

## Chapter 2

# Motivation for the Finite Element Method

**Abstract** The approach to the finite element method can be derived from different motivations. Essentially there are three ways:

- a rather descriptive way, which has its roots in the engineering working method,
- a physical or
- mathematically motivated approach.

Depending on the perspective, different formulations result, which however all result in a common principal equation of the finite element method. The different formulations will be elaborated in detail based on the following descriptions:

- matrix methods,
- physically based working and energy methods and
- weighted residual method.

The finite element method is used to solve different physical problems. Here solely finite element formulations related to structural mechanics are considered [1, 5–7, 9–12].

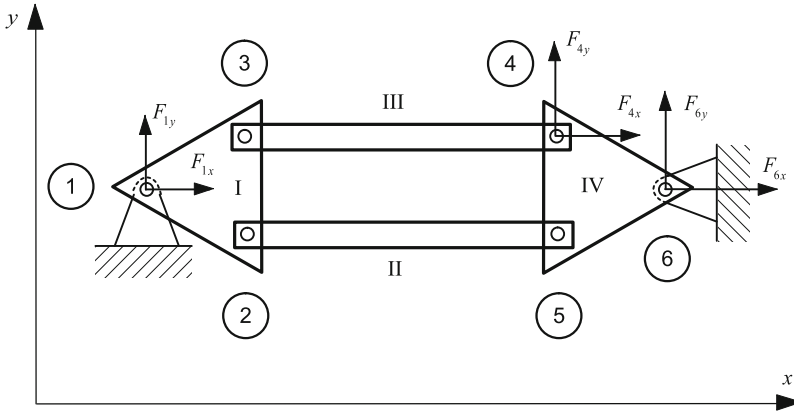
## 2.1 From the Engineering Perspective Derived Methods

Matrix methods can be regarded in elastostatics as the initial point for the application of the finite element method to analyze complex structures [2, 3]. As example a plane structure can be given (see Fig. 2.1). This example is adapted from [8].

The structure consists of various substructures I, II, III and IV. The substructures are referred to as elements. The elements are coupled at the nodes 2, 3, 4 and 5. The entire structure is supported on nodes 1 and 6, an external load affects node 4.

Unknown are

- the displacement and reaction forces on every single inner node and



**Fig. 2.1** Plane structure, adapted from [8]

- the support reactions

in consequence of the acting load.

To solve the problem, matrix methods can be used. In the matrix methods one distinguishes between the force method (static methods), which is based on a direct determination of the statically indeterminate forces, and the displacement method (kinematic method), which considers the displacements as unknown parameters.

Both methods allow the determination of the unknown parameters. The decisive advantage of the displacement method is that during the application it is not necessary to distinguish between statically determined and statically indeterminate structures. Due to the generality this method is applied in the following.

### 2.1.1 The Matrix Stiffness Method

It is the primary subgoal to establish the stiffness relation for the entire structure from Fig. 2.1. The following stiffness relation serves as the basis for the matrix displacement method:

$$\mathbf{F} = \mathbf{K} \mathbf{u} . \quad (2.1)$$

$\mathbf{F}$  and  $\mathbf{u}$  are column matrices,  $\mathbf{K}$  is a square matrix.  $\mathbf{F}$  summarizes all nodal forces and  $\mathbf{u}$  summarizes all nodal displacements. The matrix  $\mathbf{K}$  represents the stiffness matrix of the entire structure. One single element is identified as the basic unit for the problem and is characterized by the fact that it is coupled with other elements via nodes. Displacements and forces are introduced at every single node.

To solve the entire problem

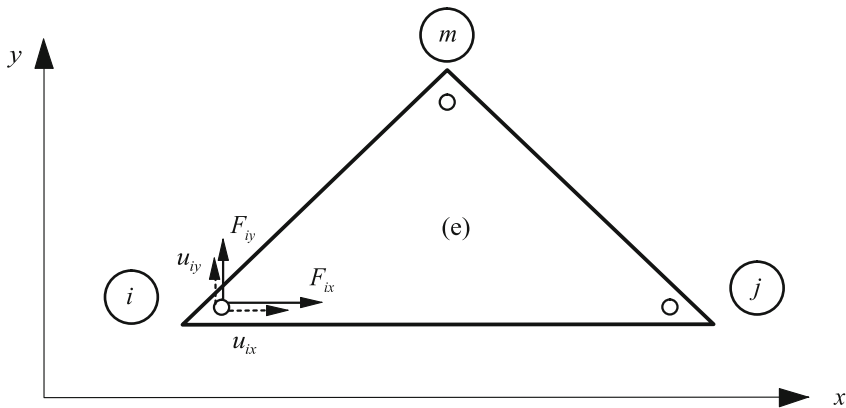
- the compatibility and
- the equilibrium

have to be fulfilled.

In the matrix displacement method one introduces the nodal displacements as essential unknowns. The displacement vector at a node is defined to be valid for all elements connected at this node. Therewith the compatibility of the entire structure *a priori* is fulfilled.

### A Single Element

Forces and displacements are introduced for each node of the single element (see Fig. 2.2).



**Fig. 2.2** Single element (e) with displacements and forces

For an entirely obvious representation the nodal forces and the nodal displacements are provided with an index 'p' to highlight that these are parameters, which are defined on nodes. The vectors of the nodal displacements  $\mathbf{u}_p$  or alternatively nodal forces  $\mathbf{F}_p$  in general consist of various components for the respective coordinates. An additional index 'e' indicates to which element the parameters relate.<sup>1</sup> Therewith the nodal forces result, according to Fig. 2.2, in

<sup>1</sup> The additional index 'e' is to be dropped at displacements since the nodal displacement is identical for each linked element in the displacement method.

$$\mathbf{F}_i^e = \begin{bmatrix} F_{ix} \\ F_{iy} \end{bmatrix}, \quad \mathbf{F}_j^e = \begin{bmatrix} F_{jx} \\ F_{jy} \end{bmatrix}, \quad \mathbf{F}_m^e = \begin{bmatrix} F_{mx} \\ F_{my} \end{bmatrix}, \quad (2.2)$$

and the nodal displacements in

$$\mathbf{u}_i = \begin{bmatrix} u_{ix} \\ u_{iy} \end{bmatrix}, \quad \mathbf{u}_j = \begin{bmatrix} u_{jx} \\ u_{jy} \end{bmatrix}, \quad \mathbf{u}_m = \begin{bmatrix} u_{mx} \\ u_{my} \end{bmatrix}. \quad (2.3)$$

If one summarizes all nodal forces and nodal displacements at one element, the

$$\text{entire node force vector } \mathbf{F}_p^e = \begin{bmatrix} \mathbf{F}_i \\ \mathbf{F}_j \\ \mathbf{F}_m \end{bmatrix} \quad (2.4)$$

as well as the

$$\text{entire node displacement vector } \mathbf{u}_p = \begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_j \\ \mathbf{u}_m \end{bmatrix} \quad (2.5)$$

for a single element is described. With the vectors for the nodal forces and displacements the stiffness relation for a single element can be defined as follows:

$$\mathbf{F}_p^e = \mathbf{k}^e \mathbf{u}_p, \quad (2.6)$$

or alternatively for each node:

$$\mathbf{F}_r^e = \mathbf{k}_{rs}^e \mathbf{u}_s \quad (r, s = i, j, m). \quad (2.7)$$

The single stiffness matrix  $\mathbf{k}^e$  connects the nodal forces. In the present example the single stiffness relation is formally defined as

