

# Chapter 2

## Asymptotics

### 2.1 Asymptotic Behavior of Student's Pdf

**Proposition 2.1** For each  $x \in R^d$ , as  $\nu \rightarrow \infty$ ,

$$f_{\nu, \Sigma, a}(x) \rightarrow g_{a, \Sigma}(x). \quad (2.1)$$

*Proof* Let  $a = 0$ . Using the well-known formula that

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}} e^{-z} z^z \left(1 + O\left(\frac{1}{z}\right)\right), \quad \text{as } z \rightarrow \infty, \quad (2.2)$$

we find that, as  $\nu \rightarrow \infty$ ,

$$\frac{\Gamma(\frac{\nu+d}{2})}{(\nu\pi)^{\frac{d}{2}} \Gamma(\frac{\nu}{2})} \sim \frac{\sqrt{\frac{4\pi}{\nu+d}} e^{-\frac{\nu+d}{2}} (\frac{\nu+d}{2})^{\frac{\nu+d}{2}}}{(\nu\pi)^{\frac{d}{2}} \sqrt{\frac{4\pi}{\nu}} e^{-\frac{\nu}{2}} (\frac{\nu}{2})^{\frac{\nu}{2}}} \rightarrow \frac{1}{(2\pi)^{\frac{d}{2}}} \quad (2.3)$$

and, obviously,

$$\left(1 + \frac{\langle x \Sigma^{-1}, x \rangle}{\nu}\right)^{-\frac{\nu+d}{2}} \rightarrow e^{-\frac{1}{2} \langle x \Sigma^{-1}, x \rangle}. \quad (2.4)$$

Here and below “ $\sim$ ” is the equivalence sign.

The statement (2.1) with  $a = 0$  follows from (1.1), (2.2), (2.3) and (2.4).

Let now  $a \neq 0$  and

$$y_\nu = \frac{2}{\nu + d} \left[ \langle a \Sigma^{-1}, a \rangle (\nu + \langle x \Sigma^{-1}, x \rangle) \right]^{\frac{1}{2}}.$$

Because, as  $\nu \rightarrow \infty$ , uniformly in  $y$  (see Appendix)

$$K_\nu(\nu y) \sim \sqrt{\frac{\pi}{2\nu}} \frac{\exp\{-\nu\sqrt{1+y^2}\}}{(1+y^2)^{\frac{1}{4}}} \left( \frac{y}{1+\sqrt{1+y^2}} \right)^{-\nu}$$

and

$$\sqrt{1+y_v^2} \sim 1 + \frac{1}{2}y_v^2,$$

we shall have that

$$\begin{aligned} K_{\frac{\nu+d}{2}} \left( \left[ \langle a\Sigma^{-1}, a \rangle (\nu + \langle x\Sigma^{-1}, x \rangle) \right]^{\frac{1}{2}} \right) &= K_{\frac{\nu+d}{2}} \left( \frac{\nu+d}{2} y_\nu \right) \\ &\sim \sqrt{\frac{\pi}{\nu+d}} \exp \left\{ -\frac{\nu+d}{2} \left( 1 + \frac{1}{2}y_\nu^2 \right) \right\} \left( \frac{y_\nu}{2 + \frac{1}{2}y_\nu^2} \right)^{-\frac{\nu+d}{2}} \\ &\sim \sqrt{\frac{\pi}{\nu+d}} e^{-\frac{\nu+d}{2}} \exp \left\{ -\frac{1}{\nu+d} \langle a\Sigma^{-1}, a \rangle (\nu + \langle x\Sigma^{-1}, x \rangle) \right\} \left( \frac{y_\nu}{2 + \frac{1}{2}y_\nu^2} \right)^{-\frac{\nu+d}{2}}. \end{aligned} \quad (2.5)$$

