Chapter 4
Weak, Strong and Very Strong Factorization

Abstract An integral domain is said to have weak factorization if each nonzero nondivisorial ideal can be factored as the product of its divisorial closure and a finite product of (not necessarily distinct) maximal ideals. An integral domain is said to have strong factorization if it has weak factorization and the maximal ideals of the factorization are distinct. If, in addition, the maximal ideals in the factorization of a nonzero nondivisorial ideal I of the domain R can be restricted to those maximal ideals M such that IR_M is not divisorial, we say that R has very strong factorization. In the present section, we study these properties with particular regard to the case of Prüfer domains or almost Dedekind domains. In the Prüfer case we provide several characterizations of domains having weak, strong or very strong factorization. We discuss the connections with h-local domains and we prove that very strong and strong factorizations are equivalent for Prüfer domains.

4.1 History

In [19], the authors introduced two factorization properties for integral domains. We start by recalling the first one, called "weak" factorization.

An integral domain R is said to have *weak factorization* if each nonzero nondivisorial ideal I can be factored as the product of its divisorial closure I^v and a finite product of (not necessarily distinct) maximal ideals; i.e.,

$$I = I^v M_1 M_2 \cdots M_n\,, \quad \text{where } M_i \in \operatorname{Max}(R) \text{ for } 1 \leq i \leq n.$$

In [19], the second factorization was called "strong" factorization and had two additional restrictions; first, the maximal ideals $\{M_1, M_2, \ldots, M_n\}$ in the factorization of a nonzero nondivisorial ideal I were required to be distinct, and, second, for $I = I^v M_1 M_2 \cdots M_n$, the M_i had to be precisely those maximal ideals M for which IR_M is not a divisorial ideal of R_M.

M. Fontana et al., *Factoring Ideals in Integral Domains*, Lecture Notes of the Unione Matematica Italiana 14, DOI 10.1007/978-3-642-31712-5_4,

It turns out that for Prüfer domains, there is no need to include this second requirement (see Theorem 4.4.8) below. Thus we now redefine two types of "strong" factorization by distinguishing, a priori, two possible situations. It is convenient to let $\mathscr{H}(I)$ denote the (possibly empty) set of maximal ideals M such that $IR_M \neq (IR_M)^v$. (So formally, $\mathscr{H}(I) = \{M \in \mathrm{Max}(R) \mid IR_M \neq I^v R_M\}$.)

An integral domain R is said to have *strong factorization* if each nonzero nondivisorial ideal I of R can be factored as follows:

$$I = I^v M_1 M_2 \cdots M_n\,, \quad \text{where } M_i \in \mathrm{Max}(R) \text{ and } M_i \neq M_j \text{ for } 1 \leq i \neq j \leq n.$$

If, in addition, the M_i in such a factorization can be restricted to those maximal ideals M such that IR_M is not divisorial, we say that R has *very strong factorization*. That is, R has very strong factorization if for each nonzero nondivisorial ideal I, I can be factored as follows:

$$I = I^v M_1 M_2 \cdots M_n, \text{ where } \mathscr{H}(I) = \{M_1, M_2, \ldots, M_n\} (\neq \emptyset).$$

Remark 4.1.1. It is rather trivial to show (by checking locally) that in any of these factorizations, if $IR_{M_i} = I^v R_{M_i}$ for some M_i, then it must be that $I^v M_i = I^v$ and thus the factor of M_i can be eliminated.

One of the main theorems of [19] is the following.

Theorem 4.1.2. [19, Theorem 1.12] *The following statements are equivalent for a Prüfer domain R.*

(i) R is h-local.
(ii) R has the very strong factorization property.
(iii) For each nonzero ideal I of R, I is divisorial if and only if IR_M is divisorial in R_M for each maximal ideal M of R.
(iv) For each nonzero ideal I of R, if IR_M is divisorial for each maximal ideal M, then I is divisorial.

In Sect. 4.4, we prove a sharper version of the equivalence (i)⇔(ii) of Theorem 4.1.2. More precisely, in Theorem 4.4.9, we show that if R is an integral domain that possesses (the new type of) strong factorization, then each nonzero finitely generated ideal is divisorial. As a corollary, we have that if R is integrally closed, then it has our redefined form of strong factorization if and only if it is an h-local Prüfer domain (Corollary 4.4.10); i.e., in this situation, very strong factorization and strong factorization coincide.

One of the key results used in the proof of the Theorem 4.1.2 is the following Proposition 4.1.3 [19, Theorem 1.10]. Its expanded version, Theorem 2.5.2, plays a significant role in proving several of the results to come.

Proposition 4.1.3. *Let R be a Prüfer domain and let P be a nonzero nonmaximal prime that is the radical of a finitely generated ideal. If I is a finitely generated ideal whose radical is P and M is a maximal ideal that contains P, then the* *ideal $J := IR_M \bigcap R$ is divisorial if and only if M is the only maximal ideal that contains P.*

As a consequence of Theorems 2.2.1, 2.4.12 and 4.1.2, we have:

Corollary 4.1.4. *An almost Dedekind domain with very strong factorization is Dedekind.*

However, there exist almost Dedekind domains with weak factorization that do not have (very) strong factorization. More precisely, we proved in [19, Theorem 1.15] the following.

Theorem 4.1.5. *Let R be an almost Dedekind domain, and let I be a nonzero ideal of R which is contained in only finitely many nondivisorial maximal ideals of R. Then $I = I^v M_1 M_2 \cdots M_n$ where the M_i are maximal ideals but are not necessarily distinct. Thus if R is an almost Dedekind domain in which each nonzero ideal is contained in only finitely many nondivisorial maximal ideals, then R has the weak factorization property.*

By using Theorem 4.1.5, [34, Example 42.6] provides an explicit example of an almost Dedekind domain with weak factorization that does not have very strong factorization. Other examples can be found in [58].

Below, in Proposition 4.2.14, we will give several ways of characterizing when an almost Dedekind domain that is not Dedekind has weak factorization, essentially establishing the converse of Corollary 4.1.4.

4.2 Weak Factorization

In Theorem 4.1.2, Prüfer domains which are h-local were characterized via the very strong factorization property. On the other hand, in the Prüfer domain case, h-local domains can be also characterized using the weak factorization property. More precisely, [19, Theorem 1.13] provides the following characterization.

Theorem 4.2.1. *Let R be a Prüfer domain. Then R is h-local if and only if R has weak factorization and finite character.*

In Theorem 4.4.8 below, we will give another proof of this, together with several other new characterizations of h-local Prüfer domains based on weak factorization-type properties.

By [19, Proposition 1.7], a Prüfer domain with weak factorization is a wTPP-domain (Sect. 2.4) and, more precisely, we have the following.

Proposition 4.2.2. *Let R be a Prüfer domain with the weak factorization property. Then the following hold.*

(1) each ideal which is primary to a nonmaximal ideal of R is divisorial (in particular, each nonmaximal prime is divisorial),

(2) if M is an idempotent maximal ideal of R and I is a nondivisorial M-primary ideal, then $I = I^v M$,
(3) each branched maximal idempotent ideal of R is sharp,
(4) R is a wTPP-domain, and
(5) each branched nonmaximal prime ideal of R is the radical of a finitely generated ideal.

The next lemma collects a few useful properties a Prüfer domain with weak factorization property has in common with one having very strong factorization (for comparison, see [67, Proposition 3.4 and Theorem 3.10] and [19, Proposition 2.10]).

Lemma 4.2.3. *Let R be a Prüfer domain with the weak factorization property.*

(1) Each nonzero prime is contained in a unique maximal ideal.
(2) Each maximal ideal of height greater than one is sharp.
(3) Each locally principal maximal ideal of height greater than one is invertible. Thus (equivalently) each unsteady maximal ideal has height one.
(4) If I is a nonzero, nondivisorial ideal with factorization $I = I^v \prod N_i \prod M_j^{r_j}$ where the N_i are the steady maximal ideals for which $IR_{N_i} \neq I^v R_{N_i}$ and the M_j are the unsteady maximal ideals for which $IR_{M_j} \neq I^v R_{M_j}$, then each N_i is idempotent and $I^v R_{N_i}$ is principal.

Proof. As each unbranched prime contains a nonzero branched prime, it suffices to prove (1) in the case $P \neq (0)$ is a nonmaximal branched prime. By Proposition 2.3.10, there is a finitely generated ideal I such that $\sqrt{I} = P$. Let M be a maximal ideal that contains P and let $J := IR_M \bigcap R$. Then by Theorem 2.5.2(2) (enlarging I if necessary), $J^v = J(P' : P') = I(P' : P')$ where P' is the largest prime that is common to all maximal ideals that contain P. Clearly, $\mathrm{Max}(R, P) = \mathrm{Max}(R, P')$. Thus P' is sharp by Lemma 2.3.9. Therefore P' is a maximal ideal of $(P' : P')$ by Corollary 2.3.21(2). Moreover, P' is the only maximal ideal of $(P' : P')$ that contains P, I and J^v (since such a maximal ideal must be extended from a prime P'' of R which must be contained in one of the maximal ideals containing I and must therefore be comparable to P'). Hence $J^v R_{P'} = IR_{P'} = JR_{P'}$ and $J^v = JR_{P'} \bigcap (P' : P')$. (The latter equality is true locally since P' is the only maximal ideal of $(P' : P')$ which contains J and J^v.) It follows that $J^v = JR_{P'} \bigcap R$.

Since R has weak factorization and $JR_{P'} = J^v R_{P'}$, there is an ideal H that is not contained in P' such that $J = J^v H$ (either with $H = R$ or H a finite product of maximal ideals). By Lemma 2.5.1(2), $JR_{P'} = J^v \Gamma(P)$ and this yields $IR_M = JR_M \supseteq J\Gamma(P) = J^v H\Gamma(P) = JHR_{P'} = JR_{P'} = IR_{P'} \supseteq IR_M$. Hence $IR_M = IR_{P'}$. Since I is finitely generated, we must have $M = P'$. This establishes (1).

For (2), (3) and (4), let M be a maximal ideal of height greater than one. Then M contains at least one nonzero branched prime. For (2), simply apply (1), Propositions 2.3.10 and 4.2.2(5), and Lemma 2.3.9 to see that M is sharp. For (3), further assume that M is locally principal, say $MR_M = aR_M$ with $a \in M$.

Since M is sharp, there is a finitely generated ideal A of R such that M is the only maximal ideal that contains A. Then the ideal $A + aR$ generates M locally, whence $M = A + aR$ is invertible.

