Chapter 25
Mechanical Interpretation of the Oberbeck–Boussinesq Equations of Motion of an Incompressible Stratified Fluid in a Gravitational Field

25.1 A Baroclinic Top

The Oberbeck–Boussinesq equations are of particular interest in connection with their extensive use in the studies of convection of an incompressible fluid, including convection of a rotating fluid, and mechanisms of baroclinic instability. In Part I we already noted that stratification of the fluid, rather than its compressibility, plays a decisive role in the mechanism of baroclinic instability. That is why in theoretical studies it does not make sense to complicate the problem by taking compressibility into account, if one does not consider near- or super-sonic motions. The Oberbeck–Boussinesq equations are written in the form

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\mathbf{u} = -\frac{1}{\rho_0}\nabla p + \frac{\rho}{\rho_0}\mathbf{g}, \tag{25.1}$$

$$\frac{\partial \rho}{\partial t} + (\mathbf{u}\nabla)\rho = 0, \qquad \operatorname{div}\mathbf{u} = 0. \tag{25.2}$$

Here $\mathbf{g}$ is the acceleration of gravity, $\rho = \rho(\mathbf{r}, t)$ is the density deviation from the average background value $\rho_0 = \text{const}$, p is the deviation of pressure from the equilibrium hydrostatic distribution $P_0 = P_0(z)$ $(dP_0/dz + g\rho_0 = 0)$. In deriving (25.1) one ignores the excess of hydrodynamical pressure $\rho d\mathbf{u}/dt$ as compared with the Archimedean forces, while $(\rho/\rho_0)\mathbf{g}$ is the total of both gravity and Archimedes forces (see Landau and Lifschitz 1986).

In terms of the vorticity $\mathbf{\Omega}$ and $\mathbf{q} = \nabla\rho/\rho_0$, the equations of motion assume the form

$$\frac{\partial \mathbf{\Omega}}{\partial t} - \{\mathbf{\Omega}, \mathbf{u}\} = -\mathbf{g} \times \mathbf{q}, \tag{25.3}$$

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{u}\nabla)\mathbf{q} = -\mathbf{q}\frac{\partial \mathbf{u}}{\partial \mathbf{r}}. \tag{25.4}$$

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They conserve the full energy of the fluid

$$E = \frac{1}{2}\rho_0 \int\!\!\int_D u^2 dxdydz - \int\!\!\int_D \rho \mathbf{g} \cdot \mathbf{r} dxdydz \tag{25.5}$$

and have two Lagrangian invariants: potential vorticity

$$\Pi = \mathbf{\Omega} \cdot \nabla \rho \tag{25.6}$$

and density ρ (by definition).

An elliptic rotation of such a stratified fluid inside an ellipsoid, arbitrarily oriented in space, can be described in the class of spatially linear velocity fields (24.4), (24.6) and density

$$\rho(r,t) = \mathbf{r} \cdot \nabla\rho = \left.\frac{\partial \rho}{\partial x_1}\right|_0 x_1 + \left.\frac{\partial \rho}{\partial x_2}\right|_0 x_2 + \left.\frac{\partial \rho}{\partial x_3}\right|_0 x_3, \quad \rho(0,t) = 0, \tag{25.7}$$

where $\nabla\rho = \nabla\rho(t)$ depends only on time. Substituting (24.6) and (24.7) into (25.3) and (25.4), one obtains the system

$$\dot{\mathbf{m}} = \boldsymbol{\omega} \times \mathbf{m} + g\boldsymbol{\sigma} \times \mathbf{l}_0, \tag{25.8}$$

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{\omega} \times \boldsymbol{\sigma}, \qquad \mathbf{m} = \mathrm{I}\boldsymbol{\omega}, \tag{25.9}$$

where components of the vector $\boldsymbol{\sigma}$ are relative differences in densities on the major semiaxes of the ellipsoid:

$$\boldsymbol{\sigma} = \frac{1}{\rho_0}\left(a_1 \left.\frac{\partial \rho}{\partial x_1}\right|_0 \mathbf{i} + a_2 \left.\frac{\partial \rho}{\partial x_2}\right|_0 \mathbf{j} + a_3 \left.\frac{\partial \rho}{\partial x_3}\right|_0 \mathbf{k} \right).$$

The vector $\mathbf{l}_0$ is a constant vector having the dimension of length. This vector is defined by the ellipsoid's orientation in space:

$$\mathbf{l}_0 = a_1 \cos\alpha_1 \mathbf{i} + a_2 \cos\alpha_2 \mathbf{j} + a_3 \cos\alpha_3 \mathbf{k},$$

where quantities $\cos\alpha_i$ $(i = 1, 2, 3)$ are cosines of the corresponding angles of the gravity acceleration vector $\mathbf{g}$ with the principal ellipsoid axes.