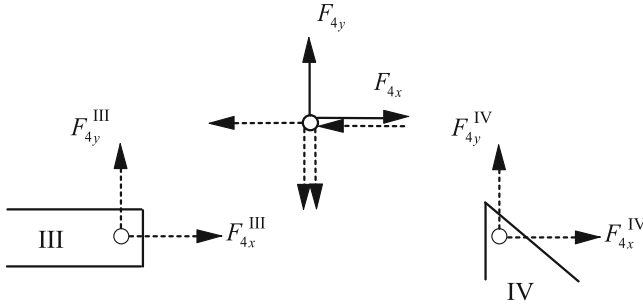
$$\begin{bmatrix} F_{ix} \\ F_{iy} \\ \hline F_{jx} \\ F_{jy} \\ \hline F_{mx} \\ F_{my} \end{bmatrix} = \begin{bmatrix} \mathbf{k}_{ii}^e & \mathbf{k}_{ij}^e & \mathbf{k}_{im}^e \\ \hline \mathbf{k}_{ji}^e & \mathbf{k}_{jj}^e & \mathbf{k}_{jm}^e \\ \hline \mathbf{k}_{mi}^e & \mathbf{k}_{mj}^e & \mathbf{k}_{mm}^e \end{bmatrix} \begin{bmatrix} u_{ix} \\ u_{iy} \\ \hline u_{jx} \\ u_{jy} \\ \hline u_{mx} \\ u_{my} \end{bmatrix}. \quad (2.8)$$

For further progression it needs to be assumed that the single stiffness matrices of the elements I, II, III and IV are known. The single stiffness relations of one-dimensional

elements will explicitly be derived in the following chapters for different loading types.

### The Overall Stiffness

The equilibrium of each single element is fulfilled via the single stiffness relation in Eq. (2.6). The overall equilibrium is satisfied by the fact that each node is set into equilibrium. As an example the equilibrium will be set for node 4 in Fig. 2.3:



**Fig. 2.3** Equilibrium on node 4 for the problem of Fig. 2.1

With

$$\mathbf{F}_4 = \begin{bmatrix} F_{4x} \\ F_{4y} \end{bmatrix} \quad (2.9)$$

the following is valid:

$$\mathbf{F}_4 = \sum_e \mathbf{F}_4^e = \mathbf{F}_4^{III} + \mathbf{F}_4^{IV}. \quad (2.10)$$

If one substitutes the nodal forces via the single stiffness relations by the nodal displacements, this yields

$$\mathbf{F}_4 = \mathbf{k}_{43}^{III} \mathbf{u}_3 + (\mathbf{k}_{44}^{III} + \mathbf{k}_{44}^{IV}) \mathbf{u}_4 + \mathbf{k}_{45}^{IV} \mathbf{u}_5 + \mathbf{k}_{46}^{IV} \mathbf{u}_6. \quad (2.11)$$

If one sets up the equilibrium on each node accordingly and notes all relations in the form of a matrix equation, the *overall* stiffness relation results

$$\mathbf{F} = \mathbf{K} \mathbf{u} \quad (2.12)$$

with

$$\mathbf{K} = \sum_e \mathbf{k}_{ij}^e, \quad (2.13)$$

or alternatively in detail

$$\begin{bmatrix} F_1 \\ \hline \mathbf{0} \\ \hline \mathbf{0} \\ \hline F_4 \\ \hline \mathbf{0} \\ \hline F_6 \end{bmatrix} = \begin{bmatrix} k_{11}^I & k_{12}^I & k_{13}^I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline k_{21}^I & k_{22}^I + k_{22}^{II} & k_{23}^I & \mathbf{0} & k_{25}^{II} & \mathbf{0} \\ \hline k_{31}^I & k_{32}^I & k_{33}^I + k_{33}^{III} & k_{34}^{III} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & k_{43}^{III} & k_{44}^{III} + k_{44}^{IV} & k_{45}^{IV} & k_{46}^{IV} \\ \hline \mathbf{0} & k_{52}^{II} & \mathbf{0} & k_{54}^{IV} & k_{55}^{II} + k_{55}^{IV} & k_{56}^{IV} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & k_{64}^{IV} & k_{65}^{IV} & k_{66}^{IV} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \hline u_2 \\ \hline u_3 \\ \hline u_4 \\ \hline u_5 \\ \hline \mathbf{0} \end{bmatrix}. \quad (2.14)$$

This equation is also referred to as the *principal equation of the finite element method*. The vector of the external loads (applied loads or support reactions) is on the left-hand side and the vector of all nodal displacements is on the right-hand side. Both are coupled via the total stiffness matrix  $\mathbf{K}$ . The elements of the total stiffness matrix result according to Eq. (2.13) by adding the appropriate elements of the single stiffness matrices.

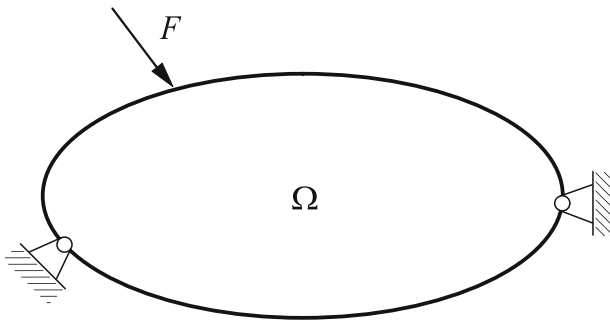
The support conditions  $u_1 = \mathbf{0}$  and  $u_6 = \mathbf{0}$  are already considered in the displacement vector. From the matrix equations 2 to 5 in (2.14) the unknown nodal displacements  $u_2$ ,  $u_3$ ,  $u_4$  and  $u_5$  can be derived. If these are known, one receives, through insertion into the matrix equations 1 and 6 in (2.14), the unknown support reactions  $F_1$  and  $F_6$ .

The matrix displacement method is precise as long as the single stiffness matrices can be defined and as long as elements are coupled in well defined nodes. This is the case for example in truss and frame structures within the heretofore valid theories.

With the so far introduced method the nodal displacements and forces in dependency on the external loads can be determined. For the analysis of the strength of a single element the strain and stress state on the inside of the element is of relevance. Usually the displacement field is described via the nodal displacements  $u_p$  and shape functions. The strain field can be defined via the kinematic relation and the stress field via the constitutive equation.

### 2.1.2 Transition to the Continuum

In the previous section, the matrix displacement method was discussed for a joint supporting structure. In contrast to this, in the continuum, the *virtual* discretized finite elements are connected at infinitely many nodal points. However, in a real application of the matrix displacement method, only a finite number of nodes can be considered. Therewith it is not possible to exactly fulfill both demanded conditions for compatibility and for equilibrium at the same time. Either the compatibility or the equilibrium will be fulfilled *on average* (Fig. 2.4).



**Fig. 2.4** Continuum with load and boundary conditions

In principle, the procedure with the force method or the displacement method can be illustrated. In the following only the *displacement method* will be considered. Here

- the compatibility is *exactly* fulfilled and
- the equilibrium *on average*.

#### The following approach results:

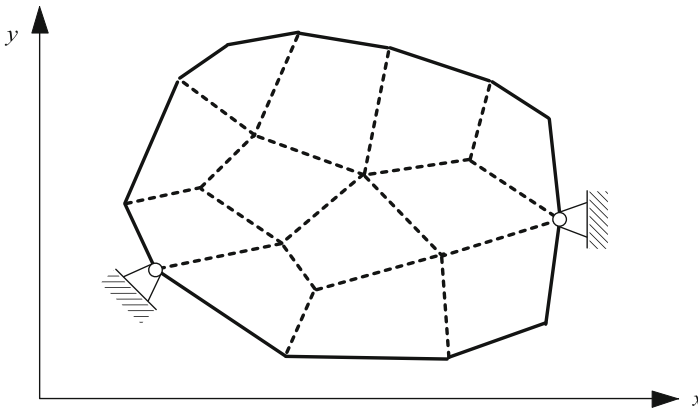
1. The continuum is discretized, meaning for two-dimensional problems it is divided by virtual lines and for three-dimensional problems through surfaces in subregions, so-called finite elements.
2. The flux of force from element to element occurs in discrete nodes. The displacements of these nodes are introduced as principal unknowns (*displacement method!*).
3. The displacement state within an element is illustrated as a function of the nodal displacement. The displacement formulations are compatible with the adjacent neighboring elements.