Finally, (4) follows from Lemma 2.5.3. □

As noted earlier in Sect. 2.4, a Prüfer domain with weak factorization is an aRTP-domain. Using Lemma 4.2.3 and several of the other results above, we are now ready to give our first alternate characterizations of Prüfer domains with weak factorization.

Theorem 4.2.4. *The following statements are equivalent for a Prüfer domain R.*

(i) R has weak factorization.
(ii) Each nonzero prime is contained in a unique maximal ideal, and R is an aRTP-domain such that for each nonzero ideal I, the set of maximal ideals N where $IR_N \neq I^v R_N$ is finite.
(iii) (a) Each steady maximal ideal is sharp,
(b) each nonzero nonmaximal prime ideal is sharp and contained in a unique maximal ideal, and
(c) for each nonzero ideal I, the set of maximal ideals N where $IR_N \neq I^v R_N$ is finite.

Proof. By Theorem 2.4.18, a Prüfer domain is an aRTP-domain if and only if each nonzero branched nonmaximal prime ideal and each steady branched maximal ideal are sharp. Under the additional assumption that each nonzero prime is contained in a unique maximal ideal, Lemma 2.3.9 guarantees that the equivalence holds true with the word "branched" removed. Hence (ii) and (iii) are equivalent.

To see that (i) implies (ii), assume that R has weak factorization. If I is a nonzero nondivisorial ideal, then we have $I = I^v \prod_{i=1}^{n} M_i^{s_i}$ for some finite set of maximal ideals $\{M_1, M_2, \ldots, M_n\}$ and positive integers $s_1, s_2, \ldots, s_n$. From this it is clear that there are at most finitely many maximal ideals N such that $IR_N \neq I^v R_N$. Also, by Lemma 4.2.3(1), each nonzero prime is contained in a unique maximal ideal. Finally, each branched idempotent maximal ideal and each nonzero nonmaximal branched prime ideal are sharp by Proposition 4.2.2(3 and 5), so that R is an aRTP domain by Theorem 2.4.18.

To complete the proof we show (iii) implies (i). Assume all three conditions in (iii) hold. By Lemma 2.3.9, statement (iii)(b) implies that each maximal ideal of height greater than one is sharp. Combined with (iii)(a), we have that the only maximal ideals that are not sharp are the height one unsteady maximal ideals.

Let I be a nonzero nondivisorial ideal. Then by (iii)(c), there is a nonempty finite set of maximal ideals $\{M_1, M_2, \ldots, M_n\}$ such that $IR_{M_i} \neq I^v R_{M_i}$ for $1 \leq i \leq n$ and $IR_M = I^v R_M$ for all maximal ideals M not in the set $\{M_1, M_2, \ldots, M_n\}$. If M_i is a height one unsteady maximal ideal, then $M_i R_{M_i}$ is principal and therefore $IR_{M_i} = M_i^{r_i} R_{M_i} \subsetneq I^v R_{M_i} = M^{t_i} R_{M_i}$ for some integers $r_i > t_i \geq 0$. In this case, $IR_{M_i} = I^v M_i^{s_i} R_{M_i}$, where $s_i = r_i - t_i$. For those M_j that are not height one unsteady maximal ideals, Theorem 2.5.4 (together with statements (iii)(a) and (iii)(b)) implies that M_j must be idempotent with $IR_{M_j} = I^v M_j R_{M_j}$. By checking locally, we have $I = I^v \prod_{i=1}^{n} M_i^{s_i}$ for some positive integers $s_1, s_2, \ldots, s_n$. Hence (iii) implies (i). □

Theorem 4.2.5. *Let R be a Prüfer domain with the weak factorization property that is not h-local. If S is an overring of R where no unsteady maximal ideal of R survives, then S is h-local.*

Proof. By Theorems 2.4.18 and 4.2.4, if P is a nonzero branched prime ideal of R that is not an unsteady maximal ideal, then P is the radical of finitely generated ideal and it is contained in a unique maximal ideal.

Assume that S is an overring of R. Then each nonzero prime of S is contained in a unique maximal ideal. If no unsteady maximal ideal of R survives in S and Q is a (nonzero) branched prime of S, then Q must be extended from either a steady maximal ideal or a branched nonmaximal prime. In either case, $Q \bigcap R$ is the radical of a finitely generated ideal and therefore so is Q. Thus S has the radical trace property by Theorem 2.4.14. Therefore S is h-local by Theorem 2.4.12. □

Corollary 4.2.6. *Let R be a Prüfer domain with the weak factorization property. If I is an ideal that is contained in no unsteady maximal ideals, then I is contained in only finitely many maximal ideals.*

Proof. Suppose I is contained in no unsteady maximal ideals. Then each maximal ideal that contains I is sharp (Theorem 4.2.4). Hence by Lemmas 2.5.1 and 2.4.19, the ring $\Gamma(I)$ has no unsteady maximal ideals and each maximal ideal of $\Gamma(I)$ contains I. Since R has the weak factorization property, $\Gamma(I)$ is h-local by Theorem 4.2.5, and so, in particular, $\Gamma(I)$ has finite character. Thus at most finitely many maximal ideals of R contain I. □

Theorem 4.2.7. *Let R be a Prüfer domain with weak factorization. If I is a radical ideal of R such that I^{-1} is a ring, then each minimal prime of I extends to a maximal ideal of I^{-1} as does each maximal ideal of R that does not contain I.*

Proof. Assume that I is a radical ideal such that I^{-1} is a ring. Since $\Theta(I)$ contains I^{-1} (Theorem 2.3.2(1)), it is always the case that a maximal ideal that does not contain I extends to a maximal ideal of I^{-1}. Suppose that P is a prime minimal over I. If P is nonmaximal, then it is (ante)sharp by Theorem 4.2.4 and Corollary 2.3.21(2). If P is maximal, then it is trivially antesharp. Hence PI^{-1} is a maximal ideal of I^{-1} by Lemma 2.5.5(1). □

We are primarily interested in applying Theorem 2.5.6 in the case that R is a Prüfer domain with weak factorization. For this situation, we can make a slight change in the hypothesis.

Theorem 4.2.8. *Let R be a Prüfer domain with weak factorization, and let P be a sharp prime of R. If I is radical ideal with $I \subseteq P$ and $\{P_\alpha\}$ is a set of minimal prime ideals such that $I = \bigcap_\alpha P_\alpha$, then P contains some $P_\beta \in \{P_\alpha\}$. If, in addition, P is not minimal over I, then $P \bigcap (\bigcap_{\alpha \neq \beta} P_\alpha)$ properly contains I.*

Proof. Simply apply Theorem 2.5.6 to the prime $Q \subseteq P$ with Q minimal over I. Such a prime is sharp since R has weak factorization (Theorem 4.2.4). □

The next corollary collects several useful consequences of Theorems 2.5.6 and 4.2.8.

Corollary 4.2.9. *Let R be a Prüfer domain with weak factorization.*

(1) If $\mathscr{W} := \{M_\alpha \mid \alpha \in \mathscr{A}\}$ is a nonempty set of unsteady maximal ideals such that $I := \bigcap_\alpha M_\alpha$ is a nonzero ideal, then no sharp prime contains I.
(2) If $\{P_\alpha\}$ is a nonempty set of pairwise incomparable sharp primes such that $J := \bigcap_\alpha P_\alpha \neq (0)$, then each P_α is minimal over J and no other sharp prime is minimal over J.
(3) No nonzero element of R is contained in infinitely many idempotent maximal ideals.

Proof. Let $I = \bigcap_\alpha M_\alpha$ be nonzero, with $\mathscr{W} = \{M_\alpha \mid \alpha \in \mathscr{A}\}$ a nonempty set of unsteady maximal ideals. Then each $M_\alpha \in \mathscr{W}$ has height one (Lemma 4.2.3(3)) and is therefore minimal over I. Moreover, the M_α are not sharp by Lemma 2.4.19. Hence no sharp prime contains I by Theorem 4.2.8, proving (1).

For (2), assume that $\{P_\alpha\}$ is a nonempty set of pairwise incomparable sharp primes such that $J = \bigcap_\alpha P_\alpha \neq (0)$. For each α, let $Q_\alpha \subseteq P_\alpha$ be a prime ideal that is minimal over J. Obviously, $J = \bigcap_\alpha Q_\alpha$ and for each P_β, $J = \bigcap_\alpha Q_\alpha \subseteq P_\beta \bigcap (\bigcap_{\alpha\neq\beta} Q_\alpha) \subseteq P_\beta \bigcap (\bigcap_{\alpha\neq\beta} P_\alpha) = J$. Thus by Theorem 4.2.8, each P_α is minimal over J, and no other sharp prime is minimal over J.

To see that (3) holds, further assume that each P_α in the intersection $J = \bigcap_\alpha P_\alpha$ of (2) is an idempotent maximal ideal. (Note that each P_α is sharp by Theorem 4.2.4.) Since $JR_{P_\alpha} = P_\alpha R_{P_\alpha}$ and P_α is idempotent, it must be that $JJ^{-1}R_{P_\alpha} = P_\alpha R_{P_\alpha}$ (since $JR_{P_\alpha} \subseteq JJ^{-1}R_{P_\alpha} \subseteq P_\alpha R_{P_\alpha}(P_\alpha R_{P_\alpha})^{-1} = P_\alpha R_{P_\alpha} = JR_{P_\alpha}$). Thus $JJ^{-1} = J$, and we have $J^{-1} = (J : J)$, and so J^{-1} is a ring. By Theorem 4.2.8, no other sharp prime can be minimal over J, whence by Lemma 4.2.3, the only other minimal primes of J must be height one (unsteady) maximal ideals. Thus $J^{-1} = \Gamma(J) \bigcap \Theta(J) = R$ (Theorem 2.3.2(2)) and so $J \neq J^v = R$. It follows that the set $\{P_\alpha\}$ is finite by Theorem 4.2.4. Hence each nonzero nonunit is contained in at most finitely many idempotent maximal ideals. □

We say that R has *finite idempotent character* if each nonzero element is contained in at most finitely many idempotent maximal ideals and *finite unsteady character* if each nonzero element is contained in at most finitely many unsteady maximal ideals. By Corollary 4.2.9, a Prüfer domain that has weak factorization also has finite idempotent character. In the next theorem, we show that a Prüfer domain with weak factorization also has finite unsteady character. A consequence of this is that a Prüfer domain has weak factorization if and only if each unsteady maximal ideal has height one, each nonzero nonunit is contained in at most finitely many noninvertible maximal ideals and $(IR_M)^{-1} = I^{-1}R_M$ for each nonzero ideal I and each sharp maximal ideal M (see Theorem 4.2.12 below).