From (1.2) and (2.5) we elementarily find that

$$\begin{aligned} f_{\nu, \Sigma, a}(x) &\sim \frac{\left(\frac{\nu}{2}\right)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \frac{2 \exp\{\langle x\Sigma^{-1}, a \rangle\}}{(2\pi)^{\frac{d}{2}} \sqrt{|\Sigma|}} \left( \frac{\langle a\Sigma^{-1}, a \rangle}{\nu + \langle x\Sigma^{-1}, x \rangle} \right)^{\frac{\nu+d}{4}} \sqrt{\frac{\pi}{\nu+d}} e^{-\frac{\nu+d}{2}} \\ &\quad \times \exp \left\{ -\frac{1}{\nu+d} \langle a\Sigma^{-1}, a \rangle (\nu + \langle x\Sigma^{-1}, x \rangle) \right\} \left( \frac{y_\nu}{2 + \frac{1}{2}y_\nu^2} \right)^{-\frac{\nu+d}{2}} \\ &\sim \frac{\exp\{\langle x\Sigma^{-1}, a \rangle\}}{(2\pi)^{\frac{d}{2}} \sqrt{|\Sigma|}} e^{-\langle a\Sigma^{-1}, a \rangle} e^{-\frac{d}{2}} \left( \frac{\nu + \langle x\Sigma^{-1}, x \rangle}{2 + \frac{1}{2}y_\nu^2} \right)^{-\frac{\nu+d}{2}} \\ &\sim \frac{\exp\{\langle x\Sigma^{-1}, a \rangle\}}{(2\pi)^{\frac{d}{2}} \sqrt{|\Sigma|}} e^{-\langle a\Sigma^{-1}, a \rangle} e^{-\frac{d}{2}} \\ &\quad \times \exp \left\{ -\frac{1}{2} (\langle x\Sigma^{-1}, x \rangle - d) \right\} \left( 1 + \frac{1}{4}y_\nu^2 \right)^{\frac{\nu+d}{2}}. \end{aligned} \quad (2.6)$$

But

$$\left( 1 + \frac{1}{4}y_\nu^2 \right)^{\frac{\nu+d}{2}} = \left( 1 + \frac{1}{(\nu+d)^2} \left[ \langle a\Sigma^{-1}, a \rangle (\nu + \langle x\Sigma^{-1}, x \rangle) \right] \right)^{\frac{\nu+d}{2}}$$

$$\rightarrow \exp \left\{ \frac{1}{2} \langle a \Sigma^{-1}, a \rangle \right\}. \quad (2.7)$$

Thus, (2.6) and (2.7) imply that, for each  $x \in R^d$ , as  $v \rightarrow \infty$ ,

$$f_{v, \Sigma, a}(x) \rightarrow \frac{\exp \{ \langle x \Sigma^{-1}, a \rangle \}}{(2\pi)^{\frac{d}{2}} \sqrt{|\Sigma|}} \exp \left\{ -\frac{1}{2} \left( \langle a \Sigma^{-1}, a \rangle + \langle x \Sigma^{-1}, x \rangle \right) \right\} = g_{a, \Sigma}(x). \quad \square$$

**Proposition 2.2** For each fixed  $x \in R^d$  and  $v > 0$ , as  $|a| \rightarrow 0$ ,

$$f_{v, \Sigma, a}(x) \rightarrow f_{v, \Sigma}(x).$$

*Proof* Indeed, as  $|a| \rightarrow 0$ ,

$$\begin{aligned} & K_{\frac{v+d}{2}} \left( \left[ \langle a \Sigma^{-1}, a \rangle (v + \langle x \Sigma^{-1}, x \rangle) \right]^{\frac{1}{2}} \right) \\ & \sim \Gamma \left( \frac{v+d}{2} \right) 2^{\frac{v+d}{2}-1} \left[ \langle a \Sigma^{-1}, a \rangle (v + \langle x \Sigma^{-1}, x \rangle) \right]^{-\frac{v+d}{4}} \end{aligned}$$

