Chapter 2
Stokes-Filtered Local Systems in Dimension One

Abstract We consider Stokes filtrations on local systems on S^1. We review some of the definitions of the previous chapter in this case and make explicit the supplementary properties coming from this particular case. This chapter can be read independently of Chap. 1.

2.1 Introduction

The notion of a (pre-)Stokes filtration is a special case of the notion of a (pre-) $\mathcal{I}$-filtration defined in Chap. 1 with a suitable sheaf $\mathcal{I}$ on the topological space $Y = S^1$. We have chosen to present this notion independently of the general results of the previous chapter, since many properties are simpler to explain in this case (see Proposition 2.7). Nevertheless, we make precise the relation with the previous chapter when we introduce new definitions. We moreover start with the non-ramified Stokes filtrations, to make easier the manipulation of such objects, and we also call it a $\mathcal{I}_1$-filtration, in accordance with the previous chapter. The (possibly ramified) Stokes filtrations are introduced in Sect. 2.4, where we define the sheaf $\mathcal{I}$ with its order. They correspond to the $\mathcal{I}$-filtrations of the previous chapter.

We also make precise the relation with the approach by Stokes data in the case of Stokes filtrations of simple exponential type.

References are [2, 17, 52] and [55, Chap. IV].

2.2 Non-ramified Stokes-Filtered Local Systems

Let $\boldsymbol{k}$ be a field. In this section, we consider local systems of finite dimensional $\boldsymbol{k}$-vector spaces on S^1. Recall (see Example 1.4) that we consider S^1 equipped with the constant sheaf $\mathcal{I}_1$ with fibre $\mathcal{P} = \mathbb{C}(\!\{x\}\!)/\mathbb{C}\{x\}$ consisting of polar parts of

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Laurent series, and the order depends on the point $e^{i\theta} = x/|x| \in S^1$ as follows. Let $\eta \in \mathcal{P}$ and let us set $\eta = u_n(x)x^{-n}$ with $n \geqslant 1$ and $u_n(0) \neq 0$ if $\eta \neq 0$. Then

$$\eta \leqslant_\theta 0 \Longleftrightarrow \eta = 0 \text{ or } \arg u_n(0) - n\theta \in (\pi/2, 3\pi/2) \mod 2\pi, \tag{1.4 $*$}$$

and $\eta <_\theta 0 \Leftrightarrow (\eta \leqslant_\theta 0$ and $\eta \neq 0)$ (see (1.4 $**$)). The order is supposed to be compatible with addition, namely, $\varphi \leqslant_\theta \psi \Leftrightarrow \varphi - \psi \leqslant_\theta 0$ and similarly for $<_\theta$. Let us rephrase Definition 1.27 in this setting.

Definition 2.1. A *non-ramified pre-Stokes filtration* on a local system $\mathscr{L}$ of finite dimensional $\boldsymbol{k}$-vector spaces on S^1 consists of the data of a family of subsheaves $\mathscr{L}_{\leqslant\varphi}$ indexed by $\mathcal{P}$ such that, for any $\theta \in S^1$, $\varphi \leqslant_\theta \psi \Rightarrow \mathscr{L}_{\leqslant\varphi,\theta} \subset \mathscr{L}_{\leqslant\psi,\theta}$.

Let us set, for any $\varphi \in \mathcal{P}$ and any $\theta \in S^1$,

$$\mathscr{L}_{<\varphi,\theta} = \sum_{\psi <_\theta \varphi} \mathscr{L}_{\leqslant\psi,\theta}. \tag{2.2}$$

This defines a subsheaf $\mathscr{L}_{<\varphi}$ of $\mathscr{L}_{\leqslant\varphi}$, and we set $\operatorname{gr}_\varphi \mathscr{L} = \mathscr{L}_{\leqslant\varphi}/\mathscr{L}_{<\varphi}$. Note that the étalé space $\mathcal{I}_1^{\text{ét}}$ is Hausdorff (see Example 1.1(1)) and the previous definition is in accordance with Definition 1.34.

Notation 2.3. We rephrase here Notation 1.3 in the present setting. Let $\varphi, \psi \in \mathcal{P}$. Recall that we denote by $S^1_{\psi\leqslant\varphi} \subset S^1$ the subset of S^1 consisting of the θ for which $\psi \leqslant_\theta \varphi$. Similarly, $S^1_{\psi<\varphi} \subset S^1$ is the subset of S^1 consisting of the θ for which $\psi <_\theta \varphi$. Both subsets are a finite union of open intervals. They are equal if $\varphi \neq \psi$. Otherwise, $S^1_{\varphi\leqslant\varphi} = S^1$ and $S^1_{\varphi<\varphi} = \varnothing$.

Given a sheaf $\mathscr{F}$ on S^1, we will denote by $\beta_{\psi\leqslant\varphi}\mathscr{F}$ the sheaf obtained by restricting $\mathscr{F}$ to the open set $S^1_{\psi\leqslant\varphi}$ and extending it by 0 as a sheaf on S^1 (for any open set $Z \subset S^1$, this operation is denoted $\mathscr{F}_Z$ in [39]). A similar definition holds for $\beta_{\psi<\varphi}\mathscr{F}$.

Definition 2.4 ((Graded) Stokes filtration). Given a finite set $\Phi \subset \mathcal{P}$, a Stokes-graded local system with Φ as set of *exponential factors* consists of the data of local systems (that we denote by) $\operatorname{gr}_\varphi \mathscr{L}$ on S^1 ($\varphi \in \Phi$). Then the graded non-ramified Stokes filtration on $\operatorname{gr}\mathscr{L} := \bigoplus_{\psi\in\Phi} \operatorname{gr}_\psi \mathscr{L}$ is given by

$$(\operatorname{gr}\mathscr{L})_{\leqslant\varphi} = \bigoplus_{\psi\in\Phi} \beta_{\psi\leqslant\varphi} \operatorname{gr}_\psi \mathscr{L}.$$

We then also have

$$(\operatorname{gr}\mathscr{L})_{<\varphi} = \bigoplus_{\psi\in\Phi} \beta_{\psi<\varphi} \operatorname{gr}_\psi \mathscr{L}.$$

A *non-ramified $\boldsymbol{k}$-Stokes filtration* on $\mathscr{L}$ is a pre-Stokes filtration which is locally on S^1 isomorphic to a graded Stokes filtration. It is denoted by $\mathscr{L}_\bullet$.

For a Stokes-filtered local system $(\mathscr{L}, \mathscr{L}_\bullet)$, each sheaf $\mathrm{gr}_\varphi \mathscr{L}$ is a (possibly zero) local system on S^1. By definition, for every φ and every $\theta_o \in S^1$, we have on some neighbourhood $\mathrm{nb}(\theta_o)$ of θ_o,

$$\begin{aligned} \mathscr{L}_{<\varphi|\mathrm{nb}(\theta_o)} &\simeq \bigoplus_{\psi\in\Phi} \beta_{\psi<\varphi}\, \mathrm{gr}_\psi \mathscr{L}_{|\mathrm{nb}(\theta_o)}, \\ \mathscr{L}_{\leqslant\varphi|\mathrm{nb}(\theta_o)} &\simeq \mathscr{L}_{<\varphi|\mathrm{nb}(\theta_o)} \oplus \mathrm{gr}_\varphi \mathscr{L}_{|\mathrm{nb}(\theta_o)} = \bigoplus_{\psi\in\Phi} \beta_{\psi\leqslant\varphi}\, \mathrm{gr}_\psi \mathscr{L}_{|\mathrm{nb}(\theta_o)}, \\ \mathscr{L}_{|\mathrm{nb}(\theta_o} &\simeq \bigoplus_{\psi\in\Phi} \mathrm{gr}_\psi \mathscr{L}_{|\mathrm{nb}(\theta_o)} \end{aligned} \tag{2.5}$$

in a way compatible with the natural inclusions.

Exercise 2.6. Show that the category of Stokes-filtered local systems has direct sums, and that any Stokes-graded local system is the direct sum of Stokes-graded local systems, each of which has exactly one exponential factor.

One can make more explicit the definition of a non-ramified Stokes-filtered local system.

Proposition 2.7. *Giving a non-ramified Stokes-filtered local system $(\mathscr{L}, \mathscr{L}_\bullet)$ is equivalent to giving, for each $\varphi \in \mathcal{P}$, a $\mathbb{R}$-constructible subsheaf $\mathscr{L}_{\leqslant\varphi} \subset \mathscr{L}$ subject to the following conditions:*

1. *For any $\theta \in S^1$, the germs $\mathscr{L}_{\leqslant\varphi,\theta}$ form an exhaustive increasing filtration of $\mathscr{L}_\theta$.*
2. *Defining $\mathscr{L}_{<\varphi}$, and therefore $\mathrm{gr}_\varphi \mathscr{L}$, from the family $\mathscr{L}_{\leqslant\psi}$ as in (2.2), the sheaf $\mathrm{gr}_\varphi \mathscr{L}$ is a local system of finite dimensional $\boldsymbol{k}$-vector spaces on S^1.*
3. *For any $\theta \in S^1$ and any $\varphi \in \mathcal{P}$, $\dim \mathscr{L}_{\leqslant\varphi,\theta} = \sum_{\psi\leqslant_\theta\varphi} \dim \mathrm{gr}_\psi \mathscr{L}_\theta$.*

We note that when Proposition 2.7(2) is satisfied, Proposition 2.7(3) is equivalent to

3′. For any $\theta \in S^1$ and any $\varphi \in \mathcal{P}$, $\dim \mathscr{L}_{<\varphi,\theta} = \sum_{\psi<_\theta\varphi} \dim \mathrm{gr}_\psi \mathscr{L}_\theta$.