Recall from Sect. 2.5 that if $P \in \mathscr{S}$ is a prime ideal of a Prüfer domain R where $\mathscr{S}$ is a set of incomparable primes of R, then P is *relatively sharp* in $\mathscr{S}$ if it contains a finitely generated ideal that is contained in no other prime of the set $\mathscr{S}$ (or equivalently, R_P does not contain $\bigcap\{R_Q \mid Q \in \mathscr{S}\setminus\{P\}\}$). The set $\mathscr{S}$ is *relatively sharp* if each prime in $\mathscr{S}$ is relatively sharp in $\mathscr{S}$. We make use of these notions in the proof of our next theorem.

Theorem 4.2.10. *Let R be a Prüfer domain. If R has weak factorization, then R has finite unsteady character.*

Proof. By way of contradiction, assume there is an infinite set of unsteady maximal ideals $\mathscr{W} := \{M_\alpha \mid \alpha \in \mathscr{A}\}$ with a nonzero intersection $I := \bigcap_\alpha M_\alpha$. Each $M_\alpha \in \mathscr{W}$ has height one by Lemma 4.2.3(3). Also, by Corollary 4.2.9(1), no sharp prime contains I, so we may assume $\mathscr{W}$ is the complete set of minimal primes of I. Note that if $I^{-1} = R$, then we cannot have weak factorization, since in this case $I^v = R$, so it would be impossible to factor I as I^v times a finite product of maximal ideals. Thus we may further assume there is an element $t \in I^{-1}\setminus R$. Since R is Prüfer, the ideal $C := (R : (1,t))$ is an invertible ideal that contains I. Thus C is contained in no maximal ideal that does not contain I. Since I is a radical ideal and each of its minimal primes is maximal, C is a radical ideal as well with $\mathrm{Max}(R,C) \subseteq \mathrm{Max}(R,I) = \mathscr{W}$. It is easy to see that the set $\mathrm{Max}(R,C)$ must be infinite since no member of $\mathscr{W}$ is sharp and C is finitely generated (Theorem 2.3.11). Hence we may further assume that I is invertible.

If either $\mathscr{W}$ or an infinite subset of $\mathscr{W}$ is a relatively sharp set, then we have a contradiction by way of Corollary 4.2.9 and Theorem 2.5.9. Hence we may further assume no infinite subset of $\mathscr{W}$ is a relatively sharp set. Using this assumption we will arrive at a contradiction by constructing an infinite subset of $\mathscr{W}$ that is a relatively sharp set.

Let $M_\beta \in \mathscr{W}$ and let $q \in M_\beta\setminus I$. Then the ideal $E := qR+I$ is an invertible ideal that properly contains I. Moreover, for each $M_\alpha \in \mathscr{W}$, ER_{M_α} contains $IR_{M_\alpha} = M_\alpha R_{M_\alpha}$. Since E is invertible, $IEE^{-1} = I$. Thus the ideal $G := IE^{-1}$ is an invertible ideal of R that is contained in each maximal ideal of $\mathscr{W}$ that does not contain E and in no maximal ideal of $\mathscr{W}$ that contains E.

Suppose $M_1, M_2, \ldots, M_n \in \mathscr{W}$ are relatively sharp in $\mathscr{W}$, $n \geq 1$. Then for each i, there is a finitely generated ideal $J_i \subsetneq M_i$ such that no other member of $\mathscr{W}$ contains J_i. Moreover, we may assume each J_i contains I and $J_i + J_k = R$ for all $i \neq k$. Then the product $J := J_1 J_2 \cdots J_n$ contains I and is contained in each M_i but in no other member of $\mathscr{W}$. Since J is invertible, it follows that the ideal IJ^{-1} of R is contained in each maximal ideal of $\mathscr{W}$ except $M_1, M_2, \ldots, M_n$. Hence IJ^{-1} is the intersection of these ideals.

Since we have assumed at most finitely many members of $\mathscr{W}$ are relatively sharp in $\mathscr{W}$, we may further assume that no member of $\mathscr{W}$ is relatively sharp. Under this assumption, if B is a finitely generated ideal with $I \subsetneq B \subseteq M_\beta$ for some $M_\beta \in \mathscr{W}$, then B is contained in infinitely many members of $\mathscr{W}$ as is IB^{-1}, and no member of $\mathscr{W}$ contains both B and IB^{-1}.

With all of these assumptions, it is now relatively easy to construct a countably infinite subset of $\mathscr{W}$ that is relatively sharp and with this arrive at a contradiction.

We construct such a subset as follows.

Let $\mathscr{A}$ be a well-ordered index set for $\mathscr{W}$ and let α_1 be the smallest member of $\mathscr{A}$. Next set $M_1 := M_{\alpha_1}$, select an element $s_1 \in M_1 \backslash I$ and set $J_1 := I + s_1 R$. Then from the above, infinitely many members of $\mathscr{W}$ contain J_1 and infinitely many do not.

For M_2, let α_2 be the smallest $\alpha \in \mathscr{A}$ such that M_α does not contain J_1, then set $M_2 := M_{\alpha_2}$. Since infinitely many members of $\mathscr{W}$ do not contain J_1, there is an element $s_2 \in M_2 \backslash I J_1^{-1}$ such that s_1 and s_2 are comaximal but $s_1 s_2$ is not in I. Set $J_2 := I + s_2 R$. Then, clearly, $I \subsetneq J_1 \bigcap J_2 = J_1 J_2$. As above, infinitely many members of $\mathscr{W}$ do not contain $C_2 := J_1 J_2$.

Recursively, for $n \geq 3$, define ideals M_n, J_n and C_n as follows. Let α_n be the smallest $\alpha \in \mathscr{A}$ such that M_α does not contain C_{n-1}, then set $M_n := M_{\alpha_n}$. For J_n, there is an element $s_n \in M_n \backslash I C_{n-1}^{-1}$ such that s_n is comaximal with the ideal C_{n-1} but $s_n C_{n-1}$ is not contained in I. Set $J_n := I + s_n R$. Then $C_n := C_{n-1} J_n = C_{n-1} \bigcap J_n \supsetneq I$.

For $n \neq m$, the elements s_n and s_m are comaximal. Thus each M_m is relatively sharp in the set $\{M_n\}_{n=1}^{\infty}$, a contradiction to our assumption that no infinite subset of $\mathscr{W}$ is relatively sharp. Therefore it must be that no nonzero element is contained in infinitely many unsteady maximal ideals. $\square$

The next result is a straightforward consequence of Corollary 4.2.9(3), Theorem 4.2.10, and Lemma 2.1.10.

Corollary 4.2.11. *Let R be a Prüfer domain. If R has weak factorization, then each nonzero nonunit is contained in at most finitely many noninvertible maximal ideals.*

Theorem 4.2.12. *The following statements are equivalent for a Prüfer domain R.*

(i) *R has weak factorization.*

(ii) *(a)* *R is an aRTP-domain,*
  (b) *each nonzero prime ideal is contained in a unique maximal ideal, and*
  (c) *each nonzero nonunit is contained in at most finitely many noninvertible maximal ideals.*

(iii) *(a)* *Each steady maximal ideal is sharp,*
  (b) *each nonzero nonmaximal prime is both sharp and contained in a unique maximal ideal, and*
  (c) *each nonzero nonunit is contained in at most finitely many noninvertible maximal ideals.*

(iv) *(a)* *Each unsteady maximal ideal has height one,*
  (b) *each nonzero ideal (or nonunit) is contained in at most finitely many noninvertible maximal ideals, and*
  (c) *$(IR_M)^{-1} = I^{-1}R_M$ for each nonzero ideal I and each steady maximal ideal M.*

(v) *For each nonzero nondivisorial ideal I, there is a finite family of primes $\{P_1, P_2, \ldots, P_n\}$ such that $I = I^v P_1 P_2 \cdots P_n$.*

(vi) *For each nonzero nondivisorial ideal I, there is a finite set of incomparable primes $\{Q_1, Q_2, \ldots, Q_m\}$ such that $I = I^v Q_1^{r_1} Q_2^{r_2} \cdots Q_m^{r_m}$ for some positive integers $r_1, r_2, \ldots, r_m$.*

Proof. For (i) implies (ii), simply apply Theorem 4.2.4 and Corollary 4.2.11. Also, the same argument used in the first part of the proof of Theorem 4.2.4 shows that (ii) and (iii) are equivalent.

To see that (iii) implies (iv), first apply Lemma 2.3.9 to see that each maximal ideal of height greater than one is sharp. Thus the only unsteady maximal ideals, if any, have height one. Also, by Theorem 2.5.4, $I^{-1}R_M = (IR_M)^{-1}$ for each nonzero ideal I and each steady maximal ideal M.

Next we show (iv) implies (i). Let I be a nonzero, nondivisorial ideal. By Theorem 2.5.4(2), if M is an invertible maximal ideal that contains I, then $IR_M = I^v R_M$. Thus there must be at least one noninvertible maximal ideal N that contains I and is such that $IR_N \neq I^v R_N$. Let $\{M_1, M_2, \ldots, M_n\}$ be the set of these maximal ideals. If M_i is idempotent, then $IR_{M_i} = I^v M_i R_{N_i}$ by Theorem 2.5.4(3). On the other hand, if M_i is locally principal (equivalently, not idempotent), then it has height one and there is a positive integer s_i such that $IR_{M_i} = I^v M_i^{s_i} R_{M_i}$ (see also the proof of Theorem 4.2.4((iii)⇒(i))). Checking locally shows that $I = I^v \prod_{i=1}^n M_i^{s_i}$ for some positive integers $s_1, s_2, \ldots, s_n$. Thus R has weak factorization.

Clearly, weak factorization implies the existence of the factorizations in (v) and (vi).

Since R is a Prüfer domain, if $Q \subsetneq P$ are distinct prime ideals, then $PQ = Q$. Hence in statement (v), if $P_i \subsetneq P_j$, then all occurrences of P_j can be removed from the factorization. It follows that (v) and (vi) are equivalent.

To complete the proof, we show that if each nonzero nondivisorial factors in the form given in (v), then R has weak factorization. For this, it suffices to show that R satisfies the criteria of Theorem 4.2.4(iii).