According to (25.9), $\boldsymbol{\sigma}^2 = \text{const}$. Therefore, by introducing the unit vector $\boldsymbol{\gamma} = \boldsymbol{\sigma}/\sigma$ and making the changes $\boldsymbol{\omega} \to -\boldsymbol{\omega}$ and $\boldsymbol{\sigma} \to -\boldsymbol{\sigma}$, the system (25.8) and (25.9) can be rewritten in the form

$$\dot{\mathbf{m}} = \mathbf{m} \times \boldsymbol{\omega} + g\sigma\boldsymbol{\gamma} \times \mathbf{l}_0, \tag{25.10}$$

$$\dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \qquad \mathbf{m} = \mathrm{I}\boldsymbol{\omega}. \tag{25.11}$$

And this is precisely the Euler–Poisson equations of motion of a heavy top, written in a coordinate system that is fixed relative to the body. In this case $\mathbf{m}$ and $\boldsymbol{\omega}$ are the angular momentum and angular velocity of the body, σ is the mass of the top,

$\boldsymbol{\gamma}$ is the unit vector in the gravity direction, and $\mathbf{l}_0$ is the radius vector of the body's center of mass. The Euler–Poisson equations have three first integrals of motion:

$$E_m = \frac{1}{2}\mathbf{m} \cdot \boldsymbol{\omega} + g\sigma \mathbf{l}_0 \cdot \boldsymbol{\gamma}, \tag{25.12}$$

$$\Pi_m = \mathbf{m} \cdot \boldsymbol{\gamma}, \qquad \boldsymbol{\gamma}^2 = \gamma_1^2 + \gamma_2^2 + \gamma_1^2. \tag{25.13}$$

The former integral is the total kinetic and potential energy of the mechanical system. The second one is the projection of angular momentum in the direction of the gravitational field, which, according to E. Noether's theorem, is preserved because of invariance of the Hamiltonian (i.e., energy) with respect to rotations around the vertical axis. The invariance of $\boldsymbol{\gamma}^2$ is a consequence of gravity's immobility relative to the space.

From the hydrodynamical point of view, E_m remains the energy, whereas Π_m can now be regarded as the potential vorticity of flows. The latter can be easily verified by a direct substitution of (24.7) and (25.7) into (25.6). It is remarkable, however, that the invariance of potential vorticity is also a consequence of E. Noether's theorem. Indeed, in the dynamics of an incompressible stratified fluid, the role of equipotential surfaces is played not by horizontal levels, as in the mechanical case, but by surfaces of constant density: any map of such a surface into itself does not change the total potential energy of a stratified fluid. Therefore, to obtain a hydrodynamical analogue of the mechanical invariant Π_m, one has to project not in the vertical direction, but in the direction that is normal to the surface of constant density, i.e., in the direction of $\nabla\rho$. Thus, there is almost a literal analogy between the mechanical and hydrodynamical invariants Π_m and Π.

The described analogy between the equations of motion for a heavy fluid and a heavy top and between their invariants remains valid for motions in the field of Coriolis forces, provided that in the case of a mechanical system, the reference frame is rotated relative to the body rather than relative to the space. In this case, one has mechanical prototypes for the equations of motion of a rotating stratified fluid,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\mathbf{u} + 2\boldsymbol{\Omega}_0 \times \mathbf{u} = -\frac{1}{\rho_0}\nabla p + \frac{\rho}{\rho_0}\mathbf{g}, \tag{25.14}$$

$$\frac{\partial \rho}{\partial t} + (\mathbf{u}\nabla)\rho = 0, \qquad \operatorname{div}\mathbf{u} = 0 \tag{25.15}$$

with the integral invariant

$$E = \frac{1}{2}\rho_0 \iint\limits_D \mathbf{u}^2\, dxdydz - \iint\limits_D \rho\mathbf{g}\cdot\mathbf{r}\, dxdydz \tag{25.16}$$

and Lagrangian invariants

$$\Pi = (\boldsymbol{\Omega} + 2\boldsymbol{\Omega}_0)\cdot\nabla\rho \quad \text{and} \quad \rho. \tag{25.17}$$

