Chapter 2
State Price Deflators and Stochastic Discounting

In this chapter, we describe stochastic discounting and valuation of random cash flows in a discrete time setting. We therefore introduce a consistent multiperiod pricing framework. This consistent multiperiod pricing framework is either based on state price deflators or on equivalent martingale measures. The connection between these two pricing concepts is then described by the market price of risk idea introduced in Sect. 2.4. Before we start with these stochastic valuation models, we explain the fundamental notion and terminology from interest rate modeling.

2.1 Zero Coupon Bonds and Term Structure of Interest Rates

To introduce the term structure of interest rates notion, we consider for the time being a continuous time setting. Thereafter we restrict to discrete time, see Sect. 2.2 onward. Throughout this book we work with one fixed reference currency.

2.1.1 Motivation for Discounting

What is discounting and why do we discount?

Discounting means attaching time values to assets and liabilities. Assume we put \$100 on a bank account, i.e. we lend out money to the bank. We expect that the value of this bank deposit grows with an annual interest rate r, say $r = 3\ \%$. Hence, we expect that in one year's time from today we can withdraw \$103 from the bank account. If the bank account would not offer a positive interest rate r then we could as well store the \$100 at home. Thus, banks attract deposits by offering positive interest rates.

This example shows that we have the expectation that money grows over time and therefore currency has a time value. The amount and speed at which it grows depends on economic factors such as growth of the economy, state of the economy,

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money supply and interest policy of the central bank, government expenditure, inflation rate, unemployment rate, foreign exchange rates, etc. All these factors interact in a non-trivial way and macro-economic theory tries to explain these interrelationships. One should also be aware of the fact that the growth of money by an annual interest rate r is very different from the real growth of money which determines the purchasing power of capital. Economists therefore consider the nominal interest rate r and the (expected) real interest rate which is the difference between the nominal interest rate and the (expected) inflation rate, see for instance Gärtner [73], p. 204, Romer [136], p. 73, or Fig. 7.2 in Ross et al. [137].

The aim of this book is to model growth of money and to value future (random) cash flows. For example, we model how the value of \$100 is growing over time using stochastic interest rate models. In particular, if we put the deposit of \$100 on a bank account and the bank guarantees a fixed (deterministic) annual interest rate of $r = 3\ \%$, then the final wealth of this investment in one year's time from today is \$103. Therefore, we call \$100 the discounted value of the final wealth \$103, and $(1 + r)^{-1} = 100/103 = 97.09\ \%$ is termed the (deterministic) discount factor. As discount factors are not known for all future periods, the future economic factors being random variables based on our knowledge today, we are going to model future interest rates and discount factors stochastically. This will lead to stochastic discounting using so-called state price deflators which can be viewed as economic indicators for the time value of money in stochastic term structure models.

2.1.2 Spot Rates and Term Structure of Interest Rates

Definition 2.1 A default-free zero coupon bond (ZCB) with maturity $m \geq 0$ is a contract that pays one unit of currency at time m. Its price at time $t \in [0, m]$ is denoted by $P(t, m)$.

By convention we set $P(m, m) = 1$. Since money grows over time, see last subsection, we expect $P(t, m) < 1$ for $t < m$.

A ZCB is a so-called *default-free* financial instrument. That is, its issuer cannot go bankrupt and hence always fulfills the ZCB contract (see also Example 2.8, below). In general, there is no default-free bond on the financial market, typically, bonds that are issued (by companies or governments) may default, i.e., there is a positive probability that the issuer is not able to fulfill the contract. In such cases, one speaks about credit risk that needs a special pricing component. This will be investigated in Sect. 5.1.2, below.

For the time being we will work in a continuous time setting and we will assume that ZCBs exist for all maturities $m \geq 0$. These ZCBs will describe the underlying dynamics of time value of money.

Definition 2.2 Choose $0 \le t < m$. The continuously-compounded spot rate for maturity m at time t is defined by

$$R(t,m) = -\frac{1}{m-t} \log P(t,m).$$

The simply-compounded spot rate for maturity m at time t is defined by

$$L(t,m) = \frac{1}{m-t} \frac{1-P(t,m)}{P(t,m)} = \frac{1}{m-t}\big(P(t,m)^{-1} - 1\big).$$

The annually-compounded spot rate for maturity m at time t is defined by

$$Y(t,m) = P(t,m)^{-\frac{1}{m-t}} - 1.$$

These are different notions to describe the ZCB price $P(t,m)$ at time $t \in [0,m)$. We have the identities

$$P(t,m) = e^{-(m-t)R(t,m)} = \big(1 + (m-t)L(t,m)\big)^{-1} = \big(1 + Y(t,m)\big)^{-(m-t)}. \quad (2.1)$$

This provides the relationships

$$R(t,m) = \frac{1}{m-t} \log\big(1 + (m-t)L(t,m)\big) = \log\big(1 + Y(t,m)\big).$$

Our aim is to model these spot rates. This requires that we calibrate the spot rates to actual financial market data and that we describe their stochastic development in the future. For the calibration we will use two different sets of data. For long times to maturity $m-t$ (more than one year) we will use government bond prices for the calibration (this is further described in Sect. 2.1.3 and Example 2.6, below). For short times to maturity $m-t$ (less than one year) the simply-compounded spot rate $L(t,m)$ is often calibrated with the LIBOR (London InterBank Offered Rate). The LIBOR is fixed daily at London market and is used for (unsecured) short term deposits that are exchanged between banks. That is, this is the rate at which highly credited financial institutions offer and borrow money at the interbank market. Therefore, in this book, we will use the LIBORs as approximation to short term risk-free rates for model calibration. We would like to mention that especially in periods of financial distress this needs to be done rather carefully. The spot rates should describe ZCB prices of *default-free* financial instruments. Therefore, these rates should not include any credit spread (default pricing component) and liquidity spread. However, credit and liquidity risks may have a major impact on prices during distress periods. The financial crisis of 2008 has demonstrated that also the interbank market can become almost illiquid and highly credited financial institutions may default. The high uncertainty at financial markets during distress periods can, for instance, be seen between the different rates of Repo-Overnight-Indexes (secured, see Example 3.9, below) and LIBOR curves (see also Figs. 3.1 and 3.2 below). We see a clear spread widening between these two curves between 2007 and 2009. This indicates high default and liquidity risks and shows that secured versus unsecured funds may behave rather differently in distress periods. As a consequence, model calibration of default-free ZCBs needs to be done carefully, and it is not always clear which data should be chosen for the calibration because typical financial market data always contain default and liquidity components that need to be isolated appropriately. This segmentation is heavily debated both in the financial and in the actuarial community, see, for instance, Das et al. [50], Mercurio [109–111], Danielsson et al. [48, 49] and Keller et al. [95]. We come back to these issues in terms of model calibration in Example 3.9 and Sect. 4.3.2.3. Moreover, there is an additional difficulty because typically we do not have observations for all maturity dates. The latter becomes relevant especially for the valuation of long-term guarantees in life insurance products, for more on this topic we refer to Sects. 2.1.3, 6.1 and 9.4.3, below.

Definition 2.3 The instantaneous spot rate (also called short rate) is, for $t \geq 0$, defined by

$$r(t) = \lim_{m \downarrow t} R(t, m).$$

Throughout this text we assume that the ZCB prices are sufficiently smooth functions so that all the necessary limits and derivatives exist. Note that we obtain from the power series expansion of $R(t, m)$

$$r(t) = \lim_{m \downarrow t} L(t, m).$$

Therefore, if we use the LIBORs as approximation to $L(t, m)$, we can calibrate the instantaneous spot rate $r(t)$ by the study of the LIBOR for small time intervals $[t, m]$.

Definition 2.4 The term structure of interest rates (yield curve) at time $t \geq 0$ is given by the graph of the function

$$m \mapsto R(t, m), \qquad\qquad m > t.$$

The yield curve $m \mapsto R(t, m)$ at time t determines the ZCB prices $P(t, m)$ for all maturities $m > t$ and vice versa, see (2.1). At any point in time $u < t$ future ZCB prices $P(t, m)$ are random and therefore need to be modeled stochastically. This stochastic term structure modeling of $R(t, m)$ and $P(t, m)$, respectively, is our aim in the subsequent sections and chapters.

Definition 2.5 The forward interest rate at time t, for $s \geq t$, is defined by

$$F(t, s+1) = -\log P(t, s+1) + \log P(t, s) = -\log \frac{P(t, s+1)}{P(t, s)}.$$

The instantaneous forward interest rate at time t, for maturity $s > t$, is defined by

$$f(t,s) = -\frac{\partial \log P(t,s)}{\partial s}.$$

In the continuous time setting, we obtain from the instantaneous forward interest rate $f(t,\cdot)$ by integration, for $m > t$,

$$P(t,m) = \exp\left\{-\int_t^m f(t,s)\, ds\right\}.$$

Note that $f(t,s)$, $s > t$, is observable at time t and hence so is $P(t,m)$.