4. Through the displacement field the strain state within the element and through the constitutive equation the stress state are known as function of the nodal displacement.
5. Via the *principle of virtual work*, statically equivalent resulting nodal forces are assigned to the stresses along the virtual element boundaries *on average*.
6. To maintain the overall equilibrium all nodal equilibria have to be fulfilled. Via this condition one gets to the total stiffness relation, from which the unknown nodal displacements can be calculated after considering the kinematic boundary conditions.
7. If the nodal displacements are known, one knows the displacement and strain field and therefore also the stress state of each single element.

### Comments to the Single Steps

#### Discretization

Through discretization the entire continuum is divided into elements. An element is in contact with one or various neighboring elements. In the two-dimensional case lines result as contact regions, in the three-dimensional case surfaces occur. Figure 2.5 illustrates a discretization for a plane case.



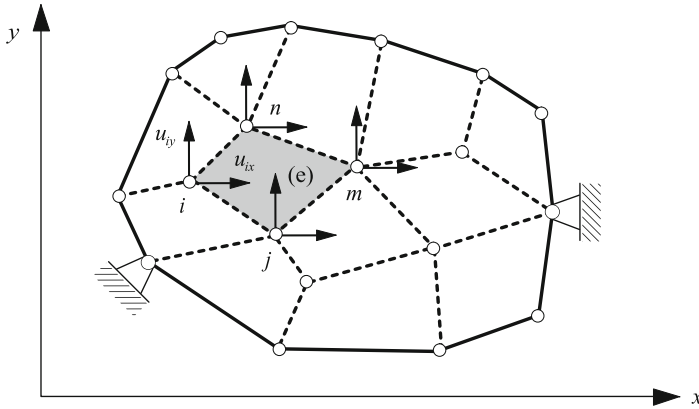
**Fig. 2.5** Discretization of a plane area

The discretization can be interpreted as follows: Single points do not change their geometric position within the continuum. The relation to the neighboring points however does change. While each point within the continuum is in interaction with its neighboring point, in the virtual discretized continuum this is only valid within one element. If two points lie within two different elements they are not directly linked.



## Nodes and Displacements

The information flow between single elements only occurs via the nodes. In the displacement method, displacements are introduced at the nodes as principal unknowns (see Fig. 2.6).

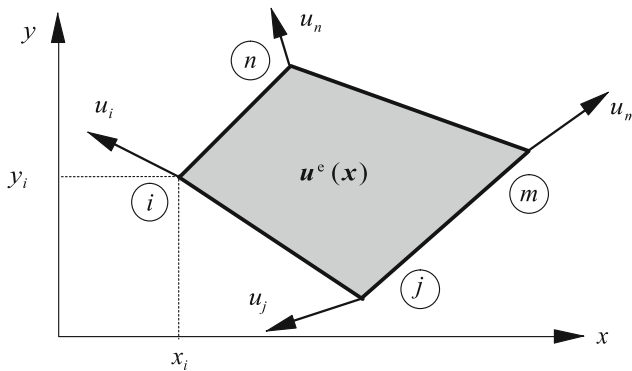


**Fig. 2.6** Nodes with displacements

The displacements are identical for each on the node neighboring elements. Forces only flow via the nodes, no forces flow via the element boundaries even though the element boundaries are geometrically identical.

## Approximation of the Displacement Field

A typical way to describe the displacement field  $u^e(x)$  on the inside of an element is to approximate the field through the displacement at the nodes and so-called shape functions (see Fig. 2.7):



**Fig. 2.7** Approximation of the displacement field in the element

$$\mathbf{u}^e(\mathbf{x}) = \mathbf{N}(\mathbf{x}) \mathbf{u}_p. \quad (2.15)$$

The discretization must not lead to holes in the continuum. To ensure the compatibility between single elements a suitable description of the displacement field has to be chosen. The choice of the shape functions has a significant influence on the quality of the approximation and will be discussed in detail in Sect. 6.4.

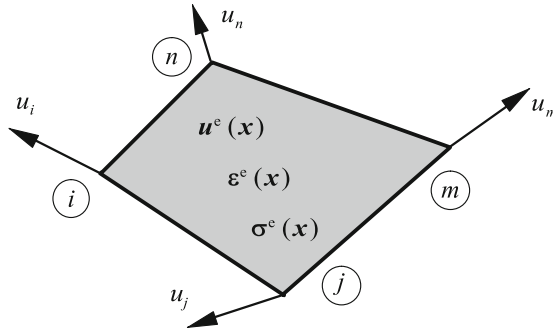
### Strain and Stress Fields

From the displacement field  $\mathbf{u}^e(\mathbf{x})$  one can get to the strain field

$$\boldsymbol{\varepsilon}^e(\mathbf{x}) = \mathcal{L}_1 \mathbf{u}^e(\mathbf{x}) \quad (2.16)$$

via the above kinematic relation. Thereby  $\mathcal{L}_1$  is a differential operator of first order.<sup>2</sup> The stress within an element can be determined via the constitutive equation (Fig. 2.8):

$$\boldsymbol{\sigma}^e(\mathbf{x}) = \mathbf{D} \boldsymbol{\varepsilon}^e(\mathbf{x}) = \mathbf{D} \mathcal{L}_1 \mathbf{N}(\mathbf{x}) \mathbf{u}_p = \mathbf{D} \mathbf{B}(\mathbf{x}) \mathbf{u}_p. \quad (2.17)$$



**Fig. 2.8** Displacement, strain and stress in the element

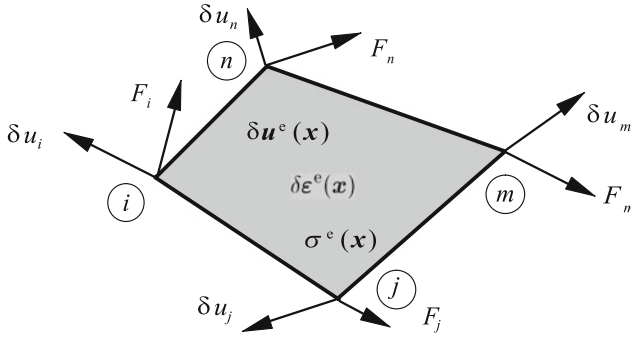
The expression  $\mathcal{L}_1 \mathbf{N}(\mathbf{x})$  contains the derivatives of the shape functions. Usually a new matrix entitled  $\mathbf{B}$  is introduced.

### Principle of Virtual Work, Single Stiffness Matrices

While any point can interact with a neighboring point within the continuum, this is only possible within an element in the discretized structure. A direct exchange beyond the element boundaries is not foreseen. The principle of virtual work represents an

<sup>2</sup> In the one-dimensional case the differential operator simplifies to the derivative  $\frac{d}{dx}$ .

appropriate tool to assign statically equivalent nodal forces to the stress along the virtual element boundaries (Fig. 2.9).