The equations of these mechanical prototypes are

$$\dot{\mathbf{m}} = \boldsymbol{\omega} \times (\mathbf{m} + 2\mathbf{m}_0) + g\boldsymbol{\sigma} \times \mathbf{l}_0, \tag{25.18}$$

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{\omega} \times \boldsymbol{\sigma}, \qquad \mathbf{m} = \mathrm{I}\boldsymbol{\omega}, \qquad \mathbf{m}_0 = \mathrm{I}\boldsymbol{\omega}_0. \tag{25.19}$$

By substituting $\boldsymbol{\omega} \to -\boldsymbol{\omega}$, $2\boldsymbol{\omega}_0 \to -\boldsymbol{\omega}_0$ and $\boldsymbol{\sigma}/\sigma \to -\boldsymbol{\gamma}$ these equations can be reduced to the heavy top equations in the Coriolis force field,

$$\dot{\mathbf{m}} = (\mathbf{m} + \mathbf{m}_0) \times \boldsymbol{\omega} + g\sigma\boldsymbol{\gamma} \times \mathbf{l}_0, \tag{25.20}$$

$$\dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \qquad \mathbf{m} = \mathrm{I}\boldsymbol{\omega}, \qquad \mathbf{m}_0 = \mathrm{I}\boldsymbol{\omega}_0 \tag{25.21}$$

with the first integrals of motion

$$E_m = \frac{1}{2}\mathbf{m} \cdot \boldsymbol{\omega} + g\sigma\mathbf{l}_0 \cdot \boldsymbol{\gamma}, \tag{25.22}$$

$$\Pi_m = (\mathbf{m} + \mathbf{m}_0) \cdot \boldsymbol{\gamma}, \quad \boldsymbol{\gamma}^2 = \gamma_1^2 + \gamma_2^2 + \gamma_1^2 = 1. \tag{25.23}$$

Below we shall use the following terminology taking into account the hydrodynamical interpretation (24.20) for equations of the classical gyroscope. We will call a *barotropic top* the equations of motion for the classical gyroscope in the Coriolis force field (24.18), while the name *baroclinic top* will stand for Eqs. (25.18) and (25.19) taking into account the stratification of the fluid medium.

25.2 Quasi-geostrophic Approximation of a Baroclinic Top

Having in mind the above analogies, from the point of view of geophysical hydrodynamics it is of special interest to construct a mechanical prototype of quasi-geostrophic equations of motion of a baroclinic atmosphere and to understand its hydrodynamical interpretation. To do this we have the perfect tool, a baroclinic top with its invariants, emphasizing fundamental symmetry properties of the equations of a rotating baroclinic fluid. First, we note that the atmospheric circulation and its laboratory analogues are convective processes. For their description, the Oberbeck–Boussinesq equations are written in terms of temperature fluctuations that are related to density fluctuations having the ratio $T/T_0 = -\rho/\rho_0$. In this case, one needs to replace the quantity $\boldsymbol{\sigma}$ in Eqs. (25.18) and (25.19) by

$$-\boldsymbol{\sigma} = \mathbf{q} = \frac{1}{T_0}\left(a_1 \frac{\partial T}{\partial x_1}\bigg|_0 \mathbf{i} + a_2 \frac{\partial T}{\partial x_2}\bigg|_0 \mathbf{j} + a_3 \frac{\partial T}{\partial x_3}\bigg|_0 \mathbf{k}\right), \tag{25.24}$$

in terms of which the invariants assume the form

$$E_m = \frac{1}{2}\mathbf{m} \cdot \boldsymbol{\omega} + g\mathbf{l}_0 \cdot \mathbf{q}, \tag{25.25}$$

$$\Pi_m = (\mathbf{m} + 2\mathbf{m}_0) \cdot \mathbf{q}, \quad \mathbf{q}^2 = q_1^2 + q_2^2 + q_1^2. \tag{25.26}$$

To derive the desired approximation, we use exactly the same scheme which was used in Part II with respect to the equations of motion of the baroclinic atmosphere. Recall that our approach was as follows.