Proof of Proposition 2.7. The point is to get the local gradedness property from the dimension property of Proposition 2.7(3). Since the local filtrations are exhaustive, the dimension property implies that the local systems $\mathscr{L}$ and $\bigoplus_\varphi \mathrm{gr}_\varphi \mathscr{L}$ are locally isomorphic, hence for each θ_o, there exists a finite family $\Phi_{\theta_o} \subset \mathcal{P}$ such that $\mathrm{gr}_\varphi \mathscr{L}_{\theta_o} \neq 0 \Rightarrow \varphi \in \Phi_{\theta_o}$. Since $\mathrm{gr}_\varphi \mathscr{L}$ is a local system, it is zero if and only if it is zero near some θ_o. We conclude that the set $\Phi_{\theta_o} \subset \mathcal{P}$ is independent of θ_o, and we simply denote it by Φ. We thus have $\mathrm{gr}_\varphi \mathscr{L} \neq 0 \Rightarrow \varphi \in \Phi$.

Lemma 2.8. *Let $\mathscr{F}$ be a $\mathbb{R}$-constructible sheaf of $\boldsymbol{k}$-vector spaces on S^1. For any $\theta_o \in S^1$, let I be an open interval containing θ_o such that $\mathscr{F}_{|I\smallsetminus\{\theta_o\}}$ is a local system of finite dimensional $\boldsymbol{k}$-vector spaces. Then $H^1(I, \mathscr{F}) = 0$.*

Proof. Let $\iota : I \smallsetminus \{\theta_o\} \hookrightarrow I$ be the inclusion. We have an exact sequence $0 \to \iota_!\iota^{-1}\mathscr{F} \to \mathscr{F} \to \mathscr{G} \to 0$, where $\mathscr{G}$ is supported at θ_o. It is therefore enough to prove the result for $\iota_! \iota^{-1} \mathscr{F}$. This reduces to the property that, if $i : (a,b) \hookrightarrow (a,b]$ is the inclusion (with $a, b \in \mathbb{R}$, $a < b$), then $H^1((a,b], i_! \boldsymbol{k}) = 0$, which is clear by Poincaré duality. □

Let us fix $\theta_o \in S^1$. Since $\mathscr{L}_{<\psi}$ is $\mathbb{R}$-constructible for any ψ, there exists an open interval $\mathrm{nb}(\theta_o)$ of S^1 containing θ_o such that, for any $\psi \in \Phi$, $\mathscr{L}_{<\psi}$ is a local system on $\mathrm{nb}(\theta_o) \smallsetminus \{\theta_o\}$. Then, for any such ψ, $H^1(\mathrm{nb}(\theta_o), \mathscr{L}_{<\psi}) = 0$, according to the previous lemma and, as $\mathrm{gr}_\psi \mathscr{L}$ is a constant local system on $\mathrm{nb}(\theta_o)$, we can lift a basis of global sections of $\mathrm{gr}_\psi \mathscr{L}_{|\mathrm{nb}(\theta_o)}$ as a family of sections of $\mathscr{L}_{\leqslant\psi,|\mathrm{nb}(\theta_o)}$. This defines a morphism $\bigoplus_\psi \mathrm{gr}_\psi \mathscr{L}_{|\mathrm{nb}(\theta_o)} \to \mathscr{L}_{|\mathrm{nb}(\theta_o)}$ sending $\bigoplus_{\psi \leqslant_\theta \varphi} \mathrm{gr}_\psi \mathscr{L}_\theta$ to $\mathscr{L}_{\leqslant\varphi,\theta}$ for any $\theta \in \mathrm{nb}(\theta_o)$ and any φ. Let us show, by induction on $\#\{\psi \in \Phi \mid \psi \leqslant_\theta \varphi\}$, that it sends $\bigoplus_{\psi \leqslant_\theta \varphi} \mathrm{gr}_\psi \mathscr{L}_\theta$ *onto* $\mathscr{L}_{\leqslant\varphi,\theta}$ for any $\theta \in \mathrm{nb}(\theta_o)$ and any φ: indeed, the assertion is clear by the dimension property if this number is zero; by the inductive assumption and according to (2.2), it sends $\bigoplus_{\psi <_\theta \varphi} \mathrm{gr}_\psi \mathscr{L}_\theta$ onto $\mathscr{L}_{<\varphi,\theta}$ for any $\theta \in \mathrm{nb}(\theta_o)$; since $\mathscr{L}_{\leqslant\varphi} = \mathscr{L}_{<\varphi} + \mathrm{image}\,\mathrm{gr}_\varphi \mathscr{L}$ in $\mathscr{L}$, the assertion follows. As both spaces $\bigoplus_{\psi \leqslant_\theta \varphi} \mathrm{gr}_\psi \mathscr{L}_\theta$ and $\mathscr{L}_{\leqslant\varphi,\theta}$ have the same dimension, due to Proposition 2.7(3), this morphism is an isomorphism. □

The finite subset $\Phi \subset \mathcal{P}$ such that $\mathrm{gr}_\varphi \mathscr{L} \neq 0 \Rightarrow \varphi \in \Phi$ is called the *set of exponential factors* of the non-ramified Stokes filtration. The following proposition is easily checked, showing more precisely exhaustivity.

Proposition 2.9. *Let $\mathscr{L}_\bullet$ be a non-ramified $\boldsymbol{k}$-Stokes filtration on $\mathscr{L}$. Then, for any $\theta \in S^1$, and any $\varphi \in \mathcal{P}$,*

- *If $\varphi <_\theta \Phi$, then $\mathscr{L}_{\leqslant\varphi,\theta} = 0$.*
- *If $\Phi <_\theta \varphi$, then $\mathscr{L}_{<\varphi,\theta} = \mathscr{L}_{\leqslant\varphi,\theta} = \mathscr{L}_\theta$.* □

Remark 2.10. One can also remark that the category of Stokes-filtered local systems with set of exponential factors contained in Φ is equivalent to the category of Φ-filtered local systems, where we regard Φ as a constant sheaf on S^1, equipped with the ordered induced by the order of $\mathcal{I}_1$ (constant sheaf with fibre $\mathcal{P}$).

Examples 2.11. 1. (Twist) Let $\eta \in \mathcal{P}_x$ and let $(\mathscr{L}, \mathscr{L}_\bullet)$ be a (pre-)Stokes-filtered local system. The twisted local system $(\mathscr{L}, \mathscr{L}_\bullet)[\eta]$ is defined by $\mathscr{L}[\eta]_{\leqslant\varphi} = \mathscr{L}_{\leqslant\varphi-\eta}$. In the Stokes-filtered case, the set of exponential factors $\Phi[\eta]$ is equal to $\Phi + \eta$. This is analogous to Definition 1.9.

2. The graded Stokes filtration with $\Phi = \{0\}$ (see Example 1.35) on the constant sheaf $\boldsymbol{k}_{S^1}$ is defined by $\boldsymbol{k}_{S^1,\leqslant\varphi} := \beta_{0\leqslant\varphi} \boldsymbol{k}_{S^1}$ for any φ, so that $\boldsymbol{k}_{S^1,\leqslant 0} = \boldsymbol{k}_{S^1}$, $\boldsymbol{k}_{S^1,<0} = 0$, and, for any $\varphi \neq 0$, $\boldsymbol{k}_{S^1,\leqslant\varphi} = \boldsymbol{k}_{S^1,<\varphi}$ has germ equal to $\boldsymbol{k}_\theta$ at $\theta \in S^1$ iff $0 \leqslant_\theta \varphi$, and has germ equal to 0 otherwise.
3. Let $\mathscr{L}_\bullet$ be any non-ramified Stokes filtration with set of exponential factors Φ reduced to one element η. According to Proposition 2.9, we have $\mathscr{L}_{<\eta} = 0$, and $\mathscr{L}_{\leqslant\eta} = \mathrm{gr}_\eta \mathscr{L} = \mathscr{L}$ is a local system on S^1. The non-ramified Stokes filtration is then described as in Example 2.11(2) above, that is, $\mathscr{L}_{\leqslant\varphi} = \beta_{\eta\leqslant\varphi} \mathscr{L}$. If we denote by $\mathscr{L}_\bullet$ this Stokes filtration of $\mathscr{L}$ then, using the twist operation

Example 2.11(1), the twisted Stokes filtration $\mathscr{L}[-\eta]_\bullet$ is nothing but the graded Stokes filtration on $\mathscr{L}$, defined as in Example 2.11(2).

4. Assume that $\#\Phi = 2$ or, equivalently (by twisting, see above), that $\Phi = \{0, \varphi_o\}$ with $\varphi_o \neq 0$. If the order of the pole of φ_o is n, then there are $2n$ Stokes directions (see Example 1.4) dividing the circle in $2n$ intervals. Given such an open interval, then 0 and φ_o are comparable (in the same way) at any θ in the interval, and the comparison changes alternatively on the intervals. Assume that $0 <_\theta \varphi_o$. Then, according to Propositions 2.9 and 2.7(3), $\mathscr{L}_{\leqslant\varphi_o,\theta} = \mathscr{L}_\theta$ and $\mathscr{L}_{<0,\theta} = 0$. Moreover, when restricted to the open interval containing θ, $\mathscr{L}_{\leqslant 0} = \mathscr{L}_{<\varphi_o}$ is a local system of rank equal to $\operatorname{rk}\operatorname{gr}_0\mathscr{L}$. On the other intervals, the roles of 0 and φ_o are exchanged.