Analogously, in the discrete time setting, we obtain from the forward interest rate $F(t,\cdot)$ by summation, for $m = t + k$, $k \in \mathbb{N}$,

$$P(t,m) = \exp\left\{-\sum_{s=t+1}^{m} F(t,s)\right\}.$$

The instantaneous forward interest rate $f(t,\cdot)$ is needed for continuous time interest rate modeling and the forward interest rate $F(t,\cdot)$ is needed for discrete time interest rate modeling. Of course, for $s \geq t$ we have

$$F(t,s+1) = \int_s^{s+1} f(t,u)\, du,$$

which says that the forward interest rate $F(t,\cdot)$ can always be obtained from the instantaneous forward interest rates $f(t,\cdot)$.

2.1.3 Estimating the Yield Curve

In general, the yield curve is not observable at the financial market and therefore needs to be estimated. This comes from the fact that there is no default-free ZCB on the market. As described above we use highly credited financial instruments for the estimation of the yield curve. Typically, this is the LIBOR for the short end of the yield curve and government bonds for the long end of the yield curve. These data are then used to fit a parametric curve. Popular parametric estimation methods for yield curve modeling are the Nelson–Siegel [121] and the Svensson [149, 150] methods. These methods are based on an exponential polynomial family with only few parameters that need to be estimated from the observable financial instruments (see Filipović [67]). Table 3.4 in Filipović [67] illustrates what method is used by which country.

The Svensson [149, 150] method is an extension of the Nelson–Siegel [121] method. It makes the following Ansatz for the instantaneous forward interest rate at time $t = 0$. Set

$$\boldsymbol{\beta} = \left(\beta^{(0)}, \beta^{(1)}, \beta^{(2)}, \beta^{(3)}, \gamma^{(1)}, \gamma^{(2)}\right).$$

We define the Svensson [149, 150] instantaneous forward interest rate for $s \geq 0$ by

$$f_S(0, s, \boldsymbol{\beta}) = \beta^{(0)} + \beta^{(1)} e^{-\gamma^{(1)} s} + \beta^{(2)} \gamma^{(1)} s e^{-\gamma^{(1)} s} + \beta^{(3)} \gamma^{(2)} s e^{-\gamma^{(2)} s}.$$

If we set $\beta^{(3)} = 0$ we obtain the Nelson–Siegel [121] formula that is given by

$$\begin{aligned} f_{NS}\big(0, s, \big(\beta^{(0)}, \beta^{(1)}, \beta^{(2)}, \gamma^{(1)}\big)\big) &= f_S\big(0, s, \big(\beta^{(0)}, \beta^{(1)}, \beta^{(2)}, 0, \gamma^{(1)}, 1\big)\big) \\ &= \beta^{(0)} + \beta^{(1)} e^{-\gamma^{(1)} s} + \beta^{(2)} \gamma^{(1)} s e^{-\gamma^{(1)} s}. \end{aligned}$$

Integration by parts leads to the Svensson yield curve $m \mapsto R_S(0, m, \boldsymbol{\beta})$ at time 0

$$\begin{aligned} R_S(0, m, \boldsymbol{\beta}) &= \frac{1}{m} \int_0^m f_S(0, s, \boldsymbol{\beta})\, ds \\ &= \beta^{(0)} + \big(\beta^{(1)} + \beta^{(2)}\big) \frac{1 - e^{-\gamma^{(1)} m}}{\gamma^{(1)} m} - \beta^{(2)} e^{-\gamma^{(1)} m} \\ &\quad + \beta^{(3)} \frac{1 - e^{-\gamma^{(2)} m}}{\gamma^{(2)} m} - \beta^{(3)} e^{-\gamma^{(2)} m}. \end{aligned}$$

The Svensson yield curve $R_S(0, m, \boldsymbol{\beta})$ allows for flexible shapes under appropriate parameter choices $\boldsymbol{\beta}$ (see e.g. Diebold–Li [57], Fig. 5, and Bolder–Stréliski [21], Fig. 1). It has the following properties for $\gamma^{(1)}, \gamma^{(2)} > 0$

$$\lim_{m \to \infty} R_S(0, m, \boldsymbol{\beta}) = \beta^{(0)} \qquad \text{and} \qquad \lim_{m \to 0} R_S(0, m, \boldsymbol{\beta}) = \beta^{(0)} + \beta^{(1)},$$

where the second limit follows by l'Hôpital's rule. The parameter $\beta^{(0)}$ is the long term rate and $\beta^{(0)} + \beta^{(1)}$ is the instantaneous spot rate $r(0)$ at time 0. The short term factor $\beta^{(1)}$ is related to the slope of the yield curve. The mid term factors $\beta^{(2)}$ and $\beta^{(3)}$ are related to the curvature of the yield curve with loadings determined by $\gamma^{(1)}$ and $\gamma^{(2)}$, respectively, see Sect. 2.2 in Diebold–Li [57] and Bolder–Stréliski [21] for more information on this topic.

For the estimation of parameter $\boldsymbol{\beta}$, minimum least squares methods are used with highly rated coupon bonds, for instance, reliable government bonds, see Müller [119]. We demonstrate this in more detail. Government bonds and corporate bonds are coupon bonds that are issued by a national government or by a corporation, respectively. These coupon bonds have a fixed maturity date m, a fixed nominal value v and they typically pay yearly a fixed coupon $c > 0$. Assume that we have two different coupon bonds with identical maturity dates m, nominal values v and coupons c. We denote their prices at time $t = 0$ by $\pi^{(1)}(0, m, c)$ and $\pi^{(2)}(0, m, c)$. In most cases we observe that $\pi^{(1)}(0, m, c) \neq \pi^{(2)}(0, m, c)$. One reason for this price inequality is that the issuers of the two coupon bonds may have different default probabilities. In case of default the holder of the coupon bond may lose both the coupon c and the nominal value v. Henceforth, if issuer (1) has a higher default probability than issuer (2), and all the other characteristics are the same, then we expect $\pi^{(1)}(0, m, c) < \pi^{(2)}(0, m, c)$ which accounts for this higher default assessment.

Because we would like to calibrate *default-free* ZCBs (and the corresponding yield curve) we need to choose coupon bonds that are highly rated or, in other words, which have a negligible default probability, and which are traded at deep and liquid markets meaning that they have transparent and reliable market prices. Typically, government bonds fulfill these requirements which makes them appropriate for calibration. However, we would like to indicate that government bonds are not always highly rated with reliable prices as, for instance, Greece, Ireland, Spain and Portugal have shown in 2010–2012.

We measure time in yearly units. The Svensson price at time $t = 0$ for a default-free coupon bond with maturity date $m \in \mathbb{N}$, nominal (principal, face) value $v = 1$ and yearly coupon $c > 0$ is given by

$$\pi_S(0, m, c, \boldsymbol{\beta}) = \sum_{t=1}^{m} c \, \exp\left\{-t \, R_S(0, t, \boldsymbol{\beta})\right\} + \exp\left\{-m \, R_S(0, m, \boldsymbol{\beta})\right\}.$$

From this Svensson price we determine the yield-to-maturity rate $y_S(m, c, \boldsymbol{\beta})$ given by the unique solution $y_S > -1$ of

$$\pi_S(0, m, c, \boldsymbol{\beta}) = \sum_{t=1}^{m} \frac{c}{(1 + y_S)^t} + \frac{1}{(1 + y_S)^m}.$$

Since we have six components in parameter $\boldsymbol{\beta}$, we choose $N > 6$ highly rated coupon bonds with maturity dates m_i, nominal values 1, coupons c_i and observed market yield-to-maturity rates $y_M^{(i)}$ for $i = 1, \ldots, N$. The minimum least squares estimator for $\boldsymbol{\beta}$ based on these observations is given by the solution

$$\widehat{\boldsymbol{\beta}}_S = \arg\min_{\boldsymbol{\beta}} \sum_{i=1}^{N} \left(y_M^{(i)} - y_S(m_i, c_i, \boldsymbol{\beta})\right)^2. \tag{2.2}$$

This provides the estimated Svensson yield curve $m \mapsto R_S(0, m, \widehat{\boldsymbol{\beta}}_S)$ for the minimum least squares estimator $\widehat{\boldsymbol{\beta}}_S$ of $\boldsymbol{\beta}$.

Depending on the purpose we could also minimize other l^2-distances or other loss functions. A slight modification of (2.2) is obtained by introducing weights for different maturity dates m_i. Another approach is to minimize the l^2-distance between other key figures like the Svensson prices $\pi_S(0, m_i, c_i, \boldsymbol{\beta})$ and the corresponding observed market prices $\pi_M^{(i)}$. Depending on the purpose different statistics may provide more appropriate results. If we perform the same estimation procedure setting $\beta^{(3)} = 0$ we obtain the Nelson–Siegel parameter estimate $\widehat{\boldsymbol{\beta}}_{NS}$ and the corresponding estimated Nelson–Siegel yield curve $m \mapsto R_{NS}(0, m, \widehat{\boldsymbol{\beta}}_{NS})$.

