

Chapter 2

Spin in Strong Interactions

In this chapter, the basic theoretical relations underlying the design and analysis of polarization experiments involving strong interactions are reported on classical examples of pion–nucleon and nucleon–nucleon scattering. After the introduction of the density matrix and reaction matrix, we discuss such notions as the complete set of experiments and the equality of the polarization \vec{P} in the direct reaction and the asymmetry \vec{A} in the inverse reaction on the example of the simple pion–nucleon system. On the example of the nucleon–nucleon system, we present the method for explicitly constructing the reaction matrix, formulate the unitarity condition, and point to the possibilities of seeking the effects of parity and time reversal violation. The presentation is primarily based on the technique of nonrelativistic quantum mechanics. Relativistic pion–nucleon and nucleon–nucleon elastic scattering matrices are discussed in the concluding sections. The inclusion of the relativistic effects insignificantly changes the nonrelativistic results.

2.1 Density Matrix

As known, a particle with spin \vec{s} is described by the wave function Ψ having $2s + 1$ components Ψ_α , where $\alpha = s, s - 1, \dots, 0, \dots, -(s - 1), -s$. If only one of these components with a certain α value is nonzero, the particle is in a pure spin state. In this case, the mean value of any operator \hat{O} is given by the expression

$$\langle \hat{O} \rangle = \langle \Psi_\alpha^* | \hat{O} | \Psi_\alpha \rangle.$$

However, components for several α values are most often nonzero in reality. In this case, the state is mixed. A simple example is a polarized proton beam. If its polarization is 100 %, this is a pure spin state: the spins of all protons are oriented identically. When the beam is partially polarized, the protons are in a mixed spin state. This means that the spins of some protons are directed upwards and the spins

of other protons are directed downwards. In this case, the mean value of an arbitrary spin operator \hat{O} in the mixed spin state Ψ is given by the expression

$$\langle \hat{O} \rangle = \sum_{\alpha} W_{\alpha} \langle \Psi_{\alpha}^* | \hat{O} | \Psi_{\alpha} \rangle, \quad (2.1)$$

where W_{α} is the weight of the pure state α . The Dirac brackets mean integration (summation) with respect to continuous (discrete) variables.

Let us consider the set of other $(2s + 1)$ components $\{\chi_m\}$, which are the orthogonal eigenfunctions of a certain spin operator as the basis functions. If this set is the complete orthogonal set, it can be used for the expansion

$$\Psi_{\alpha} = \sum_{m=-s}^s C_m^{\alpha} \chi_m. \quad (2.2)$$

The substitution of this expansion into Eq. (2.1) yields

$$\begin{aligned} \langle \hat{O} \rangle &= \sum_{\alpha mn} W_{\alpha} C_m^{\alpha*} C_n^{\alpha} (\chi_m^* \hat{O} \chi_n) = \sum_{mn} \left(\sum_{\alpha} W_{\alpha} C_n^{\alpha} C_m^{\alpha*} \right) \cdot (\chi_m^* \hat{O} \chi_n) \\ &= \sum_{mn} \rho_{nm} O_{mn}, \end{aligned} \quad (2.3)$$

where O_{mn} is the matrix element of the operator \hat{O} and

$$\rho_{nm} = \sum_{\alpha} W_{\alpha} (C_m^{\alpha*} C_n^{\alpha}) \quad (2.4)$$

is the matrix element of a certain operator $\hat{\rho}$, which is called the density matrix. The operator $\hat{\rho}$ can be represented in the matrix form

$$\hat{\rho} = \sum_{\alpha} W_{\alpha} \Psi_{\alpha} \Psi_{\alpha}^+. \quad (2.5)$$

Indeed, let us represent the matrix element of the operator $\hat{\rho}$ in the form

$$\rho_{mn} = (\chi_m^+ \hat{\rho} \chi_n) = \sum_{\alpha} W_{\alpha} (\chi_m^+ \Psi_{\alpha} \Psi_{\alpha}^+ \chi_n). \quad (2.6)$$

Since the eigenfunctions χ are orthogonal, expansion (2.2) provides

$$\chi_m^+ \Psi_{\alpha} = \sum_n C_n^{\alpha} \chi_m^+ \chi_n = C_m^{\alpha}, \quad \Psi_{\alpha}^+ \chi_m^+ = \sum_n C_n^{\alpha*} \chi_n^+ \chi_m = C_m^{\alpha*}. \quad (2.7)$$

The substitution into Eq. (2.6) finally yields the expression

$$\rho_{mn} = \sum_{\alpha} W_{\alpha} C_m^{\alpha} C_n^{\alpha*}, \quad (2.8)$$

which coincides with Eq. (2.4).

Using Eqs. (2.4) or (2.5), one can show that the matrix ρ is Hermitian:

$$\hat{\rho}^+ = \rho; \quad (2.9)$$

i.e., the mean value of ρ is a real number. Let us define the sum of the diagonal elements of an arbitrary matrix C (its trace) as

$$\text{Tr } C = \sum_i C_{ii}. \quad (2.10)$$

If the operator \hat{C} is the product of two operators \hat{A} and \hat{B} , its matrix element is given by the expression

$$C_{ij} = \sum_m A_{im} B_{mj}. \quad (2.11)$$

In this case, the sum of the diagonal elements is

$$\text{Tr } \hat{C} = \sum_{mn} A_{nm} B_{mn} = \text{Tr } \hat{A} \hat{B}. \quad (2.12)$$

Comparison with Eq. (2.3) provides

$$\langle \hat{O} \rangle = \text{Tr } \hat{O} \hat{\rho} = \text{Tr } \hat{\rho} \hat{O}. \quad (2.13)$$

The latter equality shows that two operators can be transposed in the trace even if they do not commute. However, in the general case, only a clockwise or counter-clockwise cyclic permutation without the transposition of the operators is possible (if the operators do not commute).

Thus, to determine the mean value of the operator \hat{O} , it is necessary to multiply this operator by the density matrix and to calculate the sum of the diagonal elements of the resulting matrix.

According to the theory of matrices, any matrix of rank $m = n = (2s + 1)$ can be expanded in a complete set of $(2s + 1)^2$ matrices $\{s_\nu\}$ of the same rank satisfying the orthogonality condition

$$\text{Tr } s_\nu s_\mu = \delta_{\nu\mu} (2s + 1). \quad (2.14)$$

Therefore, we can write the expansion

$$\hat{\rho} = \sum_{\mu=1}^{(2s+1)^2} C_\mu s_\mu. \quad (2.15)$$

Multiplying this relation by s_ν from the right and calculating the trace, we obtain

$$\text{Tr } \hat{\rho} s_\nu = \sum_{\mu} C_\mu \text{Tr } s_\mu s_\nu = (2s + 1) \sum_{\mu} C_\mu \delta_{\mu\nu} = (2s + 1) C_\nu. \quad (2.16)$$

The substitution of the coefficients C_μ determined from Eq. (2.16) into Eq. (2.15) finally gives

$$\hat{\rho} = \frac{1}{2s + 1} \sum_{\mu=1}^{(2s+1)^2} \text{Tr } (\hat{\rho} \hat{s}_\mu) \hat{s}_\mu. \quad (2.17)$$

Substituting $\hat{O} = \hat{s}_\mu$ (spin operator) into Eq. (2.13), we obtain the polarization vector \vec{P} :

$$\frac{1}{2}\vec{P} = \langle \hat{s}_\mu \rangle = \text{Tr}(\hat{\rho}\hat{s}_\mu). \quad (2.18)$$

Hence, the final expression for the density matrix in terms of observable $\vec{P} = \langle \hat{s} \rangle$ has the form

$$\hat{\rho} = \frac{1}{2s+1} \sum_{\mu=1}^{(2s+1)^2} \langle \hat{s}_\mu \rangle \hat{s}_\mu. \quad (2.19)$$

Thus, the density matrix is completely determined by the mean value of the operators \hat{s}_μ . With the normalization of the mean value of the density matrix to unity, we obtain

$$\langle \hat{O} \rangle = \text{Tr} \hat{\rho} \hat{O} / \text{Tr} \hat{\rho}. \quad (2.20)$$

With this normalization condition, the final expression for the density matrix has the form

$$\hat{\rho} = \frac{1}{2s+1} \sum_{\mu=1}^{(2s+1)^2} \text{Tr} \hat{\rho} \langle \hat{s}_\mu \rangle \hat{s}_\mu. \quad (2.21)$$

The above presentation was based on works Martin and Spearman (1970) and Nurushev (1983).

2.2 Reaction Matrix

When considering reactions involving particles with spin, we use the wave functions Ψ and Φ of the initial and final states, respectively. The transition matrix from the initial to final state, M , is defined as follows (Nurushev 1983):

$$\Phi = M\Psi. \quad (2.22)$$

Then, we have two density matrices:

$$\rho_i = \sum_{\alpha} W_{\alpha} \Psi_{\alpha} \Psi_{\alpha}^+ \quad (2.23)$$

for the initial state and

$$\rho_f = \sum_{\alpha} W_{\alpha} \Phi_{\alpha} \Phi_{\alpha}^+ \quad (2.24)$$

for the final state. Here, \sum_{α} stands for averaging over the initial spin states and summation over the final spin states. In view of relation (2.22), the relation between these two matrices is obtained from Eq. (2.24) in the form

$$\rho_f = \sum_{\alpha} W_{\alpha} M \Psi_{\alpha} \Psi_{\alpha}^+ M^+ = M \left(\sum_{\alpha} W_{\alpha} \Psi_{\alpha} \Psi_{\alpha}^+ \right) M^+ \quad (2.24a)$$

or

$$\rho_f = M\rho_i M^+. \quad (2.25)$$

Hence, the solution of the problem of the interaction between particles is reduced to the determination of ρ_i in terms of the mean values of the complete set of the spin matrices s_v (the mean values $\langle s_v \rangle$ are called the observables) and the operator \hat{M} . Then, the density matrix of the final state is unambiguously determined by Eq. (2.25), and a necessary observable in the final state can be calculated.

Let us find the operator $\hat{\rho}_f$ as a function of s_v and M . To this end, we multiply Eq. (2.25) by s_μ from the right and calculate the trace of the product of the matrices:

$$\begin{aligned} \text{Tr } \hat{\rho}_f s_\mu &= \text{Tr} (M \hat{\rho}_i M^+ s_\mu) \\ &= \text{Tr} \left[M \left(\frac{1}{2s+1} \sum_{v=1}^{(2s+1)^2} \text{Tr } \hat{\rho}_i \langle s_v \rangle_i s_v \right) M^+ s_\mu \right] \\ &= \frac{1}{2s+1} \text{Tr } \hat{\rho}_i \sum_{v=1}^{(2s+1)^2} \langle s_v \rangle_i \cdot \text{Tr} (M s_v M^+ s_\mu). \end{aligned} \quad (2.26)$$

In view of the relation

$$\text{Tr} (\hat{\rho}_f s_\mu) = \langle s_\mu \rangle_f \text{Tr } \hat{\rho}_f, \quad (2.27)$$

we obtain

$$\langle s_\mu \rangle_f \cdot \text{Tr } \hat{\rho}_f = \frac{1}{2s+1} \text{Tr } \hat{\rho}_i \sum_{v=1}^{(2s+1)^2} \langle s_v \rangle_i \cdot \text{Tr} (M s_v M^+ s_\mu). \quad (2.28)$$

Denoting the differential cross section as

$$I = \frac{\text{Tr } \hat{\rho}_f}{\text{Tr } \hat{\rho}_i}, \quad (2.29)$$

we arrive at the following expression for the mean value $\langle s_\mu \rangle_f$ of the spin operator s_μ final state:

$$\langle s_\mu \rangle_f \cdot I = \frac{1}{2s+1} \cdot \sum_{v=1}^{(2s+1)^2} \langle s_v \rangle_i \cdot \text{Tr} (M s_v M^+ s_\mu). \quad (2.30)$$

This expression allows one to calculate the mean value of any spin operator s_μ in the final state using the known parameters of the initial state $\langle s_v \rangle_i$ and the scattering matrix M .

Let us consider reactions of the type

$$\pi + N = \pi + N, \quad (2.31)$$

or, in the spin notation, $0 + 1/2 \rightarrow 0 + 1/2$ (spins of the pion and nucleon are 0 and $1/2$, respectively). In the corresponding two-dimensional spin space, the Pauli matrices $\vec{\sigma}$ ($\sigma_x, \sigma_y, \sigma_z$) together with the identity matrix 1 can be used as a complete

set of spin operators. In this case, the density matrix of the initial state can be written in the form

$$\hat{\rho}_i = C_0 \cdot 1 + \vec{C}_1 \cdot \vec{\sigma}. \quad (2.32)$$

Let us determine the coefficients C_0 and \vec{C}_1 . From the normalization condition of the trace of the density matrix $\hat{\rho}_i$ to unity, we obtain

$$Tr \hat{\rho}_i = C_0 \cdot Tr 1 + \vec{C}_1 \cdot Tr \vec{\sigma} = 2C_0 = 1, \quad (2.33)$$

because

$$Tr \vec{\sigma} = 0. \quad (2.34)$$

Relation (2.34) is easily verified by writing the Pauli matrices in the explicit form

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.35)$$

Let the nucleon in the initial state be polarized and have polarization vector \vec{P}_t (the subscript t means the target). For the initial state,

$$\begin{aligned} \vec{P}_t = \langle \vec{\sigma}_t \rangle &= \frac{Tr \hat{\rho}_t \vec{\sigma}_t}{Tr \hat{\rho}_t} = Tr \vec{\sigma}_t \cdot \left(\frac{1}{2} + \vec{C}_1 \cdot \vec{\sigma} \right) = Tr \vec{\sigma}_t (\vec{C}_1 \cdot \vec{\sigma}) \\ &= C_{1k} \cdot \vec{e}_k \cdot Tr \sigma_k \sigma_i = C_{1k} \cdot \vec{e}_k \cdot 2\delta_{ik} = 2\vec{C}_{1t}. \end{aligned} \quad (2.36)$$

Here, \vec{e}_k are the unit coordinate vectors in the Cartesian coordinate system and we use the relation

$$(\vec{\sigma} \cdot \vec{A})(\vec{\sigma} \cdot \vec{B}) = \vec{A} \cdot \vec{B} + i\vec{\sigma} \cdot (\vec{A} \times \vec{B}), \quad (2.37)$$

which is easily proved using relations (2.35). Thus, we obtain

$$C_0 = \frac{1}{2}, \quad \vec{C}_1 = \frac{1}{2} \vec{P}_t. \quad (2.38)$$

Then, the density matrix of the initial state $\hat{\rho}_i$, is represented in the form

$$\hat{\rho}_i = \frac{1}{2}(1 + \vec{P}_t \cdot \vec{\sigma}). \quad (2.39)$$

Hence, the density matrix $\hat{\rho}$ is completely determined by the target polarization vector \vec{P}_t (or the beam polarization vector \vec{P}_B in the case of the $1/2 + 0 \rightarrow 1/2 + 0$ reaction).

Note that, in view of the properties of the Pauli matrices, expression (2.39) cannot include operators above the first order, because all such operators are reduced to an operator maximally of the first order.

2.3 R , P , and T Transformations

In the next section, we consider nucleon–nucleon elastic scattering and, following Wolfenstein and Ashkin (1952), construct the elastic scattering matrix. Here, as a

preparation to this consideration, we discuss the constraints on this matrix that follow from the physical requirements of the isotropy of space (R operation), space inversion (P operation), and time reversal (T operation). Below in this section, we follow Bilenky et al. (1964). In the interaction (or Heisenberg) representation, the S -matrix is defined in terms of the Dirac brackets as

$$|\psi(+\infty)\rangle = S|\psi(-\infty)\rangle. \quad (2.40)$$

Here, $|\psi(-\infty)\rangle$ is the wave function of the system in the initial state at $t \rightarrow -\infty$ and $|\psi(+\infty)\rangle$ is the wave function of the system in the final state. According to this definition, the S -matrix transforms the initial state of two free nucleons to the final state with allowance for their interactions. Thus, the S -matrix contains all information on the interaction between the nucleons. If the nucleons do not interact, it is reasonable to set $S = 1$. Then, we formulate the necessary physical requirements.

1. R operation. Let $|\psi(t)\rangle$ be the wave function of the system at time t in an arbitrary reference frame called base. We introduce the second reference frame R rotated by a certain angle and denote the wave function of the system in this reference frame as $|\psi(R, t)\rangle$. The wave functions in two reference frames should be related by a unitary transformation (according to the requirement that the numbers of particles in both reference frames should be the same). The unitarity of the operator implies the equality $U^+(R) = U^{-1}(R)$.

Hence,

$$|\psi(R, t)\rangle = U(R)|\psi(t)\rangle. \quad (2.41)$$

The matrix $U(R)$ is obviously a function of the rotation angles of the R frame with respect to the base frame. The multiplication of Eq. (2.40) by $U(R)$ from the left gives

$$|\psi(R, +\infty)\rangle = U(R)S U^{-1}(R)|\psi(R, -\infty)\rangle. \quad (2.42)$$

The wave functions $|\psi(R, -\infty)\rangle$ and $|\psi(R, +\infty)\rangle$ describe the initial and final states of the nucleons, respectively, in the rotated reference frame R . Hence, in this reference frame, by the definition of the S -matrix, which is determined only by the interaction dynamics but is independent of the choice of the reference frame, the wave functions should be related by the same S -matrix as in Eq. (2.40):

$$|\psi(R, +\infty)\rangle = S|\psi(R, -\infty)\rangle. \quad (2.43)$$

Comparison of Eqs. (2.42) and (2.43) shows that

$$U(R)S U^{-1}(R) = S. \quad (2.44)$$

Since the matrix $U(R)$ is unitary, this equality can be represented in the other form

$$U^{-1}(R)S U(R) = S. \quad (2.45)$$

Relation (2.44) (or (2.45)) expresses the invariance of strong interactions under rotations of the reference frame in physical space. An application of this relation will be considered elsewhere.