   Let now θ be a Stokes direction for $(0, \varphi_o)$. As φ_o and 0 are not comparable at θ, Proposition 2.7(3′) implies that $\mathscr{L}_{<\varphi_o,\theta} = \mathscr{L}_{<0,\theta} = 0$ and, using φ such that $0, \varphi_o <_\theta \varphi$, we find, by exhaustivity, $\mathscr{L}_\theta = \mathscr{L}_{\leqslant\varphi_o,\theta} \oplus \mathscr{L}_{\leqslant 0,\theta}$. This decomposition reads as an isomorphism $\mathscr{L}_\theta \simeq \operatorname{gr}_{\varphi_o}\mathscr{L}_\theta \oplus \operatorname{gr}_0\mathscr{L}_\theta$. It extends on a neighbourhood $\operatorname{nb}(\theta)$ of θ in S^1 (we can take for $\operatorname{nb}(\theta)$ the union of the two adjacent intervals considered above ending at θ) in a unique way as an isomorphism of local systems $\mathscr{L}_{|\operatorname{nb}(\theta)} \simeq (\operatorname{gr}_{\varphi_o}\mathscr{L} \oplus \operatorname{gr}_0\mathscr{L})_{|\operatorname{nb}(\theta)}$.

   In order to end the description, we will show that the equality $\mathscr{L}_{\leqslant\varphi} = \mathscr{L}_{<\varphi}$ for $\varphi \notin \{0, \varphi_o\}$ can be deduced from the data of the corresponding sheaves for $\varphi \in \{0, \varphi_o\}$. Let us fix $\theta \in S^1$. Assume first $0 <_\theta \varphi_o$ (and argue similarly if $\varphi_o <_\theta 0$).

   - If φ is neither comparable to φ_o nor to 0, then Proposition 2.7(3) shows that $\mathscr{L}_{\leqslant\varphi,\theta} = 0$
   - If φ is comparable to φ_o but not to 0
     - If $\varphi_o <_\theta \varphi$, then $\mathscr{L}_{\leqslant\varphi,\theta} = \mathscr{L}_\theta$
     - If $\varphi <_\theta \varphi_o$, then $\mathscr{L}_{\leqslant\varphi,\theta} \subset \mathscr{L}_{<\varphi_o,\theta} = \mathscr{L}_{\leqslant 0,\theta}$, hence Proposition 2.7(3) implies $\mathscr{L}_{\leqslant\varphi,\theta} = 0$
   - If φ is comparable to 0 but not to φ_o, the result is similar
   - If φ is comparable to both φ_o and 0, then
     - If $0 <_\theta \varphi_o <_\theta \varphi$, $\mathscr{L}_{\leqslant\varphi,\theta} = \mathscr{L}_\theta$
     - If $0 <_\theta \varphi <_\theta \varphi_o$, then $\mathscr{L}_{\leqslant\varphi,\theta} = \mathscr{L}_{\leqslant 0,\theta}$
     - If $\varphi <_\theta 0 <_\theta \varphi_o$, then $\mathscr{L}_{\leqslant\varphi,\theta} = 0$

   If φ_o and 0 are not comparable at θ, then one argues similarly to determine $\mathscr{L}_{\leqslant\varphi,\theta}$.

Definition 2.12. A *morphism* $\lambda : (\mathscr{L}, \mathscr{L}_\bullet) \to (\mathscr{L}', \mathscr{L}'_\bullet)$ of non-ramified $\boldsymbol{k}$-Stokes-filtered local systems is a morphism of local systems $\mathscr{L} \to \mathscr{L}'$ on S^1 such that, for any $\varphi \in \mathcal{P}$, $\lambda(\mathscr{L}_{\leqslant\varphi}) \subset \mathscr{L}'_{\leqslant\varphi}$. According to (2.2), a morphism also satisfies $\lambda(\mathscr{L}_{<\varphi}) \subset \mathscr{L}'_{<\varphi}$. A morphism λ is said to be *strict* if, for any φ, $\lambda(\mathscr{L}_{\leqslant\varphi}) = \lambda(\mathscr{L}) \cap \mathscr{L}'_{\leqslant\varphi}$.

Definition 2.13. Given two non-ramified $\boldsymbol{k}$-Stokes-filtered local systems $(\mathscr{L}, \mathscr{L}_\bullet)$ and $(\mathscr{L}', \mathscr{L}'_\bullet)$,

- The direct sum $(\mathscr{L},\mathscr{L}_\bullet)\oplus(\mathscr{L}',\mathscr{L}'_\bullet)$ has local system $\mathscr{L}\oplus\mathscr{L}'$ and filtration $(\mathscr{L}\oplus\mathscr{L}')_{\leqslant\varphi}=\mathscr{L}_{\leqslant\varphi}\oplus\mathscr{L}'_{\leqslant\varphi}$.
- $\mathscr{H}\!om(\mathscr{L},\mathscr{L}')_{\leqslant\eta}$ is the subsheaf of $\mathscr{H}\!om(\mathscr{L},\mathscr{L}')$ consisting of local morphisms $\mathscr{L}\to\mathscr{L}'$ sending $\mathscr{L}_{\leqslant\varphi}$ into $\mathscr{L}'_{\leqslant\varphi+\eta}$ for any φ.
- The dual $(\mathscr{L},\mathscr{L}_\bullet)^\vee$ is hence defined as $(\mathscr{H}\!om(\mathscr{L},\boldsymbol{k}_{S^1}),\mathscr{H}\!om(\mathscr{L},\boldsymbol{k}_{S^1})_\bullet)$, where $\boldsymbol{k}_{S^1}$ is equipped with the graded Stokes filtration of Example 2.11(2).
- $(\mathscr{L}\otimes\mathscr{L}')_{\leqslant\eta}=\sum_\varphi\mathscr{L}_{\leqslant\varphi}\otimes\mathscr{L}'_{\leqslant\eta-\varphi}\subset\mathscr{L}\otimes\mathscr{L}'$.

In particular, a morphism of non-ramified Stokes-filtered local systems is a global section of $\mathscr{H}\!om(\mathscr{L},\mathscr{L}')_{\leqslant 0}$.

Proposition 2.14. *Given two non-ramified $\boldsymbol{k}$-Stokes filtrations $\mathscr{L}_\bullet$, $\mathscr{L}'_\bullet$ of $\mathscr{L},\mathscr{L}'$, $\mathscr{L}_\bullet\oplus\mathscr{L}'_\bullet$, $\mathscr{H}\!om(\mathscr{L},\mathscr{L}')_\bullet$, $(\mathscr{L}^\vee)_\bullet$ and $(\mathscr{L}\otimes\mathscr{L}')_\bullet$ are non-ramified $\boldsymbol{k}$-Stokes filtrations of the corresponding local systems and $\mathscr{H}\!om(\mathscr{L},\mathscr{L}')_\bullet\simeq(\mathscr{L}'^\vee\otimes\mathscr{L})_\bullet$. Moreover,*

1. *$\mathscr{H}\!om(\mathscr{L},\mathscr{L}')_{<\eta}$ is the subsheaf of $\mathscr{H}\!om(\mathscr{L},\mathscr{L}')$ consisting of local morphisms $\mathscr{L}\to\mathscr{L}'$ sending $\mathscr{L}_{\leqslant\varphi}$ into $\mathscr{L}'_{<\varphi+\eta}$ fo any φ.*
2. *$(\mathscr{L}^\vee)_{\leqslant\varphi}=(\mathscr{L}_{<-\varphi})^\perp$ and $(\mathscr{L}^\vee)_{<\varphi}=(\mathscr{L}_{\leqslant-\varphi})^\perp$ for any φ, so that $\mathrm{gr}_\varphi\mathscr{L}^\vee=(\mathrm{gr}_{-\varphi}\mathscr{L})^\vee$ (here, $(\mathscr{L}_{<-\varphi})^\perp$, resp. $(\mathscr{L}_{\leqslant-\varphi})^\perp$, consists of local morphisms $\mathscr{L}\to\boldsymbol{k}_{S^1}$ sending $\mathscr{L}_{<-\varphi}$, resp. $\mathscr{L}_{\leqslant-\varphi}$, to zero).*
3. *$(\mathscr{L}\otimes\mathscr{L}')_{<\eta}=\sum_\varphi\mathscr{L}_{\leqslant\varphi}\otimes\mathscr{L}'_{<\eta-\varphi}=\sum_\varphi\mathscr{L}_{<\varphi}\otimes\mathscr{L}'_{\leqslant\eta-\varphi}$.*

Proof. For the first assertion, let us consider the case of $\mathscr{H}\!om$ for instance. Using a local decomposition of $\mathscr{L},\mathscr{L}'$ given by the Stokes filtration condition, we find that a section of $\mathscr{H}\!om(\mathscr{L},\mathscr{L}')_\theta$ is decomposed as a section of $\bigoplus_{\varphi,\psi}\mathscr{H}\!om(\mathrm{gr}_\varphi\mathscr{L},\mathrm{gr}_\psi\mathscr{L}')_\theta$, and that it belongs to $\mathscr{H}\!om(\mathscr{L},\mathscr{L}')_{\leqslant\eta,\theta}$ if and only if its components (φ,ψ) are zero whenever $\psi-\varphi\not\leqslant_\theta\eta$. The assertion is then clear, as well as the characterization of $\mathscr{H}\!om(\mathscr{L},\mathscr{L}')_{<\eta}$.

As a consequence, a local section of $(\mathscr{L}^\vee)_{\leqslant\varphi}$ has to send $\mathscr{L}_{<-\varphi}$ to zero for any φ. The converse is also clear by using the local decomposition of $(\mathscr{L},\mathscr{L}_\bullet)$, as well as the other assertions for $\mathscr{L}^\vee$.

The assertion on the tensor product is then routine. □

Remark 2.15. One easily gets the behaviour of the set of exponential factors with respect to such operations. For instance, the direct sum corresponds to $(\Phi,\Phi')\mapsto\Phi\cup\Phi'$, the dual to $\Phi\mapsto-\Phi$ and the tensor product to $(\Phi,\Phi')\mapsto\Phi+\Phi'$.