Example 2.6 Swiss government bonds called “Eidgenossenschaft” are regarded as highly rated. Therefore we use these to estimate the yield curve for the Swiss currency CHF. We choose yield-to-maturity rates of 10 different Swiss government

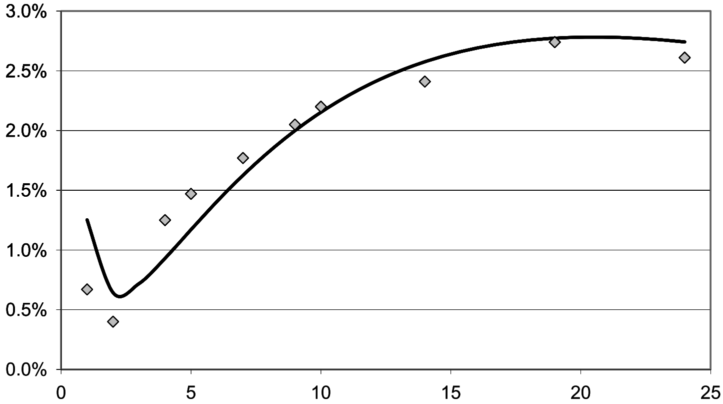

Fig. 2.1 Estimated Svensson yield curve $m \mapsto R_S(0, m, \widehat{\boldsymbol{\beta}}_S)$ as of January 1, 2009, together with the observed yield-to-maturity rates of 10 different Swiss government bonds

bonds according to their market values as of January 1, 2009 (see Fig. 2.1). If we estimate the Svensson parameter $\widehat{\boldsymbol{\beta}}_S$ by (2.2) we obtain the yield curve plotted in Fig. 2.1. Note that it is not strictly increasing. Due to the financial distress situation in 2008/2009 we obtain a non-monotonic development of the term structure of interest rates for short maturities m. This reflects current market beliefs and uncertainties about future interest rate developments.

We conclude with the following remarks. We have described how the parameters of Nelson–Siegel and Svensson yield curves can be estimated. Of course, we could also choose any other parametric curve, like cubic B-splines and exponential polynomial families and fit those to the observed data. For more on such calibration techniques we refer to Chap. 3 in Filipović [67] and Hagan–West [78].

In actuarial problems it is of special interest to have reliable estimates also at the long end of the yield curve, i.e. for long times to maturity. For example, cash flows of life insurance products can have a time horizon of 50 years. Since in practice there are no market data available for such long times to maturity, extrapolation methods are used for the long end of the yield curve. Currently, there does not exist a general agreement how this should be done and research still aims to find a robust and reliable method using different (economic) approaches.

2.2 Basic Discrete Time Stochastic Model

In the previous section, we have calibrated the yield curve $m \mapsto R(t, m)$ at a fixed point t in time. For predicting future values of assets and liabilities, we would like to know how the yield curve evolves in the future. Therefore, we aim to model the (stochastic) evolution of the yield curve. There are different modeling approaches:

(i) economic approaches that model underlying macro-economic factors (like economic growth, money supply and interest policy of the central bank, inflation rate, unemployment rate, real activity, etc.) and then the yield curve evolution is linked to these factors, (ii) purely statistical approaches that study yield curve time series, and (iii) financial mathematical approaches that are based on consistent and arbitrage-free pricing systems. We focus on the latter and give interpretations to the factors in terms of economic variables whenever possible. Moreover, statistical methods are used for model calibration. We insist on having consistent and arbitrage-free pricing systems. This is especially important in markets where we have highly correlated financial instruments as it is the case for ZCBs; we will come back to this below and we also refer to Teichmann–Wüthrich [152].

We model the yield curve behavior in a discrete time setting.

2.2.1 Valuation at Time 0

Throughout we choose a fixed finite time horizon $n \in \mathbb{N}$ and a discrete time setting with points in time $t \in \mathscr{J} = \{0, 1, \dots, n\}$. Our goal is to value discrete time cash flows $\mathbf{X} = (X_0, \dots, X_n)$ at any point in time $t \in \mathscr{J}$, where we interpret X_k to be the payment done at time $k \in \mathscr{J}$. In the sequel the notation $\mathscr{J}_- = \{0, 1, \dots, n-1\}$ will also be helpful.

Time Convention In more generality, we should assume that we have points in time $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n$, where $t_k \in \mathbb{R}$ denotes the point in time (in yearly units) when X_k is paid. In order to keep the notation simple, we assume that $t_k = k$ for all $k \in \mathscr{J}$, i.e. cash flows $\mathbf{X} = (X_0, \dots, X_n)$ are paid on a yearly grid where the span of the grid will be denoted by $\delta = 1$ (in years). On the other hand, if we work on a yearly grid, we have only a few observations to calibrate the model parameters from (e.g. 10 observations for the time period from 1999 until 2008). Therefore in examples below, we switch to a monthly grid with span $\delta = 1/12$, meaning that we have time points

$$t_k = k\,\delta \qquad \text{for } k \in \mathscr{J}.$$

If parameters relate to non-yearly grids ($\delta \neq 1$), we indicate this with a subscript δ in the parameters, see for instance Example 3.9, below.

We choose a (sufficiently rich) filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F})$ with probability measure $\mathbb{P}$ and filtration $\mathbb{F} = (\mathscr{F}_t)_{t \in \mathscr{J}}$ on the measurable space $(\Omega, \mathscr{F})$. Thus, we have an increasing sequence of σ-fields $\mathscr{F}_t$ on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ satisfying $\mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_n \subset \mathscr{F}$. The σ-field $\mathscr{F}_t$ plays the role of the information available at time $t \in \mathscr{J}$. We set $\mathscr{F}_0 = \{\emptyset, \Omega\}$ and $\mathscr{F}_n = \mathscr{F}$.

The probability measure $\mathbb{P}$ plays the role of the *real world probability measure*, also called *objective probability measure* or *physical probability measure*. It is the measure under which the cash flows and price processes are observed. We denote the expected value with respect to the real world probability measure $\mathbb{P}$ by $\mathbb{E}$.

Assumption 2.7 We assume that all cash flows $\mathbf{X} = (X_0, \ldots, X_n)$ are $\mathbb{F}$-adapted random vectors on $(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F})$ with all components X_k of $\mathbf{X}$ being integrable. We write $\mathbf{X} \in L^1_{n+1}(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F})$.

Notation $\mathbf{X} = (X_0, \ldots, X_n) \in L^1_{n+1}(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F})$ means that (i) X_k is $\mathscr{F}_k$-measurable for all $k \in \mathscr{J}$, i.e. X_k is observable w.r.t. the information $\mathscr{F}_k$ available at time k; (ii) the expected value of X_k under $\mathbb{P}$ exists for all $k \in \mathscr{J}$.

Example 2.8 (Default-free zero coupon bond) The cash flow of the default-free ZCB with maturity date $m \in \mathscr{J}$ is given by

$$\mathbf{Z}^{(m)} = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^{n+1},$$

where the "1" is at the $(m+1)$-st position. Of course, $\mathbf{Z}^{(m)} \in L^1_{n+1}(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F})$ holds true. Note that $\mathbf{Z}^{(m)}$ is a default-free ZCB meaning that the cash flow $X_m = 1$ at time m is paid with probability 1 ($\mathbb{P}$-a.s.).

Definition 2.9 An $(n+1)$-dimensional random vector $\mathbf{X} = (X_0, \ldots, X_n)$ is called

(a) non-negative ($\mathbf{X} \geq 0$) iff $X_k \geq 0$, $\mathbb{P}$-a.s., for all $k \in \mathscr{J}$;
(b) positive ($\mathbf{X} > 0$) iff $\mathbf{X} \geq 0$ and there exists $k \in \mathscr{J}$ such that $\mathbb{P}(X_k > 0) > 0$;
(c) strictly positive ($\mathbf{X} \gg 0$) iff $X_k > 0$, $\mathbb{P}$-a.s., for all $k \in \mathscr{J}$.

Observe that the default-free ZCB introduced in Example 2.8 satisfies $\mathbf{Z}^{(m)} > 0$.

We build a valuation framework for $\mathbb{F}$-adapted stochastic cash flows $\mathbf{X}$. This is done based on Bühlmann [30, 31] and Wüthrich et al. [168] using so-called state price deflators. The motivation goes as follows. Assume Q is a positive, continuous and linear (valuation) functional with normalization $Q[\mathbf{Z}^{(0)}] = 1$ on the Hilbert space $L^2_{n+1}(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F})$ of $\mathbb{F}$-adapted and square integrable cash flows $\mathbf{X}$. Then, Riesz' representation theorem says that there exists a $\mathbb{P}$-a.s. unique, $\mathbb{F}$-adapted and strictly positive random vector $\boldsymbol{\varphi} = (\varphi_0, \ldots, \varphi_n) \in L^2_{n+1}(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F})$ with $\varphi_0 \equiv 1$ such that

$$Q[\mathbf{X}] = \mathbb{E}\left[\sum_{k \in \mathscr{J}} \varphi_k \, X_k\right] \qquad \text{for all } \mathbf{X} \in L^2_{n+1}(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F}),$$

for details see Theorem 2.5 in Wüthrich et al. [168]. Thus, on the Hilbert space of square integrable cash flows there is a one-to-one correspondence between valuation functionals Q and random vectors $\boldsymbol{\varphi}$. The assumption of square integrability is often too restrictive for pricing insurance cash flows. Therefore, we relax this assumption which provides the following comprehensive valuation framework.