2. *P* operation. It is postulated that strong interactions are invariant under space inversion. This property is also called the parity conservation law. Let us consider the constraints imposed by this postulate on the S -matrix. As in the above consideration, we take the initial reference frame as base. As above, the wave function in this frame is denoted as $|\psi(t)\rangle$. We introduce the reference frame I , in which all coordinate axes are inverted, i.e., $x \rightarrow -x$, $y \rightarrow -y$, and $z \rightarrow -z$. If the base frame is left-handed, the frame I is right-handed. Let $U(I)$ be a unitary operator transforming the function $|\psi(t)\rangle$ in the base frame to the wave function in the I frame:

$$|\psi(I, t)\rangle = U(I)|\psi(t)\rangle. \quad (2.46)$$

In this case, both functions describe the same physical state, but in different coordinates. Hence, the S -matrices in two frames should be the same. Let us determine the matrix $S(I)$. By the definition,

$$|\psi(I, +\infty)\rangle = S(I)|\psi(I, -\infty)\rangle. \quad (2.47)$$

At the same time, from Eq. (2.46) we obtain

$$|\psi(I, +\infty)\rangle = U(I)|\psi(+\infty)\rangle = U(I)S|\psi(-\infty)\rangle = U(I)SU^{-1}|\psi(I, -\infty)\rangle. \quad (2.48)$$

Comparison of Eqs. (2.47) and (2.48) shows that

$$S(I) = U(I)SU^{-1}(I). \quad (2.49)$$

Thus, the postulate of the invariance of strong interactions under space inversion leads to the relation

$$S = U^{-1}(I)SU(I). \quad (2.50)$$

These two cases imply that the invariance under the R and P transformations reduces to the commutativity of the S -matrix and the corresponding transformation U matrices.

3. *T* operation. The principle of the invariance of the strong interaction under time reversal is formulated as follows. Let us consider the Schrödinger equation in the interaction representation

$$i \frac{\partial |\psi(t)\rangle}{\partial t} = H(t)|\psi(t)\rangle. \quad (2.51)$$

Changing $t \rightarrow -t$ and taking complex conjugation in this equation, we arrive at the equation

$$i \frac{\partial |\psi(-t)\rangle^*}{\partial t} = H^*(-t)|\psi(-t)\rangle^*. \quad (2.52)$$

Since $H^*(-t) \neq H(t)$ in the general case, this new equation is not a Schrödinger equation. However, let us assume that there is a unitary operator $U(T)$ providing the transformation

$$|\psi(T, t)\rangle = U(T)|\psi(-t)\rangle^*. \quad (2.53)$$

The substitution of this relation into Eq. (2.52) yields

$$i \frac{\partial |\psi(T, t)\rangle}{\partial t} = U(T)H^*(-t)U^{-1}(T)|\psi(T, t)\rangle. \quad (2.54)$$

Setting

$$H(t) = U(T)H^*(-t)U^{-1}(T), \quad (2.55)$$

we arrive at the Schrödinger equation

$$i \frac{\partial |\psi(T, t)\rangle}{\partial t} = H(t)|\psi(T, t)\rangle. \quad (2.56)$$

Thus, relation (2.55) presents a mathematical formulation of the physical postulate of the invariance of the interaction under time reversal.

According to Eq. (2.55), for each wave function $|\psi(t)\rangle$ that is a solution of Schrödinger equation (2.51), there is another function $|\psi(T, t)\rangle$ that satisfies Eq. (2.56) and describes the motion of the system in the reverse time direction. Let us obtain the requirement imposed on the S -matrix by the time reversibility condition.

We have two Schrödinger equations with two wave functions, but with the same S -matrix, because the S -matrix is independent of the initial state of the system, but is completely determined by the dynamics of interaction.

According to the above consideration and by analogy with relation (2.40), we write

$$|\psi(T, +\infty)\rangle = S|\psi(T, -\infty)\rangle.$$

Using relation (2.53), we represent this equality in the form

$$U(T)|\psi(-\infty)\rangle^* = SU(T)|\psi(+\infty)\rangle^*.$$

The multiplication of this relation by $U^{-1}(T)$ from the left yields

$$|\psi(-\infty)\rangle^* = U^{-1}(T)SU(T)|\psi(+\infty)\rangle^*. \quad (2.57)$$

Let us take into account that the S -matrix is unitary, i.e., $S^+S = 1$ and that $S^+ = \tilde{S}^*$ by the definition, where the asterisk and tilde mean the complex conjugation and transposition of the matrix, respectively. Multiplying Eq. (2.40) by S^+ from the left, taking complex conjugation, and taking into account the properties of the S -matrix, we obtain

$$|\psi(-\infty)\rangle^* = \tilde{S}|\psi(+\infty)\rangle^*. \quad (2.58)$$

The comparison of Eqs. (2.57) and (2.58) shows that

$$U^{-1}(T)SU(T) = \tilde{S}. \quad (2.59)$$

This is the requirement imposed on the S -matrix by the time-reversal invariance of the interaction.

The results are summarized as follows:

- the invariance of the interaction under rotation of the reference frame, the corresponding R operation is given by relation (2.45): $U^{-1}(R)SU(R) = S$;

- the invariance of the interaction under space inversion, the corresponding P operation is given by relation (2.50): $U^{-1}(I)SU(I) = S$;
- the invariance of the interaction under time reversal, the corresponding T operation is given by expression (2.59): $U^{-1}(T)SU(T) = \tilde{S}$.

In applications, another matrix M determined only by the interaction between the nucleons is used. The initial state of the particles at $t \rightarrow -\infty$ is denoted as $|i\rangle$, and the final state at $t \rightarrow +\infty$ is denoted as $|f\rangle$. The particles in the initial and final states do not interact and have the relative momenta \vec{p} and \vec{p}' , total momenta \vec{Q} and \vec{Q}' , and total energies E and E' , respectively. The matrix M is expressed in terms of the S matrix as

$$S - 1 = M, \quad (2.60)$$

or in terms of the matrix elements, taking into account the conservation of the energy and momentum:

$$\langle f|S|i\rangle = \langle f|i\rangle - 2\pi i \delta(\vec{Q}' - \vec{Q}) \delta(E' - E) \langle f|M|i\rangle. \quad (2.61)$$

Here, the δ functions ensure the conservation of the momentum and energy. The matrix M transforms the initial state $|i\rangle$ to the final state $|f\rangle$ and acts only in the spin space. By the definition of the matrix elements,

$$\langle f|M|i\rangle = (\chi'^+ M(\vec{p}', \vec{p}) \chi). \quad (2.62)$$

Here, χ and χ' are the spin wave functions of the initial and final states of the nucleons, respectively. The matrix M , as well as the S matrix, is determined by the dynamics of interaction and satisfies the requirements of the R , P , and T invariances. Let us consider their in more detail.

From the postulate of the invariance of the interaction under rotation of the reference frame, the following constraint on the S -matrix was obtained (see (2.45)):

$$U^{-1}(R)SU(R) = S.$$

Let us represent this relation in terms of the matrix elements in the base and rotated R frames

$$\langle f|U^{-1}(R)SU(R)|i\rangle = \langle f, R|S|R, i\rangle = \langle f|S|i\rangle. \quad (2.63)$$

Here, $|R, i\rangle$ and $|R, f\rangle$ are the wave functions in the rotated R frame. The total, Q , and relative, p , momenta in the base reference frame are related to the respective momenta Q_R and p_R in the rotated reference frame as

$$(Q_R)_i = a_{il} Q_i, \quad (p_R)_i = a_{il} p_i. \quad (2.64)$$

Here, a is the rotation matrix from the base frame to the R frame and a_{li} are its matrix elements (in this case, the cosines and sines of the rotation angle from the old to new reference frame).

Relation (2.63) indicates that the respective elements of the S matrix in different frames are the same. According to M -matrix definition (2.61), this is valid for its matrix elements:

$$(\chi'^+(R) M(\vec{p}'_R, \vec{p}_R) \chi(R)) = (\chi'^+(p) M(\vec{p}', \vec{p}) \chi(p)), \quad (2.65)$$

where $\chi(R)$ and $\chi'(R)$ are the spin wave functions in the rotated frame and χ and χ' , in the base frame. Since these sets of functions describe the same spin state, but in different frames, they should be related by a unitary transformation:

$$\chi(R) = U(R)\chi, \quad \chi'(R) = U'(R)\chi'. \quad (2.66)$$

The nucleon–nucleon scattering under consideration involves two spin particles in the initial and final states of the reaction. This means that the spin functions in the initial and final states of the reaction are the products of the functions of individual nucleons. As a result, the unitary operators $U(R)$ and $U'(R)$ are the direct products of the matrices acting on the spin functions of the individual particles.

The mean value of the spin operator in quantum mechanics is an observable, namely, the polarization vector (more precisely, the polarization vector is $\vec{P} = \vec{\sigma} = 2\vec{s}$, where $\vec{\sigma}$ is the Pauli operator and \vec{s} is the spin vector). The mean value of the spin operator should be transformed as a vector:

$$\chi^+(R)s_l\chi(R) = a_{li}\chi^+s_i\chi.$$

Here, s_l is the spin operator of one of the initial nucleons. Therefore, the matrix $U(R)$ should satisfy the condition

$$U^{-1}(R)s_lU(R) = a_{li}s_i. \quad (2.67)$$

The same condition is obviously imposed on the matrix $U'(R)$. For a given rotation angle of the frame R , the matrix U can be reconstructed from these conditions.

From Eqs. (2.65) and (2.66), it follows that

$$U'^{-1}(R)M(\vec{p}'_R, \vec{p}_R)U'(R) = M(\vec{p}', \vec{p}). \quad (2.68)$$

This is the mathematical expression of the postulate of the invariance of the interaction under space rotation.

Then, we consider the P operation, i.e., space inversion. Under this operation, the momenta, being polar vectors, change signs, whereas the spin vector, being an axial vector, does not change sign. Therefore,

$$s_l = U^{-1}(I)s_lU(I), \quad s'_l = U'^{-1}(I)s'_lU'(I), \quad (2.69)$$

where the unitary matrix $U(I)(U'(I))$ ensures the transformation of the wave function of the initial (final) state from the base frame to the inverted frame I . This transformation is written as follows:

$$\chi(I) = U(I)\chi, \quad \chi'(I) = U'(I)\chi'.$$

The condition on the S -matrix provides

$$(\chi'^+M(\vec{p}', \vec{p})\chi) = I_i I_f^* (\chi'^+M(-\vec{p}', -\vec{p})\chi(I)). \quad (2.70)$$

Here, I_i and I_f are the internal parities of two initial and two final nucleons, respectively. Using the relation between the wave functions χ and χ' , we obtain

$$M(\vec{p}', \vec{p}) = I_i I_f^* U^{-1}(I)M(-\vec{p}', -\vec{p})U(I). \quad (2.71)$$

2.4 Unitarity Condition

The Schrödinger equation for the wave function Ψ_k has the form

$$\nabla^2 \Psi_k + \frac{2\mu}{\hbar^2} (E - \hat{u}) \Psi_k = 0, \quad (2.72)$$

where μ is the reduced mass of colliding particles and \hat{u} is the potential energy of their interaction in the operator form (can include the spin and isospin operators). For the case of elastic scattering, Ψ_k can be represented in the form of the sum of two terms:

$$\Psi_k = e^{i\vec{k} \cdot \vec{r}} \chi + \frac{1}{r} e^{ikr} M(\vec{k}'', \vec{k}) \chi. \quad (2.73)$$

The first term is an incident plane wave; the second term is the divergent scattering waves; \vec{k} and \vec{k}'' are the wave vectors of the initial (before the interaction) and final states, respectively; χ is the spinor function of the initial state; and M is the reaction matrix. Since the elastic scattering is considered in the center-of-mass frame,

$$|\vec{k}''| = |\vec{k}|, \quad \hat{u}^+ = \hat{u}. \quad (2.74)$$

The second condition is the Hermitian condition imposed on the interaction potential, because its eigenvalues should be real.

Then, the interaction in the intermediate state of the colliding particles leads to the transition of the wave vector \vec{k}'' to the wave vector \vec{k}' , determining a given direction (e.g., the direction to a recording detector).

At large distances from the collision center, the following expansion can be used (to define the absolute phases, it is necessary to include a known, for example, electromagnetic or weak interaction in addition to strong interactions):

$$\begin{aligned} \Psi_k^0 &= e^{i\vec{k} \cdot \vec{r}} = e^{ikz \cos \theta} = \sum_{l=0}^{\infty} i^l (2l+1) P_l(\cos \theta) \frac{\sin(kr - \frac{1}{2}l\pi)}{kr} \\ &= \frac{1}{ikr} \sum_{l=0}^{\infty} \frac{(2l+1)}{2} P_l(\cos \theta) e^{ikr} \\ &\quad - \frac{1}{ikr} \sum_{l=0}^{\infty} (-1)^l \frac{(2l+1)}{2} P_l(\cos \theta) e^{-ikr}. \end{aligned} \quad (2.75)$$

It can be shown that

$$\begin{aligned} \sum_{l=0}^{\infty} \frac{(2l+1)}{2} P_l(\cos \theta) &= \delta(1 - \cos \theta), \\ \sum_{l=0}^{\infty} (-1)^l \frac{(2l+1)}{2} P_l(\cos \theta) &= \delta(1 + \cos \theta). \end{aligned} \quad (2.76)$$

These formulas are verified by multiplying the both sides by $P_l(\cos \theta)$ and integrating with respect to $\cos \theta$. The validity of relations (2.76) is obvious in view of the orthogonality condition of the Legendre polynomials,

$$\int_{-1}^1 P_{l'}(\cos \theta) P_l(\cos \theta) d \cos \theta = \frac{2}{2l+1} \delta_{ll'} \quad (2.77)$$

and the definition of the Dirac δ functions

$$\int f(x) \delta(x-a) dx = f(a). \quad (2.78)$$

Taking into account Eq. (2.76), from Eq. (2.75) we obtain

$$e^{i\vec{k} \cdot \vec{r}} = \frac{1}{ikr} e^{ikr} \delta(1 - \cos \theta) - \frac{1}{ikr} e^{-ikr} \delta(1 + \cos \theta). \quad (2.79)$$

Therefore,

$$\Psi_k(\vec{r}) = \frac{1}{r} e^{ikr} \cdot \chi \left(\frac{1}{ik} \delta(1 - \cos \theta) + M(\vec{k}'', \vec{k}) \right) - \frac{1}{ikr} e^{-ikr} \delta(1 + \cos \theta) \cdot \chi. \quad (2.80)$$

Similarly,

$$\Psi_{k'}(\vec{r}) = \frac{1}{r} e^{-ikr} \cdot \chi^+ \left(M(\vec{k}'', \vec{k}') - \frac{1}{ik} \delta(1 - \cos \theta) \right) + \frac{1}{ikr} e^{ikr} \delta(1 + \cos \theta) \cdot \chi^+. \quad (2.81)$$

From the Schrödinger equation

$$\nabla^2 \Psi_k + \frac{2\mu}{\hbar^2} (E - \hat{u}) \Psi_k = 0, \quad \nabla^2 \Psi_{k'}^+ + \frac{2\mu}{\hbar^2} \cdot \Psi_{k'}^+ \cdot (E - \hat{u}) = 0, \quad (2.82)$$

we can obtain the relation

$$\Psi_{k'}^+ \nabla^2 \Psi_k - \nabla^2 \Psi_{k'}^+ \Psi_k = 0, \quad (2.83)$$

which can be modified to the form

$$\nabla(\Psi_{k'}^+ \nabla \Psi_k - \nabla \Psi_{k'}^+ \Psi_k) = 0. \quad (2.84)$$

The integration over the volume V provides

$$\int \nabla(\Psi_{k'}^+ \nabla \Psi_k - \nabla \Psi_{k'}^+ \Psi_k) dv = \int \nabla(\Psi_{k'}^+ \nabla \Psi_k - \nabla \Psi_{k'}^+ \Psi_k) ds = 0. \quad (2.85)$$

Using Eqs. (2.80) and (2.81), as well as the operator $\nabla = \frac{\partial}{\partial r}$, we obtain

$$\begin{aligned} & \int \left\{ \frac{1}{kr^2} \delta(1 + \cos \theta) \delta(1 + \cos \theta') - \frac{1}{r^2} \left[M^+(\vec{k}'', \vec{k}') + \frac{i}{k} \delta(1 - \cos \theta') \right] \right. \\ & \quad \cdot \left. \left[M(\vec{k}'', \vec{k}) - \frac{1}{k} \delta(1 - \cos \theta) \right] \right\} ds = 0, \end{aligned} \quad (2.86)$$

where $ds = r^2 d\omega_{k''}$. After the integration, we arrive at the formula

$$\frac{1}{2i} [M(\vec{k}', \vec{k}) - M^+(\vec{k}, \vec{k}')] = \frac{k}{4\pi} \int M^+(\vec{k}'', \vec{k}') \cdot M(\vec{k}'', \vec{k}) d\omega_{k''}. \quad (2.87)$$

This condition can be generalized to the case of inelastic reactions:

$$\frac{1}{2i} \left[\frac{1}{k_a} M_{ab}(\vec{k}', \vec{k}) - \frac{1}{k_b} M_{ba}^+(\vec{k}, \vec{k}') \right] = \sum_C \frac{1}{4\pi} \int M_{bc}^+(\vec{k}'', \vec{k}) \cdot M_{ac}(\vec{k}'', \vec{k}') d\omega_{k''}, \quad (2.88)$$

where the summation over all possible reaction channels is implied.

Relation (2.87) provides a number of important consequences. At $\vec{k}' = \vec{k}$ (elastic scattering at zero angle), this relation gives the so-called optical theorem

$$\text{Im } a(0) = \frac{k}{4\pi} \sigma_{\text{TOT}}, \quad (2.89)$$

which is the relation between the imaginary part of the forward elastic scattering amplitude $a(0)$ and the total cross section σ_{TOT} .