(see Appendix) and, having in mind formulas (1.1), (1.2),

$$f_{v, \Sigma, a}(x) \rightarrow \frac{(\frac{v}{2})^{\frac{v}{2}}}{\Gamma(\frac{v}{2})} \frac{2^{\frac{v+d}{2}} \Gamma(\frac{v+d}{2})}{(2\pi)^{\frac{d}{2}} \sqrt{|\Sigma|}} \left( v + \langle x \Sigma^{-1}, x \rangle \right)^{-\frac{v+d}{2}} = f_{v, \Sigma}(x). \quad \square$$

**Proposition 2.3** (i) As  $|x| \rightarrow \infty$ ,

$$f_{v, \Sigma}(x) \sim c_{v, \Sigma} \left( \langle x \Sigma^{-1}, x \rangle \right)^{-\frac{v+d}{2}},$$

where

$$c_{v, \Sigma} = \frac{\Gamma \left( \frac{d+v}{2} \right)}{\pi^{\frac{d}{2}} \Gamma \left( \frac{v}{2} \right) \sqrt{|\Sigma|}}.$$

(ii) As  $|x| \rightarrow \infty$ ,  $a \neq 0$ ,

$$f_{v, \Sigma, a}(x) \sim c_{v, \Sigma, a} \left( \langle x \Sigma^{-1}, x \rangle \right)^{-\frac{v+d+1}{4}}$$

$$\times \exp \left\{ - \left[ \langle a \Sigma^{-1}, a \rangle \langle x \Sigma^{-1}, x \rangle \right]^{\frac{1}{2}} + \langle x \Sigma^{-1}, a \rangle \right\},$$

where

$$c_{v, \Sigma, a} = \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}} \left(\langle a \Sigma^{-1}, a \rangle\right)^{\frac{v+d+1}{4}}}{\Gamma\left(\frac{v}{2}\right) (2\pi)^{\frac{d-1}{2}} \sqrt{|\Sigma|}}.$$

*Proof* (i) Obviously follows from (1.1).

(ii) Because, as  $|x| \rightarrow \infty$ ,

$$\begin{aligned} & K_{\frac{v+d}{2}} \left( \left[ \langle a \Sigma^{-1}, a \rangle \left( v + \langle x \Sigma^{-1}, x \rangle \right) \right]^{\frac{1}{2}} \right) \\ & \sim \sqrt{\frac{\pi}{2}} \left[ \langle a \Sigma^{-1}, a \rangle \left( v + \langle x \Sigma^{-1}, x \rangle \right) \right]^{-\frac{1}{4}} \\ & \times \exp \left\{ - \left[ \langle a \Sigma^{-1}, a \rangle \left( v + \langle x \Sigma^{-1}, x \rangle \right) \right]^{\frac{1}{2}} \right\}, \end{aligned}$$

from (1.2) we find that, as  $|x| \rightarrow \infty$ ,

$$\begin{aligned} f_{v, \Sigma, a}(x) & \sim \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}} \left(\langle a \Sigma^{-1}, a \rangle\right)^{\frac{v+d-1}{4}}}{\Gamma\left(\frac{v}{2}\right) (2\pi)^{\frac{d-1}{2}} \sqrt{|\Sigma|}} \frac{\exp \{ \langle x \Sigma^{-1}, a \rangle \}}{\left( v + \langle x \Sigma^{-1}, x \rangle \right)^{\frac{v+d+1}{4}}} \\ & \times \exp \left\{ - \left[ \langle a \Sigma^{-1}, a \rangle \left( v + \langle x \Sigma^{-1}, x \rangle \right) \right]^{\frac{1}{2}} \right\} \\ & \sim c_{v, \Sigma, a} \left( \langle x \Sigma^{-1}, x \rangle \right)^{-\frac{v+d+1}{4}} \\ & \times \exp \left\{ - \left[ \langle a \Sigma^{-1}, a \rangle \langle x \Sigma^{-1}, x \rangle \right]^{\frac{1}{2}} + \langle x \Sigma^{-1}, a \rangle \right\}. \end{aligned}$$