Definition 2.10 (State price deflator) Assume $\boldsymbol{\varphi} = (\varphi_0, \ldots, \varphi_n) \in L^1_{n+1}(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F})$ is a strictly positive random vector with normalization $\varphi_0 \equiv 1$. Then $\boldsymbol{\varphi}$ and its components φ_k, $k \in \mathscr{J}$, are called state price deflator (actuarial mathematics), financial pricing kernel (financial mathematics) or state price density (economic theory).

We choose a fixed state price deflator $\boldsymbol{\varphi} \in L^1_{n+1}(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F})$. The set of $\mathbb{F}$-adapted cash flows which can be priced relative to the state price deflator $\boldsymbol{\varphi}$ is given by

$$\mathscr{L}_{\boldsymbol{\varphi}} = \left\{ \mathbf{X} \in L^1_{n+1}(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F});\ \mathbb{E}\left[\sum_{k \in \mathscr{J}} \varphi_k \, |X_k| \middle| \mathscr{F}_0 \right] < \infty \right\}. \tag{2.3}$$

In particular, (2.3) characterizes all risks (cash flows) that can be insured according to the choice $\boldsymbol{\varphi}$. For other cash flows no unlimited insurance cover can be offered under $\boldsymbol{\varphi}$. For instance, there is no unlimited insurance cover against earthquake events or for nuclear power accidents available at the insurance market (which should be reflected by an appropriate choice of the state price deflator $\boldsymbol{\varphi}$).

This then allows to define the *value of cash flow* $\mathbf{X} \in \mathscr{L}_{\boldsymbol{\varphi}}$ *at time 0* w.r.t. the state price deflator $\boldsymbol{\varphi}$ as follows

$$Q_0[\mathbf{X}] = \mathbb{E}\left[\sum_{k \in \mathscr{J}} \varphi_k \, X_k \middle| \mathscr{F}_0 \right]. \tag{2.4}$$

Throughout we assume that a fixed state price deflator $\boldsymbol{\varphi} \in L^1_{n+1}(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F})$ is given. Then we value the cash flows $\mathbf{X} \in \mathscr{L}_{\boldsymbol{\varphi}}$ relative to $\boldsymbol{\varphi}$ using (2.4).

At this point, we could analyze the properties of the subset $\mathscr{L}_{\boldsymbol{\varphi}} \subset L^1_{n+1}(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F})$, i.e. the properties of insurable cash flows relative to $\boldsymbol{\varphi}$. However, the only property that we will need is that $\mathbf{X}, \mathbf{Y} \in \mathscr{L}_{\boldsymbol{\varphi}}$ implies $\mathbf{X} + \mathbf{Y} \in \mathscr{L}_{\boldsymbol{\varphi}}$ which is clear.

Remarks and Outlook

- We recall the properties of state price deflators $\boldsymbol{\varphi} = (\varphi_0, \ldots, \varphi_n)$: (i) $\mathbb{F}$-adapted; (ii) integrable components φ_k; (iii) strictly positive; and (iv) normalized.
- We have fixed a state price deflator $\boldsymbol{\varphi} \in L^1_{n+1}(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F})$ for the valuation of (insurance) cash flows $\mathbf{X} \in \mathscr{L}_{\boldsymbol{\varphi}}$. In general, there are infinitely many state price deflators and the crucial question is: which one should be chosen?

  Bühlmann et al. [34] and Föllmer–Schied [71], Chap. 5, start with a financial market model that describes price processes of financial assets. Trading strategies on these financial assets generate the corresponding cash flows. On these cash flows one then constructs the pricing functional Q and $\boldsymbol{\varphi}$, respectively, so that one obtains a valuation framework free of arbitrage which explains the price formation at the financial market. This argumentation is based on cash flows generated by traded financial instruments. Our valuation framework extends this viewpoint in the sense that it allows to value also (non-traded, non-hedgeable) insurance cash flows. Basically, market risk aversion and legal constraints determine appropriate state price deflators $\boldsymbol{\varphi}$ which in turn provide the insurable cash flows $\mathbf{X} \in \mathscr{L}_{\boldsymbol{\varphi}}$ and the corresponding prices via (2.4).

- The L^2-framework as introduced in Bühlmann et al. [34] gives a nice connection between the valuation functional Q and the state price deflator $\boldsymbol{\varphi}$ using Riesz' representation theorem. Föllmer–Schied [71] use for the same valuation purpose a different approach in the sense that they directly target for the equivalent martingale measure, see Theorem 5.17 in Föllmer–Schied [71]. This will be the subject of Sect. 2.3, below.
- In Sect. 2.2.3 we introduce valuation at time $t > 0$. In Chaps. 3 and 4 we give explicit models for state price deflators and we explain how these models are used for yield curve prediction. In Chap. 5 we describe the financial market and explain how this fits into our valuation framework. This will be crucial for the valuation of insurance liabilities which is the main topic of Part II of this book.

2.2.2 Interpretation of State Price Deflators

Assume that a fixed state price deflator $\boldsymbol{\varphi} \in L^1_{n+1}(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F})$ is given.

The state price deflator component φ_k transports (random) cash amounts X_k at time k to values in time 0. This transportation is a stochastic transportation. This means that φ_k plays the role of a *stochastic discount factor*. Consider a cash flow $\mathbf{X}_k = (0, \ldots, 0, X_k, 0, \ldots, 0) \in \mathscr{L}_{\boldsymbol{\varphi}}$. Its price at time 0 is given by

$$Q_0[\mathbf{X}_k] = \mathbb{E}[\varphi_k X_k | \mathscr{F}_0].$$

This highlights the stochastic discounting mechanism of the state price deflator φ_k applied to the cash flow X_k paid at time k.

In general, the cash flows $\mathbf{X} \in \mathscr{L}_{\boldsymbol{\varphi}}$ are *not* uncorrelated from the state price deflator $\boldsymbol{\varphi}$, i.e.

$$Q_0[\mathbf{X}] = \mathbb{E}\left[\sum_{k \in \mathscr{J}} \varphi_k \; X_k \middle| \mathscr{F}_0 \right] \neq \sum_{k \in \mathscr{J}} \mathbb{E}[\varphi_k | \mathscr{F}_0] \; \mathbb{E}[X_k | \mathscr{F}_0]. \tag{2.5}$$

Therefore the evaluation of $Q_0[\mathbf{X}]$ needs to be done carefully. Assume that the state price deflator $\boldsymbol{\varphi}$ describes stochastic risk factors from the financial market and $\mathbf{X}$ is an insurance cash flow that describes the payouts to the insured. Then the valuation functional Q_0 allows for the modeling of financial guarantees and options in the insurance cash flow $\mathbf{X}$. Since these financial options and guarantees depend on the same risk drivers as the state price deflator we typically have correlation between $\mathbf{X}$ and $\boldsymbol{\varphi}$ and arrive at inequality (2.5). If the state price deflator $\boldsymbol{\varphi}$ is uncorrelated with the insurance cash flow $\mathbf{X}$ then the valuation can be done separately by taking the appropriate expected values in (2.5). This then leads to replicating portfolios for expected liabilities, represented in terms of ZCB prices given by $P(0, k) = \mathbb{E}[\varphi_k | \mathscr{F}_0]$, see also Example 2.11, below. These replicating portfolios and inequality (2.5) are further elaborated in Chap. 7.

Example 2.11 (Default-free zero coupon bond price) For the value at time 0 of the default-free ZCB with maturity $m \le n$ we obtain

$$P(0,m) = Q_0\big[\mathbf{Z}^{(m)}\big] = \mathbb{E}[\varphi_m | \mathscr{F}_0]. \tag{2.6}$$

Hence, $Q_0[\mathbf{Z}^{(m)}]$ describes the $\mathscr{F}_0$-measurable ZCB price at time 0, see also Definition 2.1. Therefore, $P(0,m)$ also transports cash amounts at time m to values in time 0. Note that $P(0,m)$ is $\mathscr{F}_0$-measurable, whereas φ_m is an $\mathscr{F}_m$-measurable random variable. This means that the discount factor $P(0,m)$ is known at the beginning of the time period $[0,m]$, whereas φ_m is only known at the end of the time period $[0,m]$. As long as we deal with deterministic cash flows $\mathbf{X}$, we can work either with ZCB prices $P(0,m)$ or with state price deflators $\boldsymbol{\varphi}$ to determine the value of $\mathbf{X}$ at time 0. But as soon as the cash flows are stochastic we need to work with state price deflators $\boldsymbol{\varphi}$ since $\mathbf{X}$ and $\boldsymbol{\varphi}$ may be influenced by the same risk factors and therefore may be dependent, see (2.5) and (8.11).