I. The Rossby number $\varepsilon = U/f_0L = \Omega_z/f_0$, together with the dimensionless parameters

$$\xi = \frac{f_0^2 L^2}{gH} = O(\varepsilon), \qquad \eta = \frac{N^2 H}{g} = O(\varepsilon) \tag{25.27}$$

are assumed to be small. Note that the same order of smallness is not necessary, and it was used only to simplify the reasoning. Here f_0 is the averaged Coriolis parameter, L and H are typical horizontal and vertical scales of the global atmospherical flows, U and Ω_z are their characteristic horizontal velocity and vertical vorticity, while $N^2 = -g\rho_0^{-1}\partial\rho/\partial z = gT_0^{-1}\partial T/\partial z$ is the square of the Brunt–Väisälä frequency, provided that $\partial T/\partial z > 0$.

II. The motion is assumed to be quasi-hydrostatic and quasi-geostrophic, i.e., relations for the thermal wind are satisfied up to $O(\varepsilon)$.

III. The desired approximation is obtained by expanding the equations for conservation of potential vorticity and temperature transport in parameter ε with accuracy up to the terms $O(\varepsilon^2)$.

Let $\mathbf{g}$ be directed in the negative direction of the axis x_3, around which the ellipsoid rotates with angular velocity $\mathbf{\Omega}_0$. For the system (25.18)–(25.19) the parameters ε, L^2 and H are defined by

$$\varepsilon = \frac{\omega}{2\omega_0}\left(\omega = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}\right), \qquad 2L^2 = a_1^2 + a_2^2, \qquad H = a_3. \tag{25.28}$$

Then

$$\xi = \frac{2\omega_0^2(a_1^2 + a_2^2)}{ga_3} = O(\varepsilon), \tag{25.29}$$

$$N^2 = \frac{g}{T_0}\frac{\partial T}{\partial x_3} = \frac{gq_3}{a_3}, \qquad \eta = \frac{N^2 a_3}{g} = q_3 = O(\varepsilon). \tag{25.30}$$

For the hydrodynamical equations (25.14) and (25.15) the thermal wind is defined by

$$-(2\mathbf{\Omega}_0\nabla)\mathbf{u} = \frac{1}{T_0}\mathbf{g} \times \nabla T + O(\varepsilon) \tag{25.31}$$

or in the coordinate form

$$\frac{\partial u}{\partial z} = -\frac{g}{2\Omega_0 T_0}\frac{\partial T}{\partial y} + O(\varepsilon), \qquad \frac{\partial v}{\partial z} = +\frac{g}{2\Omega_0 T_0}\frac{\partial T}{\partial x} + O(\varepsilon). \tag{25.32}$$

The model equations (25.18) and (25.19) are associated with the following vector relation for the thermal wind, which follows from (25.18) and (25.24):

$$\boldsymbol{\omega} \times 2\mathbf{m}_0 + g\mathbf{l}_0 \times \mathbf{q} = O(\varepsilon) \tag{25.33}$$

or in the coordinate form $\mathbf{l}_0 = (0, 0, -a_3)$ and

$$\omega_2 = -\frac{a_3 g q_2}{2I_3\omega_0} + O(\varepsilon), \qquad \omega_1 = -\frac{a_3 g q_1}{2I_3\omega_0} + O(\varepsilon). \tag{25.32$'$}$$

With the help of (24.4), (24.6), (25.32) and (25.32′) it is not difficult to show that $\omega_2 \propto \partial u/\partial z \propto -\partial T/\partial y$ and $\omega_1 \propto -\partial v/\partial z \propto -\partial T/\partial x$. Therefore ω_2 and ω_1 can be regarded as affine transformed components of the thermal wind.

According to (25.29), (25.30), and (25.32′),

$$\frac{\omega_2}{\omega_0} \propto \frac{\omega_1}{\omega_0} \propto O(\varepsilon) \propto \frac{q_2}{O(\varepsilon)} \propto \frac{q_1}{O(\varepsilon)},$$

and hence

$$q_1 \propto q_2 \propto O\big(\varepsilon^2\big). \tag{25.34}$$

The model equations (25.18) and (25.19) can be represented in the coordinate form

