in [127, Theorem 3.2 (iii)].

Corollary 2.1.43. Assume that $U : X \to \mathcal{H}$ has *a fixed point and let $\beta \ge 0$. Then U is β-strongly quasi-nonexpansive if and only if U is a $\frac{2}{\beta+1}$-relaxed cutter.*

Proof. It suffices to take $\alpha = \frac{2}{\beta+1}$ in Corollary 2.1.40. □

Remark 2.1.44. Assume that $T : X \to \mathcal{H}$ has a fixed point and is α-strongly quasi-nonexpansive, where $\alpha \ge 0$.

(i) If $\alpha = 0$, then T is quasi-nonexpansive.
(ii) T is γ-strongly quasi-nonexpansive for all $\gamma \in [0, \alpha]$.
(iii) If $\alpha > 0$, then T is strictly quasi-nonexpansive. Therefore, all properties of strictly quasi-nonexpansive operators are also valid for strongly quasi-nonexpansive operators and for cutters.

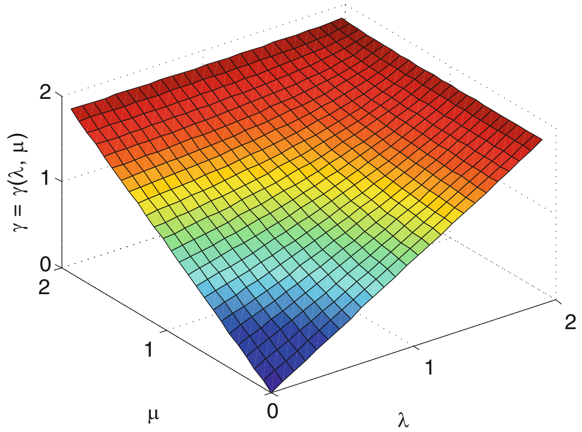


Fig. 2.5 Solution of (2.28) as a function of $\lambda, \mu \in (0, 2)$

Cutters and strongly quasi-nonexpansive operators play an important role in methods presented in further parts of the book. Therefore, we focus our attention on the properties of these operators which enable us to construct new cutters or strongly quasi-nonexpansive operators. Below, we show that a family of relaxed cutters is closed under composition and under convex combination of operators having a common fixed point. The first property of relaxed cutters follows from the lemma below whose proof is left to the reader.

Lemma 2.1.45. *Let* $\lambda, \mu \in (0, 2)$. *The unique solution* γ *of the equation*

$$\left(\frac{1-\frac{2}{\gamma}}{2}\right)^2 = \left(\frac{1}{\lambda}-\frac{1}{\gamma}\right)\left(\frac{1}{\mu}-\frac{1}{\gamma}\right) \tag{2.28}$$

is

$$\gamma = \frac{2}{\left(\frac{\lambda}{2-\lambda}+\frac{\mu}{2-\mu}\right)^{-1}+1} = \frac{4(\lambda+\mu-\lambda\mu)}{4-\lambda\mu}. \tag{2.29}$$

Moreover,

$$0 < \min\{\lambda, \mu\} < \frac{4\min\{\lambda, \mu\}}{\min\{\lambda, \mu\}+2} \le \gamma \le \frac{4\max\{\lambda, \mu\}}{\max\{\lambda, \mu\}+2} < 2.$$

A solution of (2.28) is illustrated in Fig. 2.5.

Theorem 2.1.46. *Let* $T : X \to X$ *be a* λ*-relaxed cutter,* $U : X \to X$ *be a* μ*-relaxed cutter, where* $\lambda, \mu \in (0, 2]$, *and let* $\operatorname{Fix} T \cap \operatorname{Fix} U \neq \emptyset$. *If* $\lambda, \mu \in (0, 2)$, *then* UT *is a* γ*-relaxed cutter, where* γ *is given by (2.29). If* $\lambda = 2$ *and* $\mu < 2$ *or* $\mu = 2$ *and* $\lambda < 2$, *then* UT *is a quasi-nonexpansive operator or, equivalently,* UT *is a* 2*-relaxed cutter.*

Proof. Suppose that T is a λ-relaxed cutter and that U is a μ-relaxed cutter. Take $a := Tx - x$ and $b := UTx - Tx$. Then it follows from inequality (2.22) that $\langle z - x, a\rangle \geq \frac{1}{\lambda}\|a\|^2$ and $\langle z - Tx, b\rangle \geq \frac{1}{\mu}\|b\|^2$ for any $z \in \operatorname{Fix} U \cap \operatorname{Fix} T$.

Let $\lambda, \mu \in (0, 2)$ and γ be defined by (2.29). Then Lemma 2.1.45 yields

$$\begin{aligned}
&\langle z - x, UTx - x\rangle - \frac{1}{\gamma}\|UTx - x\|^2 \\
&= \langle z - x, a + b\rangle - \frac{1}{\gamma}\|a + b\|^2 \\
&= \langle z - x, a\rangle + \langle z - x, b\rangle - \frac{1}{\gamma}\|a + b\|^2 \\
&= \langle z - x, a\rangle + \langle z - Tx, b\rangle + \langle a, b\rangle - \frac{1}{\gamma}\|a + b\|^2 \\
&\geq \frac{1}{\lambda}\|a\|^2 + \frac{1}{\mu}\|b\|^2 + \langle a, b\rangle - \frac{1}{\gamma}\|a + b\|^2 \\
&= (\frac{1}{\lambda} - \frac{1}{\gamma})\|a\|^2 + (\frac{1}{\mu} - \frac{1}{\gamma})\|b\|^2 + (1 - \frac{2}{\gamma})\langle a, b\rangle \\
&= \left\| \sqrt{\frac{1}{\lambda} - \frac{1}{\gamma}}a - \sqrt{\frac{1}{\mu} - \frac{1}{\gamma}}b \right\|^2 \geq 0.
\end{aligned}$$

Applying inequality (2.22) we obtain that UT is a γ-relaxed cutter. If $\lambda = 2$ and $\mu < 2$ or $\mu = 2$ and $\lambda < 2$, then UT is quasi-nonexpansive by Theorem 2.1.28. □

The following result is due to Yamada and Ogura (see [346, Proposition 1(d)]).

Corollary 2.1.47. *Let $T, U : X \to X$ have a common fixed point and $\rho, \sigma > 0$. If T is ρ-SQNE and U is σ-SQNE, then UT is δ-SQNE, where*

$$\delta = \frac{1}{\frac{1}{\rho} + \frac{1}{\sigma}}. \tag{2.30}$$

Proof. Suppose that T is ρ-SQNE and U is σ-SQNE. It follows from Corollary 2.1.43 that T is a λ-relaxed cutter and that U is a μ-relaxed cutter, where $\lambda = \frac{2}{1+\rho}$ and $\mu = \frac{2}{1+\sigma}$. By Theorem 2.1.46 the operator UT is a γ-relaxed cutter, where

$$\gamma = \frac{2}{(\frac{\lambda}{2-\lambda} + \frac{\mu}{2-\mu})^{-1} + 1} = \frac{2}{(\frac{1}{\rho} + \frac{1}{\sigma})^{-1} + 1}.$$

Corollary 2.1.40 yields now that UT is δ-SQNE, where δ is given by (2.30). □

Theorem 2.1.48. *Let $T_i : X \to X$ be an α_i-relaxed cutter, where $\alpha_i \in (0, 2)$, $i \in I := \{1, 2, \ldots, m\}$, or, equivalently, T_i be β_i-strongly quasi-nonexpansive, where $\beta_i = \frac{2-\alpha_i}{\alpha_i} \in (0, +\infty)$, $i \in I$. Let $\bigcap_{i \in I} \operatorname{Fix} T_i \neq \emptyset$ and $U_m := T_m T_{m-1} \ldots T_1$. Then:*

(i) The operator U_m is a γ_m-relaxed cutter, with

$$\gamma_m = \frac{2}{(\frac{\alpha_1}{2-\alpha_1} + \frac{\alpha_2}{2-\alpha_2} + \ldots + \frac{\alpha_m}{2-\alpha_m})^{-1} + 1}. \tag{2.31}$$

(ii) The operator U_m is δ_m-strongly quasi-nonexpansive, with

$$\delta_m = \frac{1}{\frac{1}{\beta_1} + \frac{1}{\beta_2} + \ldots + \frac{1}{\beta_m}}. \tag{2.32}$$

Moreover,

$$0 < \min_{i \in I} \alpha_i < \frac{2m \min_{i \in I} \alpha_i}{(m-1) \min_{i \in I} \alpha_i + 2} \leq \gamma_m \leq \frac{2m \max_{i \in I} \alpha_i}{(m-1) \max_{i \in I} \alpha_i + 2} < 2 \tag{2.33}$$

and

$$0 < \frac{\min_{i \in I} \beta_i}{m} \leq \delta_m \leq \frac{\max_{i \in I} \beta_i}{m}. \tag{2.34}$$

Proof. The assertion is obvious for $m = 1$. Note that $\gamma_m = \frac{2}{\delta_m + 1}$ and that Corollary 2.1.43 yields the equivalence of conditions (i) and (ii). We prove by induction with respect to m that these conditions hold for any $m \geq 2$.

1^0 If $m = 2$, then conditions (i) and (ii) follow directly from Theorem 2.1.46 and from Corollary 2.1.47.

2^0 Suppose that (ii) is true for some $m = k$. Consequently, U_k is δ_k-SQNE. It follows now from Corollary 2.1.47 that the operator $U_{k+1} = T_{k+1} U_k$ is δ-SQNE, where

$$\delta = \frac{1}{\frac{1}{\delta_k} + \frac{1}{\beta_{k+1}}} = \frac{1}{\frac{1}{\beta_1} + \frac{1}{\beta_2} + \ldots + \frac{1}{\beta_k} + \frac{1}{\beta_{k+1}}} = \delta_{k+1}.$$

Now, for $m = k + 1$, equality (2.31) follows from the above mentioned equivalence of (i) and (ii).

Hence, we have proved that conditions (i) and (ii) hold for all $m \geq 1$. Both inequalities in (2.34) follow immediately from equality (2.32). Now we have

$$\gamma_m = \frac{2}{\delta_m + 1} \leq \frac{2}{\frac{\min_{i \in I} \beta_i}{m} + 1} = \frac{2}{\frac{\min_{i \in I} \frac{2-\alpha_i}{\alpha_i}}{m} + 1} = \frac{2m \max_{i \in I} \alpha_i}{(m-1) \max_{i \in I} \alpha_i + 2} < 2.$$

In a similar way one can prove that

$$\gamma_m \geq \frac{2m \min_{i\in I} \alpha_i}{(m-1)\min_{i\in I} \alpha_i + 2} > \min_{i\in I} \alpha_i > 0$$

which completes the proof. □

Bauschke and Borwein proved that a composition of β_i-SQNE operators with a common fixed point is β-SQNE for $\beta := \frac{\min_{i\in I} \beta_i}{2^{m-1}}$ (see [22, Theorem 2.10 (ii)]). It is clear that this result is weaker than Theorem 2.1.48 (ii), because $\beta \leq \frac{\min_{i\in I} \beta_i}{m} \leq \delta_m$. Note that the first inequality is strict for $m > 2$ and that the other one is strict if $\beta_i \neq \beta_j$ for at least one pair $i, j \in I$.

Corollary 2.1.49. *Let $U_i : X \to \mathcal{H}$ be cutters with a common fixed point, $i \in I := \{1, 2, \ldots, m\}$, and $w : X \to \Delta_m$ be an appropriate weight function. Then the operator $U := \sum_{i\in I} \omega_i U_i$ is a cutter.*

Proof. Let $U := \sum_{i\in I} \omega_i U_i$. It is clear that a cutter is strictly quasi-nonexpansive (see Remark 2.1.44 (iii)). Therefore, it follows from Theorem 2.1.26 (i) that $\operatorname{Fix} U = \bigcap_{i\in I} \operatorname{Fix} U_i$. By Remark 2.1.31 and by the convexity of the function $\|\cdot\|^2$, we have

$$\begin{aligned}
\langle Ux - x, z - x\rangle &= \sum_{i\in I} \omega_i(x)\langle U_i x - x, z - x\rangle \\
&\geq \sum_{i\in I} \omega_i(x)\,\|U_i x - x\|^2 \\
&\geq \left\| \sum_{i\in I} \omega_i(x) U_i x - x \right\|^2 \\
&= \|Ux - x\|^2
\end{aligned}$$

for all $x \in X$ and all $z \in \operatorname{Fix} U$. Again, by Remark 2.1.31, U is a cutter. □

Theorem 2.1.50. *Let $T_i : X \to \mathcal{H}$ be an α_i-relaxed cutter, where $\alpha_i \in (0, 2)$, $i \in I := \{1, 2, \ldots, m\}$, or, equivalently, T_i be β_i-strongly quasi-nonexpansive, where $\beta_i = \frac{2-\alpha_i}{\alpha_i} \in (0, +\infty)$, $i \in I$. Let $\bigcap_{i\in I} \operatorname{Fix} T_i \neq \emptyset$ and $w \in \Delta_m$. Then the operator $T := \sum_{i\in I} \omega_i T_i$ is an α-relaxed cutter with*

$$\alpha := \sum_{i\in I} \omega_i \alpha_i. \tag{2.35}$$

Consequently, T is β-SQNE, with

$$\beta := (\sum_{i\in I} \frac{\omega_i}{\beta_i + 1})^{-1} - 1. \tag{2.36}$$

Moreover,

$$0 < \min_{i \in I} \alpha_i \leq \alpha \leq \max_{i \in I} \alpha_i < 2 \tag{2.37}$$

and

$$0 < \min_{i \in I} \beta_i \leq \beta \leq \max_{i \in I} \beta_i. \tag{2.38}$$

Proof. Without loss of generality we suppose that $w \in \operatorname{ri} \Delta_m$. Let $U_i := (T_i)_{\alpha_i^{-1}}$, i.e.,

$$U_i = \mathrm{Id} + \frac{1}{\alpha_i}(T_i - \mathrm{Id}).$$

It is clear that U_i are cutters, $i \in I$. Let α be defined by (2.35) and $\nu_i := \frac{\omega_i \alpha_i}{\alpha}$, $i \in I$. Note that $\nu = (\nu_1, \nu_2, \ldots, \nu_m) \in \operatorname{ri} \Delta_m$, consequently, ν is appropriate. Define $U := \sum_{i \in I} \nu_i U_i$. By Corollary 2.1.49, the operator U is a cutter. We have

$$U = \sum_{i \in I} \nu_i U_i = \mathrm{Id} + \sum_{i \in I} \frac{\nu_i}{\alpha_i}(T_i - \mathrm{Id}) = \mathrm{Id} + \frac{1}{\alpha} \sum_{i \in I} \omega_i (T_i - \mathrm{Id}) = \mathrm{Id} + \frac{1}{\alpha}(T - \mathrm{Id}),$$

i.e., $T = \mathrm{Id} + \alpha(U - \mathrm{Id})$ and T is an α-relaxed cutter. The second part of the theorem follows now immediately from Corollaries 2.1.40 and 2.1.43. Inequalities in (2.37) are obvious and inequalities in (2.38) follow easily from (2.36). □

Bauschke and Borwein proved that a convex combination of β_i-SQNE operators, $i \in I$, with a common fixed point is β-SQNE, where $\beta := \min_{i \in I} \beta_i$ (see [22, Proposition 2.12]). By inequality (2.38) this result is weaker than Theorem 2.1.50. Note that this inequality is strict if $\beta_i \neq \beta_j$ for at least one pair $i, j \in I$ for which ω_i and ω_j are nonzero. The second part of Theorem 2.1.50 for $m = 2$ was proved by Yamada and Ogura (see [346, Proposition 1(c)]).

The following important result extends Theorem 2.1.39.

Theorem 2.1.51. *Let $S : \mathcal{H} \to X$ be nonexpansive, $T : X \to \mathcal{H}$ be a cutter and $\lambda \in (0, 2)$. If $\operatorname{Fix} S \cap \operatorname{Fix} T \neq \emptyset$, then, for any $x \in \operatorname{Fix} S$ and $z \in \operatorname{Fix} S \cap \operatorname{Fix} T$, the following estimations hold*

$$\|ST_\lambda x - z\|^2 \leq \|x - z\|^2 - \lambda(2 - \lambda) \|Tx - x\|^2 \tag{2.39}$$

and

$$\|ST_\lambda x - z\|^2 \leq \|x - z\|^2 - \frac{2 - \lambda}{\lambda} \|ST_\lambda x - x\|^2. \tag{2.40}$$

Consequently, the operator $ST_\lambda |_{\operatorname{Fix} S}$ is $\frac{2-\lambda}{\lambda}$-strongly quasi-nonexpansive.