First, let P be a branched nonmaximal prime ideal. Then it is minimal over a finitely generated ideal I [34, Theorem 23.3(e)]. Let $J := IR_M \bigcap R$ where M is a maximal ideal that contains P. Then, clearly, $I \subseteq J \subseteq P$ and $\mathrm{Max}(R, J) = \mathrm{Max}(R, P)$. Note that $J \subsetneq P$; otherwise, $PR_M = JR_M = IR_M$ is a nonmaximal finitely generated prime in the valuation domain R_M, which is impossible. If P is not sharp, then $J \subsetneq P \subseteq P^v = J^v$ by Theorem 2.5.2(1). Now, consider a factorization $J = J^v P_1 P_2 \cdots P_n$ as in statement (v). In the valuation domain R_M, $JR_M = IR_M$ is principal, and therefore from this factorization of J, so are $J^v R_M$ and each $P_i R_M$. Hence either $P_i = M$ or $P_i R_M = R_M$. Since P is not maximal, it cannot contain any of the P_is. It follows that $P \supseteq J^v$ and hence $J^v = P$. Thus, as above, PR_M is a principal nonmaximal prime ideal of R_M, a contradiction. Therefore P is sharp.

If P is sharp but M is not the only maximal ideal that contains P, then J is not divisorial (Theorem 2.5.2(2)). Also, from the proof of Theorem 2.5.2(2), $J^v R_M = JR_{P'} = IR_{P'}$ where $P' \subsetneq M$ is the largest prime common to all maximal ideals that contain P. As above, factor J as $J = J^v P_1 P_2 \cdots P_n$.

We again have $J^v R_M$ and each $P_i R_M$ principal. However, since $IR_{P'} = J^v R_M$ is both a proper principal ideal of $R_{P'}$ and an ideal of R_M, this contradicts the fact that no proper principal ideal of a valuation domain can be an ideal in a proper overring. Thus having P sharp and in more than one maximal ideal is impossible. Therefore $J = J^v$, P is sharp and M is the only maximal ideal that contains P by Theorem 2.5.2(3).

Consequently, we have that each nonzero nonmaximal prime is sharp as is each maximal ideal of height greater than one (Lemma 2.3.9).

Thus according to Theorem 4.2.4, the only remaining case we need to consider (for sharpness) is that of a height one idempotent maximal ideal. Accordingly, let $M = M^2$ be a height one maximal ideal and let Q be a proper M-primary ideal. If Q^{-1} is a ring, then $Q^{-1} = M^{-1} = R$ by Lemma 2.3.15. Hence $Q^v = R$, and we have the (only possible) factorization $Q = Q^v M = M$, a contradiction. Thus Q^{-1} is not a ring, whence M is sharp by Theorem 2.3.17((i)$\Leftrightarrow$(iv)).

To complete the proof, we need only show that for each nonzero nondivisorial ideal B, the set of maximal ideals N for which $BR_N \neq B^v R_N$ is finite (criterion (iii)(c) of Theorem 4.2.4). Let B be a nonzero nondivisorial ideal. Then $B = B^v Q_1 Q_2 \cdots Q_m$ for some finite family of prime ideals $\{Q_1, Q_2, \ldots, Q_m\}$. Each Q_i is in a unique maximal ideal N_i and clearly $BR_N = B^v R_N$ for each maximal ideal N not in the family $\{N_1, N_2, \ldots, N_m\}$. Therefore R has weak factorization. □

We record the following simple consequence of Theorem 4.2.12 for ease of reference in Example 4.3.4 below.

Corollary 4.2.13. *Let R be a Prüfer domain with finite unsteady character. If each steady maximal ideal is invertible and each nonzero nonmaximal prime is sharp and contained in a unique maximal ideal, then R has weak factorization.*

One of the main concepts studied by Loper and Lucas in 2003 [58] is how far an almost Dedekind domain is from being Dedekind. For example, from what has already been observed in Sect. 3.2, an almost Dedekind domain R has sharp degree 2 if it is not Dedekind, but the intersection $R_2 := \bigcap R_M$ is Dedekind, where the intersection is taken over all maximal ideals M that are not invertible in R (in this case, they coincide with the dull maximal ideals of R, considered in Sect. 3.2). Note that R must have infinitely many invertible maximal ideals for this to happen. An example in [19] shows that an almost Dedekind domain with infinitely many noninvertible maximal ideals can have sharp degree 2 [19, Example 3.2]. Higher sharp degrees (including infinite ordinal degrees) can be defined recursively, as in Sect. 3.2. For example, an almost Dedekind domain R has sharp degree 3, if R_2 is not Dedekind and $R \subsetneq R_2 \subsetneq R_3$ with R_3 Dedekind where R_3 is the intersection of the localizations at the (nonempty set of) noninvertible maximal ideals of R_2. It turns out that an almost Dedekind domain that is not Dedekind has weak factorization if and only if it has sharp degree 2. More precisely, Theorem 4.1.5 essentially shows that an almost Dedekind domain with sharp degree 2 has weak factorization. In Proposition 4.2.14 below, we establish the converse, as well other ways to detect weak factorization in almost Dedekind domains.

Proposition 4.2.14. *Let R be an almost Dedekind domain that is not Dedekind. The following are equivalent.*

(i) R has weak factorization.
(ii) If $\{M_\alpha\}$ is a set of noninvertible maximal ideals such that $\bigcap_\alpha M_\alpha$ is nonzero, then the set $\{M_\alpha\}$ is finite.
(iii) Each noninvertible maximal ideal contains a finitely generated ideal that is contained in no other noninvertible maximal ideal.
(iv) If $\{M_\alpha\}$ is a nonempty set of noninvertible maximal ideals, then $\bigcap_\alpha R_{M_\alpha}$ is Dedekind.
(v) R has sharp degree 2.

Proof. Let $\{M_\gamma\}$ be the (complete) set of noninvertible maximal ideals of R. Then as observed above, R has sharp degree 2 if and only if $R_2 = \bigcap_\gamma R_{M_\gamma}$ is a Dedekind domain. Since an overring of a Dedekind domain is Dedekind, statements (iv) and (v) are equivalent. Moreover, if R_2 is Dedekind, then it is easy to see that each nonzero nonunit of R is contained in at most finitely many noninvertible maximal ideals. Thus (v) implies (i) by Theorem 4.2.12.

By Lemma 3.4.6, if (iii) holds, then each noninvertible maximal ideal becomes sharp in R_2 and from this and Theorem 2.2.1, we have that R has sharp degree 2 (so (iii) implies (v)).

Assume that R has weak factorization. Then by Theorem 4.2.12, each finitely generated nonzero ideal is contained in at most finitely many noninvertible maximal ideals. Thus an infinite intersection of noninvertible maximal ideals is zero, and we have that (i) implies (ii).

To see that (ii) implies (iii), let M be a noninvertible maximal ideal of R, and let a be a nonzero element of M. By (ii), a is contained in only finitely many noninvertible maximal ideals, say $M_1 = M, M_2, \ldots, M_n$. For each $1 < i \leq n$, pick an element $a_i \in M \setminus M_i$. Then the ideal $(a, a_2, \ldots, a_n)$ is contained in M and no other noninvertible maximal ideal of R. □

4.3 Overrings and Weak Factorization

Let R be a Prüfer domain with weak factorization. Also, let $\{M_\alpha\}$ be the set of unsteady maximal ideals, and assume that this set is nonempty. By Theorem 4.2.12, each M_α has height one, and each nonzero nonunit of R is contained in at most finitely many of the M_α. Let $T := \bigcap_\alpha R_{M_\alpha}$. By definition, sharp primes of R do not survive in T. Hence T is a one-dimensional Prüfer domain with finite character such that each localization is a rank one discrete valuation domain, that is, T is a Dedekind domain. Since an overring of a Dedekind domain is Dedekind, we have the following result.

Theorem 4.3.1. *Let R be a Prüfer domain. If R has weak factorization, then $\bigcap_\alpha R_{M_\alpha}$ is a Dedekind domain for each nonempty set of unsteady maximal ideals $\{M_\alpha\}$.*

Corollary 4.3.2. *Let R be a Prüfer domain with weak factorization. The following are equivalent for an overring T of R.*

(i) *T has weak factorization.*
(ii) *T has finite idempotent character.*
(iii) *If $\{P_\alpha\}$ is a set of incomparable idempotent primes of R with a nonzero intersection, then at most finitely many P_α's extend to maximal ideals of T.*

Proof. Let T be an overring of R, and let J be a nonzero ideal of T. Then $J = IT$ for some ideal I of R [34, Theorem 26.1(3)]. Also, if N is an idempotent maximal ideal of T, then $N = PT$ for some idempotent prime P of R. Thus statements (ii) and (iii) are equivalent.

Each unsteady maximal ideal of T is extended from an unsteady maximal ideal, and each sharp prime of R that survives in T is still sharp and contained in a unique maximal ideal of T. Note that an unsteady maximal ideal of R may extend to a steady maximal ideal of T, but in such a case the extension is an invertible height one maximal ideal (Lemma 4.2.3). Thus we have $J^{-1}T_N = (JT_N)^{-1}$ for each steady maximal ideal N of T (if N is extended from a steady maximal ideal M of R, then $I^{-1}R_M = (IR_M)^{-1}$ (Theorem 4.2.12), and so $J^{-1}T_N = (JT_N)^{-1}$; on the other hand, if N is extended from an unsteady maximal ideal M of R, then N is invertible in T with height one, in which case $J^{-1}T_N = (JT_N)^{-1}$ by Theorem 2.5.4). This shows that T satisfies condition (iv) of Theorem 4.2.12, so that T has weak factorization. Thus (ii) implies (i). Finally, (i) implies (ii) by Corollary 4.2.9(3). □

Corollary 4.3.3. *If R is Prüfer domain with weak factorization and no nonzero idempotent primes, then each overring has weak factorization.*

The next example shows that not all overrings of a Prüfer domain with weak factorization have weak factorization.

Example 4.3.4. Let $\{\mathrm{W}, \mathrm{X}, \mathrm{Y}_1, \mathrm{Y}_2, \ldots, \mathrm{Z}_1, \mathrm{Z}_2, \ldots\}$ be a countably infinite set of algebraically independent indeterminates over the field K and let $D := K[\mathrm{X}, \{\mathrm{Y}_n \mid n \geq 1\}, \{\mathrm{Z}_n^\alpha \mid n \geq 1, \alpha \in \mathbb{R}^+\}]$. For each $n \geq 1$, define a valuation v_n on the quotient field $F := K(\mathrm{X}, \{\mathrm{Y}_n \mid n \geq 1\}, \{\mathrm{Z}_n^\alpha \mid n \geq 1, \alpha \in \mathbb{R}^+\})$ of D with value group $\mathbb{R} \times \mathbb{Z}$ (lexicographically ordered) by first setting $\mathrm{v}_n(a) = (0, 0)$ for all nonzero elements in K, $\mathrm{v}_n(\mathrm{X}) = (1, 0)$, $\mathrm{v}_n(\mathrm{Y}_n) = (0, 1)$, $\mathrm{v}_n(\mathrm{Z}_n^\alpha) = (\alpha, 0)$ and $\mathrm{v}_n(\mathrm{Y}_m) = \mathrm{v}_n(\mathrm{Z}_m^\alpha) = (0, 0)$ for all $\alpha \in \mathbb{R}$ and $m \neq n$, and then extending v_n to F using "min." Let V_n be the corresponding valuation domain with quotient field F. By standard arguments, it can be shown that $\mathrm{Y}_n V_n$ is the maximal ideal of the two-dimensional valuation domain V_n. Let $R := \bigcap_n V_n(\mathrm{W})$, where $V_n(\mathrm{W})$ is the canonical (trivial) extension of V_n to the field of rational functions $F(\mathrm{W})$.