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_3 - I_2)\omega_2\omega_3 + 2I_3\omega_0\omega_2 + g a_3\sigma_2, \\ I_2\dot{\omega}_2 &= (I_1 - I_3)\omega_1\omega_3 - 2I_3\omega_0\omega_1 - g a_3\sigma_1, \end{aligned} \tag{25.35}$$

$$\begin{aligned} I_3\dot{\omega}_3 &= (I_2 - I_1)\omega_2\omega_3, \\ \dot{\sigma}_1 &= \omega_2\sigma_3 - \omega_3\sigma_2, \\ \dot{\sigma}_2 &= \omega_3\sigma_1 - \omega_1\sigma_3, \end{aligned} \tag{25.36}$$

$$\dot{\sigma}_3 = \omega_1\sigma_2 - \omega_2\sigma_1, \tag{25.37}$$

where in comparison with (25.18) and (25.19) one replaced $\boldsymbol{\sigma} \to -\boldsymbol{\sigma}$, i.e., instead of $\mathbf{q}$ in (25.33) and (25.32′) one uses $\boldsymbol{\sigma} = -\mathbf{q}$.

The system (25.35)–(25.37) has, in particular, the following families of fixed points describing the stationary states of rotations about the principal axes:

(i) $\omega_1 = \omega_2 = 0, \qquad \sigma_1 = \sigma_2 = 0, \qquad \omega_3 = \omega_{30}, \qquad \sigma_3 = \sigma_{30};$

(ii) $\omega_1 = \omega_3 = 0, \qquad \sigma_1 = \sigma_3 = 0, \qquad \omega_2 = \omega_{20}, \qquad \sigma_2 = \sigma_{20},$

$$2I_3\omega_0\omega_{20} + g a_3\sigma_{20} = 0;$$

(iii) $\omega_2 = \omega_3 = 0, \qquad \sigma_2 = \sigma_3 = 0, \qquad \omega_1 = \omega_{10}, \qquad \sigma_1 = \sigma_{10},$

$$2I_3\omega_0\omega_{10} + g a_3\sigma_{10} = 0.$$

The variables marked by the index 0 can assume arbitrary real values (these variables are not to be confused with the external parameter ω_0). It is easy to see that any representative of the family (ii) or (iii) is a *nontrivial strictly geostrophic stationary regime* of motion for any $\omega_0 \neq 0$. From Eq. (25.18), according to estimate (25.33) and relations of thermal wind (25.32′), as well as (25.29), it follows that $\dot{\sigma}_3 = o(\varepsilon^3)$. Consequently, $\sigma_3 = \sigma_{30}$ is constant with a high degree of accuracy, and the last two equations of system (25.36) with the required accuracy can be rewritten as follows:

$$\dot{\sigma}_1 = \omega_2\sigma_{30} - \omega_3\sigma_2, \qquad \dot{\sigma}_2 = \omega_3\sigma_1 - \omega_1\sigma_{30}. \tag{25.38}$$

Now eliminating from (25.36) the quantities σ_1 and σ_2 and using (25.32′), we obtain the system

$$\dot{\sigma}_1 = -\left(\frac{ga_3\sigma_{30}}{2I_3\omega_0} + \omega_3\right)\sigma_2, \qquad \dot{\sigma}_2 = \left(\frac{ga_3\sigma_{30}}{2I_3\omega_0} + \omega_3\right)\sigma_1, \tag{25.39}$$

which can be interpreted as an analogue of the equation for "potential" temperature (more precisely, the equation for its gradient, see Chap. 9), written in terms of the components of thermal wind and reduced by expansion in the parameter ε.

Now it remains to find out what the potential vorticity is in quasi-geostrophic approximation. By the above estimates, the expression for potential vorticity (see (25.26))

$$\Pi = (\mathbf{m} + 2\mathbf{m}_0)\cdot\boldsymbol{\sigma} = I_1\omega_1\sigma_1 + I_2\omega_2\sigma_2 + I_3\omega_3\sigma_3 + 2I_3\omega_0\sigma_3$$

can be rewritten in the form

$$\Pi = I_3(2\omega_0 + \omega_3)\sigma_{30} + O\left(\varepsilon^3\right).$$

Therefore, the quasi-geostrophic potential vorticity is

$$\Pi_G = I_3(2\omega_0 + \omega_3)\sigma_{30}, \qquad \dot{\Pi}_G = I_3\sigma_{30}\dot{\omega}_3, \tag{25.40}$$

and its evolution is described by the first equation of system (25.36).