The application of relation (2.88) to the pion–nucleon scattering matrix gives two relations

$$\begin{aligned} \text{Im } a(\vec{k}'', \vec{k}') &= \frac{k}{4\pi} \int [a^*(\vec{k}'', \vec{k}') \cdot a(\vec{k}'', \vec{k}') \\ &\quad + i \vec{b}^*(\vec{k}'', \vec{k}') \times \vec{b}(\vec{k}'', \vec{k}') \cdot (\vec{n}'' \times \vec{n}')] d\omega_{k''}, \end{aligned} \quad (2.90)$$

$$\begin{aligned} \text{Re } \vec{b}(\vec{k}', \vec{k}) &= \frac{k}{4\pi} \int [a^*(\vec{k}'', \vec{k}') \cdot \vec{b}(\vec{k}'', \vec{k}') + \vec{b}^*(\vec{k}'', \vec{k}') \cdot a(\vec{k}'', \vec{k}') \\ &\quad + \text{Re}[\vec{b}^*(\vec{k}'', \vec{k}') \times \vec{b}(\vec{k}'', \vec{k}') \cdot (\vec{n}'' \times \vec{n}')] d\omega_{k''}. \end{aligned} \quad (2.91)$$

Similar relations are also valid for nucleon–nucleon scattering. These relations are particularly suitable at low energies, when only the elastic channel is opened. As a result, the number of necessary experiments is halved (by two and five experiments for the πN and nucleon–nucleon scatterings, respectively). However, these statements should be considered carefully, because they are valid only under ideal conditions, which are almost inaccessible in actual experiments.

The presentation in this section follows Nurushev (1983).

2.5 Pion–Nucleon Scattering

Let us consider reactions of the type

$$\pi + N = \pi + N, \quad (2.92)$$

or, in the spin notation, $0 + 1/2 \rightarrow 0 + 1/2$. In the corresponding two-dimensional spin space, the Pauli matrices $\vec{\sigma}(\sigma_x, \sigma_y, \sigma_z)$ together with the identity matrix 1 can be used as a complete set of spin operators. In this case, the density matrix of the initial state can be written in the form (see Eq. (2.32) in Sect. 2.2)

$$\hat{\rho}_i = C_0 \cdot 1 + \vec{C}_1 \cdot \vec{\sigma}. \quad (2.93)$$

Hence, the density matrix $\hat{\rho}$ is completely determined by the target, \vec{P}_t , or beam, \vec{P}_B , polarization vector.

Then, it is necessary to determine the reaction matrix M . In the general case, it should be, first, a function of two variables, for example, momenta \vec{k}_i and \vec{k}_f before and after reactions, respectively, and, second, a two-row matrix in spin space. As the matrix, it can be expanded in the complete set consisting of the Pauli matrices and identity matrix:

$$M(\vec{k}_i, \vec{k}_f) = a(\vec{k}_i, \vec{k}_f) \cdot I + \vec{b}(\vec{k}_i, \vec{k}_f) \cdot \vec{\sigma}. \quad (2.94)$$

According to the experimental data, we require that strong interactions satisfy the following conditions (Nurushev 1983):

1. Parity conservation law. Since the parities of the initial and final systems are identical, the M -matrix should be a scalar function of the initial energy and scattering angle of the particles. This means that the vector \vec{b} , as well as $\vec{\sigma}$, should also be axial. The only axial vector that can be composed of two vectors \vec{k}_i and \vec{k}_f has the form

$$\vec{b}'(\vec{k}_i, \vec{k}_f) = b\vec{n}, \quad (2.95)$$

where $\vec{n} = \vec{k}_i \times \vec{k}_f / |\vec{k}_i \times \vec{k}_f|$ is the unit vector perpendicular to the reaction plane. The quantity $a(\vec{k}_i, \vec{k}_f)$ should obviously be a scalar function.

2. Time reversibility. Under the time reversal operation, $\vec{k}_i \rightarrow -\vec{k}_f$ and $\vec{k}_f \rightarrow -\vec{k}_i$, so that \vec{n} changes sign. Under this operation, $\vec{\sigma}$ also changes sign, so that the quantity $\vec{\sigma} \cdot \vec{n}$ is a scalar. For the $(0 + 1/2)$ system, this requirement is satisfied simultaneously with requirement 1. However, for more complex systems (e.g., $(1/2 + 1/2)$), time reversibility gives rise to additional constraints.

Thus, the πN elastic scattering matrix $M(\vec{k}_i, \vec{k}_f)$ has the form

$$M(\vec{k}_i, \vec{k}_f) = a(\vec{k}_i, \vec{k}_f) + b(\vec{k}_i, \vec{k}_f) \cdot \vec{\sigma} \cdot \vec{n}. \quad (2.96)$$

Here, b and a are called the amplitudes with and without spin flop, respectively, are complex quantities; therefore, four real functions (for a given initial energy and a given scattering angle) should be determined from experiments. A set of independent experiments necessary for the unambiguous determination of all reaction amplitudes is called the complete set of experiments. Hence, the complete set of experiments for the πN system should include at least four independent experiments.

However, reality is much more complicated than the above description. First, since the experimentally measured quantities are quadratic combinations of the amplitudes a and b , only the difference of the phases of the amplitudes a and b for strong interactions, rather than their absolute values, can be determined from an experiment. To determine the absolute values of the phases, it is necessary to include a known (for example, electromagnetic or weak) interaction in addition to strong interactions. Thus, the complete set of experiments should include more than four experiments. Second, if elastic scattering is the single allowed reaction channel, the unitarity condition gives rise to two additional relations between the amplitudes a and b ; for this reason, to determine these amplitudes, it is sufficient to perform two independent experiments.

However, at energies above the pion production threshold, the number of experiments in the complete set is larger than four.

In reality, there are three reactions induced by charged pions:

$$\begin{aligned}\pi^+ + p &\rightarrow \pi^+ + p(a), & \pi^- + p &\rightarrow \pi^- + p(b), \\ \pi^- + p &\rightarrow \pi^0 + n(c).\end{aligned}\tag{2.97}$$

These three reactions are related by the requirement of the isotopic invariance of strong interactions. As a result, the matrices of these reactions are related as

$$M(a) = M_1, \quad M(b) = \frac{1}{2}(M_1 + M_0), \quad M(c) = \frac{1}{2}(M_1 - M_0).\tag{2.98}$$

Here, the matrices M_1 and M_0 correspond to the isotopic states of the pion-nucleon system with $T = 3/2$ and $1/2$, respectively. These matrices are reconstructed similarly to the matrix M . Disregarding isotopic invariance, three reactions (2.97) are described by the set of 12 experiments. Allowance for isotopic invariance reduces this number to 8.

For the system of two particles with spin $1/2$ (example is pion + nucleon), we present the proof of the equality $P = A$, where P and A are the polarization of the particle and its asymmetry in the binary reactions, respectively; this relation is very important in applications (Bilenky et al. 1964).

Let us consider the wave function of the system with the inverted direction of the wave vector that satisfies the Schrödinger equation with reversed time ($t \rightarrow -t$). We represent it in the form

$$\begin{aligned}\Psi_{-\vec{k}'} &= e^{-i\vec{k}'\vec{r}} + \frac{1}{r}e^{ikr}M(\vec{k}'', -\vec{k}') \\ &= \frac{i}{kr}e^{-ikr} \cdot \delta(1 - \cos\theta') \\ &\quad + \frac{1}{r}e^{ikr} \left[M(\vec{k}'', -\vec{k}') - \frac{i}{k} \cdot \delta(1 - \cos\theta') \right].\end{aligned}\tag{2.99}$$

The substitution of this wave function into the following Eq. (2.85) from Sect. 2.4:

$$\int \nabla(\Psi_{\vec{k}'}^+ \nabla \Psi_{\vec{k}} - \nabla \Psi_{\vec{k}'}^+ \Psi_{\vec{k}}) dv = \int \nabla(\Psi_{\vec{k}'}^+ \nabla \Psi_{\vec{k}} - \nabla \Psi_{\vec{k}'}^+ \Psi_{\vec{k}}) ds = 0,$$

gives

$$\begin{aligned}&\int [M(\vec{k}'', \vec{k}') \cdot \delta(1 - \cos\theta') + \frac{1}{ik} \cdot \delta(1 - \cos\theta') \cdot \delta(1 - \cos\theta) \\ &\quad - M(\vec{k}'', -\vec{k}') \cdot \delta(1 + \cos\theta)] d\omega_{\vec{k}''} = 0.\end{aligned}\tag{2.100}$$

After the integration, we obtain

$$M(\vec{k}', \vec{k}) = M(-\vec{k}, -\vec{k}').\tag{2.101}$$

This relation is the condition of the time reversibility of the process.

Scattering matrix (2.96) constructed above satisfies this condition. Using the explicit form of the scattering matrix, we prove the following statement widely used in applications.

Let the polarization of the particle d be measured in the reaction

$$a(0) + b(1/2) \rightarrow c(0) + d(1/2) \quad (2.102)$$

(the spins of the particles are given in the parentheses). This is usually achieved by the rescattering of the particle d on a certain nucleus with a known analyzing power. To explain new terms, we make a brief digression.

Let a beam with a given energy and unit polarization be scattered on a nuclear target at a given angle. The number of the scattered particles per unit flux of the incident beam whose polarization is directed upward from the scattering plane is denoted as N_1 . Under the same conditions, this number for the beam polarization downwards to the scattering plane is denoted as N_2 . In this notation, the analyzing power of the target is defined by the formula

$$A_N = \frac{N_1 - N_2}{N_1 + N_2}.$$

If the beam is partially polarized, i.e., $P \neq 1$, raw asymmetry (directly measured in experiment), can be defined as

$$\varepsilon = P \cdot A.$$

The quantity ε is also called left–right asymmetry or, sometimes, “raw” asymmetry. According to the definition, the asymmetry ε coincides with the analyzing power A_N at the 100 % polarization of the beam.

We denote the polarization of the particle d as P . Let us consider the reverse reaction

$$c(0) + d_{\uparrow}(1/2) \rightarrow a(0) + b(1/2), \quad (2.103)$$

where the particle d is polarized. Let the left–right asymmetry A_N is measured in the formation of the particle a (or b). Theorem: the polarization P of the particle d in reaction (2.102) is equal to the asymmetry A of the particle b (or a) in reaction (2.103). This statement is expressed by the equality

$$P = A_N. \quad (2.104)$$

Let us prove this statement. Indeed, by the definition of polarization (Bilenky et al. 1964).

$$\vec{P} = \frac{\text{Tr}(M\rho_i M^+ \vec{\sigma})}{\text{Tr}(M\rho_i M^+)} \quad (2.105)$$

since the particle b in initial system (2.102) is unpolarized, $\rho_i = 1/2$ and, calculating P , we obtain

$$P \cdot I_0 = 2\text{Re}a^*b = \text{Tr}(MM^+ \vec{\sigma}), \quad (2.106)$$

where

$$I_0 = |a|^2 + |b|^2 \quad (2.107)$$

is the differential cross section for reaction (2.102). Since the particle d in reaction (2.103) is polarized, $\rho_i = \frac{1}{2}(1 + \vec{P}_0 \cdot \vec{\sigma})$ and the scattering cross section is given by the expression

$$I_f = \text{Tr}(M \hat{\rho}_i M^+) = \frac{1}{2} \text{Tr}(M M^+) + \frac{1}{2} \vec{P}_0 \text{Tr}(M \vec{\sigma} M^+). \quad (2.108)$$

The direct calculation shows that $M M^+ = M^+ M$ and

$$\text{Tr}(M \vec{\sigma} M^+) = \text{Tr}(M M^+ \sigma) = 2 I_0 P, \quad (2.109)$$

where P is determined by expression (2.106). Then,

$$I_f = I_0(1 + \vec{P}_0 \cdot \vec{P}). \quad (2.110)$$

By the definition of left–right asymmetry

$$A_N = \frac{1}{P_0} \frac{I_f(+) - I_f(-)}{I_f(+) + I_f(-)} = P, \quad (2.111)$$

quod erat demonstrandum.

When deriving relation (2.111), we explicitly take into account the detailed balance principle, i.e., the equality of the cross sections for the direct and inverse reactions.

Note that the positive sign in relation (2.111) appears for a reaction in which the initial and final states have the same parity. For the case of different parities, the sign in relation (2.111) is negative.

Theorem (2.104) is also proved for the case of the binary reaction when both initial (and final) particles have spin 1/2. This theorem is invalid for the inclusive reactions.

A test of the relation $P = A$ is simultaneously a test of the T invariance in strong interactions. At present, this problem is of very great interest in view of the creation of an absolute polarimeter for the RHIC collider (see Chap. 8 in the second part of the book, which is devoted to the polarimetry of beams).

It is known that a reaction with the production of hyperons is a convenient reaction for verifying the relation $P = A$. If a target is unpolarized, the polarization of a hyperon is determined from its decay. If the target is polarized, left–right asymmetry can be measured with the same instruments. The same is true for the beam. An example of such interactions is the reaction

$$\pi^- + p \rightarrow K^0 + \Lambda(\uparrow)(a), \quad \pi^- + p(\uparrow) \rightarrow K^0 + \Lambda(b). \quad (2.112)$$

This reaction is very convenient because channels (a) and (b) can be measured simultaneously if a polarized target is used. In this case, it is very important to detect both K mesons and Λ hyperons. Averaging the experimental results over the target polarizations (as if the target polarization vanishes), one can determine the polarization of the Λ hyperons (channel (a)). Averaging over the polarization of the Λ hyperons for the polarized target (channel (b)), we determine asymmetry. Comparison of these two observables ensures the direct test of the equality $P = A_N$. Such an experiment has not yet been carried out.

An important application of the relation $P = A_N$ is the measurement of asymmetry in the reaction

$$\pi^- + p(\uparrow) \rightarrow \pi^0 + n. \quad (2.113)$$

For the direct measurement of the neutron polarization on an unpolarized target, it is necessary to scatter the neutron by another target and to detect the scattered neutron. This is a difficult experimental problem owing to loss in the yields for the second scattering process and low neutron detection efficiency. Therefore, the use of the polarized proton target made it possible to measure the neutron polarization in reaction (2.113). These measurements provided first doubts in the Regge pole model very popular in the 1960s.

2.6 Nucleon–Nucleon Scattering

In this section, we present the method for determining the reaction matrix in the nonrelativistic case proposed in Wolfenstein and Ashkin (1952). We will see later that the relativistic approach does not change the results, but leads to the kinematic rotations of the observables lying in the reaction plane. The observables perpendicular to the reaction plane remain unchanged in this case.

2.6.1 Construction of the Reaction Matrix

The system of two nucleons is described by two spin operators $\vec{\sigma}_1$ and $\vec{\sigma}_2$ and two identity operators I_1 and I_2 acting in the spin spaces of the first and second particles, respectively. As a result, the scattering matrix is a four-dimensional matrix depending on the physical vectors $\vec{\sigma}_1$ and $\vec{\sigma}_2$, \vec{k}_i and \vec{k}_f (relative momenta of two nucleons in the initial and final states, respectively). When constructing the NN -scattering matrix, we follow Wolfenstein and Ashkin (1952). Since the initial (two-nucleon) and final (also two-nucleon) systems in our case have the same internal parity, the scattering matrix $M(\vec{\sigma}_1, \vec{\sigma}_2; \vec{k}_i, \vec{k}_f)$ should be a scalar function composed of the combination of spin operators and momenta. Using the spin operators, we can generally compose 16 combinations (complete set):

1	(scalar)	
$(\vec{\sigma}_1 \cdot \vec{\sigma}_2 - 1)$	(scalar)	
$(\vec{\sigma}_1 + \vec{\sigma}_2)$	(axial vector)	
$(\vec{\sigma}_1 - \vec{\sigma}_2)$	(axial vector)	
$(\vec{\sigma}_1 \times \vec{\sigma}_2)$	(axial vector)	
$l_{\alpha\beta} = (\sigma_{1\alpha}\sigma_{2\beta} + \sigma_{1\beta}\sigma_{2\alpha})$	(symmetric tensor).	(A)

In view of the properties of the sigma operators, these combinations cannot include terms of the orders higher than the first order.

The combinations that can be composed of the momenta \vec{k}_i and \vec{k}_f are as follows:

$$\begin{aligned}
 1 & \quad (\text{scalar}) \\
 \vec{k}_f - \vec{k}_i = \vec{K} & \quad (\text{polar vector}) \\
 \vec{k}_f \times \vec{k}_i = \vec{n} & \quad (\text{axial vector}) \\
 \vec{n} \times \vec{K} = \vec{P} & \quad (\text{polar vector}) \\
 K_\alpha K_\beta, n_\alpha n_\beta & \quad (\text{symmetric tensors}) \\
 P_\alpha P_\beta, K_\alpha P_\beta + K_\beta P_\alpha & \quad (\text{symmetric tensors}).
 \end{aligned} \tag{B}$$

Multiplying the quantities from sets (A) and (B), we take into account the requirement of the invariance of the matrix $M(\vec{\sigma}_1, \vec{\sigma}_2; \vec{k}_i, \vec{k}_f)$ under rotation and inversion of space. Thus, the following combinations can be included in the amplitudes of the scattering matrix:

$$1, \quad (\vec{\sigma}_1 \cdot \vec{\sigma}_2 - 1), \quad (\vec{\sigma}_1 + \vec{\sigma}_2) \cdot \vec{n}, \quad (\vec{\sigma}_1 - \vec{\sigma}_2) \cdot \vec{n}, \tag{2.114}$$

$$(\vec{\sigma}_1 \times \vec{\sigma}_2) \cdot \vec{n}, \tag{2.115}$$

$$\sum_{\alpha\beta} l_{\alpha\beta} K_\alpha K_\beta, \quad \sum_{\alpha\beta} l_{\alpha\beta} n_\alpha n_\beta, \quad \sum_{\alpha\beta} l_{\alpha\beta} P_\alpha P_\beta, \quad \sum_{\alpha\beta} l_{\alpha\beta} (K_\alpha P_\alpha + K_\beta P_\beta). \tag{2.116}$$

Combinations (2.116) can be represented in the form

$$\vec{\sigma}_1 \cdot \vec{K} \vec{\sigma}_2 \cdot \vec{K}, \quad \vec{\sigma}_1 \cdot \vec{n} \vec{\sigma}_2 \cdot \vec{n}, \quad \vec{\sigma}_1 \cdot \vec{P} \vec{\sigma}_2 \cdot \vec{P}; \tag{2.117}$$

$$\vec{\sigma}_1 \cdot \vec{K} \vec{\sigma}_2 \cdot \vec{P} + \vec{\sigma}_1 \cdot \vec{P} \vec{\sigma}_2 \cdot \vec{K}. \tag{2.118}$$

Three vectors \vec{K} , \vec{n} , and \vec{P} are mutually orthogonal. As a result, the sum of three terms in (2.117) is equal to the dot product $\vec{\sigma}_1 \cdot \vec{\sigma}_2$, as can be verified by direct calculations. Hence, only two of three terms in (2.117) are independent.