**Fig. 2.9** Principle of virtual work at one element

For this, one summarizes the nodal forces to a vector  $\mathbf{F}_p^e$ . The virtual displacements  $\delta \mathbf{u}_p$  do the external virtual work  $\delta \Pi_{\text{ext}}$  with the nodal forces, the virtual strains  $\delta \boldsymbol{\epsilon}$  do the inner work  $\delta \Pi_{\text{int}}$  with the stresses  $\boldsymbol{\sigma}^e$  inside:

$$\begin{aligned}\delta \Pi_{\text{ext}} &= (\mathbf{F}_p^e)^T \delta \mathbf{u}_p, \\ \delta \Pi_{\text{int}} &= \int_{\Omega} (\boldsymbol{\sigma}^e)^T \delta \boldsymbol{\epsilon}^e d\Omega.\end{aligned}\quad (2.18)$$

According to the principle of virtual work the following is valid:

$$\delta \Pi_{\text{ext}} = \delta \Pi_{\text{int}}. \quad (2.19)$$

If one transposes the equation

$$(\mathbf{F}_p^e)^T \delta \mathbf{u}_p = \int_{\Omega} (\boldsymbol{\sigma}^e)^T \delta \boldsymbol{\epsilon}^e d\Omega$$

and if one inserts (2.16) and (2.17) accordingly, this yields

$$(\delta \mathbf{u}_p)^T \mathbf{F}_p^e = (\delta \mathbf{u}_p)^T \int_{\Omega} \mathbf{B}^T \mathbf{D} \mathbf{B} d\Omega \mathbf{u}_p. \quad (2.20)$$

From this one receives the single stiffness relation

$$\mathbf{F}_p^e = \mathbf{k}^e \mathbf{u}_p \quad (2.21)$$

with the element stiffness matrix

$$\mathbf{k}^e = \int_{\Omega} \mathbf{B}^T \mathbf{D} \mathbf{B} d\Omega . \quad (2.22)$$

### Total Stiffness Relation

One receives the total stiffness relation

$$\mathbf{F} = \mathbf{K} \mathbf{u} \quad (2.23)$$

from the overall equilibrium. This can be achieved by setting up the equilibrium on every single node. The unknown parameters cannot be gained from the total stiffness relation yet. In the context of the equation's solution the system matrix is not regular. Only after taking at least the rigid-body motion (displacement and rotation) from the overall system, a reduced system results

$$\mathbf{F}^{\text{red}} = \mathbf{K}^{\text{red}} \mathbf{u}_p^{\text{red}} , \quad (2.24)$$

which can be solved. A description of the equation solution can be found in Sect. 7.2 and in the Appendix A.1.5.

### Determination of Element Specific Field Parameters

After the equation's solution the nodal displacements are known. Therewith the displacement, strain and stress field on the inside of every single element can be defined. In addition the support reactions can be determined.

## 2.2 Integral Principles

The derivation of the finite element method often occurs via the so-called energy principles. Therefore this chapter serves as a short summary about a few important principles. The overall potential or the total potential energy of a system can generally be written as

$$\Pi = \Pi_{\text{int}} + \Pi_{\text{ext}} \quad (2.25)$$

whereupon  $\Pi_{\text{int}}$  represents the elastic strain energy and  $\Pi_{\text{ext}}$  represents the potential of the external loads. The elastic strain energy—or work of the internal forces—

results in general for linear elastic material behavior via the column matrix of the stresses and strains into:

$$\Pi_{\text{int}} = \frac{1}{2} \int_{\Omega} \boldsymbol{\sigma}^T \boldsymbol{\varepsilon} d\Omega . \quad (2.26)$$

The potential of the external loads—which corresponds with the negative work of the external loads—can be written as follows for the column matrix of the external loads  $\mathbf{F}$  and the displacements  $\mathbf{u}$ :

$$\Pi_{\text{ext}} = -\mathbf{F}^T \mathbf{u} . \quad (2.27)$$

### • Principle of Virtual Work:

The principle of virtual work comprises the principle of virtual displacements and the principle of virtual forces. The principle of virtual displacements states that if an element is in equilibrium, the entire internal virtual work equals the entire external virtual work for arbitrary, compatible, small, virtual displacements, which fulfill the geometric boundary conditions:

$$\int_{\Omega} \boldsymbol{\sigma}^T \delta \boldsymbol{\varepsilon} d\Omega = \mathbf{F}^T \delta \mathbf{u} . \quad (2.28)$$

Accordingly the principle of virtual forces results in:

$$\int_{\Omega} \delta \boldsymbol{\sigma}^T \boldsymbol{\varepsilon} d\Omega = \delta \mathbf{F}^T \mathbf{u} . \quad (2.29)$$

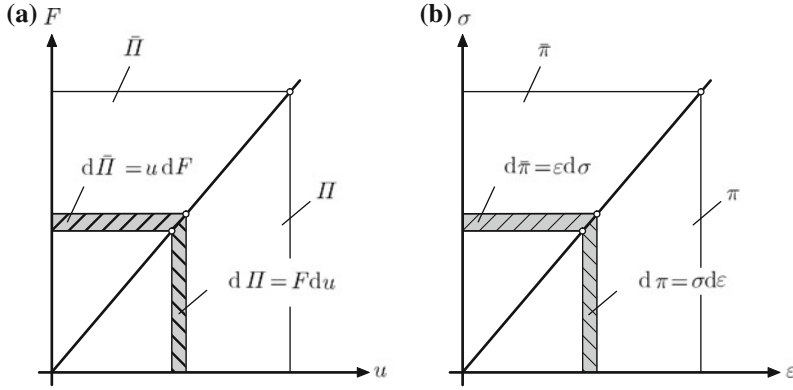
### • Principle of Minimum of Potential Energy:

According to this principle the overall potential takes an extreme value in the equilibrium position:

$$\Pi = \Pi_{\text{int}} + \Pi_{\text{ext}} = \text{minimum} . \quad (2.30)$$

### • Castigliano's Theorem:

CASTIGLIANO's first theorem states that the partial derivative of the complementary strain energy, see Fig. 2.10a with respect to an external force  $F_i$  leads to the displacement of the force application point in the direction of this force. Accordingly it results that the partial derivative of the complementary strain energy with respect to an external moment  $M_i$  leads to the rotation of the moment application point in the direction of this moment:



**Fig. 2.10** Definition of the strain energy and the complementary strain energy: **a** absolute; **b** volume specific

$$\frac{\partial \tilde{\Pi}_{\text{int}}}{\partial F_i} = u_i, \quad (2.31)$$

$$\frac{\partial \tilde{\Pi}_{\text{int}}}{\partial M_i} = \varphi_i. \quad (2.32)$$

CASTIGLIANO's second theorem states that the partial derivative of the strain energy (see Fig. 2.10a) with respect to the displacements  $u_i$  leads to the force  $F_i$  in direction to the considered displacement  $u_i$ . An analogous connection is valid for the rotation and the moment:

$$\frac{\partial \Pi_{\text{int}}}{\partial u_i} = F_i, \quad (2.33)$$

$$\frac{\partial \Pi_{\text{int}}}{\partial \varphi_i} = M_i. \quad (2.34)$$

## 2.3 Weighted Residual Method

The initial point of the weighted residual method is the differential equation, which describes the physical problem. In the one-dimensional case such a physical problem within the domain  $\Omega$  can in general be described via the differential equation

$$\mathcal{L}\{u^0(x)\} = b \quad (x \in \Omega) \quad (2.35)$$

as well as via the boundary conditions, which are prescribed on the boundary  $\Gamma$ . The differential equation is also referred to as a *strong form* of the problem since the

problem is exactly described in every point  $x$  of the domain. In Eq. (2.35)  $\mathcal{L}\{\dots\}$  represents an arbitrary differential operator, which can for example take the following forms:

$$\mathcal{L}\{\dots\} = \frac{d^2}{dx^2} \{\dots\}, \quad (2.36)$$

$$\mathcal{L}\{\dots\} = \frac{d^4}{dx^4} \{\dots\}, \quad (2.37)$$

$$\mathcal{L}\{\dots\} = \frac{d^4}{dx^4} \{\dots\} + \frac{d}{dx} \{\dots\} + \{\dots\}. \quad (2.38)$$

Furthermore  $b$  represents a given function in Eq. (2.35), whereupon one talks about a *homogeneous differential equation* in the case of  $b = 0$ :  $\mathcal{L}\{u^0(x)\} = 0$ . The exact or real solution of the problem,  $u^0(x)$ , fulfills the differential equation in every point of the domain  $x \in \Omega$  and the prescribed geometric and static boundary conditions on  $\Gamma$ . Since the exact solution for the most engineering problems cannot be calculated in general, it is the goal of the following derivation to define a best possible approximate solution

$$u(x) \approx u^0(x). \quad (2.39)$$

For the approximate solution in Eq. (2.39) in the following an approach in the form

$$u(x) = \alpha_0 + \sum_{k=1}^n \alpha_k \varphi_k(x) \quad (2.40)$$

is chosen, whereupon  $\alpha_0$  needs to fulfill the non-homogeneous boundary conditions,  $\varphi_k(x)$  represents a set of linear independent basis functions and  $\alpha_k$  are the free parameters of the approximation approach, which are defined via the approximation procedure in a way so that the exact solution  $u^0$  of the approximate solution  $u$  is approximated in the best way.