Proof. Let $x \in \operatorname{Fix} S$ and $z \in \operatorname{Fix} S \cap \operatorname{Fix} T$. Then the assumptions that S is nonexpansive and T is a cutter yield

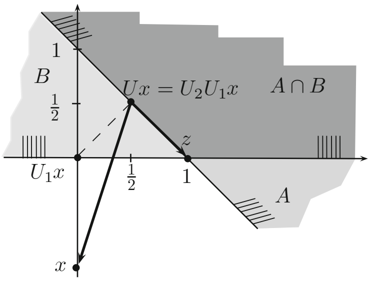


Fig. 2.6 Composition of cutters needs not to be a cutter

$$
\begin{aligned}
\|ST_\lambda x - z\|^2 &= \|ST_\lambda x - Sz\|^2 \le \|T_\lambda x - z\|^2 \\
&= \|x-z\|^2 + \lambda^2 \|Tx - x\|^2 - 2\lambda\langle z-x, Tx-x\rangle \\
&\le \|x-z\|^2 - \lambda(2-\lambda)\|Tx-x\|^2 \\
&= \|x-z\|^2 - \frac{2-\lambda}{\lambda}\|T_\lambda x - x\|^2 \\
&\le \|x-z\|^2 - \frac{2-\lambda}{\lambda}\|ST_\lambda x - Sx\|^2 \\
&= \|x-z\|^2 - \frac{2-\lambda}{\lambda}\|ST_\lambda x - x\|^2
\end{aligned}
$$

which completes the proof. □

Below we give several examples which show that a composition of quasi-nonexpansive operators does not need to be quasi-nonexpansive, that a composition of a strictly quasi-nonexpansive operator and a quasi-nonexpansive one does not need to be strictly quasi-nonexpansive and that a composition of cutters does not need to be a cutter, even if they have a common fixed point.

Example 2.1.52. Let $X := [-1,1] \subseteq \mathbb{R}$, $S, T : X \to X$, $S := -\mathrm{Id}$ and

$$Tx := \begin{cases} -x & \text{if } x = 1 \\ \frac{1}{2}x & \text{otherwise.} \end{cases}$$

One can easily check that S, T are quasi-nonexpansive, $\operatorname{Fix} S = \operatorname{Fix} T = \{0\}$ and $\operatorname{Fix} ST = \{0, 1\}$. The operator ST is not quasi-nonexpansive, because a subset of fixed points of a quasi-nonexpansive operator is convex (see Lemma 2.1.21).

Example 2.1.53. Let $X = \mathcal{H} := \mathbb{R}^2$, $A := \{x \in \mathbb{R}^2 : \langle e, x\rangle \ge 1\}$, $B := \{x \in \mathbb{R}^2 : \xi_2 \ge 0\}$, $U_1 := P_B$, $U_2 := P_A$ and $U := U_2U_1$. Then U_1 and U_2 are cutters and it follows from Theorem 2.1.26 that $\operatorname{Fix} U = \operatorname{Fix} U_1 \cap \operatorname{Fix} U_2 = A \cap B \neq \emptyset$. For $x = (0,-1)$ and $z = (1,0) \in A \cap B$, we have $Ux = (\frac{1}{2}, \frac{1}{2})$ and $\langle x - Ux, z - Ux\rangle = \frac{1}{2}$ (see Fig. 2.6). Therefore, U is not a cutter.

Example 2.1.54. Let $A, B \subseteq \mathcal{H}$ be nonempty closed convex subsets and $A \subsetneq B$. Define $S := 2P_A - \mathrm{Id}$ and $T := P_B$. We have $\operatorname{Fix} S \cap \operatorname{Fix} T = A$. It follows easily from the characterization of the metric projection that P_A and P_B are cutters. By Theorem 2.1.39, T is strictly quasi-nonexpansive and S is quasi-nonexpansive. By Theorem 2.1.28, the operator ST is quasi-nonexpansive. Unfortunately, ST is not strictly quasi-nonexpansive, because for any $x \in B \backslash A$ and for $z := P_A x$ it holds

$$\|STx - z\| = \|Sx - z\| = \|x - z\| .$$

2.2 Firmly Nonexpansive Operators

Definition 2.2.1. We say that an operator $T : X \to \mathcal{H}$ is *firmly nonexpansive* (*FNE*), if

$$\langle Tx - Ty, x - y \rangle \geq \|Tx - Ty\|^2 \tag{2.41}$$

for all $x, y \in X$. Let $\lambda \in [0, 2]$. We say that $T : X \to \mathcal{H}$ is λ*-relaxed firmly nonexpansive* (λ*-RFNE*) or, shortly, *relaxed firmly nonexpansive (RFNE)* if T is a λ-relaxation of a firmly nonexpansive operator U, i.e., $T = U_\lambda = (1-\lambda)\,\mathrm{Id} + \lambda U$. If, furthermore, $\lambda \in (0, 2)$, then we say that T is *strictly relaxed firmly nonexpansive*.

The definition of a firmly nonexpansive operator in a Hilbert space is due to Browder (see [46]), who called it a *firmly contractive operator*. Bruck introduced the name firmly nonexpansive for operators in a Banach space (see [49, Definition 6]). In Hilbert spaces both definitions coincide, as we will show in Theorem 2.2.10. Condition (vi) of this theorem is, actually, the definition of a firmly nonexpansive operator proposed by Bruck.

The following lemma is obvious.

Lemma 2.2.2. *Let $T : X \to \mathcal{H}$ and $x, y \in X$. The following inequalities are equivalent:*

(i) $\langle Tx - Ty, x - y \rangle \geq \|Tx - Ty\|^2$,
(ii) $\langle Tx - Ty, (x - Tx) - (y - Ty) \rangle \geq 0$,
(iii) $\langle Ty - Tx, x - Tx \rangle + \langle Tx - Ty, y - Ty \rangle \leq 0$,
(iv) $\langle Ty - x, Tx - x \rangle + \langle Tx - y, Ty - y \rangle \geq \|Tx - x\|^2 + \|Ty - y\|^2$.

It follows from Lemma 2.2.2 that inequality (2.41) defining a firmly nonexpansive operator can be replaced by any inequality in (i)–(iv).

Corollary 2.2.3. *Let $\lambda > 0$. An operator $S : X \to \mathcal{H}$ is λ-RFNE if and only if*

$$\langle y - x, Sx - x \rangle + \langle x - y, Sy - y \rangle \geq \frac{1}{\lambda} \|(Sx - x) - (Sy - y)\|^2 . \tag{2.42}$$

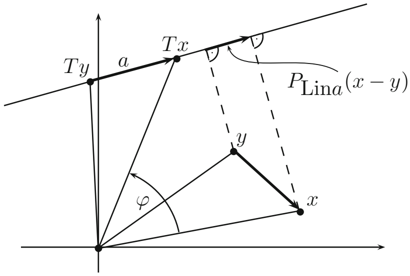


Fig. 2.7 NE and monotone operator which is not FNE

Proof. Let $S := T_\lambda = \mathrm{Id} + \lambda(T - \mathrm{Id})$ for a firmly nonexpansive operator $T : X \to \mathcal{H}$. Let $x, y \in X$. It follows from the equivalence (i)$\Leftrightarrow$(iv) in Lemma 2.2.2 and from the equality $T = S_{\lambda^{-1}}$ (see Remark 2.1.3) that S is λ-RFNE if and only if

$$\langle Ty - x, Sx - x\rangle + \langle Tx - y, Sy - y\rangle \geq \frac{1}{\lambda}(\|Sx - x\|^2 + \|Sy - y\|^2).$$

Since $Ty - x = y - x + \frac{1}{\lambda}(Sy - y)$ and $Tx - y = x - y + \frac{1}{\lambda}(Sx - x)$, the last inequality is equivalent to

$$\langle y-x, Sx-x\rangle + \langle x-y, Sy-y\rangle + \frac{2}{\lambda}\langle Sx-x, Sy-y\rangle \geq \frac{1}{\lambda}(\|Sx - x\|^2 + \|Sy - y\|^2).$$

The latter inequality is equivalent to (2.42). $\square$

2.2.1 Basic Properties of Firmly Nonexpansive Operators

Theorem 2.2.4. *A firmly nonexpansive operator $T : X \to \mathcal{H}$ is monotone and nonexpansive.*

Proof. Let T be firmly nonexpansive. By the Cauchy–Schwarz inequality, we have

$$\|Tx - Ty\| \cdot \|x - y\| \geq \langle Tx - Ty, x - y\rangle \geq \|Tx - Ty\|^2 \geq 0,$$

for all $x, y \in X$, which yields the monotonicity and the nonexpansivity of T. $\square$

The converse of Theorem 2.2.4 is not true, e.g., the operator $T : \mathbb{R}^2 \to \mathbb{R}^2$,

$$Tx := (\xi_1 \cos\varphi - \xi_2 \sin\varphi, \xi_1 \sin\varphi + \xi_2 \cos\varphi)$$

is nonexpansive and monotone for $\varphi \in (0, \pi/2)$, but T is not firmly nonexpansive (see Fig. 2.7).

Now we prove a property of firmly nonexpansive operators, which also appears in Theorem 1.2.4. In particular, the characterization of the metric projection is, actually, a corollary of the following theorem which is due to Goebel and Reich (see [185, pp. 43–44]).

Theorem 2.2.5. *Let $T : X \to \mathcal{H}$ be an operator with a fixed point.*

(i) If T is firmly nonexpansive, then T is a cutter, i.e.,

$$\langle z - Tx, x - Tx\rangle \le 0 \tag{2.43}$$

for all $x \in X$ and $z \in \operatorname{Fix} T$.

(ii) If T is a projection, i.e., $T(X) = \operatorname{Fix} T$, then the implication converse to (i) *is also true. In this case, $T = P_{\operatorname{Fix} T}$.*

Proof. (i) Let T be firmly nonexpansive, $x \in X$ and $z \in \operatorname{Fix} T$. By the equivalence (i)$\Leftrightarrow$(iii) in Lemma 2.2.2, we have

$$\langle Ty - Tx, x - Tx\rangle + \langle Tx - Ty, y - Ty\rangle \le 0,$$

and for $y = z \in \operatorname{Fix} T$ we obtain (2.43).

(ii) Suppose that T is a projection and that inequality (2.43) holds for all $x \in X$ and $z \in \operatorname{Fix} T$. Let $u, v \in X$. Taking $x = u$ and $z = Tv$ in (2.43) we get

$$\langle Tv - Tu, u - Tu\rangle \le 0, \tag{2.44}$$

and, taking $x = v$ and $z = Tu$ in (2.43), we get

$$\langle Tu - Tv, v - Tv\rangle \le 0. \tag{2.45}$$

Note that, in both cases, $z \in \operatorname{Fix} T$ because $T(X) = \operatorname{Fix} T$. Therefore, the characterization of the metric projection yields that $T = P_{\operatorname{Fix} T}$. Summing up inequalities (2.44) and (2.45) we get

$$\langle Tu - Tv, (Tu - Tv) - (u - v)\rangle \le 0,$$

i.e., T is firmly nonexpansive (see equivalence (i)$\Leftrightarrow$(ii) in Lemma 2.2.2). □

Suppose that $\operatorname{Fix} T \neq \emptyset$. It follows from the equivalence of (i) and (iii) in Lemma 2.2.2 that inequality (2.41) for $y = z \in \operatorname{Fix} T$ gives (2.43). Therefore, for T being a cutter, inequality (2.41) is required for all $x \in X$ and all $y \in \operatorname{Fix} T$, while for T being firmly nonexpansive this inequality should hold for all $x, y \in X$.

Remark 2.2.6. Neither a projection nor a separator of a nonempty subset $C \subseteq \mathcal{H}$ need to be nonexpansive (note that a separator can even be discontinuous). Furthermore, a nonexpansive separator and even a nonexpansive cutter need not to be firmly nonexpansive (see Examples 2.2.7 and 2.2.8 below).

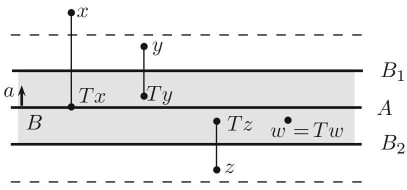


Fig. 2.8 NE separator which is not FNE

Example 2.2.7. (cf. [204] and [78, Sect. 4.10]) Let $a \in \mathcal{H}$, $\|a\| = 1$ and $\alpha > 0$. Furthermore, let $A := \{x \in \mathcal{H} : \langle a, x\rangle = 0\}$, $B_1 := \{x \in \mathcal{H} : \langle a, x\rangle = \alpha\}$, $B_2 := \{x \in \mathcal{H} : \langle a, x\rangle = -\alpha\}$ and $B := \{x \in \mathcal{H} : |\langle a, x\rangle| \leq \alpha\}$. The subset B is a band with a width of 2α and is bounded by two hyperplanes B_1 and B_2. The hyperplane A cuts the band B into two bands bounded by A and B_1 and by A and B_2. Define the operator $T : \mathcal{H} \to \mathcal{H}$ as follows

$$Tx = \begin{cases} P_A x & \text{if } |\langle a, x\rangle| \geq 2\alpha \\ 2P_{B_1}x - x & \text{if } \alpha < \langle a, x\rangle < 2\alpha \\ 2P_{B_2}x - x & \text{if } -2\alpha < \langle a, x\rangle < -\alpha \\ x & \text{if } |\langle a, x\rangle| \leq \alpha \end{cases} \tag{2.46}$$

Note that T projects onto A all points with the distance to A equal at least 2α, T reflects (with respect to the closest hyperplane B_1 or B_2) the points which do not belong to the band B with the distance to A less than 2α and T does not move the elements of the band B (see Fig. 2.8). The reader can easily check that T is nonexpansive and that T is a separator of A but T is not firmly nonexpansive. Note that $\operatorname{Fix} T = B$ and that T is not a cutter, i.e., it does not separate $\operatorname{Fix} T$, but T separates A (see Fig. 2.8).

Example 2.2.8. Let $A := \mathbb{R} \times \{0\}$ and $B := \{0\} \times \mathbb{R}$ be two subspaces of $\mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$Tx := [1 - \lambda(x)]P_A x + \lambda(x)P_B x,$$

where $\lambda(x) = \frac{\xi_1^2}{\xi_1^2+\xi_2^2}$ for $x = (\xi_1, \xi_2) \in \mathbb{R}^2$ (see Fig. 2.9). We have $P_A x = (\xi_1, 0)$, $P_B x = (0, \xi_2)$. Consequently,

$$Tx = (\frac{\xi_1\xi_2^2}{\xi_1^2 + \xi_2^2}, \frac{\xi_1^2\xi_2}{\xi_1^2 + \xi_2^2})$$

for $x \neq (0, 0)$. Note that $z := (0, 0)$ is the unique fixed point of T.
The operator T is a cutter, because

$$\langle z - Tx, x - Tx\rangle = -\frac{\xi_1^2\xi_2^2}{\xi_1^2 + \xi_2^2} \leq 0$$

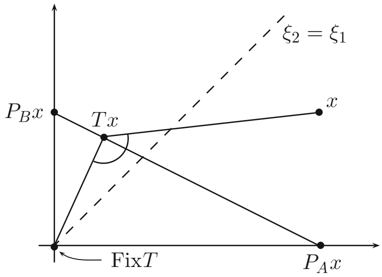


Fig. 2.9 NE cutter which is not FNE

for all $x \neq z$. (Since the weight function $w : \mathbb{R}^2 \to \Delta_2$, $w(x) := (1-\lambda(x), \lambda(x))$ is appropriate, this fact follows also from Corollary 2.1.49). Let $x, y \neq (0, 0)$. A straightforward calculation shows that

$$\frac{\|Tx - Ty\|^2}{\|x - y\|^2} = \frac{\xi_2^2\eta_2^2(\xi_1 - \eta_1)^2 + \xi_1^2\eta_1^2(\xi_2 - \eta_2)^2}{(\xi_1^2 + \xi_2^2)(\eta_1^2 + \eta_2^2)[(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2]}$$

holds for all $x = (\xi_1, \xi_2) \in \mathbb{R}^2$ and for all $y = (\eta_1, \eta_2) \in \mathbb{R}^2$, $x \neq y$. If $\xi_1\eta_1 = \xi_2\eta_2 = 0$, then, of course $Tx = Ty = (0, 0)$. Suppose that $0 < \xi_1^2\eta_1^2 \leq \xi_2^2\eta_2^2$. Then we have

$$\frac{\|Tx - Ty\|^2}{\|x - y\|^2} = \frac{(\xi_1 - \eta_1)^2 + \frac{\xi_1^2\eta_1^2}{\xi_2^2\eta_2^2}(\xi_2 - \eta_2)^2}{(1 + \frac{\xi_1^2}{\xi_2^2})(1 + \frac{\eta_1^2}{\eta_2^2})[(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2]} \leq 1.$$

If $0 < \xi_2^2\eta_2^2 \leq \xi_1^2\eta_1^2$, then we have

$$\frac{\|Tx - Ty\|^2}{\|x - y\|^2} = \frac{\frac{\xi_2^2\eta_2^2}{\xi_1^2\eta_1^2}(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2}{(1 + \frac{\xi_2^2}{\xi_1^2})(1 + \frac{\eta_2^2}{\eta_1^2})[(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2]} \leq 1.$$

Therefore, T is nonexpansive. If we take $x = (\frac{1}{2}, 1)$ and $y = (1, \frac{1}{2})$, then $Tx = (\frac{2}{5}, \frac{1}{5})$, $Ty = (\frac{1}{5}, \frac{2}{5})$ and

$$\langle Tx - Ty, x - y\rangle = -\frac{1}{5} < \frac{2}{25} = \|Tx - Ty\|^2 .$$

Therefore, T is not firmly nonexpansive.