Poincaré–Verdier duality. For a sheaf $\mathscr{F}$ on S^1, its Poincaré–Verdier dual $\boldsymbol{D}\mathscr{F}$ is $\boldsymbol{R}\,\mathscr{H}\!om_{\mathbb{C}}(\mathscr{F},\boldsymbol{k}_{S^1}[1])$ and we denote by $\boldsymbol{D}'\mathscr{F}=\boldsymbol{R}\,\mathscr{H}\!om_{\mathbb{C}}(\mathscr{F},\boldsymbol{k}_{S^1})$ the shifted complex. We clearly have $\boldsymbol{D}'\mathscr{L}=\mathscr{H}\!om_{\mathbb{C}}(\mathscr{L},\boldsymbol{k}_{S^1})=:\mathscr{L}^\vee$.

Lemma 2.16 (Poincaré duality). *For any $\varphi\in\mathcal{P}$, the complexes $\boldsymbol{D}'(\mathscr{L}_{\leqslant\varphi})$ and $\boldsymbol{D}'(\mathscr{L}/\mathscr{L}_{\leqslant\varphi})$ are sheaves. Moreover, the two subsheaves $(\mathscr{L}^\vee)_{\leqslant\varphi}$ and $\boldsymbol{D}'(\mathscr{L}/\mathscr{L}_{<-\varphi})$ of $\mathscr{L}^\vee$ coincide.*

Proof. The assertions are local on S^1, so we can assume that $(\mathscr{L},\mathscr{L}_\bullet)$ is split with respect to the Stokes filtration, and therefore that $(\mathscr{L},\mathscr{L}_\bullet)$ has only one exponential factor η, that is, $\mathscr{L}_{\leqslant\varphi} = \beta_{\eta\leqslant\varphi}\mathscr{L}$ (see Example 2.11(3). Let us denote by $\alpha_{\psi<\varphi}$ the functor which is the composition of the restriction to the open set $S^1_{\psi<\varphi}$ (see Notation 2.3) and the maximal extension to S^1, and similarly with $\leqslant$. We have an exact sequence

$$0 \longrightarrow \beta_{\eta<\varphi}\mathscr{L} \longrightarrow \mathscr{L} \longrightarrow \alpha_{\varphi\leqslant\eta}\mathscr{L} \longrightarrow 0$$

which identifies $\alpha_{\varphi\leqslant\eta}\mathscr{L}$ to $\mathscr{L}/\mathscr{L}_{<\varphi}$, and a similar one with $\beta_{\eta\leqslant\varphi}$ and $\alpha_{\varphi<\eta}$. On the other hand, $\boldsymbol{D}'(\beta_{\eta\leqslant\varphi}\mathscr{L}) = \alpha_{\eta\leqslant\varphi}\mathscr{L}^\vee$ and $\boldsymbol{D}'(\alpha_{\varphi\leqslant\eta}\mathscr{L}) = \beta_{\varphi\leqslant\eta}\mathscr{L}^\vee$. The dual of the previous exact sequence, when we replace φ with $-\varphi$, is then

$$0 \longrightarrow \beta_{-\varphi\leqslant\eta}\mathscr{L}^\vee \longrightarrow \mathscr{L}^\vee \longrightarrow \alpha_{\eta<-\varphi}\mathscr{L}^\vee \longrightarrow 0,$$

also written as

$$0 \longrightarrow \beta_{-\eta\leqslant\varphi}\mathscr{L}^\vee \longrightarrow \mathscr{L}^\vee \longrightarrow \alpha_{\varphi<-\eta}\mathscr{L}^\vee \longrightarrow 0,$$

showing that $\boldsymbol{D}'(\mathscr{L}/\mathscr{L}_{<-\varphi}) = (\mathscr{L}^\vee)_{\leqslant\varphi}$. □

2.3 Pull-Back and Push-Forward

Let $f : X' \to X$ be a holomorphic map from the disc X' (with coordinate x') to the disc X with coordinate x. We assume that both discs are small enough so that f is ramified at $x' = 0$ only. We now denote by $S^1_{x'}$ and S^1_x the circles of directions in the spaces of polar coordinates $\widetilde{X}'$ and $\widetilde{X}$ respectively. Then f induces $\widetilde{f} : S^1_{x'} \to S^1_x$, which is the composition of the multiplication by N (the index of ramification of f) and a translation (the argument of $f^{(N)}(0)$). Similarly, $\mathcal{P}_x$ and $\mathcal{P}_{x'}$ denote the polar parts in the variables x and x' respectively.

Remark 2.17. Let $\eta \in \mathcal{P}_x$ and set $f^*\eta = \eta \circ f \in \mathcal{P}_{x'}$. For any $\theta' \in S^1_{x'}$, set $\theta = \widetilde{f}(\theta')$. Then we have

$$f^*\eta \leqslant_{\theta'} 0 \Longleftrightarrow \eta \leqslant_\theta 0 \quad \text{and} \quad f^*\eta <_{\theta'} 0 \Longleftrightarrow \eta <_\theta 0.$$

(This is easily seen using the definition in terms of moderate growth in Example 1.4, since $\widetilde{f} : \widetilde{X}' \to \widetilde{X}$ is a finite covering.)

Definition 2.18 (Pull-back). Let $\mathscr{L}$ be a local system on S^1_x and let $\mathscr{L}_\bullet$ be a non-ramified $\boldsymbol{k}$-pre-Stokes filtration of $\mathscr{L}$. For any $\varphi' \in \mathcal{P}_{x'}$ and any $\theta' \in S^1_{x'}$, let us set

$$(\widetilde{f}^+\mathscr{L})_{\leqslant\varphi',\theta'} := \sum_{\substack{\psi\in\mathcal{P}_x\\ f^*\psi\leqslant_{\theta'}\varphi'}} \mathscr{L}_{\leqslant\psi,\widetilde{f}(\theta')} \subset \mathscr{L}_{\widetilde{f}(\theta')} = (\widetilde{f}^{-1}\mathscr{L})_{\theta'}.$$

Then $(\widetilde{f}^{+}\mathscr{L})_\bullet$ is a non-ramified pre-Stokes filtration on $\widetilde{f}^{-1}\mathscr{L}$, called the *pull-back* of $\mathscr{L}_\bullet$ by f. We denote by $\widetilde{f}^{+}(\mathscr{L},\mathscr{L}_\bullet)$ the pull-back pre-Stokes-filtered local system, in order to remember that the indexing set has changed (see Definition 1.33).

Proposition 2.19 (Pull-back). *The pull-back $\widetilde{f}^{+}(\mathscr{L},\mathscr{L}_\bullet)$ has the following properties:*

1. *For any $\varphi' \in \mathcal{P}_{x'}$ and any $\theta' \in S^1_{x'}$,*
$$(\widetilde{f}^{+}\mathscr{L})_{<\varphi',\theta'} = \sum_{\substack{\psi\in\mathcal{P}_x\\ f^*\psi<_{\theta'}\varphi'}} \widetilde{f}^{-1}(\mathscr{L}_{\leqslant\psi,\widetilde{f}(\theta')}).$$

2. *For any $\varphi \in \mathcal{P}_x$,*
$$(\widetilde{f}^{+}\mathscr{L})_{\leqslant f^*\varphi} = \widetilde{f}^{-1}(\mathscr{L}_{\leqslant\varphi}),$$
$$(\widetilde{f}^{+}\mathscr{L})_{< f^*\varphi} = \widetilde{f}^{-1}(\mathscr{L}_{<\varphi})$$
and
$$\mathrm{gr}_{f^*\varphi}(\widetilde{f}^{+}\mathscr{L}) = \widetilde{f}^{-1}(\mathrm{gr}_\varphi\mathscr{L}).$$

3. *In particular, if $\widetilde{f}^{+}(\mathscr{L},\mathscr{L}_\bullet)$ is a non-ramified Stokes-filtered local system for some f, then for any $\varphi \in \mathcal{P}_x$, $\mathrm{gr}_\varphi\mathscr{L}$ is a local system on S^1_x.*
4. *Let $\mathscr{L},\mathscr{L}'$ be two local systems on S^1_x equipped with non-ramified pre-Stokes filtrations and let $\lambda : \mathscr{L} \to \mathscr{L}'$ be a morphism of local systems such that, for some f, $\widetilde{f}^{-1}\lambda : \widetilde{f}^{-1}\mathscr{L} \to \widetilde{f}^{-1}\mathscr{L}'$ is compatible with the non-ramified pre-Stokes filtrations $(\widetilde{f}^{+}\mathscr{L})_\bullet, (\widetilde{f}^{+}\mathscr{L}')_\bullet$. Then λ is compatible with the non-ramified pre-Stokes filtrations $\mathscr{L}_\bullet, \mathscr{L}'_\bullet$.*
5. *Assume now that $\mathscr{L}_\bullet$ is a non-ramified $\boldsymbol{k}$-Stokes filtration (i.e., is locally graded) and let Φ be its set of exponential factors. Then $(\widetilde{f}^{+}\mathscr{L})_\bullet$ is a non-ramified $\boldsymbol{k}$-Stokes filtration on $\widetilde{f}^{-1}\mathscr{L}$ and, for any $\varphi' \in \mathcal{P}_{x'}$, $\mathrm{gr}_{\varphi'}\widetilde{f}^{+}\mathscr{L} \neq 0 \Rightarrow \varphi' \in f^*\Phi$.*
6. *The pull-back of non-ramified Stokes filtrations is compatible with $\mathscr{H}om$, duality and tensor product.*

Proof. By definition,
$$(\widetilde{f}^{+}\mathscr{L})_{<\varphi',\theta'} = \sum_{\psi'<_{\theta'}\varphi'} (\widetilde{f}^{+}\mathscr{L})_{\leqslant\psi',\theta'} = \sum_{\psi'<_{\theta'}\varphi'} \sum_{\substack{\psi\in\mathcal{P}_x\\ f^*\psi\leqslant_{\theta'}\psi'}} \mathscr{L}_{\leqslant\psi,\widetilde{f}(\theta')},$$
and this is the RHS in Proposition 2.19(1).