We close this subsection with the following remark. The standard assumption will be that the components φ_k of the state price deflator $\boldsymbol{\varphi}$ are integrable, see Definition 2.10. Below we will consider several different explicit models for state price deflators. Typically, we will start with an $(n+1)$-dimensional $\mathbb{F}$-adapted random vector for $\boldsymbol{\varphi}$ and the integrability condition is then proved by checking whether we obtain finite ZCB prices (2.6).

2.2.3 Valuation at Time $t > 0$

In the previous subsections, we have only defined valuation at time $t = 0$. We now extend the valuation to any time point $t \in \mathscr{J}$ which then leads to price processes $(Q_t[\mathbf{X}])_{t\in\mathscr{J}}$ for the cash flows $\mathbf{X} \in \mathscr{L}_{\boldsymbol{\varphi}}$. This extension should be done such that we obtain consistent or arbitrage-free price dynamics.

Definition 2.12 Assume a fixed state price deflator $\boldsymbol{\varphi} \in L^1_{n+1}(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F})$ is given. We define the price processes $(Q_t[\mathbf{X}])_{t\in\mathscr{J}}$ for cash flows $\mathbf{X} \in \mathscr{L}_{\boldsymbol{\varphi}}$ as follows:

$$Q_t[\mathbf{X}] = \frac{1}{\varphi_t}\,\mathbb{E}\Bigg[\sum_{k\in\mathscr{J}} \varphi_k\, X_k \Bigg| \mathscr{F}_t\Bigg], \qquad \text{for } t \in \mathscr{J}. \tag{2.7}$$

$Q_t[\mathbf{X}]$ denotes the value/price of the cash flow $\mathbf{X}$ at time $t \in \mathscr{J}$. This price is well-defined because φ_t is strictly positive, $\mathbb{P}$-a.s., and it is $\mathscr{F}_t$-measurable. Moreover, Definition 2.12 is in line with (2.4) at time 0 due to $\varphi_0 \equiv 1$. Note that this price process $(Q_t[\mathbf{X}])_{t\in\mathscr{J}}$ depends on the given choice of the state price deflator $\boldsymbol{\varphi}$.

An important statement is given in the following proposition:

Proposition 2.13 *Assume that a fixed state price deflator* $\boldsymbol{\varphi} \in L^1_{n+1}(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F})$ *is given and that the price processes* $(Q_t[\mathbf{X}])_{t\in\mathscr{J}}$ *of* $\mathbf{X} \in \mathscr{L}_{\boldsymbol{\varphi}}$ *are defined by* (2.7). *The deflated price processes* $(\varphi_t Q_t[\mathbf{X}])_{t\in\mathscr{J}}$ *are* $(\mathbb{P}, \mathbb{F})$*-martingales.*

Proof Integrability follows by assumption. Since $\mathscr{F}_t \subset \mathscr{F}_{t+1}$ for $t \in \mathscr{J}_-$ we have from the tower property of conditional expectations (see Williams [159])

$$\mathbb{E}\big[\varphi_{t+1} Q_{t+1}[\mathbf{X}]\big|\mathscr{F}_t\big] = \mathbb{E}\left[\mathbb{E}\left[\sum_{k\in\mathscr{J}} \varphi_k X_k \middle| \mathscr{F}_{t+1}\right]\middle|\mathscr{F}_t\right]$$

$$= \mathbb{E}\left[\sum_{k\in\mathscr{J}} \varphi_k X_k \middle| \mathscr{F}_t\right] = \varphi_t Q_t[\mathbf{X}].$$

This finishes the proof of the proposition. □

Interpretation of Proposition 2.13 Proposition 2.13 is crucial for obtaining economically meaningful pricing systems. It tells us that deflated price processes form $(\mathbb{P}, \mathbb{F})$-martingales for a fixed $\boldsymbol{\varphi}$. These martingale properties for a given state price deflator are a necessary and sufficient condition for the pricing system to be arbitrage-free, i.e. it eliminates certain gains without any downside risk. This is the key assumption for having meaningful pricing systems and in the literature it refers to the *fundamental theorem of asset pricing* (FTAP). We will come back to the FTAP in more detail in Remarks 2.21 below.

Definition 2.14 (Consistency) Choose a state price deflator $\boldsymbol{\varphi} \in L^1_{n+1}(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F})$. If the deflated price process $(\varphi_t Q_t[\mathbf{X}])_{t\in\mathscr{J}}$ is a $(\mathbb{P}, \mathbb{F})$-martingale then $(Q_t[\mathbf{X}])_{t\in\mathscr{J}}$ is called consistent w.r.t. $\boldsymbol{\varphi}$ (or $\boldsymbol{\varphi}$-consistent).

Remark The if-statement in Definition 2.14 might look strange at first sight because all price processes defined by (2.7) are $\boldsymbol{\varphi}$-consistent (see Proposition 2.13). The reason for stating Definition 2.14 in the above form is that below we also define price processes differently from (2.7) and then we first need to check the consistency condition.

Example 2.15 (Default-free zero coupon bond prices) We calculate the price of the default-free ZCB with maturity $m \le n$ at time $t \le m$. Definition 2.12 provides the $\mathscr{F}_t$-measurable and $\boldsymbol{\varphi}$-consistent price

$$P(t,m) = Q_t\big[\mathbf{Z}^{(m)}\big] = \frac{1}{\varphi_t}\mathbb{E}[\varphi_m|\mathscr{F}_t]. \tag{2.8}$$

Example 2.16 (Risk-free spot rates) In our discrete time setting the shortest time period is defined by the span $\delta = 1$ of the yearly grid size. This implies that the *one-period risk-free asset* at time t is given by the ZCB with maturity date $t+1$. Its $\boldsymbol{\varphi}$-consistent price at time t is given by

$$P(t, t+1) = Q_t\big[\mathbf{Z}^{(t+1)}\big] = \frac{1}{\varphi_t}\mathbb{E}[\varphi_{t+1}|\mathscr{F}_t].$$

The annual risk-free return at time t is described by the $\mathscr{F}_t$-measurable continuously-compounded spot rate

$$R(t, t+1) = -\log P(t, t+1) = -\log \mathbb{E}\left[\left.\frac{\varphi_{t+1}}{\varphi_t}\right| \mathscr{F}_t\right],$$

or by the $\mathscr{F}_t$-measurable annually-compounded (or simply-compounded) spot rate

$$Y(t, t+1) = L(t, t+1) = P(t, t+1)^{-1} - 1 = \mathbb{E}\left[\left.\frac{\varphi_{t+1}}{\varphi_t}\right| \mathscr{F}_t\right]^{-1} - 1.$$

Or in other words, the one-period risk-free investment at time t has price

$$P(t, t+1) = e^{-R(t,t+1)} = \big(1 + Y(t, t+1)\big)^{-1} = \frac{1}{\varphi_t}\,\mathbb{E}[\varphi_{t+1}|\mathscr{F}_t].$$

Note that $R(t, t+1)$ relates to the annual return of the bank account (see (2.12) below). That is, if we invest one unit of currency at time t into the bank account we get a risk-free value of $P(t, t+1)^{-1} = \exp\{R(t, t+1)\}$ one period later (at time $t+1$). It is important to realize that this "one period later" is strongly related to the choice of the grid size (yearly grid here). If we have a monthly grid, i.e. if we choose span $\delta = 1/12$, then the value of the bank account grows as $\exp\{\delta R(t, t+\delta)\}$ because we have reinvestment possibilities after every month. This is a first example that shows limitations of a discrete time framework, namely that the risk-free return of the bank account is only defined relative to span δ (and the smallest time interval is not well-defined if we are allowed to refine the grid size).

In this discrete time setting we define the *span-deflator* by $\breve{\varphi}_0 = 1$ and for $t \in \mathscr{J}_-$

$$\breve{\varphi}_{t+1} = \frac{\varphi_{t+1}}{\varphi_t}. \tag{2.9}$$

The span-deflator $\breve{\varphi}_{t+1}$ is $\mathscr{F}_{t+1}$-measurable and transports cash amounts (stochastically) from time $t+1$ to time t. The continuously-compounded spot rate is then given by

$$R(t, t+1) = -\log P(t, t+1) = -\log \mathbb{E}[\breve{\varphi}_{t+1}|\mathscr{F}_t].$$

Moreover, for given span-deflator $\breve{\boldsymbol{\varphi}} = (\breve{\varphi}_t)_{t\in\mathscr{J}}$ we rediscover the state price deflator $\boldsymbol{\varphi} = (\varphi_t)_{t\in\mathscr{J}}$ by considering the products

$$\varphi_t = \prod_{s=0}^{t} \breve{\varphi}_s. \tag{2.10}$$

In many models the state price deflator $\boldsymbol{\varphi}$ has a product structure (2.10) and in these cases it is natural to directly model the span-deflator $\breve{\boldsymbol{\varphi}}$. For further discussions of span-deflators we refer to Bühlmann et al. [34] and Wüthrich et al. [168].