The following property of firmly nonexpansive operators (cf. [22, Lemma 2.4 (iv)]) is often used in applications.

Corollary 2.2.9. *Let $T : X \to \mathcal{H}$ be an operator with a fixed point and $\lambda \in (0, 2]$. If T is firmly nonexpansive, then its relaxation T_λ is $\frac{2-\lambda}{\lambda}$-strongly quasi-nonexpansive, i.e.,*

$$\|T_\lambda x - z\|^2 \le \|x - z\|^2 - \frac{2-\lambda}{\lambda}\|T_\lambda x - x\|^2 \tag{2.47}$$

for all $x \in X$ and $z \in \operatorname{Fix} T$.

Proof. It follows from the first part of Theorem 2.2.5 that a firmly nonexpansive operator having a fixed point is a cutter. Therefore, T_λ is $\frac{2-\lambda}{\lambda}$-strongly quasi-nonexpansive (see Theorem 2.1.39). □

2.2.2 Relationships Between Firmly Nonexpansive and Nonexpansive Operators

One can find in the literature several equivalent definitions of firmly nonexpansive operators. The properties of these operators were studied by Zarantonello [357, Sect. 1], Bruck [49, Sects. 2 and 3], Rockafellar [299], Bruck and Reich [51, Sect. 1], Goebel and Reich [185, Chap. 1, Sect. 11], Reich and Shafrir [296], Goebel and Kirk [184, Chap. 12], Bauschke and Borwein [22, Sects. 2 and 3], Byrne [56, Sect. 2], and by Crombez [127, Sect. 2].

The class of firmly nonexpansive operators is included in the class of nonexpansive ones (see Theorem 2.2.4). Further important relationships between these two classes are also useful for the investigation of firmly nonexpansive operators. These relationships are given in the following theorem.

Theorem 2.2.10. *Let $T : X \to \mathcal{H}$. Then the following conditions are equivalent:*

(i) *T is firmly nonexpansive.*
(ii) *T_λ is nonexpansive for any $\lambda \in [0, 2]$.*
(iii) *T has the form $T = \frac{1}{2}(S + \mathrm{Id})$, where $S : X \to \mathcal{H}$ is a nonexpansive operator.*
(iv) $\mathrm{Id} - T$ *is firmly nonexpansive.*
(v) *For all $x, y \in X$ it holds*

$$\|Tx - Ty\|^2 \le \|x - y\|^2 - \|(x - Tx) - (y - Ty)\|^2. \tag{2.48}$$

(vi) *For all $x, y \in X$ and for any $\alpha \ge 0$ it holds*

$$\|Tx - Ty\| \le \|\alpha(x - y) + (1 - \alpha)(Tx - Ty)\|.$$

Proof. The equivalence (i)⇔(iv) in Theorem 2.2.10 is obvious, because both conditions can be written in the form $\langle Tx - Ty, (x - Tx) - (y - Ty)\rangle \ge 0$ for all $x, y \in X$. Nevertheless, we prove the following relations among (i)-(vi):

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(iv)} \Rightarrow \text{(v)} \Rightarrow \text{(i)} \Leftrightarrow \text{(vi)}.$$

(i)⇒(ii) Let T be firmly nonexpansive and $x, y \in X$. By the definition of a firmly nonexpansive operator, the Cauchy–Schwarz inequality and the nonexpansivity of T (see Theorem 2.2.4), we have

$$\begin{aligned}
\|T_\lambda x - T_\lambda y\|^2 &= \|\lambda Tx + (1-\lambda)x - \lambda Ty - (1-\lambda)y\|^2 \\
&= \|\lambda(Tx - Ty) + (1-\lambda)(x-y)\|^2 \\
&= \lambda^2(\|Tx - Ty\|^2 - \langle Tx - Ty, x - y\rangle) \\
&\quad + (2\lambda - \lambda^2)\langle Tx - Ty, x - y\rangle + (1-\lambda)^2 \|x - y\|^2 \\
&\le (2\lambda - \lambda^2)\langle Tx - Ty, x - y\rangle + (1-\lambda)^2 \|x - y\|^2 \\
&\le (2\lambda - \lambda^2)\|Tx - Ty\| \, \|x - y\| + (1-\lambda)^2 \|x - y\|^2 \\
&\le (2\lambda - \lambda^2)\|x - y\|^2 + (1-\lambda)^2 \|x - y\|^2 \\
&= \|x - y\|^2,
\end{aligned}$$

i.e., T_λ is nonexpansive.

(ii)⇒(iii) This implication is obvious. It suffices to take $S = T_\lambda$ for $\lambda = 2$.

(iii)⇒(iv) Let S be nonexpansive, $T := \frac{1}{2}(S + \mathrm{Id})$ and $G := \mathrm{Id} - T$. Then we have $G = \frac{1}{2}(\mathrm{Id} - S)$ and

$$\begin{aligned}
\|Gx - Gy\|^2 &= \langle Gx - Gy, x - y\rangle + \langle Gx - Gy, (Gx - Gy) - (x - y)\rangle \\
&= \langle Gx - Gy, x - y\rangle \\
&\quad + \frac{1}{4}\langle (Sx - Sy) - (x - y), (Sx - Sy) + (x - y)\rangle \\
&= \langle Gx - Gy, x - y\rangle + \frac{1}{4}(\|Sx - Sy\|^2 - \|x - y\|^2) \\
&\le \langle Gx - Gy, x - y\rangle,
\end{aligned}$$

for all $x, y \in X$.

(iv)⇒(v) Let $G := \mathrm{Id} - T$ be firmly nonexpansive. Then, for all $x, y \in X$ we have

$$\begin{aligned}
&\|Tx - Ty\|^2 + \|(\mathrm{Id} - T)x - (\mathrm{Id} - T)y\|^2 \\
&\le \|Tx - Ty\|^2 + \langle (\mathrm{Id} - T)x - (\mathrm{Id} - T)y, x - y\rangle \\
&= \|Tx - Ty\|^2 - \langle Tx - Ty, x - y\rangle + \|x - y\|^2 \\
&= -\langle Tx - Ty, (x - Tx) - (y - Ty)\rangle + \|x - y\|^2 \\
&\le \|x - y\|^2,
\end{aligned}$$

i.e., (2.48) holds.

(v)⇒(i) Let $x, y \in X$. If (2.48) holds, then, by the properties of the inner product, we have

$$\begin{aligned}\|Tx - Ty\|^2 &\le \|x - y\|^2 - \|(x - y) - (Tx - Ty)\|^2 \\ &= -\|Tx - Ty\|^2 + 2\langle Tx - Ty, x - y\rangle,\end{aligned}$$

i.e., T is firmly nonexpansive.

(i)⇔(vi) Let $x, y \in X$. The function $h : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$h(\alpha) = \frac{1}{2}\|\alpha(x - y) + (1 - \alpha)(Tx - Ty)\|^2$$

is convex as a composition of the convex function $f(\cdot) = \frac{1}{2}\|\cdot\|^2$ and an affine function $A : \mathbb{R} \to \mathcal{H}$, $A(\alpha) = \alpha(x - y) + (1 - \alpha)(Tx - Ty)$. Note that $h(0) = \frac{1}{2}\|Tx - Ty\|^2$, h is differentiable and

$$h'(0) = \langle Tx - Ty, (x - y) - (Tx - Ty)\rangle.$$

Since h is convex, we have

$$h(0) \le h(\alpha) \Longleftrightarrow h'(0) \ge 0$$

for all $\alpha \ge 0$, i.e.,

$$\begin{aligned}\|Tx - Ty\|^2 &\le \|\alpha(x - y) + (1 - \alpha)(Tx - Ty)\|^2 \\ &\Longleftrightarrow \langle Tx - Ty, x - y\rangle \ge \|Tx - Ty\|^2\end{aligned}$$

which completes the proof. □

The same kind of correspondences between firmly nonexpansive operators and nonexpansive ones (the equivalence (i)⇔(iii) in Theorem 2.2.10) and between cutters and quasi-nonexpansive operators (Corollary 2.1.33 (ii)) explain the name *firmly quasi-nonexpansive operators* for cutters (see [346, page 624]).

Condition (ii) in Theorem 2.2.10 can be formulated equivalently as follows: (ii') $T_2 := 2T - \mathrm{Id}$ is nonexpansive.

The nonexpansivity of T_2 and Lemma 2.1.12 (i) yield the nonexpansivity of T_λ for all $\lambda \in [0, 2]$, because $T_\lambda = (1 - \frac{\lambda}{2})\,\mathrm{Id} + \frac{\lambda}{2} T_2$. Moreover, the assumption that T_2 is nonexpansive is sufficient in the implication (ii)⇒(iii) as follows from the proof.

Now we present a series of corollaries of Theorem 2.2.10.

Corollary 2.2.11. *Let $T : X \to \mathcal{H}$. The operator T is firmly nonexpansive if and only if its relaxation T_λ is firmly nonexpansive for all $\lambda \in [0, 1]$.*

Proof. Let T be firmly nonexpansive and $\lambda \in [0, 1]$. By the implication (i)⇒(iii) in Theorem 2.2.10 we obtain

$$T_\lambda = (1-\lambda)\,\mathrm{Id} + \frac{\lambda}{2}(\mathrm{Id} + S) = \frac{1}{2}[\mathrm{Id} + (1-\lambda)\,\mathrm{Id} + \lambda S]$$

for a nonexpansive operator S. Note that $(1-\lambda)\,\mathrm{Id} + \lambda S$ is nonexpansive as a convex combination of nonexpansive operators (see Lemma 2.1.12 (ii)). Therefore, T_λ is firmly nonexpansive by the implication (iii)⇒(i) in Theorem 2.2.10. The sufficiency of the condition is obvious. □

Corollary 2.2.12. *Let $U : X \to \mathcal{H}$ and $\lambda \in [0,2]$. Then U is λ-RFNE if and only if U is μ-RFNE for all $\mu \in [\lambda, 2]$.*

Proof. Let $U := T_\lambda = \mathrm{Id} + \lambda(T - \mathrm{Id})$, where $T : X \to \mathcal{H}$ is a firmly nonexpansive operator, and $\mu \in [\lambda, 2]$. It is easy to see that

$$U = \mathrm{Id} + \mu(T_{\lambda/\mu} - \mathrm{Id}).$$

The corollary follows now from the fact that $T_{\lambda/\mu}$ is firmly nonexpansive (see Corollary 2.2.11). □

Corollary 2.2.13. *Let $X \subseteq \mathcal{H}$ be a closed convex subset and $S : X \to \mathcal{H}$. The following conditions are equivalent:*

(i) S is nonexpansive,
(ii) $S = 2T - \mathrm{Id}$, where $T : X \to \mathcal{H}$ is a firmly nonexpansive operator.

Proof. (ii)⇒(i) Let $S := 2T - \mathrm{Id}$ for a firmly nonexpansive operator T. It follows from the implication (i)⇒(ii) in Theorem 2.2.10 that S is nonexpansive.

(i)⇒(ii) Let S be nonexpansive and $T := \frac{1}{2}(S + \mathrm{Id})$. By the implication (iii)⇒(i) in Theorem 2.2.10 the operator F is firmly nonexpansive. Furthermore, $S = 2T - \mathrm{Id}$. □

Definition 2.2.14. (cf. [127, Definition 2.1]) We say that an operator $U : X \to \mathcal{H}$ is ν*-firmly nonexpansive* (ν*-FNE), where* $\nu > 0$, if

$$\|Ux - Uy\|^2 \le \|x - y\|^2 - \nu\,\|(x - Ux) - (y - Uy)\|^2 .$$

Vasin and Ageev call a ν-firmly nonexpansive operator for $\nu \in (0,1)$, a *pseudo-contractive operator* (see [333, Definition 2.5]). In [127, Theorem 2.3] several equivalent conditions for U to be ν-FNE are presented.

By the equivalence (i)⇔(v) of Theorem 2.2.10, an operator is firmly nonexpansive if and only if it is 1-firmly nonexpansive. Note, however, that there is a difference between a λ-RFNE operator and a λ-FNE operator. Below, we present the relationship between these two notions.

Corollary 2.2.15. *Let $\lambda \in (0,2)$. An operator $U : X \to \mathcal{H}$ is λ-relaxed firmly nonexpansive if and only if U is $\frac{2-\lambda}{\lambda}$-firmly nonexpansive, i.e.,*

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 - \frac{2-\lambda}{\lambda}\,\|(x - Ux) - (y - Uy)\|^2$$

for all $x, y \in X$. *If, furthermore,* $\operatorname{Fix} U \neq \emptyset$, *then*

$$\|Ux - z\|^2 \leq \|x - z\|^2 - \frac{2-\lambda}{\lambda}\,\|Ux - x\|^2$$

for all $x \in X$ *and* $z \in \operatorname{Fix} U$, *i.e.,* U *is* $\frac{2-\lambda}{\lambda}$*-strongly quasi-nonexpansive.*

Proof. Let $U := T_\lambda$ for a firmly nonexpansive operator T and $x, y \in X$. Applying the properties of the inner product we get for $G := \operatorname{Id} - T$

$$\begin{aligned}
\|Ux - Uy\|^2 &= \|(1-\lambda)x + \lambda Tx - (1-\lambda)y - \lambda Ty\|^2 \\
&= \|x - y - \lambda(Gx - Gy)\|^2 \\
&= \|x - y\|^2 - 2\lambda\langle x - y, Gx - Gy\rangle + \lambda^2\,\|Gx - Gy\|^2 .
\end{aligned}$$

Since $x - Ux = x - T_\lambda x = \lambda Gx$, the equalities above yield

$$\begin{aligned}
&\|Ux - Uy\|^2 - \|x - y\|^2 + \frac{2-\lambda}{\lambda}\,\|(x - Ux) - (y - Uy)\|^2 \\
&= \|Ux - Uy\|^2 - \|x - y\|^2 + \lambda(2-\lambda)\,\|Gx - Gy\|^2 \\
&= -2\lambda(\langle x - y, Gx - Gy\rangle - \|Gx - Gy\|^2).
\end{aligned}$$

The first part of the corollary follows now from the equivalence (i)$\Leftrightarrow$(iv) in Theorem 2.2.10, and now the other part follows directly from the definition of an α-strongly quasi-nonexpansive operator. □

Definition 2.2.16. Let $\alpha \in (0, 1)$. We say that an operator $T : X \to \mathcal{H}$ is α*-averaged* or, shortly, *averaged* (*AV*) if

$$T = (1-\alpha)\operatorname{Id} + \alpha S$$

holds for a nonexpansive operator $S : X \to \mathcal{H}$.

Averaged operators were studied, e.g., by Mann [252], Krasnosel'skiĭ [238], Baillon et al. [14, Sect. 2]. In [56, Sect. 2], Byrne gives relationships between averaged operators and *inverse strongly monotone operators*, i.e., operators $G : X \to \mathcal{H}$ such that

$$\langle Gx - Gy, x - y\rangle \geq \nu\,\|Gx - Gy\|^2$$

for all $x, y \in X$ and for some constant $\nu > 0$.

Definition 2.2.16 states that an operator is averaged if and only if it is an under-relaxation of a nonexpansive operator.

Corollary 2.2.17. *Let $\lambda \in (0,2)$ and $\alpha = \lambda/2$. An operator $U : X \to \mathcal{H}$ is λ-relaxed firmly nonexpansive if and only if U is α-averaged.*

Proof. ($\Rightarrow$) Let $T : X \to \mathcal{H}$ be firmly nonexpansive and $U := T_\lambda = (1-\lambda)\operatorname{Id} + \lambda T$. By the implication (i)$\Rightarrow$(iii) in Theorem 2.2.10 we have $T = \frac{1}{2}(S + \operatorname{Id})$ for a nonexpansive operator $S : X \to \mathcal{H}$. Hence, $U = (1-\alpha)\operatorname{Id} + \alpha S$, i.e., U is α-averaged.

($\Leftarrow$) Let U be α-averaged, i.e., $U = (1-\alpha)\operatorname{Id} + \alpha S$ for a nonexpansive operator S and for $\alpha = \lambda/2 \in (0,1)$. By Corollary 2.2.13 we have

$$\begin{aligned} U &= (1-\alpha)\operatorname{Id} + \alpha(2T - \operatorname{Id}) \\ &= (1-2\alpha)\operatorname{Id} + 2\alpha T \end{aligned}$$

for a firmly nonexpansive operator T. Hence, U is the λ-relaxation of $T : X \to \mathcal{H}$ with $\lambda = 2\alpha \in (0,2)$. □

Corollary 2.2.18. *Let $G : X \to \mathcal{H}$. Then G is firmly nonexpansive if and only if* $\operatorname{Id} - \mu G$ *is averaged for any* $\mu \in (0,2)$.