### 2.3.1 Procedure on Basis of the Inner Product

If one incorporates the approximate formulation for  $u^0$  into the differential equations. (2.35), one receives a local error, the so-called *residual*  $r$ :

$$r = \mathcal{L}\{u(x)\} - b \neq 0. \quad (2.41)$$

Within the weighted residual method this error is weighted with a weighting function  $W(x)$  and is integrated via the entire domain  $\Omega$ , so that the error disappears on average:

$$\int_{\Omega} r W d\Omega = \int_{\Omega} (\mathcal{L}\{u(x)\} - b) W d\Omega \stackrel{!}{=} 0. \quad (2.42)$$

This formulation is also referred to as the *inner product*. One notes that the weighting or test function  $W(x)$  allows to weigh the error differently within the domain  $\Omega$ . However the overall error must on average, meaning integrated over the domain, become zero. The structure of the weighting function is most of the time set in a similar way as with the approximate function  $u(x)$

$$W(x) = \sum_{k=1}^n \beta_k \psi_k(x), \quad (2.43)$$

whereupon  $\beta_k$  represent *arbitrary* coefficients and  $\psi_k(x)$  linear independent shape functions. The approach (2.43) includes—depending on the choice of the amount of the summands  $k$  and the functions  $\psi_k(x)$ —the class of the procedures with equal shape functions for the approximate solution and the weighting function ( $\varphi_k(k) = \psi_k(x)$ ) and the class of the procedures, at which the shape functions are chosen differently ( $\varphi_k(k) \neq \psi_k(x)$ ). Depending on the choice of the weighting function the following classic methods can be differentiated [4, 13]:

• **Point-Collocation Method:**  $\psi_k(x) = \delta(x - x_k)$

The point-collocation method takes advantage of the properties of the delta function. The error  $r$  disappears exactly on the  $n$  freely selectable points  $x_1, x_2, \dots, x_n$ , with  $x_k \in \Omega$ , the so-called collocation points and therefore the approximate solution fulfills the differential equation exactly in the collocation points. The weighting function can therefore be set as

$$W(x) = \beta_1 \underbrace{\delta(x - x_1)}_{\psi_1} + \dots + \beta_n \underbrace{\delta(x - x_n)}_{\psi_n} = \sum_{k=1}^n \beta_k \delta(x - x_k) \quad (2.44)$$

whereupon the delta function is defined as follows:

$$\delta(x - x_k) = \begin{cases} 0 & \text{for } x \neq x_k \\ \infty & \text{for } x = x_k \end{cases}. \quad (2.45)$$

If one incorporates this approach into the inner product according to Eq. (2.42) and considers the properties of the delta function,

$$\int_{-\infty}^{\infty} \delta(x - x_k) dx = \int_{x_k - \varepsilon}^{x_k + \varepsilon} \delta(x - x_k) dx = 1, \quad (2.46)$$



$$\int_{-\infty}^{\infty} f(x)\delta(x - x_k) dx = \int_{x_k-\varepsilon}^{x_k+\varepsilon} f(x)\delta(x - x_k) dx = f(x_k), \quad (2.47)$$

$n$  linear independent equations result for the calculation of the free parameters  $\alpha_k$ :

$$r(x_1) = \mathcal{L}\{u(x_1)\} - b = 0, \quad (2.48)$$

$$r(x_2) = \mathcal{L}\{u(x_2)\} - b = 0, \quad (2.49)$$

$$\vdots$$

$$r(x_n) = \mathcal{L}\{u(x_n)\} - b = 0. \quad (2.50)$$

One considers that the approximate approach has to fulfill all boundary conditions, meaning the essential and natural boundary conditions. Due to the property of the delta function,  $\int_{\Omega} r W(\delta) d\Omega = r = 0$ , no integral has to be calculated within the point-collocation procedure, meaning no integration via the inner product. One therefore does not need to do an integration and receives the approximate solution faster—compared to for example the GALERKIN procedure. A disadvantage is, however, that the collocation points can be chosen freely. These can therefore also be chosen unfavorable.

• **Subdomain-Collocation Procedure:**  $\psi_k(x) = 1$  in  $\Omega_k$  and otherwise zero

This procedure is a collocation method as well, however besides the demand that the error has to disappear on certain points, here it is demanded that the integral of the error becomes zero over the different domains, the subdomains:

$$\int_{\Omega_i} r d\Omega_i = 0 \quad \text{for a subregion } \Omega_i. \quad (2.51)$$

With this procedure the finite difference method can, for example, be derived.

• **Method of Least Squares:**  $\psi_k(x) = \frac{\partial r}{\partial \alpha_k}$

The average quadratic error is optimized at the method of least squares

$$\int_{\Omega} (\mathcal{L}\{u(x)\} - b)^2 d\Omega = \text{minimum}, \quad (2.52)$$

or alternatively

$$\frac{d}{d\alpha_k} \int_{\Omega} (\mathcal{L}\{u(x)\} - b)^2 d\Omega = 0, \quad (2.53)$$

$$\int_{\Omega} \frac{d(\mathcal{L}\{u(x)\}) - b}{d\alpha_k} (\mathcal{L}\{u(x)\} - b) d\Omega = 0. \quad (2.54)$$

• **Petrov-Galerkin Procedure:**  $\psi_k(x) \neq \varphi_k(x)$

This term summarizes all procedures, at which the shape functions of the weighting function and the approximate solution are different. Therefore, for example, the subdomain-collocation method can be allocated to this group.

• **Galerkin Procedure:**  $\psi_k(x) = \varphi_k(x)$

The basic idea of the GALERKIN or BUBNOV- GALERKIN method is to choose the *same* shape function for the approximate approach and the weighting function approach. Therefore, the weighting function results in the following for this method:

$$W(x) = \sum_{k=1}^n \beta_k \varphi_k(x). \quad (2.55)$$

Since the same shape functions  $\varphi_k(x)$  were chosen for  $u(x)$  and  $W(x)$  and the coefficients  $\beta_k$  are arbitrary, the function  $W(x)$  can be written as a variation of  $u(x)$  (with  $\delta\alpha_0 = 0$ ):

$$W(x) = \delta u(x) = \delta\alpha_1 \varphi_1(x) + \dots + \delta\alpha_n \varphi_n(x) = \sum_{k=1}^n \delta\alpha_k \times \varphi_k(x). \quad (2.56)$$

The variations can be virtual parameters, as for example virtual displacements or velocities. The incorporation of this approach into the inner product according to Eq. (2.42) yields a set of  $n$  linear independent equations for a linear operator for the definition of  $n$  unknown free parameters  $\alpha_k$ :

$$\int_{\Omega} (\mathcal{L}\{u(x)\} - b) \cdot \varphi_1(x) d\Omega = 0, \quad (2.57)$$

$$\int_{\Omega} (\mathcal{L}\{u(x)\} - b) \cdot \varphi_2(x) d\Omega = 0, \quad (2.58)$$

$$\vdots$$

$$\int_{\Omega} (\mathcal{L}\{u(x)\} - b) \cdot \varphi_n(x) d\Omega = 0. \quad (2.59)$$

Conclusion regarding the procedure based on the inner products:

These formulations demand that the shape functions—which have been assumed to be defined over the entire domain  $\Omega$ —fulfill all boundary conditions, meaning the essential and natural boundary conditions. This demand, as well as the demanded differentiation of the shape functions ( $\mathcal{L}$  operator) often lead to a difficulty finding appropriate functions in the practical application. Furthermore, in general, unsymmetric coefficient matrices occur (if the  $\mathcal{L}$  operator is symmetric the coefficient matrix of the GALERKIN method is also symmetric).