Now, we require the time-reversal invariance of these terms. Under time reversal $t \rightarrow -t$, the spin operator and momentum are transformed as follows (prime marks the quantities with reversed time):

$$\vec{\sigma}' = -\vec{\sigma}, \quad \vec{k}'_i = -\vec{k}_i, \quad \vec{k}'_f = -\vec{k}_f. \tag{2.119}$$

Using Eq. (2.119) and the definition of vectors \vec{K} , \vec{n} , and \vec{P} (see (B)), we can show that

$$\vec{K}' = \vec{K}, \quad \vec{n}' = -\vec{n}, \quad \text{and} \quad \vec{P}' = -\vec{P}. \tag{2.120}$$

Under this transformation, terms (2.115) and (2.118) change sign and, correspondingly, are rejected. Finally, the nucleon–nucleon elastic scattering matrix is expressed in the form

$$\begin{aligned}
 M(\vec{\sigma}_1, \vec{\sigma}_2; \vec{k}_i, \vec{k}_f) = & A + B(\vec{\sigma}_1 \cdot \vec{\sigma}_2 - 1) + C(\vec{\sigma}_1 + \vec{\sigma}_2) \cdot \vec{n} \\
 & + D(\vec{\sigma}_1 - \vec{\sigma}_2) \cdot \vec{n} + E(\vec{\sigma}_1 \cdot \vec{K})(\vec{\sigma}_2 \cdot \vec{K}) \\
 & + F(\vec{\sigma}_1 \cdot \vec{P})(\vec{\sigma}_2 \cdot \vec{P}).
 \end{aligned} \tag{2.121}$$

Let us introduce the triple of orthogonal unit vectors in the center-of-mass frame:

$$\vec{n} = \frac{\vec{k} \times \vec{k}'}{|\vec{k} \times \vec{k}'|}, \quad \vec{m} = \frac{\vec{k}' - \vec{k}}{|\vec{k}' - \vec{k}|}, \quad \vec{l} = \frac{\vec{k} + \vec{k}'}{|\vec{k} + \vec{k}'|}, \quad (2.122)$$

where $\vec{k} = \frac{\vec{k}_i}{|k_i|}$ and $\vec{k}' = \frac{\vec{k}_f}{|k_f|}$ are the unit vectors.

It is convenient to introduce these unit vectors, because the vectors \vec{l} and \vec{m} in the nonrelativistic approximation coincide with the directions of the momenta of the scattered and recoil particles in the laboratory frame, respectively. The scattering matrix is rewritten in the new notation as

$$\begin{aligned} M(\vec{\sigma}_1, \vec{\sigma}_2; \vec{k}_i, \vec{k}_f) = & a + b(\vec{\sigma}_1 \cdot \vec{n})(\vec{\sigma}_2 \cdot \vec{n}) + c(\vec{\sigma}_1 + \vec{\sigma}_2) \cdot \vec{n} \\ & + d(\vec{\sigma}_1 - \vec{\sigma}_2) \cdot \vec{n} + e(\vec{\sigma}_1 \cdot \vec{m})(\vec{\sigma}_2 \cdot \vec{m}) \\ & + f(\vec{\sigma}_1 \cdot \vec{l})(\vec{\sigma}_2 \cdot \vec{l}). \end{aligned} \quad (2.123)$$

The amplitudes a , b , c , d , e , and f are complex functions of the energy and scattering angle $(\vec{k}\vec{k}') = \cos\theta$.

The term with the amplitude d should be absent for nucleon–nucleon scattering. This is proved as follows (Bilenky et al. 1964). Two nucleons in the initial state have the internal parity $(-1)^l$, total spin S , and total isospin T . According to the Pauli exclusion principle, the wave function of two nucleons should be antisymmetric under permutation, i.e., should change sign:

$$P_i = (-1)^l (-1)^{S+1} (-1)^{T+1} = -1. \quad (2.124)$$

A similar relation can also be obtained for the final nucleons with the orbital angular momentum l' , spin S' , and isospin T' :

$$P_f = (-1)^{l'} (-1)^{S'+1} (-1)^{T'+1} = -1. \quad (2.125)$$

To satisfy the condition $P_i = P_f$, we take into account that the parities of the nucleons in the interaction remain unchanged and the terms containing orbital angular momenta can be canceled. We also accept the hypothesis of the isotopic invariance of the strong interaction; for this reason, the terms with isotopic spin T can also be canceled. As a result, we arrive at the condition

$$(-1)^S = (-1)^{S'}. \quad (2.126)$$

Since the possible values of S and S' are 0 (singlet state) and 1 (triplet state), then $S = S'$. This means that transitions only within triplets and singlets separately are allowed in nucleon–nucleon scattering, whereas mixed singlet–triplet and inverse transitions are forbidden. This leads to the exclusion of the term with d in the scattering matrix. The nucleon–nucleon scattering matrix has the final form

$$\begin{aligned} M(\vec{\sigma}_1, \vec{\sigma}_2; \vec{k}_i, \vec{k}_f) = & a + b(\vec{\sigma}_1 \cdot \vec{n})(\vec{\sigma}_2 \cdot \vec{n}) + c(\vec{\sigma}_1 + \vec{\sigma}_2) \cdot \vec{n} \\ & + e(\vec{\sigma}_1 \cdot \vec{m})(\vec{\sigma}_2 \cdot \vec{m}) + f(\vec{\sigma}_1 \cdot \vec{l})(\vec{\sigma}_2 \cdot \vec{l}). \end{aligned} \quad (2.127)$$

The singlet and triplet projection operators are introduced as

$$\hat{S} = \frac{1}{4}[1 - (\vec{\sigma}_1 \cdot \vec{\sigma}_2)], \quad \hat{T} = \frac{1}{4}[3 + (\vec{\sigma}_1 \cdot \vec{\sigma}_2)]. \quad (2.128)$$

Then, expression (2.127) can be represented in the form

$$\begin{aligned} M(\vec{\sigma}_1, \vec{\sigma}_2; \vec{k}_i, \vec{k}_f) = B\hat{S} + \left[C(\vec{\sigma}_1 + \vec{\sigma}_2) \cdot \vec{n} + \frac{1}{2}G(\vec{\sigma}_1 \cdot \vec{m})(\vec{\sigma}_2 \cdot \vec{m}) \right. \\ \left. + (\vec{\sigma}_1 \cdot \vec{l})(\vec{\sigma}_2 \cdot \vec{l}) + \frac{1}{2}H(\vec{\sigma}_1 \cdot \vec{m})(\vec{\sigma}_2 \cdot \vec{m}) - (\vec{\sigma}_1 \cdot \vec{l})(\vec{\sigma}_2 \cdot \vec{l}) \right. \\ \left. + N(\vec{\sigma}_1 \cdot \vec{n})(\vec{\sigma}_2 \cdot \vec{n}) \right] \hat{T}. \end{aligned} \quad (2.129)$$

The amplitude B corresponds to singlet scattering, whereas the remaining four amplitudes describe triplet scattering.

The amplitudes in expressions (2.127) and (2.129) are related as:

$$\begin{aligned} B = a - b - e - f, \quad C = c, \quad G = 2a + e + f, \\ H = e - f, \quad N = a + b. \end{aligned} \quad (2.130)$$

For joint description of all possible types of nucleon–nucleon scattering (pp , nn , and np), the general matrix can be written taking into account isotopic invariance:

$$M(\vec{\sigma}_1, \vec{\sigma}_2; \vec{k}_i, \vec{k}_f) = M_0 \hat{T}_0 + M_1 \hat{T}_1. \quad (2.131)$$

Here,

$$\hat{T}_0 = \frac{1}{4}(1 - \vec{\tau}_1 \cdot \vec{\tau}_2), \quad \hat{T}_1 = \frac{1}{4}(3 + \vec{\tau}_1 \cdot \vec{\tau}_2) \quad (2.132)$$

are the isosinglet and isotriplet projection operators, respectively; and $\vec{\tau}_1$ and $\vec{\tau}_2$ are the isospin operators of the first and second nucleons, respectively. Each of the matrices M_0 and M_1 is a scattering matrix of form (2.129).

The final wave function of the system of two nucleons can be written in the form

$$\chi_f = M(\vec{\sigma}_1, \vec{\sigma}_2; \vec{k}, \vec{k}') \chi_{iS} \chi_{iT}, \quad (2.133)$$

where χ_{iS} and χ_{iT} are the spin and isospin wave functions of the initial system of two nucleons. In view of Eqs. (2.129) and (2.132), the requirement of the antisymmetry of this function provides the following conditions on the amplitude under the change $\theta \rightarrow \pi - \theta$:

- (a) the isotriplet amplitudes B , C , and H do not change their signs, whereas G and N change their signs;
- (b) on the contrary, the isosinglet amplitudes B , C , and H change their signs, whereas G and N do not change their signs.

These relations allow one to investigate pp and nn scatterings only in the angular range $0 \leq \theta \leq 90^\circ$. Moreover, the amplitude analysis for angles 0° , 90° , and 180° can be performed only with three rather than five amplitudes; this significantly reduces the number of necessary experimental observables.

In the case of np scattering, where both isotopic matrices M_0 and M_1 are used, the measurements should be performed in a wider angular range, namely, $0 \leq \theta \leq 180^\circ$.

2.6.2 Some Ways for Experimentally Seeking P - and T -noninvariant Terms in the Matrices of the Strong Interaction

The nucleon–nucleon scattering matrix given by Eq. (2.129) can be written in the form

$$M^{(0)} = (u + v) + (u - v)(\vec{\sigma}_1 \cdot \vec{n})(\vec{\sigma}_2 \cdot \vec{n}) + C[(\vec{\sigma}_1 \cdot \vec{n}) + (\vec{\sigma}_2 \cdot \vec{n})] \\ + (g - h)(\vec{\sigma}_1 \cdot \vec{m})(\vec{\sigma}_2 \cdot \vec{m}) + (g + h)(\vec{\sigma}_1 \cdot \vec{l})(\vec{\sigma}_2 \cdot \vec{l}), \quad (2.134)$$

where \vec{l} , \vec{m} , and \vec{n} were defined in Eq. (2.122).

The aim of the complete set of experiments on nucleon–nucleon scattering is to reconstruct the amplitudes u , v , c , g , and h from the experimental data. In the case of parity violation or time reversal violation, the scattering matrix contains additional terms, which are considered below.

2.6.2.1 Parity Violation

The forward NN -scattering matrix in the case of parity violation can be written in the form

$$M = M^0 + (i/4)(M_{os} - M_{so})(\vec{\sigma}_1 \times \vec{\sigma}_2) \cdot \vec{k} \\ + (i/4)(M_{os} + M_{so})(\vec{\sigma}_1 - \vec{\sigma}_2) \cdot \vec{k}. \quad (2.135)$$

Here, M^0 is given by expression (2.134) and the amplitudes M_{os} and M_{so} determine P -violating triplet–singlet and singlet–triplet transitions, respectively. The total cross section for the interaction between polarized particles corresponding to the matrix M is written in the form (Bilenky and Ryndin 1963; Philips 1963):

$$\sigma_{P_1 P_2} = \sigma_{P_1 P_2}^{(0)} + (1/4)(\sigma_{os} - \sigma_{so})(\vec{P}_1 \times \vec{P}_2) \cdot \vec{k} \\ + (1/4)(\sigma_{os} + \sigma_{so})(\vec{P}_1 - \vec{P}_2) \cdot \vec{k}, \quad (2.136)$$

where \vec{P}_1 and \vec{P}_2 are the beam and target polarizations, respectively; and $\sigma_{os}(\sigma_{so})$ is the total cross section for the P -odd interaction with the triplet–singlet (singlet–triplet) transition. The cross section for the P -invariant interaction, $\sigma_{P_1 P_2}^{(0)}$, can be written in the form

$$\sigma_{P_1 P_2}^{(0)} = \sigma_0 + \sigma_1(\vec{P}_1 \cdot \vec{P}_2) + \sigma_2(\vec{P}_1 \cdot \vec{k})(\vec{P}_2 \cdot \vec{k}) \\ = \sigma_0 + \Delta\sigma_T \vec{P}_{1T} \cdot \vec{P}_{2T} + \Delta\sigma_L \vec{P}_{1L} \cdot \vec{P}_{2L}, \quad (2.137)$$

where the subscripts L and T mean the longitudinal and transverse polarization components of the beam. Formula (2.136) shows that, to detect a P -odd effect, it is necessary to measure the total cross section for the interaction of the longitudinally polarized beam with the unpolarized target or the unpolarized beam with the trans-

versely polarized target (the third term). Such experiments have been performed, and they will be discussed in the next sections of this book.

The second term in the formula (2.136) corresponds to simultaneous parity and time-reversal violation. To measure it, the polarized beam and polarized target, whose polarization vectors are perpendicular to each other and to the beam momentum, should be used. Such experiments have not yet been carried out.

2.6.2.2 *T*-odd Terms

If the interaction is invariant under space inversion, the term corresponding to the *T*-odd effect has the form

$$M^{(1)} = M_T(\vec{\sigma}_1 \cdot \vec{l})(\vec{\sigma}_2 \cdot \vec{m}) + (\vec{\sigma}_1 \cdot \vec{m})(\vec{\sigma}_2 \cdot \vec{l}). \quad (2.138)$$

In the case of the matrix $M^{(0)}$, it can be shown that the polarization P of the final nucleon for the case of the unpolarized initial states is equal to left–right asymmetry A_N in the scattering of the polarized nucleon on the unpolarized nucleon: $P = A_N$.

Note that this equality holds for the more general case of the direct

$$a + b \rightarrow c + d \quad (2.139)$$

and inverse

$$c + d \rightarrow a + b \quad (2.140)$$

reactions. In this case, the polarization P refers to the particle c in the direct reaction with the unpolarized particles a and b , whereas the asymmetry A refers to the particle a in the inverse reaction with polarized particles (Baz 1957).

If the interaction contains the term $M^{(1)}$, the equality $P = A_N$ is violated and is replaced by the relation (for NN elastic scattering)

$$\sigma_0(P - A) = -8\text{Im}(M_T^* \cdot h). \quad (2.141)$$

This relation should be tested at the angles at which h is noticeably nonzero. Such a test is simplified if the unambiguous phase or amplitude analysis has been performed.

2.7 Complete Set of Experiments

The idea of the complete set of experiments for the set of observables that completely and unambiguously determine the reaction matrix elements was first proposed in Puzikov et al. (1957) and Smorodinskii (1960). In application to nucleon–nucleon elastic scattering, the possible ways for reconstructing the matrix elements were first proposed in Schumacher and Bethe (1961). We use these works to reconstruct the nucleon–nucleon scattering amplitudes.

The nucleon–nucleon elastic scattering matrix was constructed in Sect. 2.6, where it was also shown that the total number of the independent complex amplitudes necessary for describing the $p + p \rightarrow p + p$ reaction at a fixed angle and a fixed energy is five. This means that ten real quantities, namely, five absolute values of the amplitudes and five their phases, should be measured at a fixed angle and a fixed initial energy. Hence, these ten observables constitute a minimum set for the complete experiment. Pion–nucleon scattering is described by two amplitudes, and the complete set should consist of no less than four observables. The complete set for pion–pion scattering consists of only two observables. The number of the components of the complete experiments generally depends on the spin of the interacting particles. The number of observables in the complete set increases with the spin. It is seen that most observables in the complete experiment are associated with spin, i.e., spin carries rich information on interaction.

Let us consider the examples of complete experiments at fixed angles and fixed initial energies.

2.7.1 Complete Experiment on pp Elastic Scattering at 0° in the Center-of-Mass Frame. The Total Cross Sections for Nucleon–Nucleon Interactions

First, the total cross section σ_T for the interaction between two particles with spin $1/2$ should be a scalar. Second, it should be a linear function of the polarizations of the initial particles, \vec{P}_1 and \vec{P}_2 . Third, it should be composed of the kinematic quantities determining the reaction (Bilenky and Ryndin 1963; Philips 1963). Thus,

$$\sigma = \sigma_0 + \sigma_1(\vec{P}_1 \cdot \vec{P}_2) + \sigma_2(\vec{P}_1 \cdot \vec{k})(\vec{P}_2 \cdot \vec{k}). \quad (2.142)$$

Here, \vec{k} is the unit vector in the incident beam direction and σ_0 , σ_1 , and σ_2 are the experimentally measured parameters depending only on the initial beam energy. Their meaning is as follows. By the definition, the polarization is the mean value of the Pauli operator; hence,

$$\begin{aligned} (\vec{P}_1 \cdot \vec{P}_2) &= \langle (\vec{\sigma}_1 \cdot \vec{\sigma}_2) \rangle = (2\vec{S}^2 - 3), \\ (\vec{P}_1 \cdot \vec{k})(\vec{P}_2 \cdot \vec{k}) &= \langle (\vec{\sigma}_1 \cdot \vec{k})(\vec{\sigma}_2 \cdot \vec{k}) \rangle = (2(\vec{S} \cdot \vec{k})^2 - 1). \end{aligned} \quad (2.143)$$

Here, $\vec{S} = \frac{1}{2}(\vec{\sigma}_1 + \vec{\sigma}_2)$ is the total spin of two initial interacting nucleons. According to Eqs. (2.142) and (2.143),

$$(\vec{P}_1 \cdot \vec{P}_2) = \sum_m w_m^t - 3w^s, \quad (\vec{P}_1 \cdot \vec{k})(\vec{P}_2 \cdot \vec{k}) = \sum_m (-1)^{1+m} w_m^t - w^s, \quad (2.144)$$

where w^s and w^t are the probabilities of finding the system of two nucleons in the singlet and triplet states, respectively. Taking into account the normalization condition $w^s + \sum_m w_m^t = 1$, it follows from Eq. (2.145) that

$$\begin{aligned} w^s &= \frac{1}{4}(1 - (\vec{P}_1 \cdot \vec{P}_2)), \\ w_0^t &= \frac{1}{4}(1 + (\vec{P}_1 \cdot \vec{P}_2) - 2(\vec{P}_1 \cdot \vec{k})(\vec{P}_2 \cdot \vec{k})), \\ w_+^t + w_-^t &= \frac{1}{2}(1 + (\vec{P}_1 \cdot \vec{k})(\vec{P}_2 \cdot \vec{k})). \end{aligned} \quad (2.145)$$

Using expression (2.142) and the definition of the triplet projection operator, we can show that $w_+^t = w_-^t$.