Proof. Necessity. Let G be firmly nonexpansive. We have

$$\operatorname{Id} - \mu G = (1-\mu/2)\operatorname{Id} + (\mu/2)[2(\operatorname{Id} - G) - \operatorname{Id}].$$

By the implications (i)$\Rightarrow$(iv) and (i)$\Rightarrow$(ii) in Theorem 2.2.10 the operator $2(\operatorname{Id} - G) - \operatorname{Id}$ is nonexpansive. Consequently, the operator $\operatorname{Id} - \mu G$ is averaged.

Sufficiency. Let $\operatorname{Id} - \mu G$ be averaged for any $\mu \in (0,2)$. Then $\operatorname{Id} - \mu G$ is nonexpansive for any $\mu \in (0,2)$ and $\operatorname{Id} - 2G$ is nonexpansive as a limit of nonexpansive operators. Now, it follows from the implication (ii)$\Rightarrow$(i) in Theorem 2.2.10 that G is firmly nonexpansive. □

Corollary 2.2.19. *Let $U : X \to \mathcal{H}$ and* $\lambda \in (0,2]$. *The operator U is λ-relaxed firmly nonexpansive if and only if its relaxation U_μ is firmly nonexpansive for* $\mu \in [0, \frac{1}{\lambda}]$.

Proof. Take $U := T_\lambda$ for a firmly nonexpansive operator $T : X \to \mathcal{H}$. Then the claim follows from the equality $U_{\lambda^{-1}} = T$ (see Remark 2.1.3) and Corollary 2.2.11. The converse implication is obvious. □

The following corollary shows that the family of firmly nonexpansive operators is closed under convex combination.

Corollary 2.2.20. *Let $T_i : X \to \mathcal{H}$, $i \in I := \{1, 2, \dots, m\}$, be firmly nonexpansive and $w = (\omega_1, \omega_2, \dots, \omega_m) \in \Delta_m$. Then the operator $T := \sum_{i \in I} \omega_i T_i$ is firmly nonexpansive.*

Proof. Let $T := \sum_{i \in I} \omega_i T_i$. By the implication (i)$\Rightarrow$(iii) in Theorem 2.2.10, we have $T_i = \frac{1}{2}(S_i + \operatorname{Id})$ for a nonexpansive operator S_i, $i \in I$. Observe

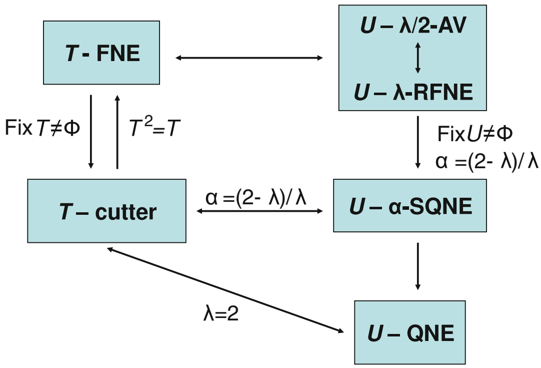


Fig. 2.10 Basic relationships among algorithmic operators

that $T = \frac{1}{2}(S + \mathrm{Id})$ for $S := \sum_{i \in I} \omega_i S_i$. By Lemma 2.1.12 (i), the operator S is nonexpansive. The corollary follows now from the implication (iii)⇒(i) in Theorem 2.2.10. □

In Fig. 2.10 we shortly present important relationships among the FNE operators, cutters, QNE operators SQNE operators and AV operators, which are proved in Sects. 2.1.3, 2.2.1 and 2.2.2. In Fig. 2.10, $T : X \to \mathcal{H}$ and $U := I_\lambda = \mathrm{Id} + \lambda(T - \mathrm{Id})$ is its λ-relaxation, where $\lambda \in (0, 2)$. We will extend this figure in Sect. 3.9.

2.2.3 Further Properties of the Metric Projection

The basic facts concerning firmly nonexpansive operators presented in the previous section yield further properties of the metric projection.

Theorem 2.2.21. *Let $C \subseteq \mathcal{H}$ be a nonempty closed convex subset and $P_C : \mathcal{H} \to \mathcal{H}$ be the metric projection onto C. Then the operator P_C is:*

(i) Idempotent, consequently Fix $P_C = C$,
(ii) A cutter,
(iii) Firmly nonexpansive,
(iv) Monotone and nonexpansive,
(v) Averaged.

Proof. (i) The property follows directly from the definition of the metric projection.
(ii) It follows from (i) and from the characterization of the metric projection (see Theorem 1.2.4) that $\langle z - P_C x, x - P_C x \rangle \leq 0$ for all $x \in \mathcal{H}$ and for all $z \in C = \mathrm{Fix}\, P_C$, which means that P_C is a cutter.
(iii) The property follows directly from (i), (ii) and from Theorem 2.2.5 (ii).

(iv) By Theorem 2.2.4, any firmly nonexpansive operator is monotone and nonexpansive. Therefore, the property follows from (iii).

(v) By the firm nonexpansivity of P_C and by the implication (i)⇒(iii) in Theorem 2.2.10, we can write $P_C = \frac{1}{2}(S + \mathrm{Id})$ for a nonexpansive operator $S : X \to \mathcal{H}$. Hence, P_C is averaged.

□

Definition 2.2.22. Let $C \subseteq \mathcal{H}$ be a nonempty closed convex subset. We call a relaxation of the metric projection $P_C : \mathcal{H} \to C$ a *relaxed metric projection* onto the subset C and we denote it by $P_{C,\lambda}$ or, shortly, by P_λ. If $\lambda < 1$, then P_λ is called an *under-projection*. If $\lambda > 1$, then P_λ is called an *over-projection*. If $\lambda = 2$, then P_λ is called the *reflection.*

We have

$$P_{C,\lambda} = P_\lambda = \mathrm{Id} + \lambda(P_C - \mathrm{Id}).$$

Corollary 2.2.23. *Let $C \subseteq \mathcal{H}$ be a nonempty closed convex subset, $\lambda \geq 0$ and $P_\lambda : \mathcal{H} \to \mathcal{H}$ be a relaxed metric projection. Then*

(i) P_λ is a nonexpansive operator for all $\lambda \in [0, 2]$,
(ii) $\operatorname{Fix} P_\lambda = C$ for all $\lambda > 0$,
(iii) For all $x \in \mathcal{H}$, $z \in C$ and $\lambda \in (0, 2]$ the following inequality holds

$$\|P_\lambda x - z\|^2 \leq \|x - z\|^2 - \frac{2-\lambda}{\lambda}\|P_\lambda x - x\|^2. \tag{2.49}$$

Consequently, P_λ is $\frac{2-\lambda}{\lambda}$-strongly quasi-nonexpansive for all $\lambda \in (0, 2]$.

Proof. Part (i) follows from the equivalence (i)⇔(ii) in Theorem 2.2.10, because P_C is firmly nonexpansive (see Theorem 2.2.21 (iii)). Part (ii) is obvious, because $\operatorname{Fix} P_C = C$. Part (iii) follows now from Corollary 2.2.9. □

Corollary 2.2.24. *Let $C \subseteq \mathcal{H}$ be a nonempty closed convex subset and $x, y \in \mathcal{H}$. Then*

$$\|P_C x - P_C y\|^2 \leq \|x - y\|^2 - \|(P_C x - x) - (P_C y - y)\|^2 \tag{2.50}$$

$$\leq \|x - y\|^2 - (\|P_C x - x\| - \|P_C y - y\|)^2. \tag{2.51}$$

In particular,

$$\|P_C x - z\|^2 \leq \|x - z\|^2 - \|P_C x - x\|^2 \tag{2.52}$$

for all $x \in \mathcal{H}$ and $z \in C$. Consequently, the metric projection $P_C : \mathcal{H} \to C$ is strongly quasi-nonexpansive.

Proof. By Theorem 2.2.21 (iii), the metric projection is firmly nonexpansive. Therefore, inequalities (2.50) and (2.51) follow directly from the implication (i)⇒(v) in Theorem 2.2.10 and from the Cauchy–Schwarz inequality. The second part follows directly from Theorem 2.2.21 (i). □

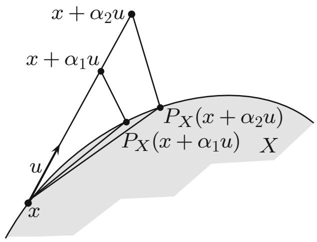


Fig. 2.11 Function $f(x) = \|P_X(x + \alpha u) - x\|$ is nondecreasing

Corollary 2.2.25. *Let $T : X \to \mathcal{H}$ and $\lambda \in (0, 2)$. If T is a cutter, then for any $x \in X$ and $z \in \operatorname{Fix} T$ the following estimations hold*

$$\|P_X T_\lambda x - z\|^2 \leq \|x - z\|^2 - \lambda(2 - \lambda)\|Tx - x\|^2$$

and

$$\|P_X T_\lambda x - z\|^2 \leq \|x - z\|^2 - \frac{2 - \lambda}{\lambda}\|P_X T_\lambda x - x\|^2. \tag{2.53}$$

Consequently, the operator $P_X T_\lambda : X \to X$ is $\frac{2-\lambda}{\lambda}$-strongly quasi-nonexpansive.

Proof. Note that P_X is a nonexpansive operator and that

$$\operatorname{Fix} P_X \cap \operatorname{Fix} T = X \cap \operatorname{Fix} T = \operatorname{Fix} T \neq \emptyset.$$

Therefore, the corollary follows from Theorem 2.1.51. □

The following corollary will be useful in further parts of the book (see also [327, Lemma 2] and [172, Lemma 1] for related results).

Corollary 2.2.26. *Let $x \in X$, $u \in \mathcal{H}$ and $0 \leq \alpha_1 < \alpha_2$. Then the following inequality holds*

$$\begin{aligned} &\|P_X(x + \alpha_2 u) - x\|^2 \\ &\geq \|P_X(x + \alpha_1 u) - x\|^2 + \|P_X(x + \alpha_2 u) - P_X(x + \alpha_1 u)\|^2. \end{aligned} \tag{2.54}$$

Consequently, the function $f : \mathbb{R}_+ \to \mathbb{R}_+$, $f(\alpha) := \|P_X(x + \alpha u) - x\|$ is nondecreasing.

Corollary 2.2.26 is illustrated in Fig. 2.11.

Proof. Inequality (2.54) is obvious for $\alpha_1 = 0$. Let now $\alpha_1 > 0$. Take $y := x + \alpha_2 u$, $z := x + \alpha_1 u$ and $\lambda = \frac{\alpha_1}{\alpha_2}$. Then we have $\lambda \in (0, 1)$ and $(x - z) = -\frac{\lambda}{1-\lambda}(y - z)$. Now inequality (2.54) can be written in the form

$$\|P_X y - x\|^2 \geq \|P_X z - x\|^2 + \|P_X y - P_X z\|^2. \tag{2.55}$$

The characterization of the metric projection (see Theorem 1.2.4) and its monotonicity (see Theorem 2.2.21 (iv)) yield

$$\begin{aligned}\langle x - P_X z, P_X y - P_X z\rangle &= \langle x - z, P_X y - P_X z\rangle + \langle z - P_X z, P_X y - P_X z\rangle \\ &\le -\frac{\lambda}{1-\lambda}\langle y - z, P_X y - P_X z\rangle \le 0,\end{aligned}$$

i.e., $\langle x - P_X z, P_X y - P_X z\rangle \le 0$, which is equivalent to (2.55), by Lemma 1.2.5. □

Let $C \subseteq \mathcal{H}$ be convex. Define the distance function $d(\cdot, C) : \mathcal{H} \to \mathbb{R}$ by $d(x, C) = \inf_{y\in C} \|x - y\|$. It follows from the continuity of the norm and from the definition of the metric projection that

$$d(x, C) = d(x, \operatorname{cl} C) = \|x - P_{\operatorname{cl} C} x\| .$$

Therefore, we suppose without loss of generality that C is closed. It turns out that the functions $d(\cdot, C)$ and $d^2(\cdot, C)$ are convex and differentiable.

Lemma 2.2.27. *Let $C \subseteq \mathcal{H}$ be a closed convex subset. Then the function $f : \mathcal{H} \to \mathbb{R}$, $f(x) := \frac{1}{2}d^2(x, C)$ is differentiable and $D_f(x) = x - P_C x$ for all $x \in \mathcal{H}$.*

Proof. (cf. [167, Proposition 2.2] and [209, Chap. IV, Example 4.1.6]) Let $x, h \in \mathcal{H}$. It follows from the definition of the metric projection and from the properties of the inner product that

$$\begin{aligned}&f(x+h) - f(x) - \langle x - P_C x, h\rangle \\ &= \frac{1}{2}\|x + h - P_C(x+h)\|^2 - \frac{1}{2}\|x - P_C x\|^2 - \langle x - P_C x, h\rangle \\ &\le \frac{1}{2}\|x + h - P_C x\|^2 - \frac{1}{2}\|x - P_C x\|^2 - \langle x - P_C x, h\rangle \\ &= \frac{1}{2}\|h\|^2 .\end{aligned}$$

Similarly, by the definition of the metric projection, the Cauchy–Schwarz inequality and the nonexpansivity of the metric projection, we obtain

$$\begin{aligned}&f(x+h) - f(x) - \langle x - P_C x, h\rangle \\ &= \frac{1}{2}\|x + h - P_C(x+h)\|^2 - \frac{1}{2}\|x - P_C x\|^2 - \langle x - P_C x, h\rangle \\ &\ge \frac{1}{2}\|x + h - P_C(x+h)\|^2 - \frac{1}{2}\|x - P_C(x+h)\|^2 - \langle x - P_C x, h\rangle \\ &= \frac{1}{2}\|h\|^2 + \langle P_C x - P_C(x+h), h\rangle\end{aligned}$$

$$\geq \frac{1}{2}\|h\|^2 - \|P_C x - P_C(x+h)\| \cdot \|h\|$$
$$\geq \frac{1}{2}\|h\|^2 - \|h\|^2 = -\frac{1}{2}\|h\|^2 .$$

Now we see that

$$-\frac{1}{2}\|h\|^2 \leq (f(x+h) - f(x) - \langle x - P_C x, h\rangle) \leq \frac{1}{2}\|h\|^2 .$$

Consequently,

$$f(x+h) = f(x) + \langle x - P_C x, h\rangle + o(\|h\|).$$

Therefore, f is differentiable and $D_f(x) = x - P_C x$. □

Lemma 2.2.28. *Let $C \subseteq \mathcal{H}$ be a closed convex subset. The function $h : \mathcal{H} \to \mathbb{R}$, $h(x) := d(x, C)$ is convex and differentiable for all $x \notin C$ and*

$$Dh(x) = \frac{x - P_C x}{\|x - P_C x\|}. \tag{2.56}$$

Proof. Since $h(x) = \inf_{y \in C} \|x - y\|$, the convexity of h follows from the fact that the function $p : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$, $p(x, y) := \|x - y\|$ is convex (as a composition of a linear function $(x, y) \to x - y$ and a convex function $z \to \|z\|$) and from the fact that for a convex function p, the function $\inf_{y \in C} p(\cdot, y)$ is convex. Since $h = \sqrt{d^2(\cdot, C)}$, the differentiability of h as well as equality (2.56) for $x \notin C$ follow from Lemma 2.2.27 and from the formula $D(\|z\|) = \frac{z}{\|z\|}$ for $z \neq 0$. □

Corollary 2.2.29. *Let $C \subseteq \mathcal{H}$ be a closed convex subset. Then the function $f : \mathcal{H} \to \mathbb{R}$, $f(x) := \frac{1}{2} d^2(x, C)$ is convex.*

Proof. The function f is convex as a composition $f = g \circ h$ of a convex function $h := d(\cdot, C)$ and of a convex and increasing function $g : \mathbb{R}^+ \to \mathbb{R}$, $g(t) := \frac{1}{2} t^2$. □

2.2.4 Metric Projection onto a Closed Subspace

Let $V \subseteq \mathcal{H}$ be a closed linear subspace. Since V is convex, the metric projection P_V is well defined. The theorem below states some properties of P_V. In particular, the first part of the theorem states that the metric projection onto V is equal to the orthogonal projection onto V.

Theorem 2.2.30. *Let $V \subseteq \mathcal{H}$ be a closed subspace and $x \in \mathcal{H}$, $y \in V$. Then*

(i) $y = P_V x$ *if and only if* $\langle x - y, z\rangle = 0$ *for all* $z \in V$,
(ii) P_V *is a bounded linear operator and* $\|P_V\| = 1$,

(iii) P_V *is self-adjoint,*
(iv) $\mathrm{Id} = P_V + P_{V^\perp}$.

Proof. (i) *Necessity*. Let $y := P_V x$. By the characterization of the metric projection (see Theorem 1.2.4), $\langle x - y, z - y\rangle \le 0$ for all $z \in V$. Suppose that $\langle x - y, w - y\rangle < 0$ for some $w \in V$. Let $u := 2y - w$. Then $u \in V$ because V is a linear subspace and we have

$$\langle x - y, u - y\rangle = \langle x - y, y - w\rangle > 0.$$

This contradiction shows that $\langle x - y, z - y\rangle = 0$ for all $z \in V$. If we take $z := 0 \in V$ in the latter equality, we obtain $\langle x - y, y\rangle = 0$. Hence, $\langle x - y, z\rangle = 0$ for all $z \in V$.
Sufficiency. Let $\langle x - y, z\rangle = 0$ for all $z \in V$. Taking $z := y \in V$ we obtain in particular $\langle x - y, y\rangle = 0$. Hence, $\langle x - y, z - y\rangle = 0$ for all $z \in V$. By the characterization of the metric projection (see Theorem 1.2.4), we have $y = P_V x$.