### 2.3.2 Procedure on Basis of the Weak Formulation

For the derivation of another class of approximate procedures the inner product is partially integrated again and again until the derivative of  $u(x)$  and  $W(x)$  has the same order and one reaches the so-called *weak formulation*. Within this formulations the demand regarding the differentiability for the approximate function is diminished, the demand regarding the weighting function however increased. If one uses the idea of the GALERKIN method, meaning equal shape functions for the approximate approach and the weighting function, the demand regarding the differentiability of the shape functions is reduced in total.

For a differential operator of second or fourth order, meaning

$$\int_{\Omega} \mathcal{L}_2\{u(x)\} W(x) d\Omega, \quad (2.60)$$

$$\int_{\Omega} \mathcal{L}_4\{u(x)\} W(x) d\Omega, \quad (2.61)$$

a one-time partial integration of Eq. (2.60) yields the weak form

$$\int_{\Omega} \mathcal{L}_1\{u(x)\} \mathcal{L}_1\{W(x)\} d\Omega = [\mathcal{L}_1\{u(x)\} W(x)]_{\Gamma}, \quad (2.62)$$

or alternatively two-times partial integration the weak form of Eq. (2.61):

$$\int_{\Omega} \mathcal{L}_2\{u(x)\} \mathcal{L}_2\{W(x)\} d\Omega = [\mathcal{L}_2\{u(x)\} \mathcal{L}_1\{W(x)\} - \mathcal{L}_3\{u(x)\} W(x)]_{\Gamma}. \quad (2.63)$$

For the derivation of the finite element method one switched to domain-wise defined shape functions. For such a domain, meaning a finite element with  $\Omega^e < \Omega$  and a local element coordinate  $x^e$  the weak formulation of (2.62), for example, results in:

$$\int_{\Omega^e} \mathcal{L}_2\{u(x^e)\} \mathcal{L}_2\{W(x^e)\} d\Omega^e = [\mathcal{L}_2\{u(x^e)\} \mathcal{L}_1\{W(x^e)\} - \mathcal{L}_3\{u(x^e)\} W(x^e)]_{\Gamma^e} . \quad (2.64)$$

Since the weak formulation contains the natural boundary conditions—for this also see sample problem 2.2—, it can be demanded in the following that the approach<sup>3</sup> for  $u(x)$  only has to fulfill the essential boundary conditions. According to the GALERKIN method it is demanded for the derivation of the principal finite element equation that the same shape functions for the approximate and weighting function are chosen. Within the framework of the finite element method the nodal values  $u_k$  are chosen for the free values  $\alpha_k$  and the shape functions  $\varphi_k(x)$  are referred to as form or shape functions  $N_k(x)$ . Therefore, the following illustrations result for the approximate solution and the weighting function:

$$u(x) = N_1(x)u_1 + N_2(x)u_2 + \cdots N_n(x)u_n = \sum_{k=1}^n N_k(x)u_k , \quad (2.65)$$

$$W(x) = \delta u_1 N_1(x) + \delta u_2 N_2(x) + \cdots \delta u_n N_n(x) = \sum_{k=1}^n \delta u_k N_k(x) , \quad (2.66)$$

whereupon  $n$  represents the number of nodes per element. It is important for this procedure that the error on the nodes, whose position has to be defined by the user, is minimized. This is a significant difference to the classic GALERKIN method on the basis of the inner product, which has found the points with  $r = 0$  itself. For the further derivation of the principal finite element equation the approaches (2.65) and (2.66) have to be written in matrix form and inserted into the weak form. For further details of the derivation refer to the explanations in Chaps. 3 and 5 at this point.

Within the framework of the finite element method the so-called RITZ method is often mentioned. The classic procedure takes into account the overall potential  $\Pi$  of a system. Within this overall potential an approximate approach in the form of (2.40) is used, which is, however, defined for the entire domain  $\Omega$  in the RITZ method. The shape functions  $\varphi_k$  have to fulfill the geometric, however not the static boundary conditions.<sup>4</sup> Via the derivative of the potential with respect to the unknown free parameters  $\alpha_k$ , meaning definition of the extremum of  $\Pi$ , a system of equations results for the definition of  $k$  free parameters, the so-called RITZ coefficients. In general, however, it is difficult to find shape functions with unknown free values, which fulfill *all* geometric boundary conditions of the problem. However, if one modifies the classic RITZ method in a way so that only the domain  $\Omega^e$  of a finite

<sup>3</sup> The index ‘e’ of the element coordinate is neglected in the following—in the case it does not affect the understanding.

<sup>4</sup> Since the static boundary conditions are implicitly integrated in the overall potential, the shape functions do not have to fulfill those. However, if the shape functions fulfill the static boundary conditions additionally, an even more precise approximation can be achieved.

element is considered and one makes use of an approximate approach according to Eq. (2.65) one also achieves the finite element method at this point.

### 2.3.3 Procedure on Basis of the Inverse Formulation

Finally it needs to be remarked that the inner product can be partially integrated again and again for the derivation of another class of approximate procedures until the derivative of  $u(x)$  can be completely shifted onto  $W(x)$ . Therewith one achieves the so-called *inverse formulation*. Depending on the choice of the weighting function one receives the following methods:

- Choice of  $W$  so that  $\mathcal{L}(W) = 0$  or  $\mathcal{L}(u) \neq 0$ .

Procedure: *Boundary element method (Boundary integral equation of the first kind)*.

- Use of a so-called fundamental solution  $W = W^*$ , meaning a solution, which fulfills the equation  $\mathcal{L}(W^*) = (-)\delta(\xi)$ .

Procedure: *Boundary element method (Boundary integral equation of the second kind)*.

The coefficient matrix of the corresponding system of equations is fully occupied and not symmetric. What is decisive for the application of the method is the knowledge about a fundamental solution for the  $\mathcal{L}$  operator (in elasticity theory such an analytical solution is known through the KELVIN solution—concentrated load at a point of an infinite elastic medium).

- Equal shape functions for approximation approach and weighting function approach. Procedure: *TREFFTZ method*.
- Equal shape functions for approximate approach and weighting function and  $\mathcal{L}(u) = \mathcal{L}(W) = 0$  is valid. Procedure: *Variation of the TREFFTZ method*.

## 2.4 Sample Problems

### 2.1. Example: Galerkin Method on Basis of the Inner Product

Since the term GALERKIN method is an often used term within the finite element method, the original GALERKIN method needs to be explained in the following within

the framework of this example. For this the differential equation, which is defined in the domain  $0 < x < 1$  is considered

$$\mathcal{L}\{u(x)\} - b = \frac{d^2 u^0}{dx^2} + x^2 = 0 \quad (0 < x < 1) \quad (2.67)$$

with the homogeneous boundary conditions  $u^0(0) = u^0(1) = 0$ . For this problem the exact solution

$$u^0(x) = \frac{x}{12} (-x^3 + 1) \quad (2.68)$$

can be defined via integration and subsequent consideration of the boundary conditions. Define the approximate solution for an approach with two free values.