Representing the total cross section in the form of the sum of the weighted cross sections for the singlet and triplet states

$$\sigma = w^s \sigma^s + \sum w_m^t \sigma_m^t \quad (2.146)$$

and substituting the expressions for w^s and w_m^t , we obtain

$$\sigma = \sigma_0 + \frac{1}{4}(\sigma_0^t - \sigma^s)(\vec{P}_1 \cdot \vec{P}_2) + \frac{1}{2}(\sigma_+^t - \sigma_0^t)(\vec{P}_1 \cdot \vec{k})(\vec{P}_2 \cdot \vec{k}). \quad (2.147)$$

Comparison of Eqs. (2.142) and (2.147) provides the following relation between the coefficients:

$$\sigma_1 = \frac{1}{4}(\sigma_0^t - \sigma^s) \quad \text{and} \quad \sigma_2 = \frac{1}{2}(\sigma_+^t - \sigma_0^t). \quad (2.148)$$

Thus, in an experiment with polarized nucleons, three total cross sections can be measured for the cases: (a) both nucleons are unpolarized, (b) both nucleons are polarized transversely to the beam, and (c) both nucleons are polarized along the beam. These three experiments constitute the complete set for determining the total cross sections for nucleon–nucleon interactions. As a result, σ_s , σ_0^t , and σ_+^t can be reconstructed and their individual contributions to the usual (unpolarized) total cross section σ_0 can be determined:

$$\sigma_0 = \frac{1}{4}\sigma^s + \frac{1}{4}\sigma_0^t + \frac{1}{2}\sigma_+^t. \quad (2.149)$$

The applications of the above relations are discussed in the section “Polarization experiments and results.”

2.7.2 Forward NN-Scattering Amplitudes

In view of the symmetry condition, the forward scattering amplitudes satisfy the conditions $c(0) = d(0) = 0$, $b(0) = e(0)$ and the forward scattering matrix has the form

$$M(\vec{\sigma}_1, \vec{\sigma}_2; \vec{k}_i, \vec{k}_f) = a(0) + e(0)(\vec{\sigma}_1 \cdot \vec{\sigma}_2) + [f(0) - e(0)](\vec{\sigma}_1 \cdot \vec{k})(\vec{\sigma}_2 \cdot \vec{k}). \quad (2.150)$$

The above unitarity condition (see matrix relation (2.87) in Sect. 2.4) applied to matrix (2.150) leads to the following relations between the imaginary parts of the amplitudes and total cross sections (optical theorem):

$$\operatorname{Im} a(0) = \frac{k}{4\pi} \sigma_T, \quad \operatorname{Im} e(0) = \frac{k}{4\pi} \sigma_1, \quad \operatorname{Im}[f(0) - e(0)] = \frac{k}{4\pi} \sigma_2. \quad (2.151)$$

Here, k is the wave number in the center-of-mass frame. Thus, the measurements of three observables σ_0 , σ_1 , and σ_2 allow one to reconstruct the imaginary parts of three amplitudes a , e , and f of the forward pp elastic scattering.

The determination of three real parts of these amplitudes is necessary to complete the reconstruction of the scattering matrix. This requires the measurements of additional three parameters at very small angles (in the so-called Coulomb–nuclear interference region). One of them is the differential cross section. Such a measurement makes it possible to reconstruct the real part of the spin-independent amplitude $a(0)$. The measurements of two other parameters, for example, σ_1 and σ_2 , allow one to reconstruct the real parts of the amplitudes e and f through the dispersion relations. Another way is to measure the spin–spin correlation parameters A_{ik} ($i, k = N, S, L$) in the Coulomb–nuclear interference region. Here, N , S , and L denote the initial-proton polarizations that are (N) perpendicular to the reaction plane, (S) transverse, and longitudinal (L) with respect to the initial momentum in the reaction plane. Thus, the complete experiment on forward pp elastic scattering is finished. This approach is obviously universal and can be applied at any initial energy.

Unfortunately, such a complete experiment has not yet been performed at any energy (except for the low-energy region, where the phase analysis has been performed).

Relations (2.151) are useful in a number of cases. They impose additional conditions on the phases in the phase analysis, which can be substantial when choosing between several sets of phase solutions. The use of the dispersion relation allows the reconstruction of the real parts $\operatorname{Re} a(0)$, $\operatorname{Re} e(0)$, and $\operatorname{Re} f(0)$, if the imaginary parts of these amplitudes are known from the experimental data (these imaginary parts appear in the integrands in the dispersion relations). The differential cross section for forward proton–proton scattering is of great interest for theoretical analysis. At the same time, a method for its direct measurement is absent and it is necessary to use an extrapolation method; i.e., the differential cross sections for elastic scattering are measured down to extremely small angles for which reliable data are yet obtained and, then, they are extrapolated to zero angle using a certain function, for example, an exponential. To test the resulting cross section, relations (2.151) are used as follows. The differential cross section for forward scattering can be written in the form

$$\frac{d\sigma}{d\Omega} = \left(\frac{k}{4\pi} \right)^2 (|a|^2 + 2|e|^2 + |f|^2). \quad (2.152)$$

This expression contains the squares of the real and imaginary parts of each of three amplitudes. The substitution of only imaginary parts from relations (2.151) gives the inequality

$$\frac{d\sigma}{d\Omega} \geq \left(\frac{k}{4\pi}\right)^2 [(\sigma_T)^2 + 2(\sigma_1)^2 + (\sigma_1 + \sigma_2)^2]. \quad (2.153)$$

This is the test relation that provides a lower limit and is widely used to measure the differential cross sections for forward scattering.

2.7.3 Complete Set of Experiments on pp Elastic Scattering at 90° in the Center-of-Mass Frame

One of the first attempts to solve this problem was made as early as in 1959 in Nurushev (1959). We briefly repeat the way proposed in that work.

The following five parameters of pp elastic scattering were measured at an energy of 660 MeV and an angle of 90° (Azhgirey et al. 1963): differential cross section $I = (2.07 \pm 0.03)$ mb/sr, spin correlation parameter $C_{nn} = (0.93 \pm 0.21)$, depolarization parameter $D = (0.93 \pm 0.17)$, transverse polarization rotation parameter $R = (0.26 \pm 0.07)$, and longitudinal polarization rotation parameter $A = (0.20 \pm 0.06)$. Using these five observables, one can reconstruct three absolute values and two relative phases of the nonzero amplitudes B , C and H (see Sect. 2.6 “Nucleon–nucleon scattering”). The phase of the amplitude B is taken to be zero. Thus, the amplitudes (dimensionless) normalized to the cross section are given by the formulas

$$\begin{aligned} |b|^2 &= \frac{|B|^2}{4I} = \frac{1}{2}(1 - C_{nn}), & |c|^2 &= \frac{2|C|^2}{I} = \frac{1}{4}(1 + C_{nn} + 2D), \\ |h|^2 &= \frac{|H|^2}{2I} = \frac{1}{4}(1 + C_{nn} - 2D), & \sin(\delta_C - \delta_B) &= -\frac{R + A}{2dc}, \\ \cos(\delta_H - \delta_B) &= \frac{A - R}{2bh}. \end{aligned} \quad (2.154)$$

The substitution of the numerical values of the observables gives (Kumekin et al. 1954)

$$\begin{aligned} |b|^2 &= 0.35 \pm 0.11, & |c|^2 &= 1.00 \pm 0.10, & |h|^2 &= 0.02 \pm 0.10, \\ \sin \delta_c &= 0.39 \pm 0.21, & \cos \delta_h &= -0.36 \pm 0.18. \end{aligned}$$

Comparison with similar data for lower energies shows (Nurushev 1959) that the contributions from the triplet amplitudes c and h prevail in the energy range under consideration, whereas the contribution from the singlet term b is smaller. The terms h (tensor interaction) and c (spin–orbit interaction) dominate in the upper and lower energy ranges, respectively.

2.7.4 Complete Set of Experiments on pp Elastic Scattering at an Arbitrary Angle in the Center-of-Mass Frame

In this section, we try to answer the question: How many and what particular observables should be measured at a given initial energy in order to reconstruct the amplitude of nucleon–nucleon elastic scattering at an arbitrary angle θ in the center-of-mass frame? More briefly, how many measurements constitute the complete set of experiments? Unfortunately, the complete set of experiments has not yet been performed for any energy above 3 GeV. The direct reconstruction of the elements of the nucleon–nucleon scattering matrix from experimental data is the single method of analysis at energies above the meson production threshold. In such an approach, one common phase in the scattering matrix remains undetermined. It can be determined at energies below the meson production threshold by means of the unitarity relation. The possibility of directly reconstructing the scattering amplitudes was discussed in Puzikov et al. (1957), Smorodinskii (1960), and Schumacher and Bethe (1961). In the last work, 11 experimental observables (the differential cross section, polarization, and components of the depolarization, polarization transfer, and polarization correlation tensors) were used and the absolute values of five amplitudes, as well as four relative phases, were unambiguously reconstructed. In agreement with the expectations, one common phase was undetermined. According to Schumacher and Bethe (1961), the complete set includes tensors up to the second order. The possibilities of simplifying the procedure for reconstructing the amplitudes with the use of the polarization tensors of the third and fourth orders were considered in Bilenky et al. (1965) and Vinternitts et al. (1965).

It is not excluded that theoretical ideas can noticeably reduce the number of necessary experiments of the complete set at asymptotic energies.

Below, we discuss the method for reconstructing the scalar amplitudes of nucleon–nucleon scattering in the relativistic case and present the particular sets of the complete set of experiments. In this presentation, we follow Bilenky et al. (1966).

We write the nucleon–nucleon scattering matrix in the form (Wolfenstein and Ashkin 1952; Dalitz 1952)

$$M(\vec{p}', \vec{p}) = (u + v) + (u - v)(\vec{\sigma}_1 \cdot \vec{n})(\vec{\sigma}_2 \cdot \vec{n}) + c(\vec{\sigma}_1 + \vec{\sigma}_2) \cdot \vec{n} \\ + (g - h)(\vec{\sigma}_1 \cdot \vec{m})(\vec{\sigma}_2 \cdot \vec{m}) + (g + h)(\vec{\sigma}_1 \cdot \vec{l})(\vec{\sigma}_2 \cdot \vec{l}). \quad (2.155)$$

Here, the complex scalar scattering amplitudes u , v , c , g , and h are functions of the energy and scattering angle θ . Our main aim is to express these amplitudes in terms of experimental quantities. The unit vectors \vec{n} , \vec{l} , \vec{m} are given by the expressions

$$\vec{l} = \frac{\vec{p}' + \vec{p}}{|\vec{p}' + \vec{p}|}, \quad \vec{m} = \frac{\vec{p}' - \vec{p}}{|\vec{p}' - \vec{p}|}, \quad \vec{n} = \vec{l} \times \vec{m} = \frac{\vec{p} \times \vec{p}'}{|\vec{p} \times \vec{p}'|}. \quad (2.156)$$

These unit vectors defined in the center-of-mass frame are mutually orthogonal. In the nonrelativistic approximation, the vectors \vec{l} and \vec{m} are directed along the

scattered and recoil particle momenta in the laboratory frame, respectively. In the relativistic case, this relation is invalid and an additional rotation angle appears.

The proton–proton scattering matrix should satisfy the Pauli exclusion principle:

$$M(\vec{p}', \vec{p}) = -\Pi(1, 2)M(-\vec{p}', \vec{p}) = -M(\vec{p}', -\vec{p})\Pi(1, 2). \quad (2.157)$$

Here,

$$\Pi(1, 2) = \frac{1}{2}(1 + \vec{\sigma}_1 \cdot \vec{\sigma}_2) \quad (2.158)$$

is the operator of the permutation of the spin variables. The substitution of pp -scattering matrix (2.155) into relation (2.157) indicates that the scalar scattering amplitudes satisfy the following symmetry conditions:

$$\begin{aligned} u(\pi - \theta) &= -u(\theta), & h(\pi - \theta) &= h(\theta), \\ c(\pi - \theta) &= c(\theta), & v(\pi - \theta) &= -g(\theta). \end{aligned} \quad (2.159)$$

It follows from these relations that, for example, three amplitudes, c , h and, e.g., g , are nonzero for an angle of 90° . Their reconstruction was considered above.

Neutron–proton elastic scattering was not discussed above. Such a discussion is most appropriate with the isotopic invariance hypothesis. In this case, two isotopic scattering matrices $M_1(\vec{p}', \vec{p})$ and $M_0(\vec{p}', \vec{p})$ with isospins 1 and 0, respectively, can be introduced. Both matrices are written in the form, where the scalar amplitudes have subscripts 1 and 0, and the matrix $M_1(\vec{p}', \vec{p})$ coincides with the pp -scattering matrix. In this case, the np -scattering matrix is determined by the expression

$$M_{np}(\vec{p}', \vec{p}) = \frac{1}{2}[M_1(\vec{p}', \vec{p}) + M_0(\vec{p}', \vec{p})]. \quad (2.160)$$

The generalized Pauli exclusion principle can be applied to the isotopic scattering matrices. Introducing the subscript $i = 0, 1$, we write the Pauli condition as follows:

$$M_i(\vec{p}', \vec{p}) = (-1)^i \Pi(1, 2)M_i(-\vec{p}', \vec{p}) = (-1)^i M_i(\vec{p}', -\vec{p})\Pi(1, 2). \quad (2.161)$$

This condition provides the following constraints on the scalar scattering amplitude in different isotopic states:

$$\begin{aligned} u_i(\pi - \theta) &= (-1)^i u_i(\theta), & h_i(\pi - \theta) &= (-1)^{i+1} h_i(\theta), \\ c_i(\pi - \theta) &= (-1)^{i+1} c_i(\theta), & v_i(\pi - \theta) &= (-1)^i g_i(\theta). \end{aligned} \quad (2.162)$$

In the nonrelativistic approximation, the problem of the joint analysis of the pp - and np -scattering data for reconstructing the scattering matrix was considered in Kazarinov (1956) and Golovin et al. (1959).

A theoretical analysis of collisions between particles is usually performed in the center-of-mass frame. However, observables such as cross sections and polarizations are measured in the laboratory frame. Therefore, it is necessary to determine the rules for the transition from one frame to the other taking into account the relativistic kinematics and the specificity of the spin transformations. Let us consider particular examples.

According to the general rules, the mean value of any spin operator $\langle \vec{\sigma}_{1L} \rangle$ is expressed as

$$\langle \vec{\sigma}_{1L} \rangle \vec{a}'_L = \text{Tr} [\vec{\sigma}_1 (\vec{a}'_L)_R \rho_f] / \text{Tr} \rho_f. \quad (2.163)$$

Here, ρ_f is the density matrix of the final state and \vec{a}'_L is an arbitrary unit vector in the laboratory frame (L frame). It is intended to measure the projection of the polarization vector of the first particle on this direction. The vector $(\vec{a}'_L)_R = R_n(\Omega') \vec{a}'_L$ includes the relativistic transformation of the spin vector and is obtained from the vector \vec{a}'_L by means of its rotation by the angle $\Omega' = \theta - 2\theta_L$ about a vector perpendicular to the scattering plane. In the nonrelativistic limit, the center-of-mass scattering angle θ is equal to $2\theta_L$, where θ_L is the laboratory scattering angle. Hence, $\Omega' = 0$ in this case.

For the measurement of the projection of the polarization vector of the recoil particle (particle 2) on the direction of the unit vector \vec{b}''_L in the L frame, we have

$$\langle \vec{\sigma}_{2L} \rangle \vec{b}''_L = \text{Tr} [\vec{\sigma}_2 (\vec{b}''_L)_R \rho_f] / \text{Tr} \rho_f. \quad (2.164)$$

It is intended to measure the projection of the polarization vector of the second particle on this direction. The vector $(\vec{b}''_L)_R = R_n(\Omega'') \vec{b}''_L$ includes the relativistic transformation of the spin vector and is obtained from the vector \vec{b}''_L by means of its rotation by the angle $\Omega'' = 2\varphi_L - \varphi$ about a vector perpendicular to the scattering plane, where $\varphi = \pi - \theta$ and φ_L are the emission angles of the second particle in the center-of-mass frame and L frame, respectively.

The correlation of polarization projections on the same unit vectors $(\vec{a}'_L, \vec{b}''_L)$ can be measured in the experiment:

$$\langle (\vec{\sigma}_1 \cdot \vec{a}'_L) (\vec{\sigma}_2 \cdot \vec{b}''_L) \rangle = \text{Tr} [(\vec{\sigma}_1 \cdot (\vec{a}'_L)_R) (\vec{\sigma}_2 \cdot (\vec{b}''_L)_R) \rho_f] / \text{Tr} \rho_f. \quad (2.165)$$

Let us introduce the following three sets of the orthonormalized unit vectors in the laboratory frame:

$$\vec{n}_L, \quad \vec{k}_L, \quad \vec{s}_L = \vec{n}_L \times \vec{k}_L, \quad (2.166)$$

$$\vec{n}_L, \quad \vec{k}'_L, \quad \vec{s}'_L = \vec{n}_L \times \vec{k}'_L, \quad (2.167)$$

$$\vec{n}_L, \quad \vec{k}''_L, \quad \vec{s}''_L = \vec{n}_L \times \vec{k}''_L. \quad (2.168)$$

Here, $\vec{k}_L, \vec{k}'_L, \vec{k}''_L$ are the unit vectors in the directions of the momenta of the incident, scattered, and recoil nucleons, respectively, and $\vec{n}_L = \vec{k}_L \times \vec{k}'_L / |\vec{k}_L \times \vec{k}'_L|$ is the unit vector perpendicular to the scattering plane; $\vec{n}_L = \vec{n}$, where \vec{n} is the unit vector perpendicular to the scattering plane in the center-of-mass frame. In the subsequent presentation, the initial, scattered, and recoil particles are described in coordinate systems (2.166), (2.167), and (2.168), respectively. When calculating observables, we use tensors up to the second rank.