(ii) Let $x_1, x_2 \in \mathcal{H}, \alpha_1, \alpha_2 \in \mathbb{R}$, $y_1 := P_V x_1$, $y_2 := P_V x_2$ and $x := \alpha_1 x_1 + \alpha_2 x_2$, $y := \alpha_1 y_1 + \alpha_2 y_2$. We show that $y = P_V x$. By (i) we have

$$\begin{aligned}\langle x - y, z\rangle &= \langle \alpha_1(x_1 - y_1) + \alpha_2(x_2 - y_2), z\rangle \\ &= \alpha_1\langle x_1 - y_1, z\rangle + \alpha_2\langle x_2 - y_2, z\rangle \\ &= 0,\end{aligned}$$

for all $z \in V$, i.e., $y = P_V x$. Since P_V is nonexpansive, it is bounded. Furthermore,

$$\|P_V x\| = \|P_V x - P_V 0\| \le \|x - 0\| = \|x\|$$

for all $x \in \mathcal{H}$ and $\|P_V x\| = \|x\|$ for $x \in V$. Hence, $\|P_V\| = 1$.

(iii) Let $x, u \in \mathcal{H}$. It follows from (i) that

$$\langle x, P_V u\rangle = \langle P_V x, P_V u\rangle$$

and

$$\langle u, P_V x\rangle = \langle P_V u, P_V x\rangle.$$

By the symmetry of the inner product

$$\langle P_V x, u\rangle = \langle x, P_V u\rangle,$$

i.e., P_V is self-adjoint.

(iv) Let $x \in \mathcal{H}$. By (i), we have $x - P_V x \in V^\perp$ and $x - P_V x = P_{V^\perp} x$. Since $x = P_V x + (x - P_V x)$, it holds $x = P_V x + P_{V^\perp} x$. □

Corollary 2.2.31. *Let $V \subseteq \mathcal{H}$ be a closed subspace and $x \in \mathcal{H}$. Then*

$$\langle P_V x, x\rangle = \|P_V x\|^2.$$

Proof. Since P_V is self-adjoint (see Theorem 2.2.30 (iii)), we have $\langle P_V x, u\rangle = \langle x, P_V u\rangle$ for all $u \in \mathcal{H}$. If we take $u := P_V x$ we obtain the desired property. □

Corollary 2.2.32. *A bounded linear operator is an orthogonal projection if and only if it is idempotent and self-adjoint.*

Proof. The necessity follows from Theorems 2.2.21 (i) and 2.2.30 (iii). Let now $T : \mathcal{H} \to \mathcal{H}$ be idempotent and self-adjoint. Let $V := T(\mathcal{H})$. It is clear that $V = \operatorname{Fix} T$ and that V is a closed subspace. Now we show that $T = P_V$. Let $x \in \mathcal{H}$ and $z \in V$. Then

$$\langle Tx, z\rangle = \langle x, Tz\rangle = \langle x, z\rangle,$$

i.e., $\langle Tx - x, z\rangle = 0$. Theorem 2.2.30 (i) implies now that $T = P_V$. □

2.2.5 Metric Projection onto a Closed Affine Subspace

Let $A \subseteq \mathcal{H}$ be a closed affine subspace and $a \in A$. Then $A - a$ is a closed linear subspace. In order to show some properties of the metric projection P_A we apply Theorem 2.2.30 together with

$$P_A x = P_{A-a}(x - a) + a \tag{2.57}$$

(see Lemma 1.2.6).

Theorem 2.2.33. *Let $A \subseteq \mathcal{H}$ be a closed affine subspace and $x, u, v, w \in \mathcal{H}$, $a, y \in A$. Then*

(i) $y = P_A x$ *if and only if* $\langle x - y, z - y\rangle = 0$ *for all* $z \in A$,
(ii) $P_A u - P_A v = P_{A-a}(u - v) = P_A(u - v) - P_A 0$,
(iii) $\langle P_A u - P_A v, w\rangle = \langle u - v, P_{A-a} w\rangle = \langle u - v, P_A w - P_A 0\rangle$,
(iv) $\langle P_A u - P_A v, u - v\rangle = \|P_A u - P_A v\|^2$,
(v) $\|u - v\|^2 = \|P_A u - P_A v\|^2 + \|(P_A u - u) - (P_A v - v)\|^2$,
(vi) P_A *is an affine operator.*

Proof. (i) Since $A - a$ is a linear subspace, $v \in A - a$ if and only if $v = z - y$, for some $z \in A$. By (2.57) and Theorem 2.2.30 (i), we have

$$y = P_A x \Leftrightarrow y - a = P_{A-a}(x - a) \Leftrightarrow \langle y - a - (x - a), z - y\rangle = 0$$

for any $z \in A$.

(ii) By (2.57) and the linearity of P_{A-a}, we have

$$\begin{aligned} P_A u - P_A v &= P_{A-a}(u-a) + a - (P_{A-a}(v-a) + a) \\ &= P_{A-a}(u-v) \\ &= P_{A-a}(u-v-a) - P_{A-a}(-a) \\ &= P_A(u-v) - P_A 0. \end{aligned}$$

(iii) Since P_{A-a} is self-adjoint, (2.57) and (ii) yield

$$\begin{aligned} \langle P_A u - P_A v, w\rangle &= \langle P_{A-a}(u-a) - (P_{A-a}(v-a), w\rangle \\ &= \langle u-v, P_{A-a}w\rangle \\ &= \langle u-v, P_{A-a}(w-a) - P_{A-a}(-a)\rangle \\ &= \langle u-v, P_A w - P_A 0\rangle. \end{aligned}$$

(iv) Property (i) yields

$$\langle P_A u - u, P_A u - P_A v\rangle = 0 \text{ and } \langle P_A v - v, P_A u - P_A v\rangle = 0$$

Therefore,

$$\langle (P_A u - u) - (P_A v - v), P_A u - P_A v\rangle = 0,$$

i.e.,

$$\langle P_A u - P_A v, u - v\rangle = \| P_A u - P_A v\|^2 .$$

(v) It follows from the properties of the inner product and from property (iv) that

$$\begin{aligned} &\|(P_A u - u) - (P_A v - v)\|^2 \\ &= \| P_A u - P_A v\|^2 + \|u-v\|^2 - 2\langle P_A u - P_A v, u - v\rangle \\ &= \|u-v\|^2 - \| P_A u - P_A v\|^2 . \end{aligned}$$

(vi) Let $\lambda \in \mathbb{R}$. Since $A - a$ is a closed subspace, (2.57) and Theorem 2.2.30 yield

$$\begin{aligned} P_A((1-\lambda)u + \lambda y) &= P_{A-a}((1-\lambda)(u-a) + \lambda(y-a)) + a \\ &= (1-\lambda)(P_{A-a}(u-a) + a) + \lambda(P_{A-a}(y-a) + a) \\ &= (1-\lambda)P_A u + \lambda P_A y \end{aligned}$$

which completes the proof.

□

2.2.6 Properties of Relaxed Firmly Nonexpansive Operators

In this section we present relationships among families of relaxed firmly nonexpansive operators, contractions, averaged operators and strongly quasi-nonexpansive operators. Furthermore, we give properties of relaxed firmly nonexpansive operators which are used in many constructions of algorithmic operators.

Theorem 2.2.34. *An α-contraction is $(1+\alpha)$-relaxed firmly nonexpansive.*

Proof. Let $T : X \to \mathcal{H}$ be an α-contraction, i.e., $\|Tx - Ty\| \leq \alpha \|x - y\|$ for all $x, y \in X$, where $\alpha \in (0, 1)$. Let $U := \frac{2}{1+\alpha} T - \frac{1-\alpha}{1+\alpha} \operatorname{Id}$, or, equivalently,

$$T = \frac{1+\alpha}{2} U + \frac{1-\alpha}{2} \operatorname{Id},$$

i.e.. T is $\frac{1+\alpha}{2}$-averaged. By the convexity of the norm and the nonexpansivity of T,

$$\begin{aligned}
\|Ux - Uy\| &= \left\| \frac{2}{1+\alpha}(Tx - Ty) - \frac{1-\alpha}{1+\alpha}(x - y) \right\| \\
&\leq \frac{2}{1+\alpha} \|Tx - Ty\| + \frac{1-\alpha}{1+\alpha} \|x - y\| \\
&\leq \frac{2\alpha}{1+\alpha} \|x - y\| + \frac{1-\alpha}{1+\alpha} \|x - y\| \\
&= \|x - y\|,
\end{aligned}$$

i.e., U is nonexpansive. Therefore, T is $(1+\alpha)$-relaxed firmly nonexpansive as a $(\frac{1+\alpha}{2})$-averaged operator (see Corollary 2.2.17). $\square$

The next results show that a family of relaxed firmly nonexpansive operators is closed under convex combination and under composition.

Theorem 2.2.35. *Let $\lambda_i \in [0, 2]$ and $U_i : X \to \mathcal{H}$ be λ_i-relaxed firmly nonexpansive, $i \in I := \{1, 2, \ldots, m\}$, $U := \sum_{i=1}^m \omega_i U_i$ for $w = (\omega_1, \ldots, \omega_m) \in \Delta_m$. Then the operator U is λ-relaxed firmly nonexpansive, where $\lambda = \sum_{j=1}^m \omega_j \lambda_j$. Consequently, U is strictly relaxed firmly nonexpansive if $\lambda_i \in (0, 2)$ for some $i \in I$ and the corresponding weight $\omega_i > 0$.*

Proof. Let $U_i := \operatorname{Id} + \lambda_i (T_i - \operatorname{Id})$, where $T_i : X \to \mathcal{H}$ are firmly nonexpansive, $\lambda_i \in [0, 2]$, $i \in I$, and $w = (\omega_1, \ldots, \omega_m) \in \Delta_m$. It is clear that $\lambda := \sum_{j=1}^m \omega_j \lambda_j \in [0, 2]$. For $\lambda = 0$ the claim is obvious, because $U = \operatorname{Id}$ in this case. Let now $\lambda \in (0, 2]$. Since

$$\sum_{i=1}^m \frac{\omega_i \lambda_i}{\sum_{j=1}^m \omega_j \lambda_j} = 1,$$

the operator

$$T := \sum_{i=1}^{m} \frac{\omega_i \lambda_i}{\sum_{j=1}^{m} \omega_j \lambda_j} T_i$$

is firmly nonexpansive as a convex combination of firmly nonexpansive operators T_i (see Corollary 2.2.20). Let $U := \sum_{i=1}^{m} \omega_i U_i$. Then we have

$$\begin{aligned} U &= \sum_{i=1}^{m} \omega_i [\mathrm{Id} + \lambda_i (T_i - \mathrm{Id})] \\ &= \mathrm{Id} + \sum_{i=1}^{m} \omega_i \lambda_i (T_i - \mathrm{Id}) \\ &= \mathrm{Id} + \left(\sum_{j=1}^{m} \omega_j \lambda_j \right) \left(\sum_{i=1}^{m} \frac{\omega_i \lambda_i}{\sum_{j=1}^{m} \omega_j \lambda_j} T_i - \sum_{i=1}^{m} \frac{\omega_i \lambda_i}{\sum_{j=1}^{m} \omega_j \lambda_j} \mathrm{Id} \right) \\ &= \mathrm{Id} + \lambda (T - \mathrm{Id}) \end{aligned}$$

and, consequently, U is λ-relaxed firmly nonexpansive. The second part of the theorem is obvious. □

Corollary 2.2.36. *A convex combination of averaged operators is an averaged operator.*

Proof. It suffices to apply Corollary 2.2.17 to Theorem 2.2.35. □

Theorem 2.2.37. *Let $T, U : X \to X$ and $\lambda, \mu \in [0, 2]$. If T is λ-RFNE and U is μ-RFNE, then the composition $V := UT$ is γ-RFNE, with*

$$\gamma = \begin{cases} 0 & \text{if } \lambda = 0 \text{ and } \mu = 0 \\ 2 & \text{if } (2-\lambda)(2-\mu) = 0 \\ \dfrac{4(\lambda + \mu - \lambda\mu)}{4 - \lambda\mu} = \dfrac{2}{(\dfrac{\lambda}{2-\lambda} + \dfrac{\mu}{2-\mu})^{-1} + 1} & \text{otherwise.} \end{cases} \tag{2.58}$$

Proof. If $\lambda = 0$ or $\mu = 0$, then $T = \mathrm{Id}$ or $U = \mathrm{Id}$, respectively, and the claim is obvious, because the operator Id is 0-RFNE. If $\lambda = 2$ or $\mu = 2$, then T and U are nonexpansive (see Theorem 2.2.10 (ii)) and UT is nonexpansive as a composition of nonexpansive operators. Therefore, UT is 2-RFNE (see Corollary 2.2.13). Let now $\lambda, \mu \in (0, 2)$ and $x, y \in \mathcal{H}$. Denote $a_1 := Tx - x$, $a_2 := Ty - y$, $b_1 := UTx - Tx$ and $b_2 := UTy - Ty$. It is clear that

$$y - x = Ty - Tx + a_1 - a_2. \tag{2.59}$$

By Corollary 2.2.3, we have

$$\langle y-x, a_1\rangle + \langle x-y, a_2\rangle \geq \frac{1}{\lambda}\|a_1-a_2\|^2$$

and

$$\langle Ty-Tx, b_1\rangle + \langle Tx-Ty, b_2\rangle \geq \frac{1}{\mu}\|b_1-b_2\|^2.$$

Therefore, the properties of the inner product, equality (2.59) and Lemma 2.1.45 yield

$$\begin{aligned}
&\langle y-x, UTx-x\rangle + \langle x-y, UTy-y\rangle - \frac{1}{\gamma}(\|(UTx-x)-(UTy-y)\|^2)\\
&= \langle y-x, a_1+b_1\rangle + \langle x-y, a_2+b_2\rangle - \frac{1}{\gamma}(\|(a_1+b_1)-(a_2+b_2)\|^2)\\
&= \langle y-x, a_1\rangle + \langle x-y, a_2\rangle + \langle Ty-Tx, b_1\rangle + \langle Tx-Ty, b_2\rangle\\
&\quad + \langle a_1-a_2, b_1-b_2\rangle - \frac{1}{\gamma}(\|(a_1+b_1)-(a_2+b_2)\|^2)\\
&\geq \frac{1}{\lambda}\|a_1-a_2\|^2 + \frac{1}{\mu}\|b_1-b_2\|^2 + \langle a_1-a_2, b_1-b_2\rangle\\
&\quad - \frac{1}{\gamma}(\|(a_1+b_1)-(a_2+b_2)\|^2\\
&= \left(\frac{1}{\lambda}-\frac{1}{\gamma}\right)\|a_1-a_2\|^2 + \left(\frac{1}{\mu}-\frac{1}{\gamma}\right)\|b_1-b_2\|^2 + \left(1-\frac{2}{\gamma}\right)\langle a_1-a_2, b_1-b_2\rangle\\
&= \left\|\sqrt{\frac{1}{\lambda}-\frac{1}{\gamma}}(a_1-a_2) - \sqrt{\frac{1}{\mu}-\frac{1}{\gamma}}(b_1-b_2)\right\|^2 \geq 0.
\end{aligned}$$

Now it follows from Corollary 2.2.3 that UT is γ-RFNE. □

Remark 2.2.38. Because of Corollary 2.2.17, Theorem 2.2.37 can be stated equivalently in terms of averaged operators:
if T is α-averaged and U is β-averaged, where $\alpha, \beta \in (0,1)$, then UT is δ-averaged, with

$$\delta := \frac{\alpha+\beta-2\alpha\beta}{1-\alpha\beta}. \tag{2.60}$$

This result is due to Ogura and Yamada (see [273, Theorem 3 (b)]). The fact that a composition of averaged operators $T := (1-\alpha)\,\mathrm{Id} + \alpha R$ and $U := (1-\beta)\,\mathrm{Id} + \beta S$ is averaged follows also from the following identity (cf. [56, Lemma 2.2 and Proposition 2.1])

$$UT = (1-\alpha)(1-\beta)\,\mathrm{Id} + (\alpha+\beta-\alpha\beta)[\frac{(1-\beta)\alpha}{\alpha+\beta-\alpha\beta}R + \frac{\beta}{\alpha+\beta-\alpha\beta}ST],$$

and from the fact that the family of nonexpansive operators is closed under compositions and convex combinations (see Lemma 2.1.12). Note, however, that [273, Theorem 3 (b)] is stronger than the result mentioned above, because

$$\delta < \alpha + \beta - \alpha\beta$$

for $\alpha, \beta \in (0, 1)$ and δ given by (2.60). It follows from Corollary 2.2.17 that the result of Ogura and Yamada is equivalent to Theorem 2.2.37 with $\lambda, \mu \in (0, 2)$. Moreover, the proof of this theorem differs from the proof of [273, Theorem 3 (b)]. Note that the property of composition of relaxed cutters with a common fixed point, expressed in Theorem 2.1.46 and the property of compositions of relaxed firmly nonexpansive operators presented in Theorem 2.2.37 are similar. Therefore, it is quite natural that the proofs of both theorems are similar. But Theorem 2.1.46 is no special case of Theorem 2.2.37 because a cutter needs not to be firmly nonexpansive, even if it is nonexpansive (see Example 2.2.8).