The first two lines of Proposition 2.19(2) are a direct consequence of Remark 2.17, and the third one is a consequence of the previous ones. Then Proposition 2.19(3) follows, as each $\mathrm{gr}_{\varphi'}(\widetilde{f}^{+}\mathscr{L})$ is a local system on $S^1_{x'}$.

Proposition 2.19(4) follows from the first line in Propositions 2.19(2) and 2.19(5) from third line and from the local gradedness condition. Then, Proposition 2.19(6) is clear. $\square$

Remark 2.20. In order to make clear the correspondence with the notion introduced in Definition 1.33 and considered in Lemma 1.40, note that the sheaf $\mathcal{I}_1$ is the constant sheaf on S^1_x with fibre $\mathcal{P}_x$, $\widetilde{f}^{-1}\mathcal{I}_1$ is the constant sheaf on $S^1_{x'}$ with fibre $\mathcal{P}_x$ and $\mathcal{I}'_1$ is the constant sheaf on $S^1_{x'}$ with fibre $\mathcal{P}_{x'}$. The map q_f is $f^* : \mathcal{P}_x \to \mathcal{P}_{x'}$.

Exercise 2.21 (Push forward). Let $\mathscr{L}'$ be a local system on $S^1_{x'}$ equipped with a non-ramified pre-Stokes filtration $\mathscr{L}'_\bullet$. Show that

1. $\widetilde{f}_*\mathscr{L}'$ is naturally equipped with a non-ramified pre-Stokes filtration defined by $(\widetilde{f}_*\mathscr{L}')_{\leqslant\varphi} = \widetilde{f}_*(\mathscr{L}'_{\leqslant f^*\varphi})$.
2. Assume moreover that $\mathscr{L}'_\bullet$ is a non-ramified Stokes filtration and let $\Phi' \subset \mathcal{P}_{x'}$ be its set of exponential factors; if there exists a finite subset $\Phi \subset \mathcal{P}_x$ such that $\Phi' = f^*\Phi$, then the push-forward pre-Stokes filtration $(\widetilde{f}_*\mathscr{L}')_\bullet$ is a Stokes filtration.

2.4 Stokes Filtrations on Local Systems

We now define the general notion of a (possibly ramified) Stokes filtration on a local system $\mathscr{L}$ on S^1.

Let d be a nonzero integer and let $\rho_d : X_d \to X$ be a holomorphic function from a disc X_d (coordinate x') to the disc X (coordinate x). For simplicity, we will assume that the coordinates are chosen so that $\rho_d(x') = x'^d$.

Definition 2.22 ((Pre-)Stokes filtration). Let $\mathscr{L}$ be a local system on S^1_x. A $\boldsymbol{k}$*-(pre-)Stokes filtration* (ramified of order $\leqslant d$) on $\mathscr{L}$ consists of a non-ramified (pre-)Stokes filtration on $\mathscr{L}' := \rho_d^{-1}\mathscr{L}$ such that, for any automorphism σ

$$\begin{tikzcd} X_d \arrow[rr, "\sigma"] \arrow[dr, "\rho_d"'] & & X_d \arrow[dl, "\rho_d"] \\ & X & \end{tikzcd}$$

and any $\varphi' \in \mathcal{P}_{x'}$, we have $\mathscr{L}'_{\leqslant\sigma^*\varphi'} = \widetilde{\sigma}^{-1}\mathscr{L}'_{\leqslant\varphi'}$ in $\mathscr{L}' = \widetilde{\sigma}^{-1}\mathscr{L}'$. Similarly, a morphism of (pre-)$\boldsymbol{k}$-Stokes-filtered local systems is a morphism of local systems which becomes a morphism of non-ramified Stokes-filtered local systems after ramification.

We will make precise the relation with the notion of a $\mathcal{I}$-filtration of Chap. 1 by defining first the sheaf $\mathcal{I}$.

The sheaf $\mathcal{I}$ *on* S^1. Let d be a positive integer. We denote by $\mathcal{I}_d$ the local system on S^1_x whose fibre at $\theta = 0$ is $\mathcal{P}_{x'}$ and whose monodromy is given by $\mathcal{P}_{x'} \ni \varphi'(x') \mapsto \varphi'(e^{2\pi i/d}x')$. If we denote by $\widetilde{\rho}_d : S^1_{x'} \to S^1_x$, $\theta' \mapsto d \cdot \theta'$ the associated map, the sheaf $\widetilde{\rho}_d^{-1}\mathcal{I}_d$ is the constant sheaf on $S^1_{x'}$ with fibre $\mathcal{P}_{x'}$. One also says that $\mathcal{I}_d$ is obtained by descent by ρ_d from $\mathcal{I}_1(x')$. In particular, $\mathcal{I}_d$ is a sheaf of ordered abelian groups on S^1_x, and the constant sheaf $\mathcal{I}_1$ with fibre $\mathcal{P}_x$ is a subsheaf of ordered abelian groups. A germ at θ_o of section of $\mathcal{I}_d$ consists of a pair (φ', θ'_o), with $\varphi' \in \mathcal{P}_{x'}$ and θ'_o such that $d \cdot \theta'_o = \theta_o \bmod 2\pi$, or equivalently of the vector $((\varphi'(x'), \theta'_o), (\varphi'(\zeta x'), \theta'_o + 2\pi/d), \dots)$. Then $(\varphi', \theta'_o) \leqslant_{\theta_o} 0$ means $\varphi' \leqslant_{\theta'_o} 0$, or equivalently $\varphi'(\zeta^k x') \leqslant_{\theta'_o + 2k\pi/d} 0$ for all k.

We will then denote by $\mathcal{I}$ the sheaf $\bigcup_{d\geqslant 1} \mathcal{I}_d$.

Remark 2.23. Let us give another description of the sheaf $\mathcal{I}_d$ which will be useful in higher dimensions. We use the notation of Example 1.4. Let us denote by j_∂ and $j_{\partial,d}$ the natural inclusions $X^* \hookrightarrow \widetilde{X}$ and $X^*_d \hookrightarrow \widetilde{X}_d$, and by $\widetilde{\rho}_d : \widetilde{X}_d \to \widetilde{X}$ the lifting of ρ_d. The natural inclusion $\mathcal{O}_{X^*} \hookrightarrow \rho_{d,*}\mathcal{O}_{X^*_d}$ induces an injective morphism $j_{\partial,*}\mathcal{O}_{X^*} \hookrightarrow j_{\partial,*}\rho_{d,*}\mathcal{O}_{X^*_d} = \widetilde{\rho}_{d,*} j_{\partial,d,*}\mathcal{O}_{X^*_d}$, that we regard as the inclusion of a subsheaf.

Let us denote by $(j_{\partial,*}\mathcal{O}_{X^*})^{\mathrm{lb}}$ the subsheaf of $j_{\partial,*}\mathcal{O}_{X^*}$ consisting of functions which are locally bounded on $\widetilde{X}$. We have $(j_{\partial,*}\mathcal{O}_{X^*})^{\mathrm{lb}} = \widetilde{\rho}_{d,*}(j_{\partial,d,*}\mathcal{O}_{X^*_d})^{\mathrm{lb}} \cap j_{\partial,*}\mathcal{O}_{X^*}$ since $\widetilde{\rho}_d$ is proper.

Let us set $\widetilde{\mathcal{I}}_1 = \varpi^{-1}\mathcal{O}_X(*0)$, that we consider as a subsheaf of $j_{\partial,*}\mathcal{O}_{X^*}$. We have $\varpi^{-1}\mathcal{O}_X = \widetilde{\mathcal{I}}_1 \cap (j_{\partial,*}\mathcal{O}_{X^*})^{\mathrm{lb}}$ in $j_{\partial,*}\mathcal{O}_{X^*}$ (a meromorphic function which is bounded in some sector is bounded everywhere, hence is holomorphic). Therefore, $\mathcal{I}_1 := \varpi^{-1}(\mathcal{O}_X(*0)/\mathcal{O}_X)$ is also equal to $\widetilde{\mathcal{I}}_1/\widetilde{\mathcal{I}}_1 \cap (j_{\partial,*}\mathcal{O}_{X^*})^{\mathrm{lb}}$.

Similarly, we can first define $\widetilde{\mathcal{I}}_d$ as the subsheaf of $\mathbb{C}$-vector spaces of $j_{\partial,*}\mathcal{O}_{X^*}$ which is the intersection of $\widetilde{\rho}_{d,*}\varpi_d^{-1}\mathcal{O}_{X_d}(*0)$ and $j_{\partial,*}\mathcal{O}_{X^*}$ in $\widetilde{\rho}_{d,*} j_{\partial,d,*}\mathcal{O}_{X^*_d}$. We then set $\mathcal{I}_d = \widetilde{\mathcal{I}}_d/\widetilde{\mathcal{I}}_d \cap (j_{\partial,*}\mathcal{O}_{X^*})^{\mathrm{lb}}$, which is a subsheaf of $j_{\partial,*}\mathcal{O}_{X^*}/(j_{\partial,*}\mathcal{O}_{X^*})^{\mathrm{lb}}$. We have $\mathcal{I}_d \subset \mathcal{I}_{d'}$ if d divides d'.

As we already noticed, $\mathcal{I}_1 = \varpi^{-1}(\mathcal{O}_X(*0)/\mathcal{O}_X)$. More generally, let us show that $\widetilde{\rho}_d^{-1}\mathcal{I}_d = \varpi_d^{-1}(\mathcal{O}_{X_d}(*0)/\mathcal{O}_{X_d})$. We will start by showing that $\widetilde{\rho}_d^{-1}\widetilde{\mathcal{I}}_d = \varpi_d^{-1}\mathcal{O}_{X_d}(*0)$.