2.3 Equivalent Martingale Measure

The price processes $(Q_t[\mathbf{X}])_{t\in\mathscr{J}}$ are expressed in the fixed reference currency chosen at the beginning, see start of Sect. 2.1 and Definition 2.1. Under the assumption that the price processes are consistent w.r.t. the given state price deflator $\boldsymbol{\varphi}$ we have martingales $(\varphi_t Q_t[\mathbf{X}])_{t\in\mathscr{J}}$ under the real world probability measure $\mathbb{P}$, see Proposition 2.13 and Definition 2.14. Theoretically, we can choose any strictly positive price process as reference unit. The choice of such a reference unit is called *choice of numeraire*. There is one specific numeraire which we are going to discuss and analyze in this section, the so-called bank account numeraire.

2.3.1 Bank Account Numeraire

If we choose as basis for discounting on the yearly grid the one-year risk-free assets described by the ZCB prices $P(t, t+1)$ at times $t \in \mathscr{J}_-$, see Example 2.16, we obtain the discrete time *bank account numeraire*. Let us describe how this is done. We define the one-year risk-free returns by the continuously-compounded spot rates as

$$r_t \stackrel{\text{def.}}{=} R(t, t+1) = -\log P(t, t+1). \tag{2.11}$$

In contrast to the instantaneous spot rate $r(t)$ in continuous time (see Definition 2.3) we denote the continuously-compounded spot rate (in discrete time for one period) with a subscript r_t. Note that r_t is $\mathscr{F}_t$-measurable, i.e. known at the beginning of the time period $(t, t+1]$. Then, in this yearly discrete time setting, we define the value of the bank account (money market account) at time $t \in \mathscr{J}$ by

$$B_t = \exp\left\{\sum_{s=0}^{t-1} r_s\right\} = \exp\left\{\sum_{s=0}^{t-1} R(s, s+1)\right\} > 0, \tag{2.12}$$

where an empty sum is defined to be zero, i.e. $B_0 = 1$. B_t describes the value at time t of an initial investment of one unit of currency at time 0 into the bank account (one-year risk-free rollover, see Example 2.16). The value of the bank account B_t is known at time $t-1$, that is, it is previsible or so-called locally riskless, see Föllmer–Schied [71], Example 5.5. For these bank account values $(B_t)_{t\in\mathscr{J}}$ we can construct an equivalent probability measure $\mathbb{P}^* \sim \mathbb{P}$ such that the $(B_t^{-1})_{t\in\mathscr{J}}$ discounted price processes are $(\mathbb{P}^*, \mathbb{F})$-martingales, see Proposition 2.18 below. This equivalent probability measure $\mathbb{P}^*$ for the bank account numeraire $(B_t)_{t\in\mathscr{J}}$ is called *equivalent martingale measure*, *risk-neutral measure* or *pricing measure*.

Remark on Time Convention In this discrete time setting the choice of the grid size is crucial. If we choose a monthly grid $\delta = 1/12$, the bank account is defined by

$$B_t^{(\delta)} = \exp\left\{ \sum_{s=0}^{t-1} \delta \; R\big(s\delta, (s+1)\delta\big) \right\} > 0. \tag{2.13}$$

This is the value at time $t\delta$ of an initial investment of one unit of currency at time 0 into the bank account. Note that $B_1^{(\delta)} = \exp\{\delta \; R(0, \delta)\} = P(0, \delta)^{-1}$ which shows that the units were chosen correctly.

If we invest one unit of currency at time 0 into the ZCB with maturity $m = 1$ (one year) then this value differs from the value of a similar investment into the bank account on the monthly grid $\delta = 1/12$, i.e. in general

$$P(0,1)^{-1} \neq B_{1/\delta}^{(\delta)}. \tag{2.14}$$

This highlights the difficulties if one works in a discrete time setting, namely that the shortest possible time interval for investments may not be well-defined if one is allowed to refine the grid size. Therefore, we first choose $\delta > 0$ and then build our theory around this choice. These difficulties can be avoided by going over to a continuous time setting. No doubt, continuous time models are mathematically much more demanding, however they often do not have richer economic properties. Therefore, we decided to work in a discrete time setting. We would like to mention that discrete time models have some limitations (compared to continuous time models). Below we will meet situations where we need a continuous time framework to obtain the full flavor of the problem.

2.3.2 Martingale Measure and the FTAP

In this subsection we construct the equivalent martingale measure $\mathbb{P}^*$ for the bank account numeraire $(B_t)_{t \in \mathscr{J}}$ and we explain how its existence is related to the fundamental theorem of asset pricing (FTAP).

Lemma 2.17 *Assume a fixed state price deflator $\boldsymbol{\varphi} \in L^1_{n+1}(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F})$ is given. The process $(\xi_t)_{t \in \mathscr{J}}$ defined by $\xi_t = \varphi_t B_t$ is a strictly positive $(\mathbb{P}, \mathbb{F})$-martingale with expected value* 1.

Proof By definition we have $\boldsymbol{\varphi} \gg 0$ and $\boldsymbol{\varphi} \in L^1_{n+1}(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F})$ which implies strict positivity of ξ_t for all $t \in \mathscr{J}$. Note that B_{t+1} is $\mathscr{F}_t$-measurable. Henceforth, for $t \in \mathscr{J}_-$ we have that

$$\mathbb{E}[\xi_{t+1} | \mathscr{F}_t] = B_{t+1} \; \mathbb{E}[\varphi_{t+1} | \mathscr{F}_t] = B_{t+1} \; \varphi_t \; P(t, t+1) = \varphi_t B_t = \xi_t,$$

which proves the martingale claim. Moreover, this implies for all $t \in \mathscr{J}_-$ (recall the normalization $\varphi_0 \equiv 1$ and $P(0,1) = B_1^{-1}$)

$$\mathbb{E}[\xi_{t+1}|\mathscr{F}_0] = \mathbb{E}[\xi_1|\mathscr{F}_0] = B_1\, \mathbb{E}[\varphi_1|\mathscr{F}_0] = B_1 P(0,1) = 1 = \xi_0,$$

see (2.8). This completes the proof. $\square$

Lemma 2.17 states that the price process $(B_t)_{t\in\mathscr{J}}$ of the bank account is consistent w.r.t. $\boldsymbol{\varphi}$ with initial value 1 (initial investment). Moreover, this price process $(B_t)_{t\in\mathscr{J}}$ is strictly positive which implies that we can use it as a numeraire, see Sect. 11.2. Thus, $(\xi_t)_{t\in\mathscr{J}}$ is a density process, see (11.1) below, and we can use it to define an equivalent probability measure $\mathbb{P}^* \sim \mathbb{P}$ via the Radon–Nikodym derivative

$$\left.\frac{d\mathbb{P}^*}{d\mathbb{P}}\right|\mathscr{F}_n = \xi_n = \varphi_n\; B_n >\; 0.$$

In the following we denote the expected value w.r.t. $\mathbb{P}^*$ by $\mathbb{E}^*$. For $A \in \mathscr{F}_n$ we have $\mathbb{P}^*[A] = \mathbb{E}^*[1_A] = \mathbb{E}[\xi_n 1_A]$, and for the calculation of conditional expectations w.r.t. $\mathbb{P}^*$ and $\mathscr{F}_t$, $t \in \mathscr{J}$, we refer to Lemma 11.3 below. Lemma 11.5 then immediately gives the following proposition.

Proposition 2.18 *Assume a fixed state price deflator $\boldsymbol{\varphi} \in L^1_{n+1}(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F})$ is given and that the price processes $(Q_t[\mathbf{X}])_{t\in\mathscr{J}}$ of $\mathbf{X} \in \mathscr{L}_{\boldsymbol{\varphi}}$ are defined by (2.7). The bank account numeraire discounted price processes $(B_t^{-1} Q_t[\mathbf{X}])_{t\in\mathscr{J}}$ are $(\mathbb{P}^*, \mathbb{F})$-martingales.*

Proposition 2.13 says that $\boldsymbol{\varphi}$ deflated price processes $(\varphi_t Q_t[\mathbf{X}])_{t\in\mathscr{J}}$ are martingales w.r.t. the real world probability measure $\mathbb{P}$, which is exactly the consistency property w.r.t. $\boldsymbol{\varphi}$, see Definition 2.14. The slightly unpleasant feature of the state price deflator $\boldsymbol{\varphi}$ is that φ_t is only observable at time t. Often, a discount factor that is previsible, i.e. $\mathscr{F}_{t-1}$-measurable, is advantageous. The bank account numeraire $(B_t)_{t\in\mathscr{J}}$ exactly provides this previsible discount factor. For this previsible discount factor we need to consider price processes under the so-called *equivalent martingale measure* $\mathbb{P}^* \sim \mathbb{P}$ (according to Proposition 2.18), i.e. bank account numeraire discounted price processes $(B_t^{-1} Q_t[\mathbf{X}])_{t\in\mathscr{J}}$ are $(\mathbb{P}^*, \mathbb{F})$-martingales. This gives the nice property

$$B_{t-1}^{-1}\; Q_{t-1}[\mathbf{X}] = \mathbb{E}^*\left[B_t^{-1} Q_t[\mathbf{X}]\middle|\mathscr{F}_{t-1}\right] = B_t^{-1}\; \mathbb{E}^*\left[Q_t[\mathbf{X}]\middle|\mathscr{F}_{t-1}\right],$$

i.e. the discount factor B_t^{-1} is already observable at time $t-1$ (previsible, locally riskless).