□

**Corollary 2.4** Let  $d=1$ .

(i) If  $a > 0$ ,  $x \rightarrow \infty$ , then

$$f_{v, \sigma^2, a}(x) \sim \frac{1}{\sigma \Gamma\left(\frac{v}{2}\right)} \left( \frac{va}{2\sigma} \right)^{\frac{v}{2}} x^{-\frac{v}{2}-1}. \quad (2.8)$$

(ii) If  $a > 0$ ,  $x \rightarrow -\infty$ , then

$$f_{v,\sigma^2,a}(x) \sim \frac{1}{\sigma \Gamma(\frac{v}{2})} \left( \frac{va}{2\sigma} \right)^{\frac{v}{2}} |x|^{-\frac{v}{2}-1} \exp \left\{ -\frac{2a|x|}{\sigma^2} \right\}. \quad (2.9)$$

(iii) If  $a < 0$ ,  $x \rightarrow \infty$ , then

$$f_{v,\sigma^2,a}(x) \sim \frac{1}{\sigma \Gamma(\frac{v}{2})} \left( \frac{v|a|}{2\sigma} \right)^{\frac{v}{2}} x^{-\frac{v}{2}-1} \exp \left\{ -\frac{2|a|x}{\sigma^2} \right\}. \quad (2.10)$$

(iv) If  $a < 0$ ,  $x \rightarrow -\infty$ , then

$$f_{v,\sigma^2,a}(x) \sim \frac{1}{\sigma \Gamma(\frac{v}{2})} \left( \frac{v|a|}{2\sigma} \right)^{\frac{v}{2}} |x|^{-\frac{v}{2}-1}. \quad (2.11)$$

## 2.2 Asymptotic Distributions for Extremal and Record Values

Let now  $d = 1$  and  $\{X_n, n \geq 1\}$  a sequence of i.i.d. random variables with common Student's  $t$ -distribution function and let  $M_n = \max_{1 \leq j \leq n} X_j$ .

**Proposition 2.5** (i) If pdf of  $\mathcal{L}(X_1)$  is  $f_{v,\sigma^2}$ , then, as  $n \rightarrow \infty$ ,

$$\mathcal{L} \left( (K_1 n)^{-\frac{1}{v}} M_n \right) \Rightarrow \Phi_v,$$

where “ $\Rightarrow$ ” means weak convergence of probability laws,  $\Phi_v$  is the Fréchet distribution

$$\Phi_v(x) = \begin{cases} \exp \{-x^{-v}\}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0, \end{cases}$$

and

$$K_1 = \frac{\Gamma(\frac{v+1}{2})\sigma^v}{v\sqrt{\pi}\Gamma(\frac{v}{2})}.$$

(ii) If pdf of  $\mathcal{L}(X_1)$  is  $f_{v,\sigma^2,a}$ ,  $a > 0$ , then, as  $n \rightarrow \infty$ ,

$$\mathcal{L} \left( (K_2 n)^{-\frac{2}{v}} M_n \right) \Rightarrow \Phi_{\frac{v}{2}},$$

where

$$K_2 = \frac{2\left(\frac{\nu a}{2\sigma}\right)^{\frac{\nu}{2}}}{\nu\sigma\Gamma\left(\frac{\nu}{2}\right)}.$$

(iii) If pdf of  $\mathcal{L}(X_1)$  is  $f_{\nu,a,\sigma^2}$ ,  $a < 0$ , then, as  $n \rightarrow \infty$ ,

$$\mathcal{L}\left(\frac{2|a|}{\sigma^2}M_n - \ln n - \left(\frac{\nu}{2} + 1\right)\ln \ln n + \ln K_3\right) \Rightarrow \Lambda,$$

where  $\Lambda$  is the Gumbel distribution

$$\Lambda(x) = e^{-e^{-x}}, \quad x \in \mathbb{R}^1,$$

and

$$K_3 = \frac{\nu^{\frac{\nu}{2}}\sigma^{\frac{\nu}{2}+3}}{2^{\nu+2}\Gamma\left(\frac{\nu}{2}\right)}.$$