Let us first note that $\rho_d^{-1}\mathcal{O}_{X^*} = \mathcal{O}_{X^*_d}$ since ρ_d is a covering, and $\widetilde{\rho}_d^{-1} j_{\partial,*}\mathcal{O}_{X^*} = j_{\partial,d,*}\rho_d^{-1}\mathcal{O}_{X^*}$ since $\widetilde{\rho}_d$ is a covering. Hence, $\widetilde{\rho}_d^{-1} j_{\partial,*}\mathcal{O}_{X^*} = j_{\partial,d,*}\mathcal{O}_{X^*_d}$.

It follows that $\widetilde{\rho}_d^{-1}\widetilde{\mathcal{I}}_d$ is equal to the intersection of $\widetilde{\rho}_d^{-1}\widetilde{\rho}_{d,*}[\varpi_d^{-1}\mathcal{O}_{X_d}(*0)]$ (since $\widetilde{\mathcal{I}}_{\widetilde{X}_d,1} = \varpi_d^{-1}\mathcal{O}_{X_d}(*0)$) and $j_{\partial,d,*}\mathcal{O}_{X^*_d}$ in $\widetilde{\rho}_d^{-1}\widetilde{\rho}_{d,*}[j_{\partial,d,*}\mathcal{O}_{X^*_d}]$. This is $\varpi_d^{-1}\mathcal{O}_{X_d}(*0)$. Indeed, a germ in $\widetilde{\rho}_d^{-1}\widetilde{\rho}_{d,*}[\varpi_d^{-1}\mathcal{O}_{X_d}(*0)]$ at θ' consists of a d-uple of germs in $\mathcal{O}_{X_d}(*0)$ at 0. This d-uple belongs to $j_{\partial,d,*}\mathcal{O}_{X^*_d,\theta'}$ iff the restrictions to X^*_d of the terms of the d-uple coincide. Then all the terms of the d-uple are equal. The argument for $\mathcal{I}_d$ is similar.

Let us express these results in terms of étalé spaces. We first note that, since $\mathcal{I}_1$ is a constant sheaf, the étalé space $\mathcal{I}_1^{\text{ét}}$ is a trivial covering of S^1_x. The previous argument shows that the fibre product $S^1_{x'} \times_{S^1_x} \mathcal{I}_d^{\text{ét}}$ is identified with $\mathcal{I}_{x',1}^{\text{ét}}$, hence is a trivial

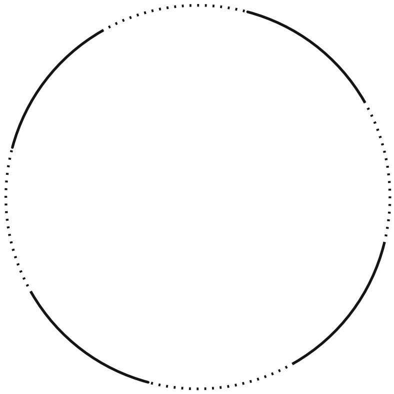

Fig. 2.1 Example with $n = 2$. The set $S^1_{\eta\leqslant 0}$ is the union of the open intervals in *full line*

covering of $S^1_{x'}$. It follows that, since $\widetilde{\rho}_d : S^1_{x'} \times_{S^1_x} \mathcal{I}^{\text{ét}}_d \to \mathcal{I}^{\text{ét}}_d$ is a finite covering of degree d, that $\mathcal{I}^{\text{ét}}_d \to S^1_x$ is a covering.

The following property will also be useful: there is a one-to-one correspondence between finite sets Φ_d of $\mathcal{P}_{x'}$ and finite coverings $\widetilde{\Sigma} \subset \mathcal{I}^{\text{ét}}_d$. Indeed, given such a $\widetilde{\Sigma}$, its pull-back $\widetilde{\Sigma}_d$ by $\widetilde{\rho}_d$ is a covering of $S^1_{x'}$ contained in the trivial covering $\widetilde{\rho}_d^{-1}\mathcal{I}^{\text{ét}}_d$, hence is trivial, and is determined by its fibre $\Phi_d \subset \mathcal{P}_{x'}$. Conversely, given such Φ_d, it defines a trivial covering $\widetilde{\Sigma}_d$ of $S^1_{x'}$ contained in $\widetilde{\rho}_d^{-1}\mathcal{I}^{\text{ét}}_d$. Let $\widetilde{\Sigma}$ be its image in $\mathcal{I}^{\text{ét}}_d$. Because the composed map $\widetilde{\Sigma}_d \to S^1_{x'} \to S^1_x$ is a covering, so are both maps $\widetilde{\Sigma}_d \to \widetilde{\Sigma}$ and $\widetilde{\Sigma} \to S^1_x$. Moreover, the degree of $\widetilde{\Sigma} \to S^1_x$ is equal to that of $\widetilde{\Sigma}_d \to S^1_{x'}$, that is, $\#\Phi_d$. Lastly, the pull-back of $\widetilde{\Sigma}$ by $\widetilde{\rho}_d$ is a covering of $S^1_{x'}$ contained in $S^1_{x'} \times_{S^1_x} \mathcal{I}^{\text{ét}}_d$, hence is a trivial covering, of degree $\#\Phi_d$, and containing $\widetilde{\Sigma}_d$, so is equal to $\widetilde{\Sigma}_d$.

Order. The sheaf $j_{\partial,*}\mathcal{O}_{X^*}$ is naturally ordered by defining $(j_{\partial,*}\mathcal{O}_{X^*})_{\leqslant 0}$ as the subsheaf of $j_{\partial,*}\mathcal{O}_{X^*}$ whose sections have an exponential with moderate growth along S^1_x. Similarly, $j_{\partial,d,*}\mathcal{O}_{X^*_d}$ is ordered. In this way, $\widetilde{\mathcal{I}}$ inherits an order: $\widetilde{\mathcal{I}}_{\leqslant 0} = \mathcal{I} \cap (j_{\partial,*}\mathcal{O}_{X^*})_{\leqslant 0}$. This order is not altered by adding a local section of $(j_{\partial,*}\mathcal{O}_{X^*})^{\text{lb}}$, and thus defines an order on $\mathcal{I}$. For each d, we also have $\widetilde{\mathcal{I}}_{d,\leqslant 0} = \widetilde{\rho}_{d,*}\big((\varpi_d^{-1}\mathcal{O}_{X_d}(*0))_{\leqslant 0}\big) \cap j_{\partial,*}\mathcal{O}_{X^*}$ and we also conclude that $\widetilde{\rho}_d^{-1}(\mathcal{I}_{d,\leqslant 0}) = \big(\varpi_d^{-1}(\mathcal{O}_{X_d}(*0)/\mathcal{O}_{X_d})\big)_{\leqslant 0}$.

To any $\varphi' \in \mathcal{P}_{x'}$ one associates a finite covering $\widetilde{\Sigma}_{\varphi'} = S^1_{x'} \subset \mathcal{I}^{\text{ét}}$ of S^1_x as above. Then $\Sigma_{\varphi'} \cap \mathcal{I}^{\text{ét}}_{\leqslant 0}$ is as in Fig. 2.1 (where the circle is $S^1_{x'}$).

Remark 2.24. The sheaf of ordered abelian groups $\mathcal{I}$ satisfies the property (1.41 $*$). The direction $\Rightarrow$ is clear. For the other direction, assume that $\psi_\theta \not\leqslant_\theta 0$. We will prove that there exists η_θ such that $0 <_\theta \eta_\theta$ and $\psi_\theta \not<_\theta \eta_\theta$. If $\psi_\theta = u_n(x)/x^n$ with $n \in \mathbb{Q}^*_+$ and $\arg u_n(0) - n\theta \in [-\pi/2, \pi/2] \bmod 2\pi$, then we take $\eta \neq 0$ having a pole order strictly less than n and a dominant coefficient such that $0 <_\theta \eta_\theta$. Then the order relation between 0 and ψ_θ is the same as the order relation between 0 and $\psi_\theta - \eta_\theta$.

Note that this argument does not hold on a subsheaf $\mathcal{I}_d$ with d fixed.

One can rephrase the definition of a (pre-)Stokes filtration by using the terminology of Chap. 1.

Lemma 2.25. *A (pre-)Stokes filtration on $\mathscr{L}$ is a (pre-)$\mathcal{I}$-filtration on $\mathscr{L}$, with $\mathcal{I}$ defined above. It is ramified of order $\leqslant d$ if the support of $\operatorname{gr}\mathscr{L}$ is contained in $\mathcal{I}_d^{\text{ét}}$.* □

Remarks 2.26. 1. The condition in Definition 2.22 can be restated by saying that, for any σ, the Stokes-filtered local system $(\mathscr{L}', \mathscr{L}'_\bullet)$ and its pull-back by $\widetilde{\sigma}$ coincide (owing to the natural identification $\mathscr{L}' = \widetilde{\sigma}^{-1}\mathscr{L}'$).

2. Given a (possibly ramified) Stokes-filtration on a local system $\mathscr{L}$, and given a section $\varphi \in \Gamma(U, \mathcal{I})$ on some open set of S^1, the subsheaf $\mathscr{L}_{\leqslant\varphi} \subset \mathscr{L}_{|U}$ is well-defined, as well as $\mathscr{L}_{<\varphi}$, and $\operatorname{gr}_\varphi \mathscr{L}$ is a local system on U. If φ is a section of $\mathcal{I}$ all over S^1, then it is non-ramified, i.e., it is a section of $\mathcal{I}_1$, and $\mathscr{L}_{\leqslant\varphi}, \mathscr{L}_{<\varphi}$ are subsheaves of $\mathscr{L}$. From the point of view of Definition 2.22, if the non-ramified Stokes filtration exists on $\mathscr{L}' = \widetilde{\rho}_d^{-1}\mathscr{L}$, one can restrict the set of indices to $\mathcal{P}_x \subset \mathcal{P}_{x'}$. Then, for $\varphi \in \mathcal{P}_x$, $\mathscr{L}'_{\leqslant\rho_d^*\varphi}$ is invariant by the automorphisms of $\mathscr{L}'$ induced by the automorphisms $\widetilde{\sigma}$, hence is the pull-back of a subsheaf $\mathscr{L}_{\leqslant\varphi}$ of $\mathscr{L}$, and similarly for $\mathscr{L}_{<\varphi}$ and $\operatorname{gr}_\varphi \mathscr{L}$. This defines a non-ramified pre-Stokes filtration on $\mathscr{L}$ for which the graded sheaves are local systems (but the dimension property 2.7(3) may not be satisfied). Note also that a morphism of Stokes-filtered local systems is compatible with this pre-Stokes filtration. Hence the category of Stokes filtrations on $\mathscr{L}$ is a subcategory of the category of non-ramified pre-Stokes filtrations on $\mathscr{L}$.