1. The zero-rank tensor is the differential cross section. It is given by the expression

$$\sigma_0 = \text{Tr} \rho_f = \frac{1}{4} \text{Tr} (MM^+) = 2(|u|^2 + |v|^2 + |c|^2 + |g|^2 + |h|^2). \quad (2.169)$$

2. The first-rank tensors:

- (a) The initial particles are unpolarized. The i -th component of the polarization of the scattered particle is measured. It is given by the expression

$$P_{1i}\sigma_0 = \frac{1}{4}\text{Tr}(\sigma_{1i}MM^+). \quad (2.170)$$

- (b) The initial particles are unpolarized. The i -th component of the polarization of the recoil particle is measured. It is given by the expression

$$P_{2i}\sigma_0 = \frac{1}{4}\text{Tr}(\sigma_{2i}MM^+) = P_{1i}\sigma_0. \quad (2.171)$$

- (c) The incident (first) particle is polarized in the i -th direction. The target (second) particle is unpolarized. The i -th component of the asymmetry is measured. It is given by the expression

$$A_{1i}\sigma_0 = \frac{1}{4}\text{Tr}(M\sigma_{1i}M^+). \quad (2.172)$$

- (d) The incident (first) particle is unpolarized. The target (second) particle is polarized in the i -th direction. The i -th component of the asymmetry of the recoil particle is measured. It is given by the expression

$$A_{2i}\sigma_0 = \frac{1}{4}\text{Tr}(M\sigma_{2i}M^+) = A_{1i}\sigma_0. \quad (2.173)$$

3. The second-rank tensors:

- (a) The depolarization tensor D_{ik} . The first particle is polarized (component i) in the initial state, and its polarization (component k) after scattering is measured. The expression for the tensor D_{ik} has the form

$$D_{ik}\sigma_0 = \frac{1}{4}\text{Tr}(\sigma_{1i}M\sigma_{1k}M^+). \quad (2.174)$$

- (b) The depolarization tensor of the second particle $D_{ik}^{(2)}$. The second particle is polarized (component i , the first particle is unpolarized) in the initial state, and its polarization (component k) after scattering is measured. The expression for the tensor $D_{ik}^{(2)}$ has the form

$$D_{ik}^{(2)}\sigma_0 = \frac{1}{4}\text{Tr}(\sigma_{2i}M\sigma_{2k}M^+) = D_{ik}\sigma_0. \quad (2.175)$$

- (c) The polarization transfer tensor from the first particle to the second K_{ik} . The first particle is polarized (the second particle is unpolarized) in the initial state, and the polarization of the second particle after scattering is measured. The expression for the tensor K_{ik} has the form

$$K_{ik}\sigma_0 = \frac{1}{4}\text{Tr}(\sigma_{1i}M\sigma_{2k}M^+). \quad (2.176)$$

- (d) The polarization transfer tensor from the second particle to the first one $K_{ik}^{(2)}$. The second particle is polarized (the first particle is unpolarized) in the initial

state, and the polarization of the first particle after scattering is measured. The expression for the tensor $K_{ik}^{(2)}$ has the form

$$K_{ik}^{(2)} \sigma_0 = \frac{1}{4} \text{Tr} (\sigma_{2i} M \sigma_{1k} M^+) = K_{ik} \sigma_0. \quad (2.177)$$

- (e) The polarization correlation tensor C_{ik} . Both particles in the initial state are unpolarized, and the correlation of their polarizations after scattering is measured. The expression for the tensor C_{ik} has the form

$$C_{ik} \sigma_0 = \frac{1}{4} \text{Tr} (\sigma_{1i} \sigma_{2k} M M^+). \quad (2.178)$$

- (f) The polarization correlation tensor $C_{ik}^{(2)}$. Both particles in the initial state are unpolarized, and the correlation of their polarizations after scattering is measured. The difference from (e) is that the Pauli operators are enclosed by M matrices. The expression for the tensor $C_{ik}^{(2)}$ has the form

$$C_{ik}^{(2)} \sigma_0 = \frac{1}{4} \text{Tr} (\sigma_{1i} \sigma_{2k} M M^+). \quad (2.179)$$

- (g) The two-spin asymmetry tensor or asymmetry correlation tensor A_{ik} . Both particles in the initial state are polarized: the first and second particles are polarized in the directions i and k , respectively. Asymmetry after scattering is measured. The expression for the tensor A_{ik} has the form

$$A_{ik} \sigma_0 = \frac{1}{4} \text{Tr} (M \sigma_{1i} \sigma_{2k} M^+). \quad (2.180)$$

- (h) The two-spin asymmetry tensor $A_{ik}^{(2)}$. Both particles in the initial state are polarized: the first and second particles are polarized in the directions k and i , respectively. Asymmetry after scattering is measured. The expression for the tensor $A_{ik}^{(2)}$ has the form

$$A_{ik}^{(2)} \sigma_0 = \frac{1}{4} \text{Tr} (M \sigma_{1k} \sigma_{2i} M^+) = A_{ik} \sigma_0. \quad (2.181)$$

The requirements of the invariance of strong interactions under a number of continuous (isotropy and uniformity of space) and discrete (space inversion and time reversal) transformations lead to relations between measured quantities. For example,

$$P_i = A_i = P n_i \quad C_{ik}(\vec{p}', \vec{p}) = A_{ik}(-\vec{p}, -\vec{p}'). \quad (2.182)$$

For the case of the beam with polarization \vec{P}_1 and the polarized target with polarization \vec{P}_2 (or two colliding polarized proton beams with the indicated polarizations), the general requirements of the invariance of the scattering matrix under the transformations listed above in the parentheses provide the following expression for the differential cross section

$$\begin{aligned} \sigma(\vec{P}_1, \vec{P}_2) = \sigma_0 \{ & 1 + P(\vec{P}_1 + \vec{P}_2) \cdot \vec{n}_L + A_{nn}(\vec{P}_1 \cdot \vec{n}_L)(\vec{P}_2 \cdot \vec{n}_L) \\ & + A_{ss}(\vec{P}_1 \cdot \vec{s}_L)(\vec{P}_2 \cdot \vec{s}_L) + A_{kk}(\vec{P}_1 \cdot \vec{k}_L)(\vec{P}_2 \cdot \vec{k}_L) \\ & + A_{sk}[(\vec{P}_1 \cdot \vec{s}_L)(\vec{P}_2 \cdot \vec{k}_L) + (\vec{P}_1 \cdot \vec{k}_L)(\vec{P}_2 \cdot \vec{s}_L)] \}. \end{aligned} \quad (2.183)$$

Here, P is the polarization appearing after the scattering of the unpolarized particles and

$$A_{ab} = (\vec{a}_L)_i A_{ik} (\vec{b}_L)_k. \quad (2.184)$$

For the measurements of the depolarization parameters D_{ik} or the polarization transfer tensor K_{ik} , additional scattering is necessary in order to determine the polarization of the scattered particles. Such experiments are very difficult because, first, luminosity in the second scattering process is low and, second, it is difficult to find analyzers with a high analyzing power at high energies. This is one of the main reasons why, for example, such experiments are not planned at the RHIC collider. In contrast to these parameters, the asymmetry correlation tensor A_{ik} is directly measured at RHIC. For the measurements of, for example, the parameter A_{ll} , the polarization of both initial particles should be oriented along their momentum; i.e., both beams should be longitudinally polarized. To measure, for example, A_{sl} , one beam of particles should be polarized along the vector \vec{s} , and the other, along the vector \vec{l} . All these possibilities can be implemented at the RHIC polarized particle collider.

As mentioned above, the measurements of observables are carried out in the laboratory frame, whereas all theoretical jobs with these quantities are performed in the center-of-mass frame. Let us determine the transformations between these two frames. First, we recall the relation between the triples of unit orthogonal vectors in the laboratory and center-of-mass frames:

$$\vec{n}_L = \vec{n}, \quad \vec{k}_L \neq \vec{k}, \quad \vec{s}_L \neq \vec{s}. \quad (2.185)$$

Here, $\vec{k} = \vec{p}/|\vec{p}|$ is the unit vector in the direction of the incident particle momentum in the center-of-mass frame, \vec{n} is the unit vector perpendicular to the scattering plane in this frame, and $\vec{s} = \vec{n} \times \vec{k}$ is the vector that is perpendicular to the momentum of the initial particles and lies in the scattering plane. Thus, we obtain the relations

$$\begin{aligned} A_{ss} &= C_+ - C_{lm} \sin \theta - C_- \cos \theta, \\ A_{kk} &= C_+ + C_{lm} \sin \theta + C_- \cos \theta, \\ A_{sk} &= -C_{lm} \cos \theta + C_- \sin \theta. \end{aligned} \quad (2.186)$$

Here,

$$C_+ = \frac{1}{2}(C_{ll} + C_{mm}), \quad C_- = \frac{1}{2}(C_{ll} - C_{mm}). \quad (2.187)$$

The components A_{nn} and C_{nn} perpendicular to the scattering plane are the same in both frames. The other components of the tensors satisfy the relations

$$C_+ = \frac{1}{2}(A_{ss} + A_{kk}), \quad (2.188)$$

$$C_{lm} = -A_{sk} \cos \theta + \frac{1}{2}(A_{kk} - A_{ss}) \sin \theta, \quad (2.189)$$

$$C_- = A_{sk} \sin \theta + \frac{1}{2}(A_{kk} - A_{ss}) \cos \theta. \quad (2.190)$$

Here, the polarization correlation tensors C_{ik} are expressed in terms of the asymmetry correlation tensors A_{nn} . This representation is reasonable. Indeed, the tensors C_{ik} can also be measured with the use of unpolarized initial particles. However, this measurement requires the analysis of the final polarization (double scattering) and, therefore, analyzing scattering. As mentioned above, such experiments are very difficult. At the same time, it is easier to measure the parameters A_{nn} (in single scattering) and to determine C_{ik} in terms of these parameters by the above formulas.

In the first experiments at low energies (beam kinetic energy 100–600 MeV), the spin correlation tensor C_{nn} was measured and other parameters were measured later. We consider these tensors:

$$\begin{aligned} C_{s's''} &= (\vec{s}'_L)_{Ri} C_{ik} (\vec{s}''_L)_{Rk}, & C_{s'k''} &= (\vec{s}'_L)_{Ri} C_{ik} (\vec{k}''_L)_{Rk}, \\ C_{k's''} &= (\vec{k}'_L)_{Ri} C_{ik} (\vec{s}''_L)_{Rk}, & C_{k'k''} &= (\vec{k}'_L)_{Ri} C_{ik} (\vec{k}''_L)_{Rk}. \end{aligned} \quad (2.191)$$

Let us apply the rotation operator about the vector \vec{n} to an arbitrary vector \vec{a} and represent the result in terms of three vectors:

$$R_n(\Omega)\vec{a} = (\vec{a} \cdot \vec{n})\vec{n}(1 - \cos \Omega) + \vec{a} \cos \Omega + (\vec{n} \times \vec{a}) \sin \Omega. \quad (2.192)$$

Using this relation and formulas (2.183), we can find that

$$(\vec{k}'_L)_R = R_n(\Omega')\vec{k}'_L = \vec{l} \cos \alpha + \vec{m} \sin \alpha, \quad (2.193)$$

$$(\vec{s}'_L)_R = R_n(\Omega')\vec{s}'_L = -\vec{l} \sin \alpha + \vec{m} \cos \alpha. \quad (2.194)$$

Here, the relativistic spin rotation angle is $\alpha = \theta/2 - \theta_L$, where θ and θ_L are the particle scattering angles in the center-of-mass and laboratory frames, respectively.

Similar transformation formulas can be obtained for the recoil particle:

$$(\vec{k}''_L)_R = R_n(\Omega'')\vec{k}''_L = -\vec{l} \sin \alpha' - \vec{m} \cos \alpha', \quad (2.195)$$

$$(\vec{s}''_L)_R = R_n(\Omega'')\vec{s}''_L = \vec{l} \cos \alpha' - \vec{m} \sin \alpha'. \quad (2.196)$$

Here, the relativistic spin rotation angle is $\alpha' = \varphi/2 - \varphi_L$, where φ and φ_L are the scattering angles of the recoil particle in the center-of-mass and laboratory frames, respectively.

We emphasize two features. First, relativistic spin precession (Thomas precession) concerns only the polarization components lying in the scattering plane and does not involve its normal component. Second, in the nonrelativistic limit, angles are $\alpha = \alpha' = 0$ and Thomas precession at low energies does not affect observables. In the nonrelativistic case, we have the equalities

$$(\vec{k}'_L)_R = \vec{l}, \quad (\vec{s}'_L)_R = \vec{m}, \quad (\vec{k}''_L)_R = -\vec{m}, \quad (\vec{s}''_L)_R = \vec{l}. \quad (2.197)$$

Let us express experimentally measured quantities (2.189) in terms of the tensors in the center-of-mass frame using relations (2.191)–(2.194). The desired formulas have the form

$$C_{s's''} = -C_+ \sin(\alpha + \alpha') + C_{lm} \cos(\alpha - \alpha') - C_- \sin(\alpha - \alpha'). \quad (2.198)$$

$$C_{s'k''} = -C_+ \cos(\alpha + \alpha') + C_{lm} \sin(\alpha - \alpha') + C_- \cos(\alpha - \alpha'). \quad (2.199)$$

$$C_{k's''} = C_+ \cos(\alpha + \alpha') + C_{lm} \sin(\alpha - \alpha') + C_- \cos(\alpha - \alpha'). \quad (2.200)$$

$$C_{k'k''} = -C_+ \sin(\alpha + \alpha') - C_{lm} \cos(\alpha - \alpha') + C_- \sin(\alpha - \alpha'). \quad (2.201)$$

It is easy to verify that the observables are related as

$$(C_{s's''} + C_{k'k''})/(C_{s'k''} - C_{k's''}) = \tan(\alpha + \alpha'). \quad (2.202)$$

Hence, the number of the independent observables is three rather than four.

Let us solve the inverse problem, i.e., express three parameters C_+ , C_- , C_{lm} in terms of the observables $C_{s's''}$, $C_{s'k''}$, and $C_{k's''}$. Using relations (2.196)–(2.199), we obtain

$$C_+ = (C_{k's''} - C_{s'k''})/2 \cos(\alpha + \alpha'), \quad (2.203)$$

$$C_- = \frac{1}{2}(C_{s'k''} + C_{k's''}) \cos(\alpha - \alpha') - \left[C_{s's''} + \frac{1}{2} \tan(\alpha + \alpha') (C_{k's''} - C_{s'k''}) \right] \sin(\alpha - \alpha'), \quad (2.204)$$

$$C_{lm} = \frac{1}{2}(C_{s'k''} + C_{k's''}) \sin(\alpha - \alpha') + \left[C_{s's''} + \frac{1}{2} \tan(\alpha + \alpha') (C_{k's''} - C_{s'k''}) \right] \cos(\alpha - \alpha'). \quad (2.205)$$

Now, we consider the case with the polarized beam and unpolarized target. The polarization components of the scattered particles are measured after collisions. In view of invariance, these components can be written in the form

$$\sigma(\vec{P}_1) \langle \vec{\sigma}_1 \rangle_L \cdot \vec{n}_L = \sigma_0 (P + D_{nn}(\vec{P}_1 \cdot \vec{n}_L)), \quad (2.206)$$

$$\sigma(\vec{P}_1) \langle \vec{\sigma}_1 \rangle_L \cdot \vec{k}'_L = \sigma_0 (D_{k'k}(\vec{P}_1 \cdot \vec{k}_L) + D_{k's}(\vec{P}_1 \cdot \vec{s}_L)), \quad (2.207)$$

$$\sigma(\vec{P}_1) \langle \vec{\sigma}_1 \rangle_L \cdot \vec{s}'_L = \sigma_0 (D_{s'k}(\vec{P}_1 \cdot \vec{k}_L) + D_{s's}(\vec{P}_1 \cdot \vec{s}_L)). \quad (2.208)$$

Here, $\sigma(\vec{P}_1)$ is the differential cross section for the scattering of particles with polarization \vec{P}_1 arbitrarily oriented in space on the unpolarized target; it is given by the expression

$$\sigma(\vec{P}_1) = \sigma_0 (1 + P(\vec{P}_1 \cdot \vec{n})). \quad (2.209)$$

Taking certain components of the initial and final polarizations, we arrive at the known Wolfenstein parameters:

$$D_{nn} = D = (\vec{n}_L)_i D_{ik}(\vec{n}_L)_k, \quad D_{s's} = R = (\vec{s}'_L)_{Ri} D_{ik}(\vec{s}_L)_k; \quad (2.210)$$

$$D_{s'k} = A = (\vec{s}'_L)_{Ri} D_{ik}(\vec{k}_L)_k, \quad D_{k's} = R' = (\vec{k}'_L)_{Ri} D_{ik}(\vec{s}_L)_k; \quad (2.211)$$

$$D_{k'k} = A' = (\vec{k}'_L)_{Ri} D_{ik}(\vec{k}_L)_k. \quad (2.212)$$

The spin rotation parameters (A, R, A', R') contain the final unit vectors with the subscript R , which means the necessity of the inclusion of the relativistic spin rotation in the reaction plane.

Referring readers interested in the details of the derivation of the following formulas to Bilenky et al. (1965), we present the expressions of physical observables in terms of the scattering amplitude in the relativistic case:

$$\sigma_0 = 2(|u|^2 + |v|^2 + |c|^2 + |g|^2 + |h|^2), \quad (2.213)$$

$$\sigma_0 D_{nn} = 2(|u|^2 + |v|^2 + |c|^2 - |g|^2 - |h|^2), \quad (2.214)$$

$$\sigma_0 K_{nn} = 2(|u|^2 - |v|^2 + |c|^2 + |g|^2 - |h|^2), \quad (2.215)$$

$$\sigma_0 C_{nn} = 2(|u|^2 - |v|^2 + |c|^2 - |g|^2 + |h|^2), \quad (2.216)$$

$$\sigma_0 P = 4 \operatorname{Re} c u^*, \quad (2.217)$$

$$\sigma_0 D_+ = 4 \operatorname{Re} u v^*, \quad (2.218)$$

$$\sigma_0 D_- = 4 \operatorname{Re} g h^*, \quad (2.219)$$

$$\sigma_0 D_{lm} = 4 \operatorname{Im} c v^*, \quad (2.220)$$

$$\sigma_0 K_+ = 4 \operatorname{Re} u g^*, \quad (2.221)$$

$$\sigma_0 K_- = 4 \operatorname{Re} v h^*, \quad (2.222)$$

$$\sigma_0 K_{lm} = 4 \operatorname{Im} c g^*, \quad (2.223)$$

$$\sigma_0 C_+ = 4 \operatorname{Re} v g^*, \quad (2.224)$$

$$\sigma_0 C_- = 4 \operatorname{Re} u h^*, \quad (2.225)$$

$$\sigma_0 C_{lm} = -4 \operatorname{Im} c h^*. \quad (2.226)$$

These 14 experimental observables are used to reconstruct five amplitudes and their phases. In this case, one common phase remains undetermined. Fundamentally, it also can be reconstructed. However, this problem is not discussed here.