(1) R is a Bézout domain.
(2) For each n, the ideal $M_n := \mathrm{Y}_n R$ is an invertible height two maximal ideal.
(3) For each n, $P_n := \sqrt{\mathrm{Z}_n R}$ is a height one idempotent prime that is sharp and M_n is the only maximal ideal of R that contains P_n.
(4) Let J be the ideal generated by the set $\{\mathrm{X}/\mathrm{Z}_n \mid n \geq 1\}$. Then $M := \sqrt{J}$ is an unsteady height one maximal ideal.
(5) There are no other nonzero prime ideals in R.
(6) R has weak factorization. Moreover, each nonzero nondivisorial ideal I factors as $I^v M^k$ for some positive integer k.
(7) Let $\mathscr{S} := \{\mathrm{Y}_n^k \mid n \geq 1, k \geq 0\}$. The ring $R_{\mathscr{S}}$ is a one-dimensional Prüfer domain that does not have weak factorization.
(8) If $\mathscr{Q}$ is an infinite proper subset of $\{P_n\}_{n=1}^{\infty}$, then the ideal $H := \bigcap\{P_m \mid P_m \in \mathscr{Q}\}$ is a (nonzero) radical ideal such that $(H : H)$ does not have weak factorization.

Proof. For each n, let v_n^* denote the trivial extension of v_n to $F(\mathrm{W})$ (set the value of W to be 0, and extend to $F(\mathrm{W})$ using "min") [34, page 218]. The corresponding valuation domain is $V_n(\mathrm{W})$ [34, Propositions 18.7 and 33.1]. That R is a Bézout domain with $M_n = \mathrm{Y}_n R$ a maximal ideal, that $R_{M_n} = V_n(\mathrm{W})$ and that M_n has height two for each n follow from results on eab-operations and Kronecker function rings in [34, Chap. 32] (cf. also Halter–Koch [40, Theorem 2.2(2)]).

For each n, let $D_n := K[\mathrm{X}, \{\mathrm{Y}_m \mid 1 \leq m \leq n\}, \{\mathrm{Z}_m^{\alpha} \mid 1 \leq m \leq n, \alpha \in \mathbb{R}^+\}]$ and let $T_n := D_n[\mathrm{W}]$.

Let z be a nonzero element of the quotient field $F(\mathrm{W})$ of R. Then there is a pair of integers s and n such that z can be factored as a product $\mathrm{X}^s(g/f)$ where $g, f \in T_n \backslash \mathrm{X}T_n$. For each $k > n$, z is in $V_k(\mathrm{W})$ if and only if $s \geq 0$. Also, for $k > n$, z is a unit of $V_k(\mathrm{W})$ if and only if $s = 0$. Hence $z \in R$ implies $s \geq 0$, and $z \in \bigcap_n M_n$ implies $s > 0$.

By Theorem 2.5.10, $\bigcap_n M_n$ is the Jacobson radical of R and no maximal ideal other than one of the M_n's is sharp. Also, by Corollary 4.2.9, the only sharp primes that contain $\bigcap_n M_n$ are the M_n's. Since X is in each M_n, each maximal ideal contains X.

Let $t := a/b \in R \backslash M_n$ be a nonunit of R with $a, b \in D[\mathrm{W}]$ and let k be a positive integer greater than the largest power of Z_n that appears in a term of a. Then no "cancellation" can occur in the numerator of $t + \mathrm{Z}_n^k = (a + b\mathrm{Z}_n^k)/b$. In $V_n(\mathrm{W})$, $t + \mathrm{Z}_n$ is a unit since $t \notin M_n$. For $m \neq n$, $\mathrm{v}_m^*(a) \geq \mathrm{v}_m^*(b)$ and $\mathrm{v}_m^*(\mathrm{Z}_n^k) = \mathrm{v}_m(\mathrm{Z}_n^k) = (0,0)$. From the definition of (the original) v_m, $\mathrm{v}_m^*(a + b\mathrm{Z}_n^k) = \mathrm{v}_m^*(b)$ and therefore $\mathrm{v}_m^*(t + \mathrm{Z}_n^k) = (0,0)$. Hence $t + \mathrm{Z}_n^k$ is a unit of R.

Let N be a maximal ideal of R that does not contain some particular X/Z_n. Since $\mathrm{X} \in N$, then N must contain Z_n, and all positive powers of Z_n. From the argument in the previous paragraph, we must have $N = M_n$. This not only shows that M_n is the only maximal ideal that contains Z_n, it also shows that $P_n = \sqrt{\mathrm{Z}_n R}$ is a sharp prime and M_n is the only maximal ideal that contains P_n.

Let J be the ideal of R generated by the set $\{\mathrm{X}/\mathrm{Z}_n \mid n \geq 1\}$. Since each P_n contains the set $\{\mathrm{X}/\mathrm{Z}_m \mid m \neq n\}$, each element of J is in infinitely many P_n's. On the other hand, P_n does not contain X/Z_n. Thus no P_n contains J, nor does any M_n.

Let $E := K[\mathrm{W}, \{\mathrm{Y}_n \mid n \geq 1\}, \{\mathrm{Z}_n^\alpha \mid n \geq 1, \alpha \in \mathbb{R}^+\}]$ and let $h \in E \backslash \{0\}$. Then there is an integer n such that $h \in E_n := K[\mathrm{W}, \{\mathrm{Y}_m \mid 1 \leq m \leq n\}, \{\mathrm{Z}_m^\alpha \mid 1 \leq m \leq n, \alpha \in \mathbb{R}^+\}]$. Let $s_n := \sum_{m=1}^n \mathrm{X}/\mathrm{Z}_m$ and consider the element $h + s_n$. For $k > n$, h is a unit of $V_k(\mathrm{W})$ and s_n is a nonunit, and for $m \leq n$, $(0, 0) = \mathrm{v}_m(s_n) = \mathrm{v}_m^*(s_n) = \mathrm{v}_m^*(h + s_n)$. Thus $h + s_n$ is a unit of R. This implies that if M is a maximal ideal that contains J, then R_M contains L the quotient field of E. It follows that $R_M = L[\mathrm{X}]_{(\mathrm{X})}$ is a discrete rank one valuation domain. We also have that M is the only prime that contains J. Hence M is height one, unsteady and the only other nonzero prime besides the M_n's and P_n's. Thus R has weak factorization by Corollary 4.2.13. By Theorem 2.5.4, if I is a nonzero nondivisorial ideal, then $IR_{M_n} = I^v R_{M_n}$ for each M_n and therefore $I = I^v M^k$ for some positive integer k.

Obviously, each M_n blows up in $R_{\mathscr{S}}$, but each P_n survives. Hence $R_{\mathscr{S}}$ is a one-dimensional Prüfer domain. As each element in M is in all but finitely many P_n's, the extension of M to $R_{\mathscr{S}}$ remains unsteady. Thus $R_{\mathscr{S}} = \bigcap_n R_{P_n}$. Let $B := \bigcap_n P_n R_{\mathscr{S}}$. Since X is in B, $P_n R_{\mathscr{S}}$ is minimal over B. As $P_n R_{P_n}$ is idempotent and $BR_{P_n} = P_n R_{P_n}$, $R_{P_n} \supseteq B(R_{\mathscr{S}} : B)R_{P_n} = P_n(R_{\mathscr{S}} : B)R_{P_n} = P_n R_{P_n}$. Hence $B(R_{\mathscr{S}} : B) = B$. Since $R_{\mathscr{S}} = \bigcap_n R_{P_n}$, $(R_{\mathscr{S}} : B) = R_{\mathscr{S}}$. Clearly, we cannot factor B as B^v $(= R_{\mathscr{S}})$ times a finite product of powers of maximal ideals. Hence $R_{\mathscr{S}}$ does not have weak factorization.

Finally, let $\mathscr{Q}$ be an infinite proper subset of $\{P_n\}_{n=1}^\infty$ and let $H := \bigcap\{P_m \mid P_m \in \mathscr{Q}\}$. As above $\mathrm{X} \in H$. Thus by Corollary 4.2.9(2), $\mathscr{Q}$ is the complete set of sharp primes that are minimal over H. Also $H^{-1} = (H : H)$ as above since $HR_{P_m} = P_m R_{P_m}$ being idempotent implies $HH^{-1}R_{P_m} = HR_{P_m}$ for each $P_m \in \mathscr{Q}$. Hence each $P_m \in \mathscr{Q}$ extends to a (idempotent) maximal ideal of $(H : H)$ (Theorem 4.2.7). But this means that H is contained in infinitely many idempotent maximal ideals of $(H : H)$. Then Corollary 4.2.9(3) shows that $(H : H)$ does not have weak factorization. $\square$

This example also shows that a ring of quotients of a Prüfer domain with weak factorization need not have weak factorization.

Theorem 4.3.5. *Let R be a Prüfer domain with weak factorization. If M is an unsteady maximal ideal, then there is a finitely generated ideal I and an infinite set of steady maximal ideals $\{M_\alpha\}$ each containing I, such that MT is the only unsteady maximal ideal of $T := \bigcap_\alpha R_{M_\alpha}$ and the only other maximal ideals of T are those of the form $M_\alpha T$.*

Proof. Let M be an unsteady maximal ideal and let J be a nonzero finitely generated ideal that is contained in M. Since R has finite unsteady character (Theorem 4.2.10), at most finitely many other unsteady maximal ideals contain J, say $M_1, M_2, \ldots, M_n$. For each i, there is an element $b_i \in M$ such that $b_i R + M_i = R$. Let $I := J + b_1 R + b_2 R + \cdots + b_n R$. Then M is the only unsteady maximal ideal that contains I.