An equivalent formulation of the following result can be found in [349, Lemma 1].

Corollary 2.2.39. *Let $T, U : \mathcal{H} \to \mathcal{H}$ be firmly nonexpansive. Then the composition $V := UT$ is $\frac{4}{3}$-relaxed firmly nonexpansive. Consequently, V_λ is firmly nonexpansive for all $\lambda \in [0, \frac{3}{4}]$ and nonexpansive for all $\lambda \in [0, \frac{3}{2}]$. If, furthermore, V has a fixed point, then V_λ is strongly quasi-nonexpansive for all $\lambda \in (0, \frac{3}{2})$.*

Proof. If we take $\lambda = \mu = 1$ in Theorem 2.2.37, we obtain that V is $\frac{4}{3}$-relaxed firmly nonexpansive. Recall that $(V_\lambda)_\mu = V_{\lambda\mu}$ (see Remark 2.1.3). Corollary 2.2.19 yields the firm nonexpansivity of V_γ for all $\gamma \in [0, \frac{3}{4}]$. By the implication (i)⇒(ii) in Theorem 2.2.10, V_γ is nonexpansive for all $\gamma \in [0, \frac{3}{2}]$. Now let $\operatorname{Fix} V \neq \emptyset$ and $\gamma \in (0, \frac{3}{2})$. Then V_γ is strongly quasi-nonexpansive, by Corollary 2.2.9. □

Yamada et al. also proved that, for any $\lambda > \frac{3}{2}$, there exist firmly nonexpansive operators T, U such that V_λ is not nonexpansive, where $V := UT$ (see [349, Remark 1 (b)]). This means that the constant $\frac{3}{2}$ is optimal in Corollary 2.2.39.

Remark 2.2.40. Let $T, U : \mathcal{H} \to \mathcal{H}$ be firmly nonexpansive having a common fixed point. Then it follows from Corollaries 2.2.39 and 2.2.15 that UT is $\frac{1}{2}$-strongly quasi nonexpansive. A special case of this property was proved in [152, Proposition 1] for T, U being orthogonal projections onto subspaces of $\mathcal{H}$.

Corollary 2.2.41. *Let $T : \mathcal{H} \to \mathcal{H}$ be firmly nonexpansive and $\lambda \in [0, 2]$. If V is a closed affine subspace, then the operator $U := (1 - \lambda)P_V + \lambda P_V T$ is $\frac{4}{4-\lambda}$-relaxed firmly nonexpansive.*

Proof. Let V be a closed affine subspace. By Theorem 2.2.33 (vi), the operator P_V is affine, consequently,

$$(1 - \lambda)P_V + \lambda P_V T = P_V T_\lambda.$$

Now it follows from Theorem 2.2.37 that U is $\frac{4}{4-\lambda}$-relaxed firmly nonexpansive, because the metric projection is firmly nonexpansive. □

A weaker formulation of Corollary 2.2.41 can be found in [349, Lemma 2], where $T := P_C$ for a closed convex subset C.

Theorem 2.2.42. *Let $T_i : X \to X$ be λ_i-relaxed firmly nonexpansive, where $\alpha_i \in [0, 2]$, $i \in I$. Then the composition $S_m := T_m T_{m-1} \dots T_1$ is γ_m-relaxed firmly nonexpansive, where $\gamma_m = 0$ if $\lambda_i = 0$ for all $i \in I$, $\gamma_m = 2$ if $\lambda_i = 2$ for at least one $i \in I$ and*

$$\gamma_m = \frac{2}{\left(\frac{\lambda_1}{2-\lambda_1} + \frac{\lambda_2}{2-\lambda_2} + \dots + \frac{\lambda_m}{2-\lambda_m}\right)^{-1} + 1}, \tag{2.61}$$

otherwise. Moreover,

$$\frac{2m \min_{i \in I} \lambda_i}{(m-1)\min_{i \in I} \lambda_i + 2} \leq \gamma_m \leq \frac{2m \max_{i \in I} \lambda_i}{(m-1)\max_{i \in I} \lambda_i + 2}, \tag{2.62}$$

consequently, $\gamma_m < 2$ if $\lambda_i < 2$ for all $i \in I$.

Proof. Let $\lambda_i = 0$ for all $i \in I$. In this case, $S_m = \mathrm{Id}$, i.e., S_m is 0-relaxed firmly nonexpansive. Let $\lambda_i = 2$ for some $i \in I$. Then S_m is nonexpansive as a composition of nonexpansive operators, i.e., S_m is 2-RFNE (see Corollary 2.2.13).

Let now $\lambda_i \in [0, 2)$ for all $i \in I$ and $\lambda_j > 0$ for at least one $j \in I$. We prove by induction with respect to m that S_m is γ_m-RFNE, where γ_m is given by (2.61). Note that (2.61) is equivalent to

$$\frac{\gamma_m}{2-\gamma_m} = \frac{\lambda_1}{2-\lambda_1} + \frac{\lambda_2}{2-\lambda_2} + \dots + \frac{\lambda_m}{2-\lambda_m}. \tag{2.63}$$

1^0 For $m = 2$ the above fact follows directly from Theorem 2.2.37.

2^0 Suppose that, for some $m = k$, the operator S_m is γ_m-RFNE. We prove that S_{k+1} is γ_{k+1}-RFNE. If $\lambda_{k+1} = 0$, then $T_{k+1} = \mathrm{Id}$, S_{k+1} is a composition of k operators which are relaxed firmly nonexpansive and the claim follows from the induction assumption. Let now $\lambda_{k+1} \in (0, 2)$, then we have $S_{k+1} = T_{k+1} S_k$, where T_{k+1} is λ_{k+1}-RFNE and S_k is γ_k-RFNE. It follows from Theorem 2.2.37 that S_{k+1} is γ-RFNE, where

$$\gamma = \frac{2}{\left(\frac{\gamma_k}{2-\gamma_k} + \frac{\lambda_{k+1}}{2-\lambda_{k+1}}\right)^{-1} + 1},$$

and, together with (2.63), this gives for $m = k$

$$\begin{aligned} \frac{\gamma}{2-\gamma} &= \frac{\gamma_k}{2-\gamma_k} + \frac{\lambda_{k+1}}{2-\lambda_{k+1}} \\ &= \frac{\lambda_1}{2-\lambda_1} + \frac{\lambda_2}{2-\lambda_2} + \dots + \frac{\lambda_k}{2-\lambda_k} + \frac{\lambda_{k+1}}{2-\lambda_{k+1}}, \end{aligned}$$

consequently, $\gamma = \gamma_{k+1}$. We have proved that, for any $m \in \mathbb{N}$, the operator S_m is γ_m-RFNE, where γ_m is given by (2.61).

Now we prove (2.62). By (2.63), we have

$$m \frac{\min_{i \in I} \lambda_i}{2 - \min_{i \in I} \lambda_i} \leq \frac{\gamma_m}{2 - \gamma_m} \leq m \frac{\max_{i \in I} \lambda_i}{2 - \max_{i \in I} \lambda_i},$$

which is equivalent to (2.62). □

A part of the results presented in Theorem 2.2.42 can be found in [122, Lemma 2.2 (iii)], where it was proved that a composition of λ_i-RFNE operators T_i, where $\lambda_i \in [0, 2]$, $i \in I$, is $\frac{2m \max_{i \in I} \lambda_i}{(m-1) \max_{i \in I} \lambda_i + 2}$-SQNE.

Corollary 2.2.43. *Let $T_i : X \to X$, $i \in I$, be firmly nonexpansive. Then the operator $S_m = T_m \ldots T_1$ is γ_m-relaxed firmly nonexpansive with $\gamma_m = \frac{2m}{m+1}$. Consequently, S_m is $\frac{1}{m}$-strongly quasi-nonexpansive.*

Proof. It suffices to take $\lambda_i = 1$, $i \in I$, in (2.61). The second part of the corollary follows from Corollary 2.2.9. □

Dye and Reich obtained a result which is a special case of the second part of Corollary 2.2.43 with T_i, $i \in I$, being orthogonal projections onto one-dimensional subspace of a Hilbert space (see [152, Theorem on page 109]).

Corollary 2.2.44. *Let $T_i : X \to X$ be firmly nonexpansive, $S_i := T_i \ldots T_1$, $i \in I$, and $w = (\omega_1, \ldots, \omega_m) \in \Delta_m$. Then the operator $S := \sum_{i=1}^m \omega_i S_i$ is λ-relaxed firmly nonexpansive, where*

$$\lambda = \sum_{i=1}^{m} \omega_i \frac{2i}{i+1}. \tag{2.64}$$

Proof. By Corollary 2.2.43, the operators S_i are γ_i-relaxed firmly nonexpansive with $\gamma_i = \frac{2i}{i+1}$. By Theorem 2.2.35, S is λ-relaxed firmly nonexpansive, where λ is given by (2.64). □

The composition of firmly nonexpansive operators needs not to be firmly nonexpansive (see Exercise 2.5.10).

Definition 2.2.45. Let $T : X \to \mathcal{H}$, $\lambda \in [0, 2]$. The operator $R_\lambda : X \to \mathcal{H}$, $R_\lambda := P_X T_\lambda$ is called a *projected relaxation* of T.

The theorem below gives important properties of the projected relaxation of a firmly nonexpansive operator.

Theorem 2.2.46. *Let $T : X \to \mathcal{H}$ be firmly nonexpansive, $R_\lambda := P_X T_\lambda$, be the projected relaxation of T, where $\lambda \in (0, 2)$. Then:*

(i) *R_λ is $\frac{4}{4-\lambda}$-relaxed firmly nonexpansive.*
(ii) Fix $R_\lambda = \mathrm{Fix}(P_X T)$.

(iii) If $\operatorname{Fix}(P_X T) \neq \emptyset$*, then the operator* R_λ *is* $\frac{2-\lambda}{2}$*-SQNE, i.e.,*

$$\|R_\lambda x - z\|^2 \leq \|x - z\|^2 - \frac{2-\lambda}{2} \|R_\lambda x - x\|^2 \tag{2.65}$$

for all $x \in X$ *and for all* $z \in \operatorname{Fix}(P_X T)$.

(iv) If $\operatorname{Fix} T \neq \emptyset$*, then the operator* R_λ *is* $\frac{2-\lambda}{\lambda}$*-SQNE.*

Proof. (i) Since the metric projection P_X is firmly nonexpansive, it is 1-relaxed firmly nonexpansive. By Theorem 2.2.37, the operator R_λ is $\frac{4}{4-\lambda}$-RFNE.

(ii) This property follows from Corollary 1.2.10.

(iii) Since R_λ is μ-RFNE, where $\mu = \frac{4}{4-\lambda}$ (see (i)), Corollary 2.2.9 yields

$$\begin{aligned}\|R_\lambda x - z\|^2 &\leq \|x - z\|^2 - \frac{2-\mu}{\mu} \|R_\lambda x - x\|^2 \\ &= \|x - z\|^2 - \frac{2-\lambda}{2} \|R_\lambda x - x\|^2 .\end{aligned}$$

(iv) The claim follows from Corollary 2.2.25.

□

If $X = \mathcal{H}$, then $R_\lambda = T_\lambda$, nevertheless, estimation (2.65) is weaker than estimation (2.47). Furthermore, estimation (2.65) is weaker than estimation (2.53). Note, however, that we have supposed in Corollary 2.2.25 that the operator $T : X \to \mathcal{H}$ is a cutter, consequently $\operatorname{Fix} T \neq \emptyset$, while in Theorem 2.2.46 (iii) we have supposed that $\operatorname{Fix}(P_X T) \neq \emptyset$, which is weaker than the assumption $\operatorname{Fix} T \neq \emptyset$.

2.2.7 Fixed Points of Firmly Nonexpansive Operators

A firmly nonexpansive operator is nonexpansive (see Theorem 2.2.4), therefore, the subset of its fixed points is closed and convex (see Proposition 2.1.11). In this section we show that the subsets $\operatorname{Fix} T$ for FNE- and for NE-operators are intersections of half-spaces, which also yields the closedness and convexity of $\operatorname{Fix} T$. Equivalent formulations to the results below can be found in [185, Equalities (11.3) and (11.4)].

Theorem 2.2.47. *Let* $X \subseteq \mathcal{H}$ *be closed convex and* $T : X \to \mathcal{H}$ *be firmly nonexpansive. Then*

$$\operatorname{Fix} T = \bigcap_{x \in X} \{z \in X : \quad \langle Tx - x, Tx - z\rangle \leq 0\}.$$

Consequently, $\operatorname{Fix} T$ *is a closed convex subset.*

Proof. Since a firmly nonexpansive operator with a fixed point is a cutter (see Theorem 2.2.5), the theorem follows from Lemmas 2.1.36 and 2.1.35. □

Corollary 2.2.48. *Let $X \subseteq \mathcal{H}$ be closed and convex. The subset of fixed points of a nonexpansive operator $S : X \to \mathcal{H}$ has the form*

$$\operatorname{Fix} S = \bigcap_{x \in X} \{z \in X : \quad 2\langle z - x, Sx - x\rangle \geq \|Sx - x\|^2\}, \tag{2.66}$$

consequently, Fix S *is a closed convex subset.*

Proof. Let $S : X \to \mathcal{H}$ be nonexpansive. By Corollary 2.2.13, we have $S = 2T - \mathrm{Id}$ for a firmly nonexpansive operator T. It is clear that $\operatorname{Fix} S = \operatorname{Fix} T$. Theorem 2.2.47 yields now

$$\operatorname{Fix} S = \bigcap_{x \in X} \{z \in X : \quad \langle \frac{1}{2}(Sx + x) - x, \frac{1}{2}(Sx + x) - z\rangle \leq 0\}$$

which is equivalent to (2.66). □

2.3 Strongly Nonexpansive Operators

Definition 2.3.1. An operator $T : X \to \mathcal{H}$ is called *strongly nonexpansive* (SNE), if T is nonexpansive and for all sequences $\{x^k\}_{k=0}^{\infty}, \{y^k\}_{k=0}^{\infty} \subseteq X$ the following implication is true

$$\left.\begin{array}{c}(x^k - y^k) \text{ is bounded and} \\ \|x^k - y^k\| - \|Tx^k - Ty^k\| \to 0\end{array}\right\} \Longrightarrow (x^k - y^k) - (Tx^k - Ty^k) \to 0,$$

The notion of strongly nonexpansive operators in Banach spaces was proposed by Bruck and Reich in [51, Sect. 1], where also properties of these operators are proved (see also [23, Sect. 4.3]).

Remark 2.3.2. It is clear that a contraction is a strongly nonexpansive operator. Indeed, let T be a contraction, i.e., $\|Tx - Ty\| \leq \alpha \|x - y\|$ for all $x, y \in X$ and for a constant $\alpha \in (0, 1)$, and $(x^k - y^k)$ be bounded and such that $\|x^k - y^k\| - \|Tx^k - Ty^k\| \to 0$. Then we have

$$\|x^k - y^k\| - \|Tx^k - Ty^k\| \geq (1 - \alpha)\|x^k - y^k\| \to 0.$$

Consequently, $x^k - y^k \to 0$ and $Tx^k - Ty^k \to 0$, i.e., T is strongly nonexpansive.

Remark 2.3.3. (S. Reich, A private communication (2009)) Let $X \subseteq \mathcal{H}$ be compact. Then a strictly nonexpansive operator defined on X is strongly nonexpansive. Indeed, let $T : X \to \mathcal{H}$ be strictly nonexpansive, i.e.,

$$\|Tx - Ty\| < \|x - y\| \text{ or } x - y = Tx - Ty$$

for all $x, y \in X$, and X be compact. We show that T is strongly nonexpansive. Suppose that sequences $\{x^k\}_{k=0}^{\infty}$ and $\{y^k\}_{k=0}^{\infty}$ are given such that $\|x^k - y^k\| - \|Tx^k - Ty^k\| \to 0$ and that there exist subsequences $\{x^{n_k}\}_{k=0}^{\infty} \subseteq \{x^k\}_{k=0}^{\infty}$ and $\{y^{n_k}\}_{k=0}^{\infty} \subseteq \{y^k\}_{k=0}^{\infty}$ and a constant $\varepsilon > 0$ such that

$$\|(x^{n_k} - y^{n_k}) - (Tx^{n_k} - Ty^{n_k})\| \geq \varepsilon.$$

Since X is compact, we can suppose without loss of generality that $x^{n_k} \to x$ and $y^{n_k} \to y$. Since T is continuous as a nonexpansive operator, we have $Tx^{n_k} \to Tx$ and $Ty^{n_k} \to Ty$. Hence, we obtain in the limit $\|x - y\| = \|Tx - Ty\|$, which yields, due to strict nonexpansivity of T, that $x - y = Tx - Ty$. On the other hand, we have

$$\|(x - y) - (Tx - Ty)\| = \lim_k \|(x^{n_k} - y^{n_k}) - (Tx^{n_k} - Ty^{n_k})\| \geq \varepsilon,$$

a contradiction, which shows that T is strongly nonexpansive.