One of the variants of reconstruction of the amplitudes is as follows (for details, see Bilenky et al. 1966):

$$|g|^2 = \frac{1}{8} \sigma_0 (1 + K_{nn} - D_{nn} - C_{nn}); \quad (2.227)$$

$$|h|^2 = \frac{1}{8} \sigma_0 (1 - K_{nn} - D_{nn} + C_{nn}); \quad (2.228)$$

$$|v|^2 = \frac{1}{8} \sigma_0 (1 - K_{nn} + D_{nn} - C_{nn}); \quad (2.229)$$

$$|u|^2 + |c|^2 = \frac{1}{8} \sigma_0 (1 + K_{nn} + D_{nn} + C_{nn}). \quad (2.230)$$

For further analysis, it is necessary to fix the phase of any of five amplitudes. For example, let the amplitude c be real positive. This means that the scattering matrix is determined up to the phase of the amplitude c . Under this condition, we obtain

$$\operatorname{Re} u = \frac{1}{4c} \sigma_0 P, \quad \operatorname{Im} h = \frac{1}{4c} \sigma_0 C_{lm}, \quad (2.231)$$

$$\operatorname{Im} v = -\frac{1}{4c} \sigma_0 D_{lm}, \quad \operatorname{Im} g = -\frac{1}{4c} \sigma_0 K_{lm}. \quad (2.232)$$

For further calculations, we write the following identity for any two complex quantities:

$$|x|^2|y|^2 - (\operatorname{Re} xy^*)^2 = |x|^2(\operatorname{Im} y)^2 + |y|^2(\operatorname{Im} x)^2 - 2\operatorname{Re} xy^* \operatorname{Im} x \operatorname{Im} y. \quad (2.233)$$

Taking $x = g$ and $y = h$ and using Eqs. (2.231) and (2.232), we obtain

$$c^2 = \frac{|g|^2 M^2 - |h|^2 N^2 - 2\operatorname{Re} gh^* MN}{|g|^2|h|^2 - (\operatorname{Re} gh^*)^2}. \quad (2.234)$$

Here, the quantities $|g|^2$, $|h|^2$, and $\operatorname{Re} gh^*$ were determined above and

$$M = \frac{1}{4}\sigma_0 C_{lm}, \quad N = -\frac{1}{4}\sigma_0 K_{lm}. \quad (2.235)$$

It remains to determine the signs of the quantities

$$\operatorname{Im} u, \quad \operatorname{Re} h, \quad \operatorname{Re} v, \quad \operatorname{Re} g. \quad (2.236)$$

One relation can be written immediately in the form

$$\operatorname{Re} gh^* = \operatorname{Re} g \operatorname{Re} h + \operatorname{Im} g \operatorname{Im} h = \frac{1}{4}\sigma_0 D_-. \quad (2.237)$$

Using Eq. (2.218), we can determine the signs of $\operatorname{Im} u$ and $\operatorname{Re} v$. Any unused equation from the set of Eqs. (2.213)–(2.226) can be used to eliminate the remaining ambiguity.

Thus, the problem of the relativistic reconstruction of the nucleon–nucleon scattering matrix has been solved in the general form. Since the volume of the book is limited, we omit such interesting problems as the reconstruction of the amplitudes by means of the measurements of the polarization parameters of the recoil particles and the joint analysis of pp and np scatterings using isotopic invariance.

2.8 Partial Wave Analysis

The results of polarization experiments on the elastic scattering of nucleons in the low-energy range (0.1–10 GeV) were considered using the partial wave analysis (Hoshizaki 1968; Matsuda 1993). In this method, the scattering amplitude is expanded in terms of the eigenfunctions of the complete set of conserving operators and the expansion coefficients are the elements of the scattering matrix S .

These elements are expressed in terms of the phase shifts, which contain complete information on the interaction process. There are several reasons to apply this method. First, the number of the phases directly depends on the maximum orbital angular momentum L_{\max} in the interaction. According to nonrelativistic quantum mechanics, L_{\max} is related to the impact parameter b as $(L_{\max} + \frac{1}{2})\hbar \approx bp_i$, where p_i is the incident particle momentum in the center-of-mass frame. According to this relation, L_{\max} increases with energy; thus, the phase analysis becomes impossible when the number of free parameters is equal to or larger than the number of experimental points. The situation is further complicated at energies above the meson

production threshold, when the phases become complex. This is the main reason why the phase analysis is not applied for high energies.

Nevertheless, the phase analysis was widely used for low energies in 1950–1960 for several reasons. First, the number of the phases and, correspondingly, the number of free parameters are small for low energies. For example, taking $b = 1.5 \frac{\hbar}{m_\pi c}$ for the impact parameter, we can estimate $L_{\max} = 1$ and 3 at the kinetic energy $T = 50$ and 300 MeV, respectively. Hence, it is easy to perform the phase analysis. The second important reason to perform this analysis is that the phases, being dependent on the impact parameter, make it possible to scan the internal structure of the nucleon. If the particles pass at a large distance from each other, the interaction between them is weak and the phases are small. On the contrary, for the central collision ($L = 0$), the phases are expected to be large. Deviations from this picture are obviously possible, for example, in the presence of repulsion forces or when resonances are formed. The third reason to perform the phase analysis at low energies is that the angular dependence of any observable can be calculated (predicting its behavior) using a few phases and, then, the calculation can be compared with experimental data. Fourth, any theoretical model should be tested on the phase analysis data if the phase analysis is unambiguous.

Let us express the scattering matrix elements in terms of the phase.

In view of the unitarity of the S -matrix, it can be written in terms of the phase operator δ as follows:

$$S = e^{i\delta}. \quad (2.238)$$

The nucleon–nucleon interaction matrix should conserve the total angular momentum J , total spin S , and parity $\Pi = (-1)^L$. The requirement of the antisymmetry under the permutation of two nucleons leads to the relation

$$(-1)^{S+1+T+1} \Pi = (-1)^{S+T+L} = -1. \quad (2.239)$$

Here, T is the isospin of the system of two nucleons. Relation (2.239) should be applied separately for the system of two nucleons in the initial and final states. Taking into account this relation, the elements of the S -matrix can be characterized by three quantum numbers: the total angular momentum J , the total spin of the system of two nucleons S , and the orbital angular momentum L , because T is unambiguously determined from the above relation; i.e., T can be omitted when specifying the elements of the S -matrix.

Let us consider the matrix M defined by the expression

$$M(\vec{p}_f, \vec{p}_i) = \frac{2\pi}{ik} \langle \theta_f \varphi_f | S - 1 | \theta_i \varphi_i \rangle. \quad (2.240)$$

Here, $|\theta_i \varphi_i\rangle$ and $\langle \theta_f \varphi_f|$ are the wave functions of the initial and final system of two nucleons, respectively; and θ_n, φ_n , where $n = i, f$, are the angles of the momenta \vec{p}_i, \vec{p}_f . The elements of this matrix in spin space are given by the expression

$$\langle S m_s | M(\vec{p}_f, \vec{p}_i) | S' m'_s \rangle = \frac{2\pi}{ik} \langle \theta_f \varphi_f, S m_s | e^{2i\delta} - 1 | S' m'_s, \theta_i \varphi_i \rangle. \quad (2.241)$$

This expression can be rewritten in terms of the spherical functions of the angles using the properties of the completeness and orthogonality of the wave functions

$$\begin{aligned} & \frac{2\pi}{ik} \sum \langle \theta_f \varphi_f | L m_L \rangle \langle L m_L S m_S | L S J m_J \rangle \langle L S J m_J | e^{2i\delta} - 1 | L' S' J' m'_J \rangle \\ & \times \langle L' S' J' m'_J | L' m'_L S' m'_S \rangle \langle L' m'_L | \theta_i \varphi_i \rangle. \end{aligned} \quad (2.242)$$

Here,

$$\langle \theta \varphi | L m_L \rangle = Y_L^{m_L}(\theta, \varphi). \quad (2.243)$$

Below, we use the following notation for the Clebsch–Gordan coefficients appearing in expression (2.242):

$$C_{LS}(J m_J m_L m_S) = \langle L m_L S m_S | L S J m_J \rangle. \quad (2.244)$$

The quantization axis is usually taken in the direction of the incident particle momentum; in this case, the angles θ_i and φ_i are zero and the angles θ_f and φ_f are the scattering angles of the final particles. After these simplifications, since S , J , and m_J are conserving quantum numbers, expression (2.242) can be represented in the form

$$\begin{aligned} & \langle S m_S | M(\vec{p}_f, \vec{p}_i) | S' m'_S \rangle \\ & = \delta_{SS'} \frac{4\pi}{ik} \sum_L \sum_{J=|L-S|}^{L+S} \sum_{L'=|J-S|}^{L+S} \sqrt{\frac{2L'+1}{4\pi}} Y_L^{m'_S - m_S}(\theta, \varphi) \\ & \times C_{LS}(J, m'_S, m'_S - m_S, m_S) C_{LS}(J, m'_S, 0, m_S) \langle L S J m'_S | e^{2i\delta} - 1 | L' S J m'_S \rangle. \end{aligned} \quad (2.245)$$

The summation should be performed taking into account antisymmetry condition (2.239) and that the scattering matrix elements are independent of the projection of the total angular momentum, m_J , due to the isotropy of space. The nonzero elements of the matrix $S - 1$ are denoted as

$$\begin{aligned} R_L &= \langle L 0 L m_J | e^{2i\delta} - 1 | L 0 L m_J \rangle; \\ R_{LJ} &= \langle L 1 J m_J | e^{2i\delta} - 1 | L 1 J m_J \rangle; \\ -R_{\pm}^J &= R^J = \langle J \pm 1, 1, J, m_J | e^{2i\delta} - 1 | J \mp 1, 1, J, m_J \rangle. \end{aligned} \quad (2.246)$$

Time reversal invariance leads to the equality $R_+^J = R_-^J = R^J$. For further simplification, we introduce the following notation for the triplet and singlet elements:

$$M_{m_S m'_S} = \langle 1 m_S | M | 1 m'_S \rangle, \quad M_{ss} = \langle 00 | M | 00 \rangle. \quad (2.247)$$

As a result, the nonzero matrix elements for the singlet and triplet states can be written in the form

$$M_{ss} = \frac{2\pi}{ik} \sum \sqrt{\frac{2L+1}{4\pi}} R_L Y_L^0(\theta, \varphi) \quad (2.248)$$

for the singlet transitions and

$$\begin{aligned}
M_{m_s m'_s} = & \frac{2\pi}{ik} \sum_L \left[\sum_{J=L-1}^{L+1} \sqrt{\frac{2L+1}{4\pi}} C_{L1}(J, m'_s, m'_s - m_s, m_s) C_{L1}(J, m'_s, 0, m_s) R_{LJ} \right. \\
& - \sum_{J=L\pm 1}^1 \sqrt{\frac{2L'+1}{4\pi}} C_{L1}(J, m'_s, m'_s - m_s, m_s) C_{L1}(J, m'_s, 0, m_s) R^J \left. \right] \\
& \times Y_{L\pm 1}^{m'_s - m_s}(\theta, \varphi)
\end{aligned} \tag{2.249}$$

for the triplet transitions. Here, $L' = 2J - L$ in the second term. When applying these formulas to pp elastic scattering, it is necessary to take into account two circumstances. First, two protons are identical; therefore, measuring instruments cannot determine which proton, from the beam or from the target, is detected. As a result, the number of counts is twice as large as that following from the above consideration. Thus, to compare with the theoretical cross section, the measured cross sections should be halved. The second circumstance is associated with the antisymmetry condition (Pauli exclusion principle). Owing to this circumstance, the partial amplitudes with spins $S = 0$ and 1 include only even and odd orbital angular momenta, respectively. Since the orbital angular momentum is not conserved, for a given total angular momentum J in the triplet state, there are two states differing in the orbital angular momentum. As an example, we point to the states ${}^3P_2 \leftrightarrow {}^3F_2$, ${}^3F_4 \leftrightarrow {}^3H_4$, etc. of the pp system. For each pair of such states, the mixed transitions ${}^3P_2 \leftrightarrow {}^3F_2$, ${}^3F_4 \leftrightarrow {}^3H_4$ are possible in addition to the direct transitions ${}^3P_2 \rightarrow {}^3P_2$, ${}^3F_2 \rightarrow {}^3F_2$. Correspondingly, the phases describing the direct transitions should be supplemented by additional parameters for describing the mixed transitions. Such parameters are called mixing parameters and are usually denoted as ε_J . To describe the mixed transitions in the absence of inelastic channels, the following two-dimensional symmetric unitary submatrix is introduced:

$$S_J - 1 = \begin{vmatrix} R_{J-1,J} & -R^J \\ -R^J & R_{J+1,J} \end{vmatrix}. \tag{2.250}$$

An unambiguous method for parameterizing this matrix in terms of the phase is absent. One of the methods was proposed by Blatt and Weisskopf (1952) and Blatt and Biedenharn (1952). This method is the diagonalization of the matrix by means of a unitary transformation

$$S_J = G S'_J G^{-1}, \tag{2.251}$$

where

$$S'_J = \begin{vmatrix} e^{2i\delta_{J-1,J}} & 0 \\ 0 & e^{2i\delta_{J+1,J}} \end{vmatrix}, \quad G = \begin{vmatrix} \cos \varepsilon_J & -\sin \varepsilon_J \\ \sin \varepsilon_J & \cos \varepsilon_J \end{vmatrix}. \tag{2.252}$$

This set of phase shifts is called the proper phase shift and is convenient if the Coulomb interaction can be neglected.

Another parameterization variant was proposed in Stapp et al. (1957) in the form

$$S_J = \tilde{S}'_J \tilde{G} \tilde{S}'_J, \tag{2.253}$$

where

$$\tilde{S}'_J = \begin{vmatrix} e^{i\bar{\delta}_{J-1,j}} & 0 \\ 0 & e^{i\bar{\delta}_{J+1,j}} \end{vmatrix}, \quad \tilde{G} = \begin{vmatrix} \cos 2\bar{\varepsilon}_J & i \sin 2\bar{\varepsilon}_J \\ i \sin 2\bar{\varepsilon}_J & \cos 2\bar{\varepsilon}_J \end{vmatrix}. \quad (2.254)$$

This parameterization is favorable over parameterization (2.252), because it provides the estimation of the mixing parameters for low orbital angular momenta (where the nuclear interaction prevails over the Coulomb interaction) in the pure form (without the Coulomb contribution). An additional advantage of this parameterization is that it allows one to more clearly separate the nuclear and Coulomb contributions. For this reason, this parameterization is more often used in the phase analysis.

The relation between representations (2.251) and (2.253) can be found by equating them to each other, because they are the elements of the same matrix. This relation is expressed as follows:

$$\begin{aligned} \delta_{J-1,J} + \delta_{J+1,J} &= \bar{\delta}_{J-1,J} + \bar{\delta}_{J+1,J}, \\ \sin(\delta_{J-1,J} - \delta_{J+1,J}) &= \sin 2\bar{\varepsilon}_J / \sin 2\varepsilon_J, \\ \sin(\bar{\delta}_{J-1,J} - \bar{\delta}_{J+1,J}) &= \tan 2\bar{\varepsilon}_J / \tan 2\varepsilon_J. \end{aligned} \quad (2.255)$$

Table 2.1 presents the elements of the matrix M in terms of the partial waves h . These matrix elements can also be used in the case of neutron–proton elastic scattering with three changes: (a) the Coulomb amplitudes are neglected, (b) all sums over the even or odd L values are extended over all L values (even and odd), and (c) the resulting sums are multiplied by a factor of $1/2$.

The partial nuclear amplitudes h are expressed in terms of the scattering phases by the formulas

$$2ikh_l = (e^{2i\bar{\delta}_l^N} - 1)e^{2i\Phi_l} \quad (2.256)$$

for the singlet states and

$$2ikh_{lj} = (e^{2i\bar{\delta}_{lj}^N} - 1)e^{2i\Phi_l} \quad (2.257)$$

for the triplet states.

For the mixed singlet–triplet states,

$$2ikh_{j\pm 1,j} = (\cos 2\varepsilon_j^N e^{2i\bar{\delta}_{j\pm 1,j}^N} - 1)e^{2i\Phi_l} \quad (2.258)$$

$$2kh^j = \sin 2\varepsilon_j^N e^{i(\bar{\delta}_{j-1,j}^N + \bar{\delta}_{j+1,j}^N)}, \quad (2.259)$$

where ε_j^N is the mixing parameter for the total angular momentum j and the superscript N means that this parameter refers to pure nuclear scattering.

The Coulomb amplitudes are defined as follows:

$$f_c(\theta) = \frac{-n}{k(1 - \cos \theta)} e^{-in \log[(1 - \cos \theta)/2]}, \quad (2.260)$$

where $n = \frac{e^2}{\hbar v}$ and v is the relative velocity in the center-of-mass frame. The symmetrized and antisymmetrized Coulomb amplitudes used in the partial wave analysis are presented in Table 2.1.