*Proof* (i) From Proposition 2.3 (i) with  $d = 1$  and the l'Hospital's rule we have, as  $x \rightarrow \infty$ ,

$$\int_x^\infty f_{\nu,\sigma^2}(u)du \sim \frac{c_{\nu,\sigma}}{\nu\sigma} \left(\frac{x}{\sigma}\right)^{-\nu} = K_1 x^{-\nu}, \quad (2.12)$$

where

$$c_{\nu,\sigma} = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)\sigma}.$$

The statement (i) is standard for Pareto-like distributions (see, e.g., [1, 2]).

(ii) From Corollary 2.4 (i) and the l'Hospital's rule we have that, as  $x \rightarrow \infty$ ,

$$\int_x^\infty f_{\nu,\sigma^2,a}(u)du \sim K_2 x^{-\frac{\nu}{2}} \quad (2.13)$$

and the conclusion is analogs to (i).

(iii) From Corollary 2.4 (iii) and the l'Hospital's rule we find that, as  $x \rightarrow \infty$ ,

$$\int_x^\infty f_{\nu,\sigma^2,a}(u)du \sim \frac{\sigma}{2|a|\Gamma\left(\frac{\nu}{2}\right)} \left(\frac{\nu|a|}{2\sigma}\right)^{\frac{\nu}{2}} x^{-\frac{\nu}{2}-1} \exp\left\{-\frac{2|a|x}{\sigma^2}\right\}. \quad (2.14)$$

The statement (iii) is standard for gamma-like distributions (see, e.g., [1, 2]).

□

Now let us recall main results on limit theorems for record values in the sequences of i.i.d. random variables  $\{X_n, n \geq 1\}$  with a common continuous distribution function  $F$  which will be applied to the case of Student's  $t$ -distributions.

The record times are  $L_1 = 1$ ,  $L_{n+1} = \min\{k : k > n, X_k > X_{L_n}\}$  for  $n = 1, 2, \dots$ , and the record values are  $R_n = X_{L_n}$ ,  $n = 1, 2, \dots$ . Let  $W(x) = -\log(1 - F(x))$  be the integrated hazard function and the associate distribution function  $A(x) = 1 - e^{-\sqrt{W(x)}}$ ,  $x \in R^1$ . Let  $l_{a,b}(x) = ax + b$ ,  $a > 0$ ,  $b \in R^1$ , be a group of affine homeomorphisms of  $R^1$  with the composition law

$$l_{a_1, b_1} * l_{a_2, b_2} = l_{a_1 a_2, a_1 b_2 + b_1},$$

the unit element  $l_{1,0}$  and the inverse  $l_{a,b}^{-1} = l_{a^{-1}, a^{-1}b}$ .

The domain of attraction problem for record values using linear normalization was solved by Resnick (see [3] also [4]). It was proved that the class of all possible non-degenerated weak limit laws  $\mathcal{Q}$  such that for suitable constants  $a_n > 0$ ,  $b_n \in R^1$ , as  $n \rightarrow \infty$ ,

$$\mathcal{L}(l_{a_n, b_n}^{-1}(R_n)) \Rightarrow \mathcal{Q}$$

coincide with the class of laws  $\Phi(-\log(-\log G(\cdot)))$ , where  $\Phi$  is a standard normal distribution and  $G$  is a  $l$ -max stable law, i. e. a non-degenerated distribution on  $R^1$  such that for any  $n \geq 2$  there exist constants  $a_n > 0$ ,  $b_n \in R^1$  satisfying

$$G^n(x) = G(l_{a_n, b_n}(x)), \quad x \in R^1.$$

As in the classical extreme value theory this class can be factorized into three linear types, saying that probability distributions  $F_1$  and  $F_2$  are of the same linear type if there exist constants  $a > 0$ ,  $b \in R^1$  such that