In financial mathematics one usually works under the bank account numeraire and $\mathbb{P}^*$ since many derivations of price processes are easier and more straightforward under $\mathbb{P}^*$. However, for actuarial purposes one always needs to keep track of the real world probability measure $\mathbb{P}$ because insurance benefits and parameter choices can only be understood and modeled under $\mathbb{P}$. We will always use the representation that is practically more useful for the explicit problems considered.

Under the assumptions of Proposition 2.13, we obtain the following two corollaries from Lemma 11.4. The first corollary tells us how we can calculate the price of a ZCB at time $t \leq m$ under the real world probability measure $\mathbb{P}$ or under the equivalent martingale measure $\mathbb{P}^*$:

Corollary 2.19 *The price of the ZCB with maturity $m \leq n$ at time $t \leq m$ is given by*

$$P(t,m) = \frac{1}{\varphi_t}\,\mathbb{E}[\varphi_m|\mathscr{F}_t] = \frac{1}{B_t^{-1}}\,\mathbb{E}^*\left[B_m^{-1}|\mathscr{F}_t\right] = \mathbb{E}^*\left[\exp\left\{-\sum_{s=t}^{m-1} r_s\right\}\middle|\mathscr{F}_t\right].$$

The second corollary shows us how we can calculate the price of a general cash flow X_k at time $t \leq k \in \mathscr{J}$ under the probability measures $\mathbb{P}$ and $\mathbb{P}^*$, respectively:

Corollary 2.20 *The price of a cash flow $\mathbf{X}_k = (0, \ldots, 0, X_k, 0, \ldots, 0) \in \mathscr{L}_{\boldsymbol{\varphi}}$ at time $t \leq k$ is given by*

$$Q_t[\mathbf{X}_k] = \frac{1}{\varphi_t}\,\mathbb{E}[\varphi_k\; X_k|\mathscr{F}_t] = \mathbb{E}^*\left[\exp\left\{-\sum_{s=t}^{k-1} r_s\right\} X_k\middle|\mathscr{F}_t\right].$$

That is, if we choose an $\mathscr{F}_k$-measurable contingent claim X_k, we can easily calculate its price at time $t \leq k$ under the valuation functional Q_t. This can either be done under the real world probability measure $\mathbb{P}$ using the $\mathbb{F}$-adapted state price deflator $\boldsymbol{\varphi}$ for stochastic discounting or under the equivalent martingale measure $\mathbb{P}^*$ using the previsible bank account numeraire $(B_t)_{t\in\mathscr{J}}$ for discounting.

In the next remark we are putting the existence of an equivalent martingale measure $\mathbb{P}^*$ into the general pricing context. This gives the link to the literature in financial mathematics and to the concept of no-arbitrage. Due to its seminal importance the FTAP deserves more than (just) a remark, so for an adequate and comprehensive treatment we refer to the literature cited in Remarks 2.21.

Remarks 2.21 (Fundamental theorem of asset pricing, FTAP)

- In Definition 2.12 we have defined the price processes $(Q_t[\mathbf{X}])_{t\in\mathscr{J}}$ such that $\boldsymbol{\varphi}$ deflated price processes become $(\mathbb{P},\mathbb{F})$-martingales or equivalently that the $(B_t^{-1})_{t\in\mathscr{J}}$ discounted price processes become $(\mathbb{P}^*,\mathbb{F})$-martingales, see Proposition 2.18. These martingale properties are crucial and imply that we have a "consistent pricing system" which corresponds to one implication of the FTAP (see Sect. 1.6 in Delbaen–Schachermayer [56], Theorem 2.2 in Cairns [38] or Theorem 5.17 in Föllmer–Schied [71]).

  The FTAP (see Delbaen–Schachermayer [55]) says that the existence of an equivalent martingale measure is equivalent to the appropriately defined no-arbitrage condition. This implies that the existence of an equivalent martingale measure rules out appropriately defined arbitrage opportunities (see also Sect. 6.C in Duffie [59]). In general, this is the easier implication of the FTAP and this is also the one that we are using here (having a consistent pricing system). The argument roughly shows that if we have initial value 0, then we can prove, using the numeraire invariance theorem, that this implies that we cannot have certain gains without any downside risk.

The opposite question, however, is much more delicate. In the general continuous time setting, Delbaen–Schachermayer [55] gave the argument that if no-arbitrage is defined the right way then it implies the existence of an equivalent martingale measure. In the discrete and finite time horizon model the same argument was first given by Dalang et al. [47], an elegant proof was provided by Schachermayer [142], and derivations based on the Esscher transform are found in Rogers [132] and Bühlmann et al. [34].

As a consequence, in the literature many authors use the existence of an equivalent martingale measure $\mathbb{P}^*$ as being equal to the no-arbitrage assumption.

- As mentioned above, the existence of an equivalent martingale measure rules out appropriately defined arbitrage. In general, there are infinitely many equivalent martingale measures which implies non-uniqueness of prices. If there is only one equivalent martingale measure then the market is called complete, for instance the Black–Scholes model leads to an arbitrage-free and complete market model in continuous time (see e.g. Björk [13], Sect. 7.2).
- Note that we have only assumed the existence of a state price deflator $\boldsymbol{\varphi}$ on $(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F})$. Under this assumption we have defined a consistent pricing system via martingales. However, this does not tell us anything about financial risks (of asset prices) and technical risks (of insurance claims), nor does it tell us anything about how we should organize the hedging and risk bearing. At the current stage the filtration $\mathbb{F}$ contains all available economic, financial and insurance technical information. Moreover, we assume that $\boldsymbol{\varphi}$ is $\mathbb{F}$-adapted and hence enables to deflate according to financial market information as well as for probability distortion of insurance technical risks. This will be the key construction in Chap. 6 where we decouple financial and insurance technical risks.
- We mention once more that working in a discrete and finite time setting has the advantage that the mathematics become simpler and we can concentrate on the intuitive properties behind the models. The drawback is that the smallest time interval is not well-defined. This may cause problems when we define the risk-free asset, see Example 2.16 and (2.14). In the context of hedging of financial risks we go over to a continuous time model, see Sect. 9.3.5.

We conclude that state price deflated price processes $(\varphi_t Q_t[\mathbf{X}])_{t\in\mathscr{J}}$ need to be $(\mathbb{P}, \mathbb{F})$-martingales and bank account numeraire discounted price processes $(B_t^{-1} Q_t[\mathbf{X}])_{t\in\mathscr{J}}$ need to be $(\mathbb{P}^*, \mathbb{F})$-martingales. $\mathbb{P}$ is called real world probability measure and $\mathbb{P}^*$ is called equivalent martingale measure.

2.4 Market Price of Risk

The goal of this section is to describe the difference between the real world probability measure $\mathbb{P}$ and the equivalent martingale measure $\mathbb{P}^*$ introduced in the previous section. In doing so we give the foundations for explicit state price deflator constructions. The basic idea is to consider the span-deflator introduced in (2.9). We try to identify its dynamics under both probability measures $\mathbb{P}$ and $\mathbb{P}^*$. This consideration is then closely related to the consideration of the continuously-compounded spot rate $(r_t)_{t \in \mathscr{J}_-} = (R(t, t+1))_{t \in \mathscr{J}_-}$. We target for models that have an appealing structure and which allow to calculate ZCB prices in closed form. One particular family of models will lead to so-called affine term structures, which are of the form

$$P(t, m) = \frac{1}{\varphi_t} \mathbb{E}[\varphi_m | \mathscr{F}_t] = \exp\left\{A(t, m) - r_t \; B(t, m)\right\}, \tag{2.15}$$

for appropriate functions $A(\cdot, \cdot)$ and $B(\cdot, \cdot)$. Affine term structures have the advantage that they allow for simple analytical calculations. Therefore, many contributions in the literature describe affine term structures, see e.g. Vasicek [155], Cox et al. [44], Dai–Singleton [46] and Filipović [67].