Let $\{M_\alpha\}$ be the set of maximal ideals, other than M, that contain I. This set is infinite since M is not sharp. On the other hand, each M_α is steady and therefore sharp. By Lemma 2.5.1, the maximal ideals of $\Gamma(I)$ are the ideals of the form $M_\alpha\Gamma(I)$ and $M\Gamma(I)$. Since at most finitely many of the M_α are idempotent, $\Gamma(I)$ has weak factorization (Proposition 4.2.14). It also has infinitely many sharp maximal ideals and each maximal ideal contains the nonzero finitely generated ideal $I\Gamma(I)$. Hence $M\Gamma(I)$ must be the unique unsteady maximal ideal of $\Gamma(I)$. Moreover, $\Gamma(I) = \bigcap_\alpha R_{M_\alpha}$. □

Recall that for a nonzero ideal I of a domain R, $\mathrm{Min}(R, I)$ denotes the set of minimal primes of I (in R) and $\Phi(I) = \bigcap\{R_{P_\alpha} \mid P_\alpha \in \mathrm{Min}(R, I)\}$.

Lemma 4.3.6. *Let R be a Prüfer domain with weak factorization and let I be a nonzero ideal of R. Then $Max(\Phi(I)) = \{P\Phi(I) \mid P \in Min(R, I)\}$.*

Proof. Since R has weak factorization, the only nonzero primes that are not sharp are the unsteady maximal ideals, each of which has height one. If M is a maximal ideal that is minimal over I, then $M\Phi(I)$ is a maximal ideal of $\Phi(I)$. Let $P \subsetneq Q$ be primes with $P \in \mathrm{Min}(R, I)$. Since P is sharp, there is a finitely generated ideal $J \subseteq Q$ such that $P \subsetneq J \subseteq Q$ (Proposition 2.3.20). Obviously, no other minimal prime of I contains J. Hence R_Q does not contain $\Phi(I)$. It follows that $P\Phi(I)$ is a maximal ideal of $\Phi(I)$. □

Lemma 4.3.7. *Let R be a Prüfer domain with weak factorization and let I be a nonzero ideal that is not contained in the Jacobson radical of R. If N is a maximal ideal of $\Theta(I)$, then either $N \bigcap R$ is a maximal ideal of R that does not contain I or $N \bigcap R$ is an unsteady maximal ideal of R.*

Proof. Let P be a nonzero prime of R that is neither comaximal with I nor an unsteady maximal ideal of R. Since R has weak factorization, P is contained in a unique maximal ideal M and it is sharp (Theorem 4.2.4). Hence there is a (nonzero) finitely generated ideal $J \subseteq P$ such that M is the only maximal ideal that contains J. It follows that R_P does not contain $\Theta(I)$ $(= \bigcap\{R_Q \mid Q \in \mathrm{Max}(R)\backslash\mathrm{Max}(R, I)\})$. Since R is a Prüfer domain, $P\Theta(I) = \Theta(I)$. As each prime of $\Theta(I)$ is extended from a prime of R, if N is a maximal ideal of $\Theta(I)$, then either $N \bigcap R$ is a maximal ideal of R that does not contain I or it is an unsteady maximal ideal. □

Theorem 4.3.8. *Let R be a Prüfer domain with weak factorization, and let I be a nonzero ideal of R.*

(1) Both $\Gamma(I)$ and $\Theta(I)$ have weak factorization.
(2) $\Phi(I)$ has weak factorization if and only if at most finitely many minimal primes of I are idempotent.

Proof. By Lemma 2.5.1, $\mathrm{Max}(\Gamma(I)) = \{M\Gamma(I) \mid M \in \mathrm{Max}(R, I)\}$. As no nonzero ideal is contained in infinitely many idempotent maximal ideals (Corollary 4.2.9), the same occurs in $\Gamma(I)$. Hence $\Gamma(I)$ has finite idempotent character. Thus $\Gamma(I)$ has weak factorization by Corollary 4.3.2.

From Lemma 4.3.7, each maximal ideal of $\Theta(I)$ is extended from a maximal ideal of R. Hence $\Theta(I)$ also has finite idempotent character. Another application of Corollary 4.3.2 yields that $\Theta(I)$ has weak factorization.

For $\Phi(I)$, $\mathrm{Max}(\Phi(I)) = \{P\Phi(I) \mid P \in \mathrm{Min}(R, I)\}$ by Lemma 4.3.6. Thus $\Phi(I)$ has finite unsteady character if and only if at most finitely many minimal primes of I are idempotent. It follows that $\Phi(I)$ has weak factorization if and only if at most finitely many minimal primes of I are idempotent. $\square$

Theorem 4.3.9. *If R is a Prüfer domain with weak factorization, then each overring has weak factorization if and only if there is no nonzero ideal with infinitely many idempotent minimal primes.*

Proof. From the previous theorem, if $I \neq (0)$ has infinitely many idempotent minimal primes, then $\Phi(I)$ does not have weak factorization. Conversely, suppose $T \supsetneq R$ is an overring that does not have weak factorization. Then by Corollary 4.3.2, there is a nonzero ideal B of T that is contained in infinitely many idempotent maximal ideals of T. It follows that the ring $\Gamma_T(B)$ has infinitely many idempotent maximal ideals, each of which contains B. Thus we may assume B is contained in each maximal ideal of T. Let M be a maximal ideal of T. Then $M = PR$ for some prime P of R. If M is idempotent, then P is a sharp prime of R. It follows that M is a sharp prime of T. By Corollary 2.5.7, M is minimal over the Jacobson radical of T. Hence the Jacobson radical has infinitely many idempotent minimal primes. $\square$

4.4 Finite Divisorial Closure

If R is an h-local Prüfer domain and I is a nonzero nondivisorial ideal, then there is a finitely generated ideal $J \subseteq I^v$ such that $I + J = I^v$ [19, Proposition 2.10]. In the next lemma, we generalize this result by showing that a Prüfer domain with weak factorization has the same property.

Lemma 4.4.1. *Let R be a Prüfer domain with the weak factorization property. If I is a nonzero ideal of R, then there is a finitely generated ideal J such that $I + J = I^v$.*

Proof. If I is divisorial, there is nothing to prove. Hence we assume that I is not divisorial with factorization $I = I^v \prod_i N_i \prod_j M_j^{r_j}$. We may further assume that each N_i is steady with $I^v N_i \neq I^v$ and each M_j is unsteady with $r_j > 0$. At least one of the (finite) sets $\{N_i\}$ or $\{M_j\}$ is nonempty.

Since $M_j R_{M_j}$ is principal and has height one by Lemma 4.2.3, there is an element $a_j \in I^v$ such that $a_j R_{M_j} = I^v R_{M_j}$. Now, consider an N_i. Since it is steady, if it is locally principal, then it is invertible. But, in that case, $I^v N_i$ is a divisorial ideal that contains I and is properly contained in I^v, which is impossible. Thus it must be that N_i is idempotent with $I \subseteq I^v N_i \subsetneq I^v$. Hence $I^v R_{N_i}$ is principal by Lemma 2.5.3, and we may choose $b_i \in I^v$ such that $I^v R_{N_i} = b_i R_{N_i}$. Now, set $J := (a_1, \ldots, a_m, b_1, \ldots, b_n)$. To see that $I^v = I + J$ simply check locally.

By construction, we have $I + J \subseteq I^v$, $I^v R_{M_j} = a_j R_{M_j} \subseteq (I + J)R_{M_j}$ for each M_j, $I^v R_{N_i} = b_i R_{N_i} \subseteq (I + J)R_{N_i}$ for each N_i, and $I^v R_N = IR_N \subseteq (I + J)R_N$ for all other maximal ideals N (if any). Hence $I^v = I + J$. □

We say that R has the *finite divisorial closure property* if, for each nondivisorial ideal $I \neq (0)$, there is a finitely generated ideal J such that $I^v = I + J$. A Prüfer domain with the finite divisorial closure property need not have weak factorization. For example, in the ring of entire functions the only divisorial ideals are the principal ones, so that this Prüfer domain has the finite divisorial closure property trivially. However, the primes reverse roles from what occurs with weak factorization—the only sharp primes are the height one invertible maximal ideals, all other primes have infinite height, and none of these is sharp, but each is contained in a unique maximal ideal. (For the properties of the ring of entire functions mentioned above see, for example, [47], [34, Pages 146–148 and Exercise 19, page 256] and [24, Sect. 8.1].)

In Theorem 4.4.7, we combine the finite divisorial closure property with another to obtain yet another characterization for Prüfer domains with weak factorization. In Theorem 4.4.8, we do the same for h-local Prüfer domains.

Lemma 4.4.2. *Let R be a Prüfer domain and let I be a nonzero nondivisorial ideal. If there is a finitely generated ideal J such that $I + J = I^v$, then for each maximal ideal M containing I with $IR_M \neq I^v R_M$, $I^v R_M$ is principal.*

Proof. Assume that $I^v = I + J$ for some finitely generated ideal J, and let M be a maximal ideal such that $IR_M \neq I^v R_M$. Since R_M is a valuation domain, $I^v R_M = JR_M$ is principal. □

Theorem 4.4.3. *Let R be a Prüfer domain with the finite divisorial closure property.*

(1) If P is a nonzero nondivisorial prime, then $P^v = R$.
(2) If P is a nonzero divisorial prime, then P is sharp and contained in a unique maximal ideal.

Proof. Let P be a nonzero prime ideal of R. We may assume that P is not maximal since both parts of the theorem hold trivially if P is maximal (Lemma 2.1.1(1) and Remark 2.1.2(1)).

For (1), we assume that P is not divisorial. In this case, there is a finitely generated ideal A such that $P^v = P + A \supsetneq P$. Then $P^{-1} = (P : P)$ (Theorem 2.3.2(2)), and, since R is integrally closed, $P^{-1} = (P^v)^{-1} = (\sqrt{P^v})^{-1}$ by Remark 2.3.3. Thus $(\sqrt{P^v})P^{-1} = (\sqrt{P^v})(\sqrt{P^v})^{-1} \subseteq R$, whence $(\sqrt{P^v}) \subseteq P^v$, and we have that P^v is a radical ideal. Now, by way of contradiction, suppose that $P^v \neq R$. If M is a maximal ideal that contains P^v, then $P^v R_M$ is a prime ideal of the valuation domain R_M that properly contains PR_M. Hence $P^v R_M = (P + A)R_M = AR_M$ is a principal prime ideal, and we must have $P^v R_M = MR_M$. It follows that each minimal prime of P^v is a maximal ideal of R and therefore $\Gamma(P^v) = \Phi(P^v)$. But in this case, $P^{-1} = (P^v)^{-1} = \Gamma(P^v) \bigcap \Theta(P^v) = R$ (Theorem 2.3.2(1, b)), whence $P^v = R$, the desired contradiction.