$$F_1(x) = F_2(l_{a,b}(x)), \quad x \in R^1.$$

In the classical case these types are generated by the Fréchet distribution  $\Phi_\gamma$ , the Gumbel distribution  $\Lambda$  and the Weibull distribution

$$\Psi_\gamma(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ \exp\{-(x)^\gamma\}, & \text{if } x < 0, \quad \gamma > 0, \end{cases}$$

which correspond to generators of three types of the limiting laws for  $\mathcal{L}(l_{a_n, b_n}(R_n))$ :

$$\tilde{\Phi}_\gamma(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ \Phi(\log x^\gamma), & \text{if } x > 0, \quad \gamma > 0, \end{cases}$$

$$\tilde{\Psi}_\gamma(x) = \begin{cases} \Phi(\log(-x)^\gamma), & \text{if } x < 0, \\ 1, & \text{if } x \geq 0, \quad \gamma > 0, \end{cases}$$

and the standard normal distribution  $\Phi(x)$ ,  $x \in R^1$ .

We say that  $F$  belongs to the record domain of attraction under linear normalization of the non-degenerated distribution  $Q$  ( $F \in \text{RDA}_I(Q)$  for short) if there exist constants  $a_n > 0$  and  $b_n \in R^1$  such that  $\mathcal{L}(l_{a_n, b_n}^{-1}(R_n)) \Rightarrow Q$ , as  $n \rightarrow \infty$ .

Duality theorem of Resnick says that  $F \in \text{RDA}_I(\tilde{\Phi}_\gamma) \Leftrightarrow A \in \text{MDA}_I(\Phi_{\frac{\gamma}{2}})$ ,  $F \in \text{RDA}_I(\tilde{\Psi}_\gamma) \Leftrightarrow A \in \text{MDA}_I(\Psi_{\frac{\gamma}{2}})$  and  $F \in \text{RDA}_I(\Phi) \Leftrightarrow A \in \text{MDA}_I(\Lambda)$ , where  $\text{MDA}_I(Q)$  denotes the maximum domain of attraction under linear normalization of the non-degenerated distribution  $Q$  (see, e.g., [3]). As a corollary we find that in the case of heavy-tailed distributions  $F$  the record values cannot have non-degenerate limiting distributions if we use linear normalization. Indeed, for the Pareto-like distributions  $F$ , satisfying, as  $x \rightarrow \infty$ ,

$$1 - F(x) \sim Kx^{-\delta}, \quad \delta > 0,$$

the associate distributions  $A$  satisfy, as  $x \rightarrow \infty$ ,

$$1 - A(x) \sim e^{-\sqrt{\delta \log x}}.$$

In this case  $A \in \text{MDA}_I(\Phi_{\frac{\gamma}{2}}) \cup \text{MDA}_I(\Psi_{\frac{\gamma}{2}}) \cup \text{MDA}_I(\Lambda)$ . This fact is an argument to consider limit theorems for the record values using power normalization.

Let

$$p_{\alpha, \beta}(x) = \alpha |x|^\beta \text{sign} x, \quad \alpha > 0, \quad \beta > 0, \quad x \in R^1.$$

Observe that this class of functions form a group of homeomorphisms of  $R^1$  with the composition law

$$p_{\alpha_1, \beta_1} * p_{\alpha_2, \beta_2} = p_{\alpha_1 \alpha_2^{\beta_1}, \beta_1 \beta_2},$$

the unit element  $p_{1,1}$  and the inverse

$$p_{\alpha, \beta}^{-1} = p_{\alpha^{-\beta-1}, \beta^{-1}}.$$

We say that  $F$  belongs to the record domain of attraction under power normalization of the non-degenerate distribution  $Q$  ( $F \in \text{RDA}_p(Q)$  for short) if there exist constants  $\alpha_n > 0$ ,  $\beta_n > 0$  such that, as  $n \rightarrow \infty$ ,  $\mathcal{L}(p_{\alpha_n, \beta_n}^{-1}(R_n)) \Rightarrow Q$ .