Theorem 2.3.4. *Let $T : X \to \mathcal{H}$ be firmly nonexpansive and $\lambda \in (0, 2)$. Then the relaxation T_λ of T is strongly nonexpansive.*

Proof. Let $\{x^k\}_{k=0}^{\infty}, \{y^k\}_{k=0}^{\infty} \subseteq X$ be such that $\|x^k - y^k\|$ is bounded and

$$\|x^k - y^k\| - \|T_\lambda x^k - T_\lambda y^k\| \to 0.$$

The firm nonexpansivity of T yields the nonexpansivity of T_λ (see Theorem 2.2.10 (ii)), consequently, the sequence $\left\{\|x^k - y^k\| + \|T_\lambda x^k - T_\lambda y^k\|\right\}_{k=0}^{\infty}$ is bounded. Therefore, by the obvious equality $T_\lambda x - x = \lambda(Tx - x)$ and by Corollary 2.2.15, we have

$$\begin{aligned}
&\left\|(x^k - y^k) - (T_\lambda x^k - T_\lambda y^k)\right\|^2 \\
&= \left\|(T_\lambda x^k - x^k) - (T_\lambda y^k - y^k)\right\|^2 \\
&\leq \frac{\lambda}{2-\lambda}\left(\|x^k - y^k\|^2 - \|T_\lambda x^k - T_\lambda y^k\|^2\right) \\
&= \frac{\lambda}{2-\lambda}\left(\|x^k - y^k\| - \|T_\lambda x^k - T_\lambda y^k\|\right)\left(\|x^k - y^k\| + \|T_\lambda x^k - T_\lambda y^k\|\right) \to 0,
\end{aligned}$$

i.e., $\left\|(x^k - y^k) - (T_\lambda x^k - T_\lambda y^k)\right\| \to 0$ and T_λ is strongly nonexpansive. □

In the previous sections we have proved that the following classes of operators are closed under composition and under convex combination:

(a) The class of strictly relaxed cutters with a common fixed point (see Theorems 2.1.46 and 2.1.50),
(b) The class of strongly quasi-nonexpansive operators with a common fixed point (see Corollary 2.1.47 and Theorem 2.1.50),
(c) The class of strictly relaxed firmly nonexpansive operators (see Theorems 2.2.37 and 2.2.35)
(d) The class of averaged operators (see Remark 2.2.38 and Corollary 2.2.36).

It turns out that the class of strongly nonexpansive operators has the same properties. The first part of the theorem below was proved by Bruck and Reich in [51, Proposition 1.1] and the other one by Reich in [295, Lemma 1.3].

Theorem 2.3.5. *Let $T_1, T_2 : X \to X$ be strongly nonexpansive and T have one of the following forms:*

(i) $T := T_2 T_1$,
(ii) $T := (1-\lambda)T_1 + \lambda T_2$, *where* $\lambda \in [0,1]$.

Then T is strongly nonexpansive.

Proof. By Lemma 2.1.12, the operator T is nonexpansive. Let the sequences $\{x^k\}_{k=0}^\infty, \{y^k\}_{k=0}^\infty \subseteq X$ be such that $(x^k - y^k)$ is bounded and $\|x^k - y^k\| - \|Tx^k - Ty^k\| \to 0$.

(i) By the nonexpansivity of T_1 and T_2, we have

$$\|Tx^k - Ty^k\| = \|T_2(T_1x^k) - T_2(T_1y^k)\| \le \|T_1x^k - T_1y^k\| \le \|x^k - y^k\|,$$

$k \ge 0$, consequently,

$$\|x^k - y^k\| - \|T_1x^k - T_1y^k\| \to 0$$

and

$$\|T_1x^k - T_1y^k\| - \|T_2(T_1x^k) - T_2(T_1y^k)\| \to 0.$$

Since T_1 and T_2 are strongly nonexpansive, we have

$$\begin{aligned}&(x^k - y^k) - (Tx^k - Ty^k) = \\ &(x^k - y^k) - (T_1x^k - T_1y^k) + (T_1x^k - T_1y^k) - (T_2(T_1x^k) - T_2(T_1y^k)) \to 0,\end{aligned}$$

i.e., T is strongly nonexpansive.

(ii) The assertion is clear when $\lambda = 0$ or $\lambda = 1$. Let $\lambda \in (0,1)$. By the convexity of the norm and the nonexpansivity of T_1 and T_2, we have

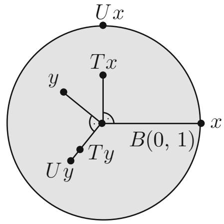


Fig. 2.12 SNE operator which is not AV

$$\begin{aligned}\left\|Tx^k - Ty^k\right\| &= \left\|(1-\lambda)T_1x^k + \lambda T_2x^k - (1-\lambda)T_1y^k - \lambda T_2y^k\right\| \\ &\le (1-\lambda)\left\|T_1x^k - T_1y^k\right\| + \lambda\left\|T_2x^k - T_2y^k\right\| \\ &\le (1-\lambda)\left\|x^k - y^k\right\| + \lambda\left\|x^k - y^k\right\| = \left\|x^k - y^k\right\|,\end{aligned}$$

consequently,

$$\begin{aligned}&\left\|x^k - y^k\right\| - \left\|Tx^k - Ty^k\right\| \\ &\ge (1-\lambda)(\left\|x^k - y^k\right\| - \left\|T_1x^k - T_1y^k\right\|) + \lambda(\left\|x^k - y^k\right\| - \left\|T_2x^k - T_2y^k\right\|).\end{aligned}$$

Therefore,

$$\left\|x^k - y^k\right\| - \left\|T_1x^k - T_1y^k\right\| \to 0$$

and

$$\left\|x^k - y^k\right\| - \left\|T_2x^k - T_2y^k\right\| \to 0.$$

By the strong nonexpansivity of T_1 and T_2, we have now

$$\begin{aligned}&(x^k - y^k) - (Tx^k - Ty^k) \\ &= (1-\lambda)((x^k - y^k) - (T_1x^k - T_1y^k)) + \lambda((x^k - y^k) - (T_2x^k - T_2y^k)) \to 0,\end{aligned}$$

i.e., T is strongly nonexpansive.

□

The following example shows that the class of averaged operators or, equivalently, the class of strictly relaxed firmly nonexpansive operators is a proper subclass of the class of strongly nonexpansive operators.

Example 2.3.6. Let $X := B(0,1) \subseteq \mathcal{H}$ be a unit ball, $U : \mathcal{H} \to \mathcal{H}$ be a unitary operator such that $\langle Ux, x\rangle = 0$ for all $x \in \mathcal{H}$ (e.g., $U : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $Ux := (-\xi_2, \xi_1)$ for $x = (\xi_1, \xi_2) \in \mathbb{R}^2$ with the standard inner product) and the operator $T : X \to X$ be defined by

$$Tx := \alpha(x)Ux.$$

with $\alpha(x) := 1 - \frac{1}{2}\|x\|$ (see Fig. 2.12).

It is clear that $\alpha(x)Ux = U(\alpha(x)x)$, consequently,

$$\begin{aligned}\|Tx - Ty\| &= \|\alpha(x)Ux - \alpha(y)Uy\| \\ &= \|U(\alpha(x)x) - U(\alpha(y)y)\| \\ &= \|\alpha(x)x - \alpha(y)y\| .\end{aligned}$$

A straightforward calculation shows that

$$\begin{aligned}&\|x - y\|^2 - \|Tx - Ty\|^2 \\ &= \|x - y\|^2 - \|\alpha(x)x - \alpha(y)y\|^2 \\ &= -\frac{1}{4}\|x\|^4 - \frac{1}{4}\|y\|^4 + \|x\|^3 + \|y\|^3 \\ &\quad -\langle x, y\rangle(\|x\| + \|y\| - \frac{1}{2}\|x\| \cdot \|y\|) \\ &= (\|x\| - \|y\|)^2(\|x\| + \|y\|)(1 - \frac{1}{4}(\|x\| + \|y\|)) \\ &\quad +(\|x\| \cdot \|y\| - \langle x, y\rangle)(\|x\| + \|y\| - \frac{1}{2}\|x\| \cdot \|y\|).\end{aligned}$$

We have $\|x\| + \|y\| \geq \|x\| \cdot \|y\|$, since $\|x\|, \|y\| \in [0, 1]$. This fact and the Cauchy–Schwarz inequality yield

$$\|x\| + \|y\| - \frac{1}{2}\|x\| \cdot \|y\| \geq \frac{1}{2}\|x\| \cdot \|y\| \geq \frac{1}{4}\|x\| \cdot \|y\| - \frac{1}{4}\langle x, y\rangle,$$

consequently,

$$\begin{aligned}&\|x - y\|^2 - \|Tx - Ty\|^2 \\ &\geq (\|x\| - \|y\|)^2(\|x\| + \|y\|)(1 - \frac{1}{4}(\|x\| + \|y\|)) + \frac{1}{4}(\|x\| \cdot \|y\| - \langle x, y\rangle)^2\end{aligned}$$

and T is nonexpansive. We apply the above inequalities to $x = x^k$ and $y = y^k$. Suppose that $x^k, y^k \in X$ and that $\left\|x^k - y^k\right\| - \left\|Tx^k - Ty^k\right\| \to 0$. Then, of course,

$$\left\|x^k - y^k\right\|^2 - \left\|Tx^k - Ty^k\right\|^2 \to 0,$$

because $\left\|x^k - y^k\right\| + \left\|Tx^k - Ty^k\right\|$ is bounded. Therefore,

$$\left\|x^k\right\| - \left\|y^k\right\| \to 0$$

(note that $1 - \frac{1}{4}(\|x^k\| + \|y^k\|) \geq \frac{1}{2}$) and

$$\|x^k\| \cdot \|y^k\| - \langle x^k, y^k \rangle \to 0.$$

Now we have

$$\|x^k - y^k\|^2 = \left(\|x^k\| - \|y^k\|\right)^2 + 2\left(\|x^k\| \cdot \|y^k\| - \langle x^k, y^k \rangle\right) \to 0,$$

i.e., $(x^k - y^k) \to 0$. Furthermore, $(Tx^k - Ty^k) \to 0$, by the nonexpansivity of T, consequently,

$$(x^k - y^k) - (Tx^k - Ty^k) \to 0,$$

i.e., T is strongly nonexpansive. Note that $z = 0$ is the unique fixed point of T. Suppose that T is α-averaged, for a constant $\alpha \in (0, 1)$. By Corollary 2.2.17, the operator T is (2α)-relaxed firmly nonexpansive. Consequently, the operator $V = T_\mu$, where $\mu = (2\alpha)^{-1} \in (\frac{1}{2}, +\infty)$ is firmly nonexpansive (see Corollary 2.2.19) and V is a cutter (see Theorem 2.2.5), i.e.,

$$\begin{aligned}
&-\mu\langle x, x - Tx\rangle + \mu^2 \|x - Tx\|^2 \\
&= -\mu\langle x + \mu(Tx - x), x - Tx\rangle \\
&= \langle -T_\mu x, x - T_\mu x\rangle \\
&= \langle 0 - Vx, x - Vx\rangle \leq 0.
\end{aligned}$$

Dividing the inequalities above by $\mu > 0$, we obtain, for all $x \neq z$,

$$\frac{1}{2} < \mu \leq \frac{\langle x, x - Tx\rangle}{\|x - Tx\|^2} = \frac{\|x\|^2}{\|x\|^2 + \alpha^2(x)\|x\|^2} = \frac{1}{1 + (1 - \frac{1}{2}\|x\|)^2}.$$

Applying the inequalities above to a sequence $\{x^k\}_{k=0}^\infty$ with $\lim_k x^k = 0$, we obtain

$$\frac{1}{2} < \mu \leq \lim_k \frac{1}{1 + (1 - \frac{1}{2}\|x^k\|)^2} = \frac{1}{2},$$

a contradiction, which proves that T is not averaged.

2.4 Generalized Relaxations of Algorithmic Operators

In the definition of a relaxation T_λ of an operator $T : X \to \mathcal{H}$ we have supposed that the relaxation parameter $\lambda \in [0, 2]$ (see Definition 2.1.2). Furthermore, the assumption $\lambda \in (0, 2)$ is necessary for the strong quasi nonexpansivity of the λ-relaxation of a firmly nonexpansive operator T with Fix $\neq \emptyset$ (see proof of

Theorem 2.1.39). However, in some applications, relaxations of operators (e.g., of firmly nonexpansive ones) with the relaxation parameter which are greater than 2 are successfully used. In general, the convergence of sequences generated by such operators is not guaranteed. It turns out that, if we allow to vary the relaxation parameter in dependence on the current point, in such a way that the relaxed operator is a cutter, then we can apply the usual convergence analysis for sequences generated by such an operator. Below we define a generalization of a relaxation of an operator, which permits us to extend the convergence results to sequences generated by the generalized relaxation.

Definition 2.4.1. Let $T : X \to \mathcal{H}$, $\lambda \in [0, 2]$ and $\sigma : X \to (0, +\infty)$. The operator $T_{\sigma,\lambda} : X \to \mathcal{H}$,

$$T_{\sigma,\lambda}x := x + \lambda\sigma(x)(Tx - x) \tag{2.67}$$

is called the *generalized relaxation* of T, the value λ is called the *relaxation parameter* and σ is called the *step size function*. If $\sigma(x) \geq 1$ for all $x \in X$, then the operator $T_{\sigma,\lambda}$ is called an *extrapolation* of T_λ.

Some special cases of generalized relaxations of some classes of nonexpansive operators, presented in various forms and applied in most cases to the convex feasibility problems, were studied by Gurin et al. [196, Sect. 3], Pierra [284, Sect. 1], Cegielski [62, Sect. 4.3], Kiwiel [229, Sect. 3], Bauschke [17, Sects. 7.3 and 8.3], Combettes [118, Sects. 5.4–5.8], [120, Sect. IV], Bauschke et al. [30, Sect. 3] Bauschke et al. [25] and by Cegielski and Suchocka in [76].

In this section we present properties of generalized relaxations of cutters and give conditions for a generalized relaxation to be strongly quasi-nonexpansive. These properties will be applied in one of the next chapters in order to prove the convergence of sequences generated by such operators.

Denote $T_\sigma = T_{\sigma,1}$.

Remark 2.4.2. Let $T : X \to \mathcal{H}$, $\lambda \in [0, 2]$ and $\sigma : X \to (0, +\infty)$.

(a) If $\sigma(x) = 1$ for all $x \in X$, then $T_{\sigma,\lambda} = T_\lambda$, i.e., the generalized relaxation of T is reduced to the classical relaxation of T.
(b) The values of the step size function σ for $x \in \operatorname{Fix} T$ have no influence on the form of an operator $T_{\sigma,\lambda}$ because $T_{\sigma,\lambda}\,|_{\operatorname{Fix} T} = \mathrm{Id}$ for any step size function σ and for any $\lambda \in (0, 2]$. Therefore, we can suppose without loss of generality that $\sigma(x) = 1$ for all $x \in \operatorname{Fix} T$.
(c) For any $x \in X$ the following equalities hold

$$T_{\sigma,\lambda}x - x = \lambda\sigma(x)(Tx - x) = \lambda(T_\sigma x - x), \tag{2.68}$$

i.e., $T_{\sigma,\lambda}$ is a λ-relaxation of an operator T_σ.
(d) For any $\lambda \neq 0$ it holds $\operatorname{Fix} T_{\sigma,\lambda} = \operatorname{Fix} T$ (cf. Remark 2.1.4).

The corollary below is a version of Theorem 2.1.39.

Corollary 2.4.3. *Let* $T : X \to \mathcal{H}$ *have a fixed point,* $\sigma : X \to (0, +\infty)$ *be a step size function and* $\lambda \in (0, 2)$*. Then* T_σ *is a cutter if and only if* $T_{\sigma,\lambda}$ *is* $\frac{2-\lambda}{\lambda}$*-strongly quasi-nonexpansive. In both cases*

$$\|T_{\sigma,\lambda}x - z\|^2 \le \|x - z\|^2 - \lambda(2-\lambda)\sigma^2(x)\,\|Tx - x\|^2 \tag{2.69}$$

for all $x \in X$ *and* $z \in \operatorname{Fix} T$.