For (2), first assume P is both divisorial and branched. Then there is a proper P-primary ideal Q. If P is not sharp, then $\Theta(P) = (P : P) \subseteq P^{-1} \subseteq Q^{-1} = \Theta(P)$ (Corollary 2.3.18). Hence $Q^v = P^v = P$. Thus we have a finitely generated ideal B such that $P = Q^v = Q + B$. As $Q \neq Q^v$, there is a maximal ideal M that (properly) contains P with $QR_M \subsetneq PR_M = Q^v R_M = BR_M$, which is impossible since P is not maximal. Thus it must be that P is sharp.

Continuing with the assumption that P is both divisorial and branched, and now sharp as well, let A be a finitely generated ideal with radical P (Proposition 2.3.10). Also, let M be a maximal ideal that contains P and let $C := AR_M \bigcap R$. Then by Theorem 2.5.2, $C^v = A(P' : P')$, where P' is the largest prime common to all maximal ideals that contain P. If M is not the only maximal ideal that contains P, then C is not divisorial (Theorem 2.5.2 again), and so there is a finitely generated ideal $J \nsubseteq C$ such that $C^v = C + J$. We may assume that $A \subseteq J$, which implies that $CR_M = AR_M \subseteq JR_M$. Hence $JR_M = C^v R_M = A(P' : P')R_M = AR_{P'}$, the latter equality following from the fact that P' is a maximal ideal of $(P' : P') = \Theta(P) \bigcap R_{P'}$ (Lemma 2.3.9 and Theorem 2.3.2(2)(b)). But, since P' is properly contained in M, $AR_{P'}$ cannot be an invertible ideal of R_M. Thus M must be the only maximal ideal that contains P.

The only case left is when P is a (nonmaximal) prime ideal that is both divisorial and unbranched. In this case, P contains a nonzero branched prime P_0 which, from (1), cannot be nondivisorial. Thus P_0 is divisorial and branched, and therefore by the above, it is sharp and contained in a unique maximal ideal. It follows that this same maximal ideal is the only one that contains P. Hence P is sharp by Lemma 2.3.9.

□

Corollary 4.4.4. *Let R be a Prüfer domain. If R has the finite divisorial closure property, then the following statements are equivalent.*

(i) Each nonzero nonmaximal prime is sharp and contained in a unique maximal ideal, and each maximal ideal of height greater than one is sharp.
(ii) Each nonzero nonmaximal branched prime is sharp (i.e., R is a wTPP-domain by Theorem 2.4.17).
(iii) Each nonzero nonmaximal branched prime is divisorial.

Proof. Obviously, (i) implies (ii). Also, (ii) implies (iii) since a nonmaximal sharp prime in a Prüfer domain must be divisorial (Corollary 2.3.21).

Assume that R has the finite divisorial closure property. If each nonzero nonmaximal branched prime is divisorial, then each is sharp and contained in a unique maximal ideal by Theorem 4.4.3(2). A maximal ideal of height greater than one contains a nonzero nonmaximal branched prime as does an unbranched (nonzero) prime. But such a branched prime is contained in a unique maximal ideal. Thus each maximal ideal of height greater than one is sharp as is each unbranched prime (Lemma 2.3.9). □

From Theorem 2.4.18 and the previous corollary, we immediately deduce the following.

Corollary 4.4.5. *Let R be a Prüfer domain. If R is an aRTP-domain and has the finite divisorial closure property, then each nonzero prime is contained in a unique maximal ideal.*

We do not know whether the conclusion of Corollary 4.4.5 can be strengthened to "R has the weak factorization property" or not.

Lemma 4.4.6. *Let I be a nonzero nondivisorial ideal in a Prüfer domain R. If there is a finitely generated ideal $J \subseteq I^v$ and a finite set of maximal ideals $\{M_1, M_2, \ldots, M_n\}$ such that $I^v = I + J$ and $J \prod M_i^{r_i} \subseteq I$ for some positive integers $r_1, r_2, \ldots, r_n$, then $I = I^v \prod M_i^{s_i}$ for some nonnegative integers $s_1, s_2, \ldots, s_n$ with $s_i \leq r_i$ for each i.*

Proof. Assume that there is a finitely generated ideal $J \subseteq I^v$ and a finite set of maximal ideals $\{M_1, M_2, \ldots, M_n\}$ such that $I^v = I + J$ and $J \prod M_i^{r_i} \subseteq I$ for some positive integers $r_1, r_2, \ldots, r_n$. For a maximal ideal M outside the set $\{M_1, M_2, \ldots, M_n\}$, we have $JR_M \subseteq IR_M \subseteq I^v R_M = (I + J)R_M$. Thus $IR_M = I^v R_M$.

We may divide the set $\{M_1, M_2, \ldots, M_n\}$ into three disjoint subsets: $\mathscr{A}_1(I) := \{M_i \mid IR_{M_i} = I^v R_{M_i}\}$, $\mathscr{A}_2(I) := \{M_i \mid IR_{M_i} \neq I^v R_{M_i}$ and M_i is idempotent$\}$ and $\mathscr{A}_3(I) := \{M_i \mid IR_{M_i} \neq I^v R_{M_i}$ and M_i is locally principal$\}$. While $\mathscr{A}_1(I)$ may be empty, at least one of $\mathscr{A}_2(I)$ and $\mathscr{A}_3(I)$ must be nonempty since I is not divisorial. Note that $\prod M_i^{r_i} R_{M_j} = M_j^{r_j} R_{M_j}$ for each j. Hence we have $JM_j^{r_j} R_{M_j} \subseteq IR_{M_j}$. Also, since each R_{M_j} is a valuation domain, $IR_{M_j} \subsetneq I^v R_{M_j} = JR_{M_j}$ for each $M_j \in \mathscr{A}_2(I) \bigcup \mathscr{A}_3(I)$.

The maximal ideals in $\mathscr{A}_1(I)$ and $\mathscr{A}_2(I)$ are quite easy to deal with. For $M_i \in \mathscr{A}_1(I)$, we set $s_i = 0$. For $M_j \in \mathscr{A}_2(I)$, $M_j^r = M_j$ for each positive integer r. Also, there can be no ideals properly between $JM_j R_{M_j} = I^v M_j R_{M_j}$ and $JR_{M_j} = I^v R_{M_j}$ for $M_j \in \mathscr{A}_2(I)$. Since $I^v M_j R_{M_j} = JM_j R_{M_j} \subseteq IR_{M_j} \subsetneq I^v R_{M_j}$, we have $IR_{M_j} = I^v M_j R_{M_j}$ and we may set $s_j = 1$.

Finally, for $M_k \in \mathscr{A}_3(I)$, the only ideals between $JM_k^{r_k} R_{M_k} = I^v M_k^{r_k} R_{M_k}$ and $JR_{M_k} = I^v R_{M_k}$ are the ideals of the form $JM_k^s R_{M_k} = I^v M_k^s R_{M_k}$ for each nonnegative integer $s \leq r_k$. As observed above, one such ideal is IR_{M_k}. Thus $IR_{M_k} = I^v M_k^{s_k} R_{M_k}$ for some positive integer $s_k \leq r_k$. Checking locally at each maximal ideal now yields $I = I^v \prod M_i^{s_i}$ for the nonnegative integers $s_1, s_2, \ldots, s_n$ (in each case with $s_i \leq r_i$). □

Theorem 4.4.7. *Let R be a Prüfer domain. Then R has weak factorization if and only if for each nonzero nondivisorial ideal I, there is an invertible ideal $J \subseteq I^v$, a finite set of maximal ideals $\{M_1, M_2, \ldots, M_n\}$ and positive integers $r_1, r_2, \ldots, r_n$ such that $I^v = I + J$ and $J \prod M_i^{r_i} \subseteq I$.*

Proof. If I is a nondivisorial ideal and R has weak factorization, then there is a finite set of maximal ideals $\{M_1, M_2, \ldots, M_n\}$ and positive integers $r_1, r_2, \ldots, r_n$ such that $I = I^v \prod M_i^{r_i}$. By Lemma 4.4.1, there is an invertible ideal $J \subseteq I^v$ such that $I^v = I + J$. Obviously, we also have $J \prod M_i^{r_i} \subseteq I$.

For the converse, simply apply Lemma 4.4.6. □

As we recalled in Theorem 4.2.1, the equivalence of (1) and (6) in the following theorem originally appeared in [19] as part of Theorem 1.13. Using Theorem 4.2.4, we will give an alternate proof that (6) implies ((5) implies) (1).

Theorem 4.4.8. *The following statements are equivalent for a Prüfer domain R.*

(i) R is h-local.
(ii) For each nonzero nondivisorial ideal I, there is a finite set of distinct maximal ideals $\{M_1, M_2, \ldots, M_k\}$ such that $I = I^v M_1 M_2 \cdots M_k$.
(iii) For each nonzero nondivisorial ideal I, there is a finite set of incomparable primes $\{Q_1, Q_2, \ldots, Q_m\}$ such that $I = I^v Q_1 Q_2 \cdots Q_m$.
(iv) For each nonzero nondivisorial ideal I, there is a finite set of distinct prime ideals $\{P_1, P_2, \ldots, P_n\}$ such that $I = I^v P_1 P_2 \cdots P_n$.
(v) R has the weak factorization property and no unsteady maximal ideals.
(vi) R has both the weak factorization property and finite character.
(vii) R has the weak factorization property, and each maximal ideal of R is sharp.
(viii) R has the weak factorization property, and each nonzero nonunit is contained in only finitely many invertible maximal ideals.
(ix) R has both finite character and the finite divisorial closure property.
(x) R has both the radical trace property and the finite divisorial closure property.
(xi) For each nonzero nondivisorial ideal I of R, there is a finite nonempty set of maximal ideals $\{M_1, M_2, \ldots, M_n\}$ and a finitely generated ideal $J \subseteq I^v$ such that $I^v = I + J$ and $J \prod M_i \subseteq I$.

Proof. We establish the following sets of implications: (vi)⇔ (viii); (iii)⇔ (iv); (i)⇒ (xi)⇒ (ii)⇒ (iii)⇒ (v)⇒ (i); (i)⇒ (ix)⇒ (x)⇒ (i); and (i)⇒ (vi)⇒ (vii)⇒ (v). We note that (i) ⇒ (ii) is clear from Theorem 4.1.2.

It is clear that (vi) implies (viii). Also, we have that (viii) implies (vi) since weak factorization implies that each nonzero nonunit of R is contained in only finitely many noninvertible maximal ideals (Theorem 4.2.12). Several of the other implications are also easy to deal with. The equivalence of (iii) and (iv) follows from the fact that, checking locally, $QP = Q$ if $Q \subsetneq P$ are primes (in a Prüfer domain). Thus a factorization as in (iv) can simply be reduced to one with incomparable primes.