Thus, the *quasi-geostrophic approximation of system* (25.35)–(25.37) *of the sixth order describing the motion of a baroclinic top is reduced to the dynamical system of order three*:

$$\begin{aligned} I_3\dot{\omega}_3 &= (I_2 - I_1)\omega_1\omega_2, \\ \dot{\omega}_1 &= -\left(\frac{ga_3\sigma_{30}}{2I_3\omega_0} + \omega_3\right)\omega_2, \\ \dot{\omega}_2 &= \left(\frac{ga_3\sigma_{30}}{2I_3\omega_0} + \omega_3\right)\omega_1, \end{aligned} \tag{25.41}$$

in which one employs equations (25.39) and, for uniformity of notation, one makes a formal substitution $\sigma_1 \to \omega_1$, $\sigma_2 \to \omega_2$. System (25.41) corresponds to equations for slow variables in the theory of relaxation oscillations (see, e.g., Arnold et al. 1986), and in this case it describes the slow evolution of the principal components of global geophysical flows, namely, the vertical vorticity ω_3 and the thermal wind (ω_1, ω_2).

The system is written in terms of the defining characteristics of global geophysical flows: namely, the vertical vorticity, the components of the thermal wind, and the vertical stratification. Note that the latter is invariant in this approximation and it enters the equations of motion as an a priori given parameter. This is similar to the case of the quasi-geostrophic approximation for the equations of motion for the real baroclinic atmosphere.

After dividing each of Eqs. (25.41) by ω_0^2 and introducing slow time and new dependent variables

$$\tau = \omega_0 t, \qquad X = \frac{\omega_1}{\omega_0}, \qquad Y = \frac{\omega_2}{\omega_0}, \qquad Z = S + \frac{\omega_3}{\omega_0},$$

system (25.41) can be written in the exceptionally simple form:

$$\dot{X} = -YZ, \qquad \dot{Y} = ZX, \qquad \dot{Z} = \varGamma XY, \tag{25.42}$$

$$\varGamma = \frac{I_2 - I_1}{I_3} = \frac{a_1^2 - a_2^2}{a_1^2 + a_2^2}, \qquad S = \frac{g a_3 \sigma_{30}}{2 I_3 \omega_0^2}. \tag{25.43}$$

Here S is nothing but the stratification parameter S known in geophysical fluid dynamics (see Pedlosky, 1987) and it is related to the parameter of baroclinicity $\alpha^2 = L_R^2 / L_0^2$ (see Chap. 11) as follows:

$$S = \frac{N^2 H^2}{f_0^2 L^2} = \frac{L_R^2}{L^2} = \alpha^2 \frac{L_0^2}{L^2}. \tag{25.44}$$

Here, L_0 is the Rossby–Obukhov scale and L_R is the Rossby internal deformation radius.

Below, without loss of generality, one can set $a_1 > a_2$. System (25.42) has two quadratic first integrals of motion:

$$E_G = \frac{1}{2}\left(\varGamma X^2 + Z^2\right), \qquad \varTheta_G = X^2 + Y^2, \tag{25.45}$$

which can be treated as full energy and entropy when $S = 0$ (neutral stratification). As we shall see below, $S \neq 0$ characterizes the degree of deviation from a quasi-geostrophic motion. According to the Obukhov theorem (Gledzer et al. 1981), system (25.42) having two quadratic positive invariants is equivalent to the Euler equations of motion for the classical gyroscope.

Thus, we obtained the following somewhat unexpected result:

The quasi-geostrophic approximation of the equations of motion for a heavy fluid top in the field of Coriolis forces is the Euler equation of motions of a rigid body with a fixed point, formulated in terms of the defining characteristics of global geophysical flows, i.e., in terms of its vertical vorticity and components of thermal wind.

25.3 Exercises

1. Find the fixed points of system (25.35)–(25.37). How do they correspond to the fixed points of system (25.41)? What stationary motions do they describe?
2. Sketch the phase portrait of system (25.42) in the space (X, Y, Z), using the invariants (25.45).
3. Study stability of the regimes (i)–(iii) in the framework of the original and truncated systems (25.35)–(25.37) and (25.41), respectively. What is the impact of the vertical stratification σ_{30}?
4. Find the first integrals of the system (25.41). What is their physical meaning and how do they relate to the first integrals of the quasi-geostrophic equations of motion of a baroclinic atmosphere?

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