## 2.1 Solution

For the construction of the approximate solution  $u(x)$  according to the GALERKIN method the following approach with two free parameters can be made use of:

$$u^0(x) \approx u(x) = \alpha_1 \varphi_1(x) + \alpha_2 \varphi_2(x), \quad (2.69)$$

$$= \alpha_1 x(1 - x) + \alpha_2 x^2(1 - x), \quad (2.70)$$

$$= \alpha_1 x + (\alpha_2 - \alpha_1)x^2 - \alpha_2 x^3. \quad (2.71)$$

One needs to consider that the functions  $\varphi_1(x)$  and  $\varphi_2(x)$  are chosen in a way so that the boundary conditions, meaning  $u(0) = u(1) = 0$ , are fulfilled. Therefore, polynomials of first order are eliminated since a linear slope could only connect the two zero points as a horizontal line. Furthermore, both functions are chosen in a way so that they are linearly independent. The first derivatives of the approximation approach result in

$$\frac{du(x)}{dx} = \alpha_1 + 2(\alpha_2 - \alpha_1)x - 3\alpha_2 x^2, \quad (2.72)$$

$$\frac{d^2 u(x)}{dx^2} = 2(\alpha_2 - \alpha_1) - 6\alpha_2 x, \quad (2.73)$$

and the error function results in the following via the second derivative from Eq. (2.41):

$$r(x) = \frac{d^2 u}{dx^2} + x^2 = 2(\alpha_2 - \alpha_1) - 6\alpha_2 x + x^2. \quad (2.74)$$

The insertion of the weighting function, meaning

$$W(x) = \delta u(x) = \delta \alpha_1 x(1 - x) + \delta \alpha_2 x^2(1 - x), \quad (2.75)$$

into the residual equation yields

$$\int_0^1 \underbrace{\left(2(\alpha_2 - \alpha_1) - 6\alpha_2 x + x^2\right)}_{r(x)} \times \underbrace{\left(\delta\alpha_1 x(1-x) + \delta\alpha_2 x^2(1-x)\right)}_{W(x)} dx = 0 \quad (2.76)$$

or generally split into two integrals:

$$\delta\alpha_1 \int_0^1 r(x)\varphi_1(x)dx + \delta\alpha_2 \int_0^1 r(x)\varphi_2(x)dx = 0. \quad (2.77)$$

Since the  $\delta\alpha_i$  are arbitrary coefficients and the shape functions  $\varphi_i(x)$  are linearly independent, the following system of equations results herefrom:

$$\delta\alpha_1 \int_0^1 \left(2(\alpha_2 - \alpha_1) - 6\alpha_2 x + x^2\right) \times (x(1-x)) dx = 0, \quad (2.78)$$

$$\delta\alpha_2 \int_0^1 \left(2(\alpha_2 - \alpha_1) - 6\alpha_2 x + x^2\right) \times (x^2(1-x)) dx = 0. \quad (2.79)$$

After the integration, a system of equations results for the definition of the two unknown free parameters  $\alpha_1$  and  $\alpha_2$

$$\frac{1}{20} - \frac{1}{6}\alpha_2 - \frac{1}{3}\alpha_1 = 0, \quad (2.80)$$

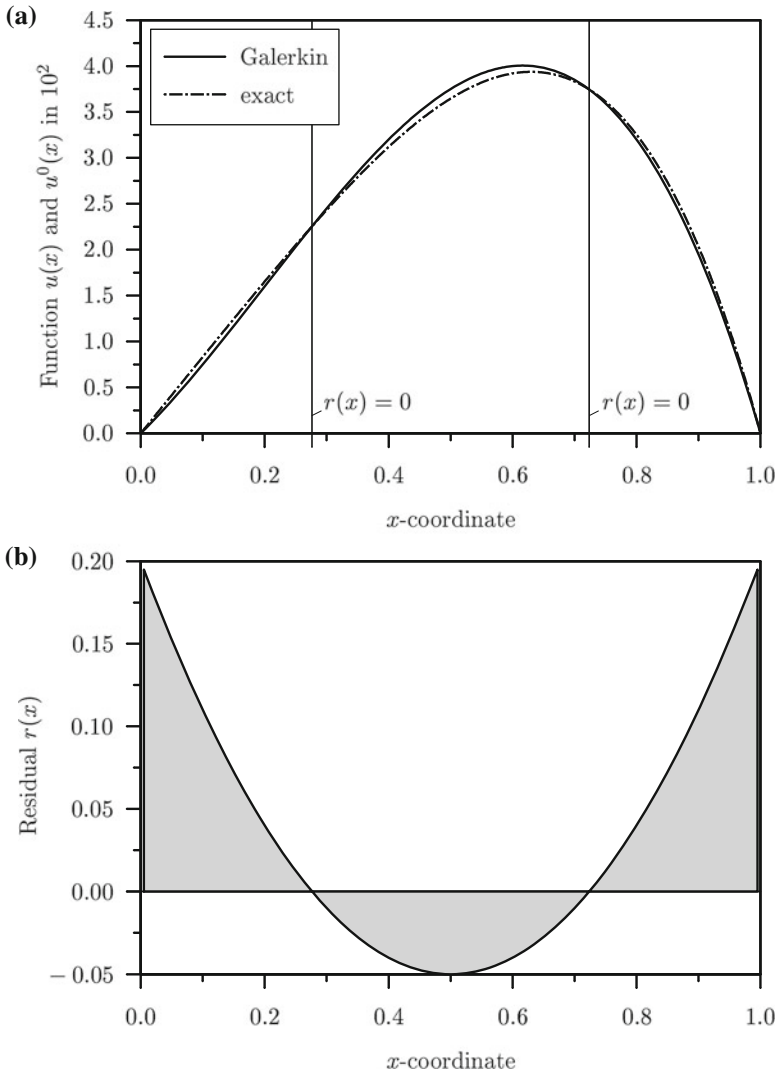
$$\frac{1}{30} - \frac{2}{15}\alpha_2 - \frac{1}{6}\alpha_1 = 0, \quad (2.81)$$

or alternatively in matrix notation:

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{15} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{20} \\ \frac{1}{30} \end{bmatrix}. \quad (2.82)$$

From this system of equations the free parameters result in  $\alpha_1 = \frac{1}{15}$  and  $\alpha_2 = \frac{1}{6}$ . Therefore, the approximate solution and the error function finally result in:

$$u(x) = x \left( -\frac{1}{6}x^2 + \frac{1}{10}x + \frac{1}{15} \right), \quad (2.83)$$

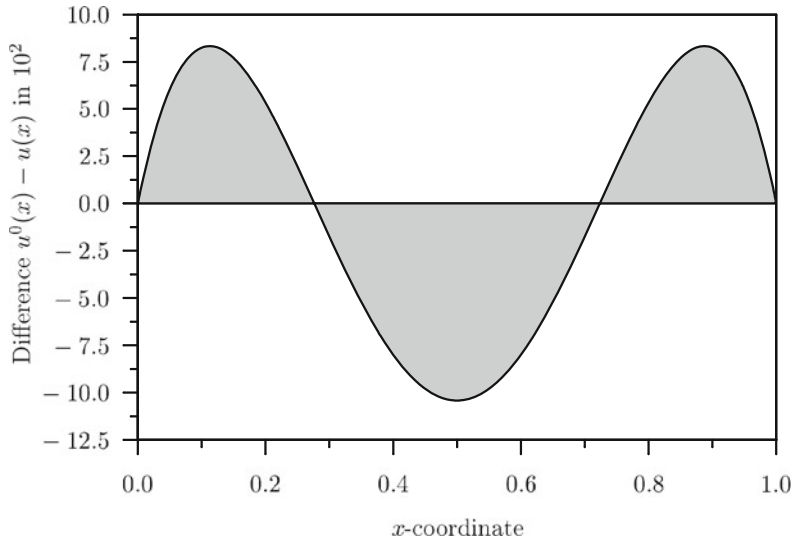


**Fig. 2.11** Approximate solution according to the GALERKIN method, **a** exact solution and **b** residual as a function of the coordinate

$$r(x) = x^2 - x + \frac{1}{5}. \quad (2.84)$$

The comparison between the approximate solution and exact solution is illustrated in Fig. 2.11a. One can see that the two solutions coincide on the boundaries—one needs to consider that the approximate approach has to fulfill the boundary conditions—as well as on two other locations.





**Fig. 2.12** Absolute difference between exact solution and approximate solution as function of the coordinate

It needs to be remarked at this point that the error function—see Fig. 2.11b—does not illustrate the difference between the exact solution and the approximate solution. Rather it is about the error, which results from inserting the approximate solution into the differential equation. To illustrate this, Fig. 2.12 shows the absolute difference between exact solution and approximate solution.