   Notice however that the non-ramified Stokes-filtered local system $(\widetilde{f}^{-1}\mathscr{L}, (\widetilde{f}^{-1}\mathscr{L})_\bullet)$ is not (in general) equal to the pull-back $\widetilde{f}^+(\mathscr{L}, \mathscr{L}_\bullet)$ where $\mathscr{L}_\bullet$ is this pre-Stokes filtration.

3. We will still denote by $\mathscr{L}_\bullet$ a (possibly ramified) Stokes filtration on $\mathscr{L}$ and by $(\mathscr{L}, \mathscr{L}_\bullet)$ a (possibly ramified) Stokes-filtered local system, although the previous remark makes it clear that we do not understand $\mathscr{L}_\bullet$ as a family of subsheaves of $\mathscr{L}$ on S^1.

4. The "set of exponential factors of the Stokes-filtered local system" is now replaced by a subset $\widetilde{\Sigma} \subset \mathcal{I}^{\text{ét}}$ such that the projection to $\partial\widetilde{X}$ is a finite covering. It corresponds to a finite subset $\Phi_d \subset \mathcal{P}_{x'}$ for a suitable ramified covering ρ_d (see the last part of Remark 2.23), which is the set of exponential factors of the non-ramified Stokes filtration of $\widetilde{\rho}_d^{-1}\mathscr{L}$.

5. The category of Stokes-filtered local systems $(\mathscr{L}, \mathscr{L}_\bullet)$ with associated covering contained in $\widetilde{\Sigma}$ is equivalent to the category of $\widetilde{\Sigma}^{\text{sh}}$-filtered local systems (see Remark 2.10).

6. Proposition 2.14 holds for $\boldsymbol{k}$-Stokes filtrations.

7. Lemma 2.16 holds for $\boldsymbol{k}$-Stokes filtrations, that is, the family $\boldsymbol{D}'(\mathscr{L}/\mathscr{L}_{<-\varphi})$ of local subsheaves of $\mathscr{L}^\vee$ indexed by local sections of $\mathcal{I}$ forms a Stokes filtration of $\mathscr{L}^\vee$.

8. The category of non-ramified $\boldsymbol{k}$-Stokes-filtered local systems on S^1_x is a full subcategory of that of $\boldsymbol{k}$-Stokes-filtered local systems. Indeed, given a non-ramified Stokes-filtered local system on S^1_x, one extends it as a ramified Stokes-filtered local system of order d from $\mathcal{I}_1^{\text{ét}}$ to $\mathcal{I}_d^{\text{ét}}$ using a formula analogous to that of Proposition 2.19(1).
9. If the set Φ_d of exponential factors of $\widetilde{\rho}_d(\mathscr{L}, \mathscr{L}_\bullet)$ takes the form $\rho_d^*\Phi$ for some finite subset $\Phi \subset \mathcal{P}_x$ (equivalently, the finite covering $\widetilde{\Sigma}$ of $\partial\widetilde{X}$ is trivial, see Remark 2.26(4)), then the Stokes filtration is non-ramified.

2.5 Extension of Scalars

Let $(\mathscr{L}, \mathscr{L}_\bullet)$ be a $\boldsymbol{k}$-Stokes-filtered local system and let $\boldsymbol{k}'$ be an extension of $\boldsymbol{k}$. Then $(\boldsymbol{k}' \otimes_{\boldsymbol{k}} \mathscr{L}, \boldsymbol{k}' \otimes_{\boldsymbol{k}} \mathscr{L}_\bullet)$ is a $\boldsymbol{k}'$- Stokes-filtered local system defined over $\boldsymbol{k}'$. The following properties are satisfied for any local section φ of $\mathcal{I}$:

- $(\boldsymbol{k}' \otimes_{\boldsymbol{k}} \mathscr{L})_{<\varphi} = \boldsymbol{k}' \otimes_{\boldsymbol{k}} \mathscr{L}_{<\varphi}$, and $\operatorname{gr}_\varphi(\boldsymbol{k}' \otimes_{\boldsymbol{k}} \mathscr{L}) = \boldsymbol{k}' \otimes \operatorname{gr}_\varphi \mathscr{L}$, so the set of exponential factors of $(\boldsymbol{k}' \otimes_{\boldsymbol{k}} \mathscr{L}, \boldsymbol{k}' \otimes_{\boldsymbol{k}} \mathscr{L}_\bullet)$ is equal to that of $(\mathscr{L}, \mathscr{L}_\bullet)$.
- $\mathscr{L}_{\leqslant\varphi} = (\boldsymbol{k}' \otimes_{\boldsymbol{k}} \mathscr{L}_{\leqslant\varphi}) \cap \mathscr{L}$ in $\boldsymbol{k}' \otimes_{\boldsymbol{k}} \mathscr{L}$.

In such a case, the $\boldsymbol{k}'$-Stokes-filtered local system $(\boldsymbol{k}' \otimes_{\boldsymbol{k}} \mathscr{L}, \boldsymbol{k}' \otimes_{\boldsymbol{k}} \mathscr{L}_\bullet)$ is said to be *defined over* $\boldsymbol{k}$.

Conversely, let now $(\mathscr{L}, \mathscr{L}_\bullet)$ be a $\boldsymbol{k}'$-Stokes-filtered local system and let $\widetilde{\Sigma} \subset \mathcal{I}$ be its covering of exponential factors. We wish to find sufficient conditions to ensure that it comes from a $\boldsymbol{k}$-Stokes-filtered local system by extension of scalars.

Proposition 2.27. *Assume that the local system $\mathscr{L}$ is defined over $\boldsymbol{k}$, that is, $\mathscr{L} = \boldsymbol{k}' \otimes \mathscr{L}_{\boldsymbol{k}}$ for some $\boldsymbol{k}$-local system $\mathscr{L}_{\boldsymbol{k}}$ (regarded as a subsheaf of $\mathscr{L}$), and that, for any local section φ of $\widetilde{\Sigma}$,*

$$\mathscr{L}_{\leqslant\varphi} = \boldsymbol{k}' \otimes_{\boldsymbol{k}} (\mathscr{L}_{\boldsymbol{k}} \cap \mathscr{L}_{\leqslant\varphi}),$$

where the intersection is taken in $\mathscr{L}$. Then $(\mathscr{L}, \mathscr{L}_\bullet)$ is a $\boldsymbol{k}'$-Stokes-filtered local system defined over $\boldsymbol{k}$.

Proof. It is not difficult to reduce to the non-ramified case, so we will assume below that $\mathcal{I} = \mathcal{I}_1$ and replace $\widetilde{\Sigma}$ by $\Phi \subset \mathcal{P}_x$. We set, for any local section ψ of $\mathcal{P}_x$, $\mathscr{L}_{\boldsymbol{k},\leqslant\psi} := \mathscr{L}_{\boldsymbol{k}} \cap \mathscr{L}_{\leqslant\psi}$, so that the condition reads $\mathscr{L}_{\leqslant\varphi} = \boldsymbol{k}' \otimes_{\boldsymbol{k}} \mathscr{L}_{\boldsymbol{k},\leqslant\varphi}$ for $\varphi \in \Phi$. This defines a pre-$\mathcal{I}$-filtration of $\mathscr{L}_{\boldsymbol{k}}$, and we will show that this is indeed a $\mathcal{I}$-filtration.

1. We start with a general property of Stokes-filtered local systems. Let $(\mathscr{L}, \mathscr{L}_\bullet)$ and Φ be as above, and let $\psi \in \mathcal{P}_x$. Set $\Psi = \Phi \cup \{\psi\}$ and denote by $\operatorname{St}(\Psi, \Psi)$ the (finite) set of Stokes directions of pairs $\varphi, \eta \in \Psi$. The sheaves $\mathscr{L}_{\leqslant\psi}$ and $\mathscr{L}_{<\psi}$ can be described as triples consisting of their restrictions to the open set

$S^1 \smallsetminus \mathrm{St}(\Psi,\Psi)$, the closed set $\mathrm{St}(\Psi,\Psi)$, and a gluing map from the latter to the restriction to this closed set of the push-forward of the former. We will make this description explicit.

On any connected component I of $S^1 \smallsetminus \mathrm{St}(\Psi,\Psi)$, the set Ψ is totally ordered, and there exists $\varphi = \varphi(I,\psi) \in \Phi$ such that $\mathscr{L}_{\leqslant\psi|I} = \mathscr{L}_{\leqslant\varphi|I}$. Similarly, there exists $\eta = \eta(I,\psi) \in \Phi$ such that $\mathscr{L}_{<\psi|I} = \mathscr{L}_{<\eta|I}$.

Let us fix $\theta_o \in \mathrm{St}(\Psi,\Psi)$ and denote by I_1, I_2 the two connected components of $S^1 \smallsetminus \mathrm{St}(\Psi,\Psi)$ containing θ_o in their closure, with corresponding inclusions $j_i : I_i \hookrightarrow S^1$, $i = 1,2$. We also denote by $i_o : \{\theta_o\} \hookrightarrow S^1$ the closed inclusion and set $\varphi_i := \varphi(I_i,\psi)$, $i = 1,2$ (resp. $\eta_i := \eta(I_i,\psi)$).