A non-degenerate distribution function  $\tilde{G}$  on  $R^1$  is called  $p$ -max stable if for any  $n \geq 2$  there exist constants  $\tilde{\alpha}_n > 0$ ,  $\tilde{\beta}_n > 0$  such that

$$\tilde{G}^n(x) = \tilde{G}(p_{\tilde{\alpha}_n, \tilde{\beta}_n}(x)), \quad x \in R^1.$$



Probability distributions  $F_1$  and  $F_2$  are of the same power type if there exist constants  $\alpha > 0$ ,  $\beta > 0$  such that  $F_1(x) = F_2(p_{\alpha,\beta}(x))$ ,  $x \in R^1$ .

The class of non-degenerated limiting distributions for  $\mathcal{L}(p_{\alpha_n, \beta_n}^{-1}(R_n))$ , as  $n \rightarrow \infty$ , is equal to the class of law  $\hat{\Phi}(-\log(-\log \hat{G}(\cdot)))$ , where  $\hat{G}$  is a  $p$ -max stable law  $K$ , and is factorized to the six power types, generated by the distribution functions (see [5, 6]):

$$\begin{aligned}\hat{\Phi}_{1,\gamma}(x) &= \begin{cases} 0, & \text{if } x \leq 1, \\ \Phi(\gamma \log \log x), & \text{if } x > 1, \quad \gamma > 0, \end{cases} \\ \hat{\Phi}_{2,\gamma}(x) &= \begin{cases} 0, & \text{if } x \leq 0, \\ \Phi(-\gamma \log |\log x|), & \text{if } 0 < x < 1, \\ 1, & \text{if } x \geq 1, \quad \gamma > 0, \end{cases} \\ \hat{\Phi}_{3,\gamma}(x) &= \begin{cases} 0, & \text{if } x \leq -1, \\ \Phi(-\gamma \log |\log |x||), & \text{if } -1 < x < 0, \\ 1, & \text{if } x \geq 0, \quad \gamma > 0, \end{cases} \\ \hat{\Phi}_{4,\gamma}(x) &= \begin{cases} \Phi(-\gamma \log \log |x|), & \text{if } x < -1, \\ 1, & \text{if } x \geq -1, \quad \gamma > 0, \end{cases} \\ \hat{\Phi}_5(x) &= \begin{cases} 0, & \text{if } x \leq 0, \\ \Phi(\log x), & \text{if } x > 0, \end{cases}\end{aligned}$$

and

$$\hat{\Phi}_6(x) = \begin{cases} \Phi(-\log |x|), & \text{if } x < 0, \\ 1, & \text{if } x \geq 0. \end{cases}$$

There are the valid analog of Resnick's duality theorem and the principle of equivalent tails, which says that if continuous distribution functions  $F_1$  and  $F_2$  are such that  $r(F_1) = r(F_2)$  and  $1 - F_1(x) \sim 1 - F_2(x)$ , as  $x \uparrow r(F_1)$ , then  $F_1 \in \text{RDA}_p(Q)$  if and only if  $F_2 \in \text{RDA}_p(Q)$  with the same normalizing constants, where  $r(F) = \sup\{x : F(x) < 1\}$  and  $Q$  is a non-degenerate limiting distribution for record values using power normalization.

The following analog of classical R. von Mises theorem [7] holds true.

**Theorem 2.6** [8]. *Assume that the integrated hazard function  $W(x)$  is differentiable in some neighborhood of  $r(F)$ . Then:*

(i) *if  $r(F) = \infty$  and*

$$\lim_{x \rightarrow \infty} \frac{W'(x)x \log x}{\sqrt{W(x)}} = \gamma, \quad \gamma > 0,$$

*then  $F \in \text{RDA}_p(\hat{\Phi}_{1,\gamma})$ ;*

(ii) if  $0 < r(F) < \infty$  and

$$\lim_{x \uparrow r(F)} \frac{W'(x)x \log \left( \frac{r(F)}{x} \right)}{\sqrt{W(x)}} = \gamma, \quad \gamma > 0,$$

then  $F \in RDA_p(\hat{\Phi}_{2,\gamma})$ ;