Next, we establish the series of implications (i)⇒ (xi)⇒ (ii)⇒ (iii)⇒ (v)⇒ (i). It is clear that (ii) implies (iii). For (xi) implies (ii), just apply Lemma 4.4.6. To see that (iii) implies (v), assume that (iii) holds. Then R has weak factorization by Theorem 4.2.12 ((vi)⇒ (i)). In order to prove (v), it remains to show that R has no unsteady maximal ideals. For this, suppose that M is a maximal ideal that is locally principal. Then $M^2 \neq M$. If M is not invertible, then $(R : M^2) = ((R : M) : M) = (R : M) = R$ (Remark 2.1.2(1)), in which case by (iii) we must have $M^2 = (M^2)^v M = R \cdot M = M$, a contradiction. Thus we must have M invertible. Hence (iii) implies (v).

Next, we show that (v) implies (i). Hence we assume that R has weak factorization and no unsteady maximal ideals. By Theorem 4.2.12, each nonzero prime is contained in a unique maximal ideal, and, since there are no unsteady maximal ideals, each nonzero (branched) prime ideal is sharp. Thus R is an RTP-domain by Theorem 2.4.10. Therefore R is h-local by Theorem 2.4.12.

To complete the sequence of implications (i)$\Rightarrow$(xi)$\Rightarrow$ (ii)$\Rightarrow$ (iii)$\Rightarrow$ (v)$\Rightarrow$ (i), we show (i)$\Rightarrow$ (xi). Assume that R is h-local, and let I be a nondivisorial ideal of R. By Lemma 4.4.1 and the implications (i)$\Rightarrow$ (ii)$\Rightarrow$ (iii)$\Rightarrow$ (v) (or, directly, by [19, Proposition 2.10]), there is a finitely generated ideal J such that $I^v = I + J$. Since R also has very strong factorization (Theorem 4.1.2), there is a finite set of maximal ideals $\{M_1, M_2, \ldots, M_n\}$ with $I = I^v \prod M_i = (I + J) \prod M_i$. Of course, this yields $J \prod M_i \subseteq I$, and we have that (i) implies (xi), completing the set. This also gives (i) implies (ix) since an h-local domain has finite character by definition.

To complete the set (i)$\Rightarrow$ (ix)$\Rightarrow$ (x)$\Rightarrow$ (i), we need (ix)$\Rightarrow$ (x) and (x)$\Rightarrow$ (i). If R has finite character, then it has the radical trace property by Theorem 2.4.11(2) and Lemma 2.3.4(2). Hence (ix) implies (x). Since a RTP-domain is clearly an aRTP-domain, (x) implies (i) by Corollary 4.4.5 and Theorem 2.4.12.

Finally, we consider the set (i)$\Rightarrow$ (vi)$\Rightarrow$ (vii)$\Rightarrow$ (v). By definition, if R is h-local, then it has finite character. Also it has very strong, hence weak, factorization by Theorem 4.1.2. Thus we have (i) implies (vi). An unsteady maximal ideal is not sharp. Thus (vii) implies (v).

Assume (vi). If M is a maximal ideal of R, then using finite character (and prime avoidance), it is easy to produce a finitely generated ideal I such that $\sqrt{I} = M$. Hence M is sharp (Proposition 2.3.10). Also, recall that a sharp maximal ideal must be steady by Lemma 2.4.19. Therefore (vi) implies (vii). □

By Theorem 4.1.2((i)$\Leftrightarrow$(ii)), the equivalence of (i) and (ii) in Theorem 4.4.8 means that, *for Prüfer domains, very strong factorization and strong factorization are equivalent*. Without the Prüfer assumption, we still have the following general result.

Theorem 4.4.9. *Let R be an integral domain. If R has strong factorization, then each nonzero finitely generated ideal of R is divisorial (equivalently, each nonzero ideal is a t-ideal).*

Proof. Suppose that R has strong factorization. We first observe, by checking locally, that if A is a nonzero ideal of R with factorization $A = A^v N_1 N_2 \cdots N_k$ with, say, $AR_{N_1} = A^v R_{N_1}$, then N_1 can be omitted from the factorization. Now, by way of contradiction, assume that I is a nonzero finitely generated ideal of R which is not divisorial and write $I = I^v \prod M_i$ for distinct maximal ideals $M_1, M_2, \ldots, M_n$ (and $n \geq 1$). We may assume no M_i can be omitted. Let $Q := \prod M_i$ and consider the ideal IQ. Calculating the divisorial closure, we have $(IQ)^v = (I^v Q)^v = I^v$. If IQ is divisorial, then $I^v = IQ \subseteq I$ and so I is also divisorial, a contradiction. Thus IQ is not divisorial and must therefore have a factorization as I^v times a finite product of distinct maximal ideals. Clearly, $IQR_M = IR_M = I^v R_M$ for each maximal ideal M outside the set $\{M_1, M_2, \ldots, M_n\}$. Hence the only possible way to factor IQ is as $IQ = I^v Q_1$, where Q_1 is a product of a subset of the M_i. However, this yields $I \subseteq IQ$, which contradicts Nakayama's Lemma. Hence each nonzero finitely generated ideal of R is divisorial. The parenthetical statement follows easily from the definition of the t-operation. $\square$

One of many characterizations of Prüfer domains is that an integrally closed domain is Prüfer if and only if each nonzero finitely generated ideal is divisorial (see, for example, [34, Proposition 34.12]). Also, a celebrated result of Heinzer states that in an integrally closed domain R, each nonzero ideal is divisorial if and only if R is Prüfer h-local with all maximal ideals invertible (Theorem 2.1.6). The next result shows that, in the class of integrally closed integral domains, Prüfer domains with strong factorization lie in between the domains in which all nonzero ideals are divisorial and those in which all nonzero finitely generated ideals are divisorial.

Corollary 4.4.10. *Let R be an integrally closed domain. Then R has strong factorization if and only if R is an h-local Prüfer domain.*

Proof. Simply apply [34, Proposition 34.12] and Theorems 4.4.9 and 4.4.8. $\square$

We close this section by showing that, in the integrally closed case, a domain with weak factorization must also be Prüfer. We need a preliminary result.

Lemma 4.4.11. *Let R be a domain with weak factorization. Then R_M is a valuation domain for each nondivisorial maximal ideal M of R.*

For a nonzero fractional ideal J of R, there is a nonzero element $r \in R$ such that rJ is an (integral) ideal of R. Moreover, $(rJ)^v = rJ^v$ so that J is divisorial if and only if rJ is divisorial. Hence we may easily extend weak (and strong) factorization to fractional ideals.

Proof. Let M be a nondivisorial maximal ideal, and assume $x \in K \setminus R_M$ where K is the quotient field of R. We shall show that $x^{-1} \in R_M$. Consider the fractional ideal $M + Rx$. Since M is nondivisorial, we have $(M + Rx)^v = (M^v + Rx)^v = (R + Rx)^v$. If $M + Rx$ is divisorial, this yields $1 \in (R + Rx)^v = M + Rx$, and we can write $1 = m + rx$ with $m \in M, r \in R$. In this case, we have $x^{-1} = r(1-m)^{-1} \in R_M$, as desired. Hence we assume that $M + Rx$ is not divisorial, in which case we have a factorization $M+Rx = (M+Rx)^v Q = (R+Rx)^v Q$, where Q is a product of (not necessarily distinct) maximal ideals. Suppose that $R + Rx$ is divisorial. Then $M + Rx = (R + Rx)Q$. If $Q \subseteq M$, then $M + Rx \subseteq M + Mx$, and we can write $x = a + bx$ with $a, b \in M$, whence $x = a(1-b)^{-1} \in R_M$, a contradiction. On the other hand, if $Q \not\subseteq M$, we have $MR_M + R_M x = R_M + R_M x$, which yields $1 \in MR_M + R_M x$, and we obtain $x^{-1} \in R_M$, as above.

It remains to consider the case where $R+Rx$ is not divisorial (and $M+Rx$ is also not divisorial). Recall that we have $M+Rx = (R+Rx)^v Q$. Since (we are assuming that) $R + Rx$ is not divisorial, we have a factorization $R + Rx = (R + Rx)^v Q'$, where Q' is a product of maximal ideals. Let $I := (R + Rx)^v$. Locally, we have $MR_M + R_M x = IR_M(MR_M)^i$ and $R_M + R_M x = IR_M(MR_M)^j$, for some nonnegative integers i, j. If $i \leq j$, then $MR_M + R_M x \supseteq R_M + R_M x$, and we obtain

$x^{-1} \in R_M$ as above. If $i > j$, then $MR_M + R_M x = IR_M(MR_M)^j(MR_M)^{i-j} = (R_M + MR_M x)(MR_M)^{i-j} \subseteq MR_M + MR_M x$, and, as before, we obtain the contradiction $x \in R_M$. This completes the proof. □

Theorem 4.4.12. *If R is an integrally closed domain with weak factorization, then R is a Prüfer domain.*

Proof. As in Theorem 4.4.9, we show that each nonzero finitely generated ideal is divisorial. Suppose, on the contrary, that I is a nonzero finitely generated ideal which is not divisorial. Then we may write $I = I^v Q$ with Q a product of (not necessarily distinct) maximal ideals. We still have $(IQ)^v = (I^v Q)^v = I^v$ as in the proof of Theorem 4.4.9. This yields $Q^{-1}I^v = Q^{-1}(QI)^v \subseteq (Q^{-1}QI)^v \subseteq I^v$. Hence $Q^{-1}I = Q^{-1}I^v Q \subseteq I^v Q = I$. Since I is finitely generated and R is integrally closed, then $R \subseteq Q^{-1} \subseteq (I : I) = R$ [34, Proposition 34.7], thus we must have $Q^{-1} = R$. It follows that each factor M of Q must be a nondivisorial maximal ideal. By Lemma 4.4.11, R_M is a valuation domain for each such M. Hence IR_M is a principal ideal of R_M. Thus $I^v R_M = (I^{-1})^{-1} R_M \subseteq (I^{-1}R_M)^{-1} = (IR_M)^v = IR_M$, and we have $IR_M = I^v R_M$. Since this equality obviously holds for any maximal ideal which is not a factor of Q, we obtain the contradiction that $I = I^v$. □

Springer

http://www.springer.com/978-3-642-31711-8

Factoring Ideals in Integral Domains
Fontana, M.; Houston, E.; Lucas, Th.
2013, VIII, 164 p., Softcover
ISBN: 978-3-642-31711-8