Spot Rate Dynamics Under the Real World Probability Measure Assume that, under $\mathbb{P}$, the continuously-compounded spot rate process $(r_t)_{t \in \mathscr{J}_-}$ satisfies: $r_0 > 0$ (fixed) and for $t = 1, \ldots, n-1$

$$r_t = f(t, r_{t-1}) + \mathbf{g}(t, r_{t-1}) \; \boldsymbol{\varepsilon}_t, \tag{2.16}$$

where f and $\mathbf{g}$ are sufficiently well-behaved real- and $\mathbb{R}^N$-valued functions, respectively. Moreover, $(\boldsymbol{\varepsilon}_t)_{t \in \mathscr{J}}$ is an N-dimensional $\mathbb{F}$-adapted process and $\boldsymbol{\varepsilon}_{t+1}$ is independent of $\mathscr{F}_t$ under $\mathbb{P}$ for $t \in \mathscr{J}_-$. Note that $\boldsymbol{\varepsilon}_t$ and $\mathbf{g}$ are N-dimensional and their product in (2.16) is meant in the inner product sense on $\mathbb{R}^N$. We define the range of r_t by $\mathscr{Z}_t$, i.e., for all $t \in \mathscr{J}_-$ we have $r_t \in \mathscr{Z}_t$, $\mathbb{P}$-a.s.

Remark Observe that we choose an $\mathbb{F}$-adapted random sequence $(\boldsymbol{\varepsilon}_t)_{t \in \mathscr{J}}$. However, $\boldsymbol{\varepsilon}_0$ and $\boldsymbol{\varepsilon}_n$ from this sequence are not used for the spot rate $(r_t)_{t \in \mathscr{J}_-}$ modeling. But choosing the full sequence for $t \in \mathscr{J}$ sometimes simplifies the notation.

Definition of the State Price Deflator Next we choose an $\mathbb{R}^N$-valued function $\boldsymbol{\lambda}(t+1, z)$ that is sufficiently well-behaved for $t \in \mathscr{J}_-$ and $z \in \mathscr{Z}_t$. The function $\boldsymbol{\lambda}$ plays the role of the *market price of risk*. It models the aggregate market risk aversion and expresses the difference between the real world probability measure $\mathbb{P}$ and the equivalent martingale measure $\mathbb{P}^*$. Here we do an exogenous choice for the market price of risk $\boldsymbol{\lambda}$. However, in a fully-fledged economic model the market price of risk should by induced endogenously by a market equilibrium condition.

Choose an $\mathbb{F}$-adapted N-dimensional process $(\boldsymbol{\delta}_t)_{t \in \mathscr{J}}$ such that $\boldsymbol{\delta}_{t+1}$ is independent of $\mathscr{F}_t$ under $\mathbb{P}$. The process $(\boldsymbol{\delta}_t)_{t \in \mathscr{J}}$ is often called *deflator innovation*.

Assume that for all $t \in \mathscr{J}_-$ and $z \in \mathscr{Z}_t$ the expected value

$$\mathbb{E}\left[\exp\left\{\boldsymbol{\lambda}(t+1,z)\,\boldsymbol{\delta}_{t+1}\right\}\right]$$

is finite. We define the span-deflator by $\breve{\varphi}_0 \equiv 1$ and for $t \in \mathscr{J}_-$ we choose

$$\breve{\varphi}_{t+1} = c_t\ \exp\left\{-r_t + \boldsymbol{\lambda}(t+1,r_t)\,\boldsymbol{\delta}_{t+1}\right\},$$

for an appropriate $\mathscr{F}_t$-measurable variable $c_t > 0$, $\mathbb{P}$-a.s. Note that the span-deflator has the property

$$r_t = -\log P(t,t+1) = -\log \mathbb{E}\left[\left.\frac{\varphi_{t+1}}{\varphi_t}\right|\mathscr{F}_t\right] = -\log \mathbb{E}[\breve{\varphi}_{t+1}|\mathscr{F}_t]. \tag{2.17}$$

This gives the normalizing requirement for c_t. Namely, a straightforward calculation provides

$$c_t = \mathbb{E}\left[\exp\left\{\boldsymbol{\lambda}(t+1,r_t)\,\boldsymbol{\delta}_{t+1}\right\}\middle|\mathscr{F}_t\right]^{-1}.$$

This leads us to the following definition

$$h(t+1,z) = \log \mathbb{E}\left[\exp\left\{\boldsymbol{\lambda}(t+1,z)\,\boldsymbol{\delta}_{t+1}\right\}\middle|\mathscr{F}_t\right] < \infty, \qquad \mathbb{P}\text{-a.s.} \tag{2.18}$$

Then we define the $\mathbb{F}$-adapted state price deflator $\boldsymbol{\varphi}$ by

$$\varphi_t = \prod_{s=0}^{t} \breve{\varphi}_s = \exp\left\{-\sum_{s=1}^{t}\left[r_{s-1} + h(s,r_{s-1})\right] + \sum_{s=1}^{t}\boldsymbol{\lambda}(s,r_{s-1})\,\boldsymbol{\delta}_s\right\}. \tag{2.19}$$

The general model assumption now is that the distributions of $\boldsymbol{\delta}_t$ and $\boldsymbol{\varepsilon}_t$ and the functions f, $\boldsymbol{g}$ and $\boldsymbol{\lambda}$ are chosen such that $\boldsymbol{\varphi}$ is a state price deflator in $L^1_{n+1}(\Omega,\mathscr{F},\mathbb{P},\mathbb{F})$: $\mathbb{F}$-adapted is clear; normalization $\varphi_0 \equiv 1$ follows because an empty sum is defined to be equal to 0; strict positivity follows from $h < \infty$ and $\boldsymbol{\lambda}\boldsymbol{\delta}_s > -\infty$, $\mathbb{P}$-a.s.; the L^1-property depends on an appropriate choice of the functions f, $\boldsymbol{g}$ and $\boldsymbol{\lambda}$ and the stochastic processes $(\boldsymbol{\delta}_t)_{t\in\mathscr{J}}$ and $(\boldsymbol{\varepsilon}_t)_{t\in\mathscr{J}}$. We refrain from explicitly giving sufficient conditions for the L^1-property at the current stage. Below we will define explicit models and then we will provide these conditions for every model considered.

Note that $(\boldsymbol{\varepsilon}_t,\boldsymbol{\delta}_t)$ does not necessarily have a multivariate Gaussian distribution. However, Gaussian assumptions often lead to closed form solutions. Moreover, we did not assume anything on the dependence structure between $\boldsymbol{\delta}_t$ and $\boldsymbol{\varepsilon}_t$. Often they are assumed to be identical, see (3.5), but the theory holds true in more generality.

> Definitions (2.16) and (2.19) provide a framework for explicit models for state price deflators $\boldsymbol{\varphi}$. In the next chapter we provide such explicit models. These are often of the affine term structure type (2.15).

Equivalent Martingale Measure Finally, we calculate the equivalent martingale measure $\mathbb{P}^* \sim \mathbb{P}$. The density process $(\xi_t)_{t\in\mathscr{J}}$ is given by (see also Lemma 2.17)

$$\xi_t = \varphi_t\ B_t = \prod_{s=1}^{t} \breve{\varphi}_s\ \exp\{r_{s-1}\} = \prod_{s=1}^{t} \exp\left\{-h(s, r_{s-1}) + \boldsymbol{\lambda}(s, r_{s-1})\ \boldsymbol{\delta}_s\right\}$$

$$= \prod_{s=1}^{t} \mathbb{E}\left[\exp\left\{\boldsymbol{\lambda}(s, r_{s-1})\ \boldsymbol{\delta}_s\right\}\middle|\mathscr{F}_{s-1}\right]^{-1} \exp\left\{\boldsymbol{\lambda}(s, r_{s-1})\ \boldsymbol{\delta}_s\right\}. \tag{2.20}$$

This gives the desired $(\mathbb{P}, \mathbb{F})$-martingale $(\xi_t)_{t\in\mathscr{J}}$ with $\mathbb{P}$-expected value 1. Note that for $\boldsymbol{\lambda} \equiv 0$ we obtain $\xi_t \equiv 1$ and the two probability measures $\mathbb{P}$ and $\mathbb{P}^*$ coincide. From this we can see that the market price of risk $\boldsymbol{\lambda}$ describes the "difference" between the real world probability measure $\mathbb{P}$ and the equivalent martingale measure $\mathbb{P}^*$. The fact that $\mathbb{P}$ and $\mathbb{P}^*$ coincide for $\boldsymbol{\lambda} \equiv 0$ implies in this case that $\varphi_t = B_t^{-1}$.

Outlook In this chapter we have introduced the valuation framework for (insurance) cash flows $\mathbf{X}$. This was done by choosing a state price deflator $\boldsymbol{\varphi}$, Definition 2.12 then leads to consistent price processes according to Definition 2.14. In Part II we are going to separate these cash flows and their valuation into a financial part and an insurance technical part by introducing the valuation portfolio. This will require the explicit introduction of a financial market, which will become important for the understanding of asset-and-liability management of insurance cash flows $\mathbf{X}$.

In the next chapters, we give explicit models for state price deflators. This is done in the spirit of this section. We start with Gaussian distributions for $\boldsymbol{\delta}_t$ and $\boldsymbol{\varepsilon}_t$ and later on we extend this to other distributional models. These models will also provide the corresponding term structures of interest rates.

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