We claim that, in the neighbourhood of θ_o (and more precisely, on $I_1 \cup I_2 \cup \{\theta_o\}$), the sheaf $\mathscr{L}_{\leqslant\psi}$ is described by the data $\mathscr{L}_{\leqslant\psi|I_i} = \mathscr{L}_{\leqslant\varphi_i|I_i}$, $i = 1,2$, $\mathscr{L}_{\leqslant\psi,\theta_o} = i_o^{-1} j_{1,*}\mathscr{L}_{\leqslant\varphi_1|I_1} \cap i_o^{-1} j_{2,*}\mathscr{L}_{\leqslant\varphi_2|I_2}$, where the intersection is taken in $i_o^{-1} j_{1,*}\mathscr{L} = i_o^{-1} j_{2,*}\mathscr{L} = \mathscr{L}_{\theta_o}$, and the gluing map is the natural inclusion map of the intersection into each of its components. A similar statement holds for $\mathscr{L}_{<\psi}$. This easily follows from the local description $\mathscr{L}_{\leqslant\psi} = \bigoplus_{\varphi\in\Phi} \beta_{\varphi\leqslant\psi}\, \mathrm{gr}_\varphi \mathscr{L}$, and similarly for $\mathscr{L}_{<\psi}$.

2. We now claim that $\mathscr{L}_{\boldsymbol{k},\leqslant\psi} := \mathscr{L}_{\boldsymbol{k}} \cap \mathscr{L}_{\leqslant\psi}$ satisfies $\boldsymbol{k}' \otimes \mathscr{L}_{\boldsymbol{k},\leqslant\psi} = \mathscr{L}_{\leqslant\psi}$. This is by assumption on $S^1 \smallsetminus \mathrm{St}(\Psi,\Psi)$, according to the previous description, and it remains to check this at any $\theta_o \in \mathrm{St}(\Psi,\Psi)$. The previous description gives $i_o^{-1}\mathscr{L}_{\boldsymbol{k},\leqslant\psi} = i_o^{-1} j_{1,*}\mathscr{L}_{\boldsymbol{k},\leqslant\varphi_1|I_1} \cap i_o^{-1} j_{2,*}\mathscr{L}_{\boldsymbol{k},\leqslant\varphi_2|I_2}$ and the result follows easily (by considering a suitable basis of $\mathscr{L}_{\boldsymbol{k},\theta_o}$ for instance).
3. Let us now define $\mathscr{L}_{\boldsymbol{k},<\psi}$ as $\sum_{\psi'\in\mathcal{P}_x} \beta_{\psi'<\psi}\mathscr{L}_{\boldsymbol{k},\psi'}$, as in (2.2). Then the previous description also shows that $\mathscr{L}_{\boldsymbol{k},<\psi} = \mathscr{L}_{\boldsymbol{k}} \cap \mathscr{L}_{<\psi}$ and that $\mathscr{L}_{<\psi} = \boldsymbol{k}' \otimes_{\boldsymbol{k}} \mathscr{L}_{\boldsymbol{k},<\psi}$.
4. As a consequence, we obtain that $\mathrm{gr}_\psi \mathscr{L} = \boldsymbol{k}' \otimes_{\boldsymbol{k}} \mathrm{gr}_\psi \mathscr{L}_{\boldsymbol{k}}$ for any ψ, from which the proposition follows. □

Remark 2.28. The condition considered in the proposition is that considered in [40] in order to define a $\boldsymbol{k}$-structure on a Stokes-filtered local system defined over $\boldsymbol{k}'$ (e.g. $\boldsymbol{k} = \mathbb{Q}$ and $\boldsymbol{k}' = \mathbb{C}$). This proposition shows that there is no difference with the notion of Stokes-filtered $\boldsymbol{k}$-local system.

2.6 Stokes-Filtered Local Systems and Stokes Data

In this section, we make explicit the relationship between Stokes filtrations and the more conventional approach with Stokes data in the simple case of a Stokes-filtered local system of *exponential type*.

Stokes-filtered local systems of exponential type.

Definition 2.29 (see [40, 55]). We say that a Stokes-filtered local system $(\mathscr{L}, \mathscr{L}_\bullet)$ is of *exponential type* if it is non-ramified and its exponential factors have a pole of order one at most.

In such a case, we can replace $\mathcal{P}$ with $\mathbb{C} \cdot x^{-1}$, and thus with $\mathbb{C}$, and for each $\theta \in S^1$, the partial order $\leqslant_\theta$ on $\mathbb{C}$ is compatible with addition and satisfies

$$c \leqslant_\theta 0 \iff c = 0 \text{ or } \arg c - \theta \in (\pi/2, 3\pi/2) \mod 2\pi.$$

We will use notation of Sect. 2.2 by replacing $\varphi = c/x \in \mathcal{P}$ with $c \in \mathbb{C}$. For each pair $c \neq c' \in \mathbb{C}$, there are exactly two values of θ mod 2π, say $\theta_{c,c'}$ and $\theta'_{c,c'}$, such that c and c' are not comparable at θ. We have $\theta'_{c,c'} = \theta_{c,c'} + \pi$. These are the Stokes directions of the pair (c, c'). For any θ in one component of $S^1 \smallsetminus \{\theta_{c,c'}, \theta'_{c,c'}\}$, we have $c <_\theta c'$, and the reverse inequality for any θ in the other component.

Stokes data. These are linear data which provide a description of Stokes-filtered local system. Given a finite set $C \subset \mathbb{C}$ and given $\theta_o \in S^1$ which is not a Stokes direction of any pair $c \neq c' \in C$, θ_o defines a total ordering on C, that we write $c_1 <_{\theta_o} c_2 <_{\theta_o} \cdots <_{\theta_o} c_n$.

Definition 2.30. Let C be a finite subset of $\mathbb{C}$ totally ordered by θ_o. The category of *Stokes data of type* (C, θ_o) has objects consisting of two families of $\boldsymbol{k}$-vector spaces $(G_{c,1}, G_{c,2})_{c \in C}$ and a diagram of morphisms

$$\bigoplus_{c \in C} G_{c,1} \underset{S'}{\overset{S}{\rightrightarrows}} \bigoplus_{c \in C} G_{c,2} \tag{2.30 $*$}$$

such that, for the numbering $C = \{c_1, \ldots, c_n\}$ given by θ_o,

1. $S = (S_{ij})_{i,j=1,\ldots,n}$ is block-upper triangular, i.e., $S_{ij} : G_{c_i,1} \to G_{c_j,2}$ is zero unless $i \leqslant j$, and S_{ii} is invertible (so $\dim G_{c_i,1} = \dim G_{c_i,2}$, and S itself is invertible).
2. $S' = (S'_{ij})_{i,j=1,\ldots,n}$ is block-lower triangular, i.e., $S'_{ij} : G_{c_i,1} \to G_{c_j,2}$ is zero unless $i \geqslant j$, and S'_{ii} is invertible (so S' itself is invertible).

A morphism of Stokes data consists of morphisms of $\boldsymbol{k}$-vector spaces $\lambda_{c,\ell} : G_{c,\ell} \to G'_{c,\ell}$, $c \in C$, $\ell = 1, 2$ which are compatible with the diagrams (2.30 $*$).

Fixing bases in the spaces $G_{c,\ell}$, $c \in C$, $\ell = 1, 2$, allows one to present Stokes data by matrices (Σ, Σ') where $\Sigma = (\Sigma_{ij})_{i,j=1,\ldots,n}$ (resp. $\Sigma' = (\Sigma'_{ij})_{i,j=1,\ldots,n}$) is block-lower (resp. -upper) triangular and each Σ_{ii} (resp. Σ'_{ii}) is invertible.

The following is a translation of a classical result (see [52] and the references given therein, see also [27] for applications):

Proposition 2.31. *There is a natural functor from the category of Stokes-filtered local systems with exponential factors contained in C and the category of Stokes data of type (C, θ_o), which is an equivalence of categories.* □

The proof of this proposition, that we will not reproduce here, mainly uses Theorem 3.5 of the next chapter, and more precisely Lemma 3.12 to define the functor.

Duality. Let $(\mathscr{L}, \mathscr{L}_\bullet)$ be a Stokes-filtered local system. Recall (see Definition 2.13) the dual local system $\mathscr{L}^\vee$ comes equipped with a Stokes filtration $(\mathscr{L}^\vee)_\bullet$ defined by

$$(\mathscr{L}^\vee)_{\leqslant c} = (\mathscr{L}_{<-c})^\perp,$$

where the orthogonality is relative to duality. In particular, $\mathrm{gr}_c(\mathscr{L}^\vee) = (\mathrm{gr}_{-c}\mathscr{L})^\vee$. Similarly, given Stokes data $((G_{c,1}, G_{c,2})_{c\in C}, S, S')$ of type (C, θ_0), let us denote by ${}^tS$ the adjoint of S by duality. Define Stokes data $((G_{c,1}, G_{c,2})_{c\in C}, S, S')^\vee$ of type $(-C, \theta_o)$ by the formula $G^\vee_{-c,i} = (G_{c,i})^\vee$ $(i = 1, 2)$ and $S^\vee = {}^tS^{-1}$, $S'^\vee = {}^tS'^{-1}$, so that the diagram (2.30 $*$) becomes

$$\bigoplus_{i=1}^r (G_{c_i,1})^\vee \underset{{}^tS'^{-1}}{\overset{{}^tS^{-1}}{\rightrightarrows}} \bigoplus_{i=1}^r (G_{c_i,2})^\vee \qquad (2.30\,*)^\vee$$

Then the equivalence of Proposition 2.31 is compatible with duality (see [27]).

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