(iii) if  $r(F) = 0$  and

$$\lim_{x \uparrow 0} \frac{W'(x)x \log |x|}{\sqrt{W(x)}} = \gamma, \quad \gamma > 0,$$

then  $F \in RDA_p(\hat{\Phi}_{3,\gamma})$ ;

(iv) if  $r(F) < 0$  and

$$\lim_{x \uparrow r(F)} \frac{W'(x)|x| \log \left( \frac{x}{r(F)} \right)}{\sqrt{W(x)}} = \gamma, \quad \gamma > 0,$$

then  $F \in RDA_p(\hat{\Phi}_{4,\gamma})$ ;

(v) if  $W$  is twice differentiable in some neighborhood of  $r(F)$  and

$$\lim_{x \uparrow r(F)} W(x) \left( \frac{W''(x)}{(W'(x))^2} + \frac{1}{xW'(x)} \right) = 0, \quad (2.15)$$

then for  $0 < r(F) \leq \infty$   $F \in RDA_p(\hat{\Phi}_5)$  and for  $r(F) \leq 0$   $F \in RDA_p(\hat{\Phi}_6)$ .

### Proposition 2.7

- (i) If pdf of  $F$  is  $f_{\nu, \sigma^2}$ , then  $F \in RDA_p(\hat{\Phi}_5)$ .
- (ii) If pdf of  $F$  is  $f_{\nu, \sigma^2, a}$ ,  $a > 0$ , then  $F \in RDA_p(\hat{\Phi}_5)$ .
- (iii) If pdf of  $F$  is  $f_{\nu, \sigma^2, a}$ ,  $a < 0$ , then  $F \in RDA_l(\Phi)$ .

*Proof*

- (i) From the principle of equivalent tails and (2.12) it is enough to check (2.15) with  $r(F) = \infty$  and the integrated hazard function

$$W(x) = \nu \ln x - \ln K_1.$$

Indeed,

$$\frac{W''(x)}{(W'(x))^2} + \frac{1}{xW'(x)} = \frac{-\frac{\nu}{x^2}}{\left(\frac{\nu}{x}\right)^2} + \frac{1}{\nu} \equiv 0.$$

- (ii) From the principle of equivalent tails and (2.13) it is enough to check (2.15) with  $r(F) = \infty$  and the integrated hazard function

$$W(x) = \frac{\nu}{2} \ln x - \ln K_2.$$

Again we find that

$$\frac{W''(x)}{(W'(x))^2} + \frac{1}{x W'(x)} = \frac{-\frac{\nu}{2x^2}}{\left(\frac{\nu}{2x}\right)^2} + \frac{2}{\nu} \equiv 0.$$

- (iii) From (2.14) and the principle of equivalent tails it is enough to consider the integrated hazard function

$$W(x) = \left(\frac{\nu}{2} + 1\right) \ln x + \frac{2|a|}{\sigma^2} x - \ln K_3,$$

where

$$K_3 = \frac{\sigma}{2|a|\Gamma\left(\frac{\nu}{2}\right)} \left(\frac{\nu|a|}{2\sigma}\right)^{\frac{\nu}{2}}.$$

The corresponding associated distribution

$$\begin{aligned} 1 - A(x) &= \exp \left\{ -\sqrt{\left(\frac{\nu}{2} + 1\right) \ln x + \frac{2|a|}{\sigma^2} x - \ln K_3} \right\} \\ &\sim \exp \left\{ -\sqrt{\frac{2|a|}{\sigma^2} x} \right\}, \quad \text{as } x \rightarrow \infty. \end{aligned}$$

Using again the principle of equivalent tails, Resnick's duality theorem and criteria from the classical extreme value theory we easily find that  $A \in \text{MDA}_I(\Lambda)$  and thus  $F \in \text{RDA}_I(\Phi)$ .  $\square$

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