Finally it can be summarized that the advantage of the GALERKIN method is that the procedure itself is in search of the points with  $r = 0$ . This is quite an advantage in comparison to the collocation method. However within the GALERKIN method the integration needs to be performed and therefore this method is in comparison to the collocation more complex and slower.

## 2.2 Example: Finite Element Method

For the differential equations (2.67) and the given boundary conditions one needs to calculate, based on the weak formulation, a finite element solution, based on two equidistant elements with linear shape functions.

### Solution

The partial integration of the inner product yields the following formulation:

$$\int_0^1 \left( \frac{d^2 u(x)}{dx^2} + x^2 \right) W(x) dx = 0, \quad (2.85)$$

$$\int_0^1 \frac{d^2 u(x)}{dx^2} W(x) dx + \int_0^1 x^2 W(x) dx = 0, \quad (2.86)$$

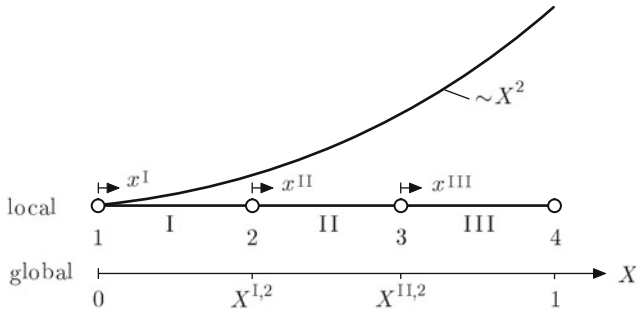
$$\left[ \frac{du(x)}{dx} W(x) \right]_0^1 - \int_0^1 \frac{du(x)}{dx} \frac{dW(x)}{dx} dx + \int_0^1 x^2 W(x) dx = 0, \quad (2.87)$$

or alternatively the weak form in its final form:

$$\int_0^1 \frac{du(x)}{dx} \frac{dW(x)}{dx} dx = \left[ \frac{du(x)}{dx} W(x) \right]_0^1 + \int_0^1 x^2 W(x) dx. \quad (2.88)$$

For the derivation of the finite element method one merges into domain-wise defined shape functions. For such a domain  $\Omega^e < \Omega$ , namely a finite element<sup>5</sup> of the length  $L^e$ , the weak formulation results in:

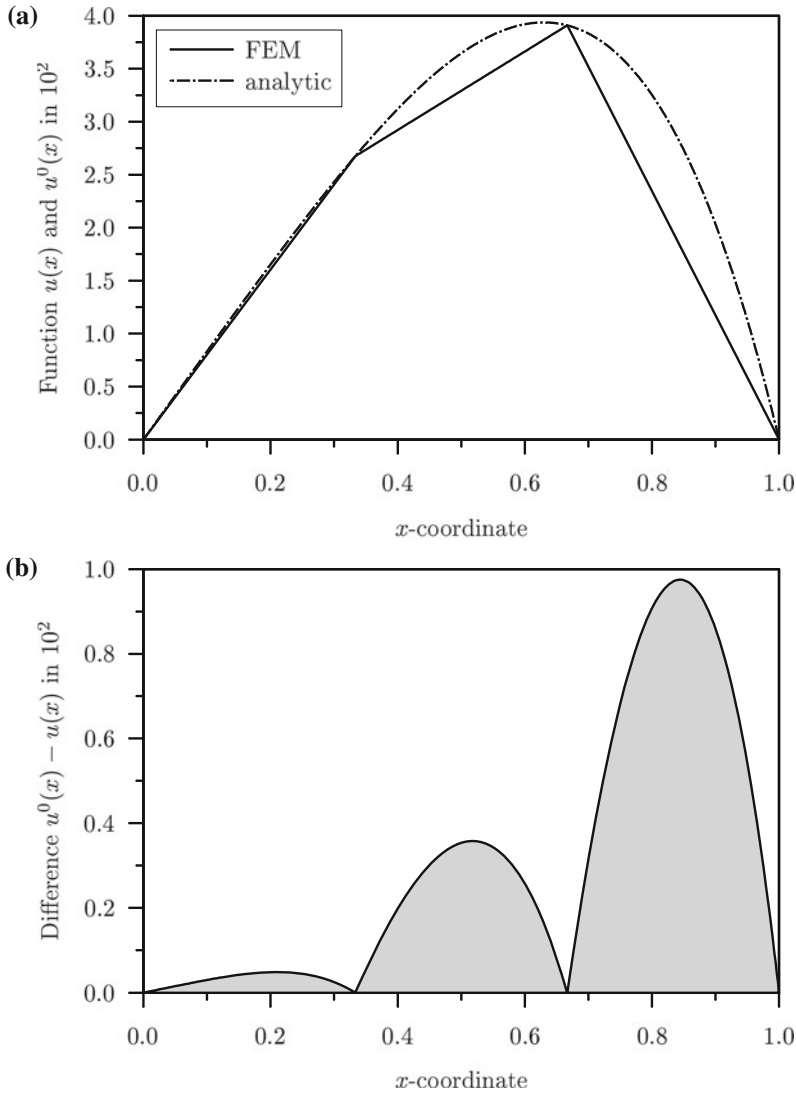
$$\int_0^{L^e} \frac{du(x^e)}{dx^e} \frac{dW(x^e)}{dx^e} dx^e = \left[ \frac{du(x^e)}{dx^e} W(x^e) \right]_0^{L^e} + \int_0^{L^e} (x^e + c^e)^2 W(x^e) dx^e. \quad (2.89)$$



**Fig. 2.13** Global coordinate system  $X$  and local coordinate system  $x_i$  for every element

In the transition from Eqs. (2.88) to (2.89), meaning from the global formulation to the consideration on the element level, in particular the quadratic expression on the right-hand side of Eq. (2.88) needs to be considered. To ensure that the in the

<sup>5</sup> Usually a separate *local* coordinate system  $0 \leq x^e \leq L^e$  is introduced for each element 'e'. The coordinate in Eq. (2.88) is then referred to as global coordinate and receives the symbol  $X$ .



**Fig. 2.14** **a** Exact solution and approximate solution and **b** absolute difference between exact solution and finite element solution as a function of the coordinate

global coordinate system defined expression  $X^2$  is considered appropriately in the description on the element level, a coordinate transformation has to be performed for every element 'e' via a term  $c^e$ . From Fig. 2.13 it can be seen that the term  $c$  turns zero for the first element (I) since global and local coordinate system coincide. For the second element (II)  $c^{\text{II}} = X^{1,2} = \frac{1}{3}$  results with an equidistant division and  $c^{\text{III}} = X^{\text{II},2} = \frac{2}{3}$  results accordingly for the third element (III).

Since the weak formulation contains the natural boundary conditions—for this see the boundary expression in Eq. (2.89)—it can be demanded in the following that the approach for  $u(x)$  has to fulfill the essential boundary conditions only. According to the GALERKIN method it is demanded for the derivation of the principal finite element equation that the same shape functions for the approximate and weighting function are chosen. Within the framework of the finite element method the nodal values  $u_k$  are chosen for the free parameters  $\alpha_k$  and the shape functions  $\varphi_k(x)$  are referred to as form or shape functions  $N_k(x)$ . For linear shape functions the following illustrations result for the approximate solution and the weighting function:

$$u(x) = N_1(x)u_1 + N_2(x)u_2, \quad (2.90)$$

$$W(x) = \delta u_1 N_1(x) + \delta u_2 N_2(x). \quad (2.91)$$

For the chosen linear shape functions, an element-wise linear course of the approximate function and a difference between the exact solution and the approximate approach as shown in Fig. 2.14 are obtained. It is obvious that the error is minimal on the nodes, at the best identical with the exact solution.

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