Proof. By Remark 2.4.2 (c), $T_{\sigma,\lambda}$ is the λ-relaxation of T_σ. The first part of the theorem follows now from Theorem 2.1.39. The $\frac{2-\lambda}{\lambda}$-strong quasi nonexpansivity of $T_{\sigma,\lambda}$ means

$$\|T_{\sigma,\lambda}x - z\|^2 \le \|x - z\|^2 - \frac{2-\lambda}{\lambda}\,\|T_{\sigma,\lambda}x - x\|^2 .$$

Applying now (2.68) to the inequality above we obtain (2.69). □

Let $T : X \to \mathcal{H}$ be an operator with a fixed point. Our aim is to give sufficient conditions for the step size function $\sigma : X \to (0, +\infty)$, at which T_σ is a cutter. The following definition was proposed in [70, Definition 9.17].

Definition 2.4.4. We say that an operator $T : X \to \mathcal{H}$ with a fixed point is *oriented* if for all $x \notin \operatorname{Fix} T$

$$\delta(x) := \inf_{z \in \operatorname{Fix} T} \frac{\langle z - x, Tx - x\rangle}{\|Tx - x\|^2} > 0. \tag{2.70}$$

If $\delta(x) \ge \delta > 0$ for all $x \notin \operatorname{Fix} T$, then we say that T is *strongly oriented*.

It follows from Remark 2.1.31 that $T : X \to \mathcal{H}$ is strongly oriented if and only if T is an α-relaxed cutter for some $\alpha > 0$.

Corollary 2.4.5. *Let* $T : X \to \mathcal{H}$ *be an oriented operator with* $\operatorname{Fix} T \ne \emptyset$*. If a step size function* $\sigma : X \to (0, +\infty)$ *satisfies the inequality*

$$\sigma(x) \le \frac{\langle z - x, Tx - x\rangle}{\|Tx - x\|^2} \tag{2.71}$$

for all $x \notin \operatorname{Fix} T$ *and* $z \in \operatorname{Fix} T$*, then* T_σ *is a cutter. Consequently, for any* $\lambda \in (0, 2)$*, the generalized relaxation* $T_{\sigma,\lambda}$ *of* T *is* $\frac{2-\lambda}{\lambda}$*-strongly quasi-nonexpansive.*

Proof. Let $x \notin \operatorname{Fix} T$, $z \in \operatorname{Fix} T$ and $\sigma : X \to (0, +\infty)$ be a step size function satisfying (2.71). The existence of σ follows from the assumption that T is oriented. Then (2.68) and inequality (2.71) yield

$$\begin{aligned}
\langle z - x, T_\sigma x - x\rangle &= \langle z - x, \sigma(x)(Tx - x)\rangle \\
&\ge \|\sigma(x)(Tx - x)\|^2 \\
&= \|T_\sigma x - x\|^2 .
\end{aligned}$$

By the equivalence (a)⇔(b) in Lemma 1.2.5, we have

$$\langle z - T_\sigma x, x - T_\sigma x\rangle \leq 0,$$

i.e., T_σ is a cutter. The $\frac{2-\lambda}{\lambda}$-strong quasi nonexpansivity of $T_{\sigma,\lambda}$ follows now from Corollary 2.4.3. □

The convergence of sequences generated by generalized relaxations of an algorithmic operator U, which we present in the next chapter, requires a stronger condition than (2.71). As we will see, the convergence holds if we additionally suppose that U is strongly oriented, or, equivalently, that the step size $\sigma(x) \geq \alpha$ for all $x \in X$ and for a constant $\alpha > 0$. This leads to α-relaxed cutters (see Remark 2.1.31). It is clear that if an operator $T : X \to \mathcal{H}$ with a fixed point is an α-relaxed cutter for some $\alpha > 0$, then there exists a step size function $\sigma : X \to (0, +\infty)$ satisfying inequality (2.71), e.g., $\sigma(x) = \alpha^{-1}$ for all $x \in X$ (cf. (2.22)). In practice, however, it is important to determine a step size $\sigma(x)$ for which the difference between the right- and the left-hand side of inequality (2.71) is as small as possible for all $z \in \operatorname{Fix} T$. Theoretically, the best possibility would be $\sigma(x) = \delta(x)$ for $x \notin \operatorname{Fix} U$, where $\delta(x)$ is defined by (2.70), but the computation of $\delta(x)$ is, in most cases, impossible, because we usually do not know $\operatorname{Fix} T$ explicitly.

Having an α-relaxed cutter T we can construct its generalized relaxation $T_{\sigma,\lambda}$ with the range of the step size function σ contained in $[\alpha, +\infty)$ and satisfying assumptions of Corollary 2.4.5. The corollary below gives a collection of operators which are α-relaxed cutters.

Corollary 2.4.6. *Let $U : X \to \mathcal{H}$ have a fixed point. Then U is an α-relaxed cutter with:*

(a) $\alpha = 1$ *if U is firmly nonexpansive,*
(b) $\alpha = \lambda$ *if U is λ-relaxed firmly nonexpansive, where* $\lambda \in (0, 2]$,
(c) $\alpha = 2$ *if U is nonexpansive,*
(d) $\alpha = 2\nu$ *if U is ν-averaged, where* $\nu \in (0, 1)$,
(e) $\alpha = \frac{2}{1+\beta}$ *if U is β-strongly quasi-nonexpansive, where* $\beta > 0$.

Proof. (a) Let U be firmly nonexpansive. Then it follows from the first part of Theorem 2.2.5 that T is a cutter, i.e., T is a 1-relaxed cutter.

(b) Let $\lambda \in (0, 2]$ and $U := \operatorname{Id} + \lambda(T - \operatorname{Id})$ for a firmly nonexpansive operator T. Then, by (a), we have

$$\begin{aligned}\langle z - x, Ux - x\rangle &= \lambda\langle z - x, Tx - x\rangle \\ &\geq \lambda \|Tx - x\|^2 = \frac{1}{\lambda}\|Ux - x\|^2 .\end{aligned}$$

(c) Let U be nonexpansive. Then $U = 2T - \operatorname{Id}$ for a firmly nonexpansive operator T (see Corollary 2.2.13) and this case is covered by (b) for $\lambda = 2$.

(d) Let $\nu \in (0, 1)$ and $U := (1 - \nu)\operatorname{Id} + \nu S$ for a nonexpansive operator S. Then U is 2ν-relaxed firmly nonexpansive (see Corollary 2.2.17). The claim follows now from (b) with $\lambda = 2\nu$.

(e) Let $\beta > 0$ and U be β-strongly quasi-nonexpansive. It follows from Corollary 2.1.43 that U is a $\frac{2}{1+\beta}$-relaxed cutter.

□

In the lemma below we state some obvious properties of the generalized relaxation.

Lemma 2.4.7. *Let $T : X \to \mathcal{H}$ be an operator with a fixed point, and $\{\sigma_j\}_{j\in J} : X \to (0, +\infty)$ be a family of step size functions.*

(i) If T_{σ_j}, $j \in J$, are cutters, then $T_{\sup_{j\in J}\sigma_j}$ is a cutter.
(ii) If $\sigma_i \le \sigma_j$ for some $i, j \in J$ and T_{σ_j} is a cutter, then T_{σ_i} is a cutter.

If T is a cutter, then there exists a step size function σ with $\sigma(x) \ge 1$ for all $x \notin \operatorname{Fix} T$, for which T_σ is a cutter, e.g., a step size function σ defined by $\sigma(x) = \delta(x)$, where $\delta(x)$ is given by (2.70) for $x \notin \operatorname{Fix} T$. Consequently, the generalized relaxation $T_{\sigma,\lambda}$ is strongly quasi-nonexpansive for any $\lambda \in (0, 2)$ (see Theorem 2.4.5). Note that $\sigma(x) \ge 1$, by Remark 2.1.31. The following example shows, however, that there is a cutter T such that the generalized relaxation $T_{\sigma,\lambda}$ is strongly quasi-nonexpansive for all $\lambda \in (0, 2)$ if and only if $\sigma(x) \le 1$ for all $x \notin \operatorname{Fix} T$.

Example 2.4.8. Let $T_{\sigma,\lambda}$ be a generalized relaxation of the metric projection $P_C : \mathcal{H} \to \mathcal{H}$, where $C \subseteq \mathcal{H}$ is a nonempty closed convex subset, i.e., $T_{\sigma,\lambda}(x) = x + \lambda\sigma(x)(P_C x - x)$ for a relaxation parameter $\lambda \in (0, 2)$ and for some step size function $\sigma : \mathcal{H} \to (0, +\infty)$. For any $x \in \mathcal{H}$ we have

$$\begin{aligned}
\|T_{\sigma,\lambda}x - P_C x\|^2 &= \|x + \lambda\sigma(x)(P_C x - x) - P_C x\|^2 \\
&= \|x - P_C x\|^2 + \lambda^2\sigma^2(x)\,\|P_C x - x\|^2 - 2\lambda\sigma(x)\,\|P_C x - x\|^2 \\
&= \|x - P_C x\|^2 - \lambda\sigma(x)(2 - \lambda\sigma(x))\,\|P_C x - x\|^2,
\end{aligned}$$

consequently,

$$\|T_{\sigma,\lambda}x - P_C x\|^2 = \|x - P_C x\|^2 - \frac{2 - \lambda\sigma(x)}{\lambda\sigma(x)}\,\|T_{\sigma,\lambda}x - x\|^2. \tag{2.72}$$

Let $\lambda \in (0, 2)$. Suppose that $T_{\sigma,\lambda}$ is strongly quasi-nonexpansive, i.e.,

$$\|T_{\sigma,\lambda}x - z\|^2 \le \|x - z\|^2 - \alpha\,\|T_{\sigma,\lambda}(x) - x\|^2 \tag{2.73}$$

for some $\alpha > 0$, for all $x \in \mathcal{H}$ and $z \in C := \operatorname{Fix} P_C$. Note that α can depend on λ. Let $x \notin C$ and $z = P_C x$. Then (2.72) and (2.73) yield

$$0 < \alpha \le \frac{2 - \lambda\sigma(x)}{\lambda\sigma(x)}.$$

There exists a constant α satisfying the above inequalities for all $\lambda \in (0, 2)$ if and only if $\sigma(x) \le 1$.

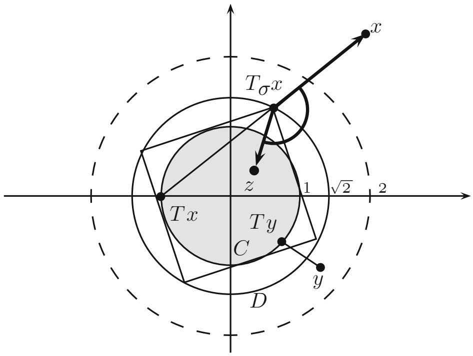


Fig. 2.13 Operators T and T_σ from Example 2.4.9

If $T : X \to \mathcal{H}$ is firmly nonexpansive with a fixed point, then T is oriented and for the function δ defined by (2.70) it holds $\delta(x) \geq 1$ for all $x \notin \operatorname{Fix} T$. Therefore, Corollary 2.4.5 applied to a firmly nonexpansive operator T with the step size $\sigma(x) := \delta(x)$ for $x \notin \operatorname{Fix} T$ is an extension of Theorem 2.2.5 (i) for generalized relaxations. Unfortunately, Theorem 2.2.5 (ii) cannot be analogously extended. The fact that $T : X \to \mathcal{H}$ is a projection and T_σ is a cutter for some step size function $\sigma : X \to (0, +\infty)$ does not yield the firm nonexpansivity of T. Even if we additionally suppose that T is nonexpansive, T needs not to be firmly nonexpansive (see Example 2.2.7). Moreover, a projection T for which T_σ is a cutter needs not to be continuous.

Example 2.4.9. Let $\mathcal{H} = \mathbb{R}^2$, $C := B(0,1)$, $D := \operatorname{bd} B(0, \sqrt{2})$, $a = (1,0)$. Define the operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$Tx := \begin{cases} P_C x & \text{for } \|x\| \leq 2 \\ -a & \text{for } \|x\| > 2, \xi_1 \geq 0 \\ a & \text{for } \|x\| > 2, \xi_1 < 0. \end{cases}$$

It is clear that T is a projection with $\operatorname{Fix} T = C$. For $\|x\| > 2$, let Ux be the unique common point of the segment $[x, Tx]$ and the circle D. Define the function $\sigma : \mathbb{R}^2 \to \mathbb{R}$ by

$$\sigma(x) := \begin{cases} 1 & \text{if } \|x\| \leq 2 \\ \frac{\|Ux - x\|}{\|Tx - x\|} & \text{if } \|x\| > 2. \end{cases}$$

Observe that for $\|x\| > 2$ it holds $T_\sigma x = Ux$. It follows from geometrical considerations (note that the square circumscribed on the circle $\operatorname{bd} B(0,1)$ is inscribed in the circle $\operatorname{bd} B(0, \sqrt{2})$) that for all $x \in \mathbb{R}^2$ and $z \in C = \operatorname{Fix} T$ it holds

$$\langle x - T_\sigma(x), z - T_\sigma(x) \rangle \leq 0$$

(see Fig. 2.13). Therefore, T_σ is a cutter. Note that T is not continuous, therefore, T cannot be firmly nonexpansive.

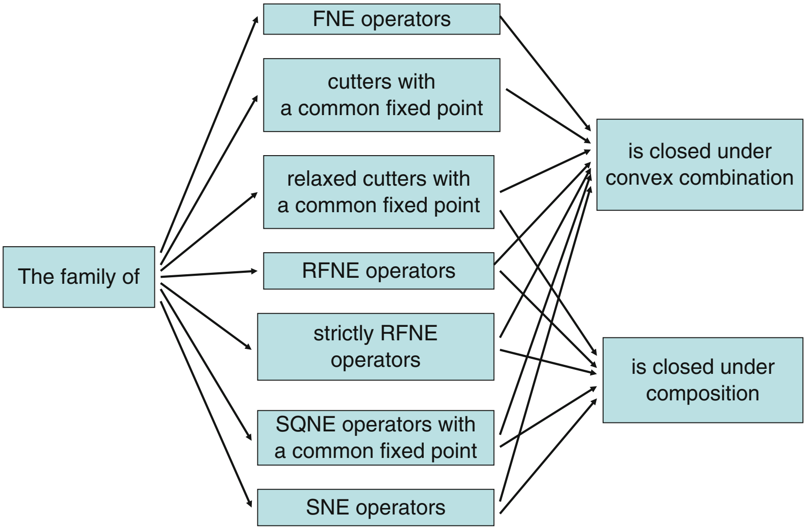


Fig. 2.14 Closedness of families of algorithmic operators

SUMMARY

In Fig. 2.14 we recall in a short form the properties of algorithmic operators which were presented in this chapter. These properties are useful in construction of projection methods. We will describe these constructions in Chaps. 4 and 5.

2.5 Exercises

Exercise 2.5.1. Show that $(T_\lambda)_\mu = T_{\lambda\mu}$ for all $\lambda, \mu \in \mathbb{R}$.

Exercise 2.5.2. Let $T : \mathbb{R} \to \mathbb{R}$,

$$Tx = \begin{cases} x^2 & \text{if } |x| \leq \frac{3}{4} \\ |x| - \frac{3}{16} & \text{if } |x| > \frac{3}{4}. \end{cases}$$

Show that T is quasi-nonexpansive and continuous, but T is not a nonexpansive operator.

Exercise 2.5.3. Let $\{U_i\}_{i \in I}$ be a finite family of operators, $U_i : X \to \mathcal{H}$, $i \in I$. Let $w : X \to \Delta_m$ be a weight function satisfying $\omega_i(x) > 0$ for some $i(x) = \operatorname{argmax}_{i \in I} \|U_i x - x\|$ for all $x \in X$. Prove that w is appropriate with respect to the family $\{U_i\}_{i \in I}$.

Exercise 2.5.4. Prove that the assumption on the C-strict quasi nonexpansivity in Theorem 2.1.26 (i) can be weakened. In this case it suffices to suppose that all U_i are quasi-nonexpansive, $i \in I$, and at least one of them is C-strictly quasi-nonexpansive. The assumption that the weight function w is appropriate should be replaced in this case by a stronger one, namely: $w_j(x) > 0$ for all x such that $I(x) \neq \emptyset$ and for all $j \in I(x)$.

Exercise 2.5.5. Prove Corollary 2.1.29.

Exercise 2.5.6. Prove Lemma 2.1.45.

Exercise 2.5.7. Show that the operator $T : \mathbb{R}^2 \to \mathbb{R}^2$,

$$Tx := (\xi_1 \cos\varphi - \xi_2 \sin\varphi, \xi_1 \sin\varphi + \xi_2 \cos\varphi)$$

is nonexpansive and monotone for $\varphi \in (0, \pi/2)$, but T is not firmly nonexpansive.

Exercise 2.5.8. Prove Lemma 2.2.2.

Exercise 2.5.9. Show that the operator T presented in Example 2.2.7 is nonexpansive and that T is a separator of A, but T is not firmly nonexpansive.

Exercise 2.5.10. Let $\mathcal{H} = \mathbb{R}^2$, $A := \{x \in \mathbb{R}^2 : \xi_2 = 0\}$ and $B := \{x \in \mathbb{R}^2 : \xi_1 = \xi_2\}$. By Theorem 2.2.21 (iii) P_A and P_B are firmly nonexpansive. Check that $T := P_B P_A$ is not firmly nonexpansive.

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