Table 2.1 Singlet–triplet matrix elements for pp elastic scattering in terms of the partial amplitudes h . The Coulomb interaction contributions are presented in the explicit form with allowance for the identity of protons

1	$M_{ss} = f_{c,s} + 2 \sum_{\text{even } l} (2l+1) h_l P_l$
2	$M_{11} = f_{c,a} + \sum_{\text{odd } l} [(l+2) h_{l,l+1} + (2l+1) h_{l,l} + (l-1) h_{l,l-1} - \sqrt{(l+1)(l+2)} h^{l+1} - \sqrt{(l-1)l} h^{l-1}] P_l$
3	$M_{00} = f_{c,a} + 2 \sum_{\text{odd } l} [(l+1) h_{l,l+1} + l h_{l,l-1} + (l-1) h_{l,l-1} + \sqrt{(l+1)(l+2)} h^{l+1} + \sqrt{(l-1)l} h^{l-1}] P_l$
4	$M_{01} = \sqrt{2} \sum_{\text{odd } l} [-\frac{l+2}{l+1} h_{l,l+1} + \frac{2l+1}{l(l+1)} h_{l,l} + \frac{l-1}{l} h_{l,l-1} + \sqrt{\frac{l+2}{l+1}} h^{l+1} - \sqrt{\frac{l-1}{l}} h^{l-1}] P_l^1$
5	$M_{10} = \sqrt{2} \sum_{\text{odd } l} [h_{l,l+1} - h_{l,l-1} + \sqrt{\frac{l+2}{l+1}} h^{l+1} - \sqrt{\frac{l-1}{l}} h^{l-1}] P_l^1$
6	$M_{1-1} = \sum_{\text{odd } l} [\frac{1}{l+1} h_{l,l+1} - \frac{2l+1}{l(l+1)} h_{l,l} + \frac{1}{l} h_{l,l-1} - \frac{1}{\sqrt{(l+1)(l+2)}} h^{l+1} - \frac{1}{\sqrt{(l-1)l}} h^{l-1}] P_l^2$
7	$M_{11} - M_{00} - M_{1-1} - \sqrt{2} \text{ctg} \theta (M_{10} + M_{01}) = 0$
8	$f_{c,s} = f_c(\theta) + f_c(\pi - \theta)$, $f_{c,a} = f_c(\theta) - f_c(\pi - \theta)$, where f_c is the Coulomb amplitude

P_l , P_l^1 , and P_l^2 are the associated Legendre polynomials of the zeroth, first, and second orders, respectively

Let us consider the separation of the Coulomb and nuclear contributions in the phase analysis. Since Coulomb forces are long-range and nuclear forces are short-range, i.e., they weakly overlap, it is usually accepted that the pure nuclear, $\bar{\delta}^N$, and Coulomb, ϕ , phases are added. In this case,

$$\bar{\delta}_L^N = \bar{\delta}_L - \phi_L, \quad \bar{\delta}_{JL}^N = \bar{\delta}_{JL} - \phi_L, \quad \bar{\varepsilon}_J^N = \bar{\varepsilon}_J. \quad (2.261)$$

These phases denoted by the overlined symbols with the superscript N are called pure nuclear phases. The Coulomb phases are calculated by the formula (Stapp et al. 1957):

$$\phi_L \equiv \eta_L - \eta_0 = \sum_{x=1}^L \text{atan} \left(\frac{n}{x} \right). \quad (2.262)$$

Here, $n = \frac{e^2}{\hbar v}$ and v is the relative velocity. Let us introduce the Coulomb scattering matrix by the formula $R_c = S_c - 1$. Then, the general reaction matrix is written in the form

$$R = S - 1 = \varepsilon + R_c, \quad \alpha = S - R_c, \quad (2.263)$$

where ε is the mixing parameter for a given j value.

The matrix α corresponding to pure nuclear scattering can be expanded in partial waves, whereas R_c is calculated exactly and is given by the expression

Table 2.2 Expressions for experimentally measured quantities in terms of the amplitude of the pp elastic scattering matrix in the nonrelativistic case

1	$\sigma_0 = a ^2 + b ^2 + 2 c ^2 + e ^2 + f ^2$
2	$\sigma_0 D_{nn} = a ^2 + b ^2 + 2 c ^2 - e ^2 - f ^2$
3	$\sigma_0 D_{ll} = a ^2 - b ^2 - e ^2 + f ^2$
4	$\sigma_0 D_{mm} = a ^2 - b ^2 + e ^2 - f ^2$
5	$\sigma_0 D_{ml} = 2 \operatorname{Im} c^*(a - b)$
6	$\sigma_0 P_0 = 2 \operatorname{Re} c^*(a + b)$
7	$\sigma_0 C_{ml} = 2 \operatorname{Im} c^*(e - f)$
8	$\sigma_0 K_{ml} = 2 \operatorname{Im} c^*(e + f)$
9	$\sigma_0 C_{nn} \setminus 2 = \operatorname{Re} ab^* + c ^2 - \operatorname{Re} ef^*$
10	$\sigma_0 K_{nn} \setminus 2 = \operatorname{Re} ab^* + c ^2 + \operatorname{Re} ef^*$
11	$\sigma_0 C_{ll} \setminus 2 = \operatorname{Re} af^* - \operatorname{Re} be^*$
12	$\sigma_0 K_{ll} \setminus 2 = \operatorname{Re} af^* + \operatorname{Re} be^*$
13	$\sigma_0 C_{mm} \setminus 2 = \operatorname{Re} ae^* - \operatorname{Re} bf^*$
14	$\sigma_0 K_{mm} \setminus 2 = \operatorname{Re} ae^* + \operatorname{Re} bf^*$

$$\langle f | R_c | i \rangle = \frac{ik}{2\pi} f_c(\theta), \quad f_c(\theta) = -\frac{n}{k(1 - \cos \theta)} \exp \left[-in \log \left(\frac{1 - \cos \theta}{2} \right) \right]. \quad (2.264)$$

The partial amplitudes h are calculated using formulas (2.246), (2.253), and (2.261), as well as the relation $\alpha = 2ikh$. These expressions have the form

$$h_L = \frac{1}{2ik} [\exp(2i\delta_L^N) - 1] \exp(2i\phi_L) \quad (2.265)$$

for the singlet state and

$$\begin{aligned} h_{LJ} &= \frac{1}{2ik} [\exp(2i\bar{\delta}_{LJ}^N) - 1] \exp(2i\phi_L), \\ h_{J\pm 1, J} &= \frac{1}{2ik} [\cos 2\varepsilon_J^N \exp(2i\bar{\delta}_{J\pm 1, J}^N) - 1] \exp(2i\phi_{J\pm 1}), \\ h^J &= \frac{1}{2ik} [\sin 2\varepsilon_J^N \exp(i\bar{\delta}_{J-1, J}^N + i\bar{\delta}_{J+1, J}^N)] \end{aligned} \quad (2.266)$$

for the triplet state.

These relations make it possible to express the elements of the matrix M in terms of the phase shifts; hence, experimental observables are determined in terms of the phases. Thus, the phase analysis is possible.

Table 2.2 presents the expressions for measured quantities in terms of the amplitudes a, b, c, e , and f expressed through linear relations with the matrix elements presented in Table 2.1. Hence, experimental data can be used either for the phase analysis or for direct reconstruction of the amplitudes.

2.9 Relativistic Pion–Nucleon Scattering Matrix

In the preceding sections, we consider pion–nucleon scattering in the nonrelativistic case, where the kinetic energy of a particle is much lower than its rest energy. This approach was appropriate in the early 1950s, when synchrocyclotrons accelerated protons up to energies 200–300 MeV. However, the kinetic energy of accelerated protons reached the rest energy in the mid-1950s and, then, became much higher than the rest energy. Theoreticians foresaw this situation and developed the covariant formulation for the density and scattering matrices, which made it possible to analyze processes at relativistic energies. The main results of such an analysis confirmed the applicability of the nonrelativistic approach in the center-of-mass frame, and the relativistic corrections were reduced to an additional rotation angle. Such corrections refer to the observables that involve the polarization components in the scattering plane (parameters A and R), whereas the parameters that involved only polarization components perpendicular to the scattering plane (parameters P , D_{NN} , and C_{NN}) remain unchanged.

Below, we apply the relativistic description to the reaction

$$a(0) + b(1/2) = a(0) + b(1/2) \quad (2.267)$$

(the spins of the particles are given in the parentheses) proposed in Stapp (1956). Since the particle b is a Dirac particle with spin $1/2$, it is described by a four-component wave function ψ . The wave function of a free incident (or initial) particle (with positive energy) can be written in the form

$$\psi = \exp(if \cdot x) \sum_{i=1}^2 A_i U_i, \quad (2.268)$$

whereas the wave function of an antiparticle (with negative energy) has the form

$$\psi = \exp(-if \cdot x) \sum_{i=3}^4 A_i U_i. \quad (2.269)$$

Here, $f(\vec{f}, f_0)$ is the four-momentum of the particle in the base frame, where it is measured (for example, in the laboratory frame), such that $f_0 > 0$; x are four-dimensional space coordinates. Each spinor U_i has four components U_{si} given by the expressions

$$U_{si}(f) = (\mp if \cdot \gamma_{si} + m) / [2m(f_0 + m)]. \quad (2.270)$$

Hereinafter, the upper sign (–) refers to the subscripts $i = 1, 2$ (positive energy), whereas the lower sign (+), to the subscripts $i = 3, 4$ (negative energy). The subscripts in the four-vector $\gamma(-i\beta\vec{\alpha}, \beta)$ denote its matrix elements and m is the mass of the Dirac particle. The spinors are normalized in the covariant form as follows:

$$U_i^+(f) U_j(f) = U_i^*(f) \beta U_j(f) = \pm \delta_{ij}. \quad (2.271)$$

The sign (+) in the spinor means complex conjugation and transposition (interchange of the columns and rows of the matrix), i.e., Hermitian conjugation

$U^+ = U^* \beta$. Using formula (2.270), it is easy to verify that the spinors U_i satisfy the Dirac equation

$$(\pm i f \cdot \gamma + m) U_i(f) = 0. \quad (2.272)$$

To make the below expressions shorter, we introduce the notation

$$\gamma(v) = (\gamma \cdot v) / \sqrt{(v \cdot v)}. \quad (2.273)$$

The denominator on the right-hand side can be either a positive real number or a positive imaginary number. With this symbol, the Dirac equation is represented in the shorter form

$$\gamma(f) U_i = \pm U_i. \quad (2.274)$$

The wave function of the initial state of the pion–nucleon system was introduced above (see Eqs. (2.268) and (2.269)). Let Φ be the wave function of the final state of the same system. The relation between these two functions is determined by the reaction matrix $S(f', t, f)$:

$$\Phi(f') = S(f', t, f) \Psi(f). \quad (2.275)$$

The theory of “holes” requires that the Dirac particle described by a plane wave with the momentum f at time $T = -\infty$ and by a plane wave with the momentum f' at time $T = +\infty$ has the same sign of energy. This means that the transition of the particle to the antiparticle and vice versa is forbidden. This does not mean that the scattering matrix cannot describe the production of a particle. However, in this particular case, we analyze elastic processes and imply this exclusion, which is mathematically written in the form

$$S(f', t, f) = \gamma(f') S(f', t, f) \gamma(f). \quad (2.276)$$

Introducing the new symbol

$$\gamma(u, w) = \gamma(u / \sqrt{|u \cdot u|} + w / \sqrt{|w \cdot w|}) = [\gamma(u) + \gamma(w)] \quad (2.277)$$

and taking into account the relations

$$\gamma(u) \gamma(u) = 1 = \gamma(w) \gamma(w), \quad (2.278)$$

we obtain

$$\gamma(u) \gamma(u, w) = \gamma(u, w) \gamma(w). \quad (2.279)$$

The new scattering matrix $S_q(k', t, k)$ is introduced through the relation

$$S(f', t, f) = \gamma(f', t) S_q(k', t, k) \gamma(t, f). \quad (2.280)$$

This matrix $S_q(k', t, k)$ is the scattering matrix in the center-of-mass frame with the relative momenta \vec{k} and \vec{k}' before and after scattering, respectively, and the total energy t in the center-of-mass frame.

Then, the substitution of Eq. (2.280) into Eq. (2.276) provides the condition of the exclusion of the particle–antiparticle transition in the form

$$S_q(k', t, k) = \gamma(t) S_q(k', t, k) \gamma(t). \quad (2.281)$$

For a deeper insight into the meaning of this matrix, we substitute relation (2.281) into relation (2.280) and obtain

$$S(f', t, f) = (\gamma(f', t)\gamma(t))S_q(k', t, k)(\gamma(t)\gamma(t, f)). \quad (2.282)$$

The transformation $(\gamma(t)\gamma(t, f))$ relates the rest frame of the initial particle (R_i frame) with the center-of-mass frame (C frame for brevity). The transformation $(\gamma(f', t)\gamma(t))$ relates the $R_{f'}$ frame to the C frame. To verify this, we write the Lorentz transformation

$$L(f) = \exp\left[-\frac{1}{2}\theta(\vec{\alpha} \cdot \vec{f})/|\vec{f}|\right]. \quad (2.283)$$

This expression is modified as

$$L(f) = \beta(-i\gamma \cdot f + m\beta)/[2m(f_0 + m)]. \quad (2.284)$$

In the C frame, $t(0, t_0)$ and $\gamma(t) = \beta$ and it can be shown that the following equality takes place:

$$\gamma(t_1)\gamma(t_1, f_1) = L(f_1), \quad \gamma(f'_1, t_1)\gamma(t_1) = L^{-1}(f'_1). \quad (2.285)$$

Subscript 1 means that the quantities are taken in the C frame. Thus, the scattering matrix can be written in the form

$$S(f'_1, t_1, f_1) = L^{-1}(f'_1)S_q(k'_1, t_1, k_1)L(f_1). \quad (2.286)$$

Now, this expression can be interpreted as follows. The S -matrix in the C frame is the product of two Lorentz transformations and the scattering matrix S_q . The Lorentz transformation $L(f_1)$ transforms the spinor of the initial particle from the C frame to the R_{f_1} frame, i.e., to the rest frame of the initial particle, where spin is physically defined. Then, the unitary operator S_q describes the effect of scattering on this spinor. Finally, the second Lorentz transformation transforms the spinor of the final particles to the C frame.

Since the S -matrix specified by expression (2.275), as well as the γ matrix, has the covariant form, the new matrix $S_q(k', t, k)$ should also be covariant.

It is a 4×4 matrix and can be expanded in 16 Dirac matrices constituting a complete set:

$$S_q(k', t, k) = A + B_\mu \gamma_\mu + \frac{1}{2}C_{\mu\nu}\sigma_{\mu\nu} + D_\mu(i\gamma_5\gamma_\mu) + E\gamma_5. \quad (2.287)$$

As expected in the general case, we have 16 coefficients, which are functions of the relative momenta \vec{k} and \vec{k}' and the total energy in the center-of-mass frame, t . These 16 coefficients are the scalar parameter A , pseudoscalar E , 4 components of the vector B , 4 components of the pseudovector D , and 6 components of the antisymmetric tensor C . Condition (2.281) implying the conservation of the sign of

the total energy before and after reaction (a particle cannot become the antiparticle and vice versa) leads to the following constraints on the coefficients:

$$\begin{aligned} B_\mu &= -i N_B (B t_\mu), \\ C_{\mu\nu} &= N_C C \left\{ k_\mu k'_\nu - k_\nu k'_\mu \frac{(m^2 - \mu^2)}{|t \cdot t|} \cdot [t_\mu (k'_\nu - k_\nu) - t_\nu (k'_\mu - k_\mu)] \right\}, \\ D_\mu &= N_D D (-i) k_\lambda k'_\rho t_\sigma \varepsilon_{\mu\lambda\rho\sigma} \equiv D n_\mu, \quad E = 0. \end{aligned} \quad (2.288)$$

Here, $\varepsilon_{\mu\lambda\rho\sigma}$ is the antisymmetric tensor and n is the four-dimensional unit pseudovector, which is the generalization of the nonrelativistic three-dimensional pseudovector \vec{n} perpendicular to the scattering plane. This relativistic pseudovector n has the properties

$$(k \cdot n) = (k' \cdot n) = (t \cdot n) = (1 - n \cdot n) = 0. \quad (2.289)$$

The coefficients B , C , and D , as well as A , are scalar functions of the energy and scattering angle and μ is the pion mass. The normalization coefficients N_B , N_C , and N_D are chosen so that the following equalities are satisfied:

$$B_\mu B^\mu = B^2, \quad C_\mu C^\mu = 2C^2, \quad D_\mu D^\mu = D^2. \quad (2.290)$$

In the center-of-mass frame, expressions (22) are strongly simplified:

$$B_\mu \gamma_\mu = B\beta, \quad \frac{1}{2} C_{\eta\nu} \sigma_{\mu\nu} = C(\vec{\sigma} \cdot \vec{N}), \quad D_\mu i \gamma_5 \gamma_\mu = D\beta(\vec{\sigma} \cdot \vec{N}), \quad E = 0. \quad (2.291)$$

Here, $(\vec{\sigma} \cdot \vec{N})$ is the dot product of the three-dimensional vectors $\vec{\sigma}$ and \vec{N} , where \vec{N} is the unit vector perpendicular to the scattering plane in the center-of-mass frame. The 4×4 Dirac matrix σ_i ($i = 1, 2, 3$) has the form

$$\sigma_i = \begin{vmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{vmatrix}. \quad (2.292)$$

Here, the two-dimensional Pauli matrices appear in the parentheses. Combining the terms in expressions (2.291) with the term A (see Eq. (2.287)), we obtain the following expression for the scattering matrix:

$$S_q(k', t, k) = \begin{vmatrix} (F^+ + G^+(\vec{\sigma} \cdot \vec{N})) & 0 \\ 0 & (F^- + G^-(\vec{\sigma} \cdot \vec{N})) \end{vmatrix}, \quad (2.293)$$

where

$$F^\pm = A \pm B, \quad G^\pm = D \pm C. \quad (2.294)$$

Thus, pion-nucleon scattering in the relativistic case is also described by two complex amplitudes F and G ; the upper (+) and lower (−) superscripts refer to the scattering of particles and antiparticles, respectively. With the use of the projection operator

$$\Lambda^\pm(t) = \frac{1}{2} [1 \pm \gamma(t)], \quad (2.295)$$

the expression for pion–nucleon scattering matrix can be easily transformed to the covariant form

$$S_q(k', t, k) = \sum_{\pm} \Lambda^{\pm}(t) [F^{\pm} + G^{\pm} i \gamma_5 (\gamma \cdot n)]. \quad (2.296)$$

After the substitution of this relation into Eq. (2.287), the resulting S -matrix is covariant, and the states with positive and negative energies are clearly separated. This form of the scattering matrix is often used when discussing the problem of spin interactions.

2.9.1 Covariant Density Matrix

As mentioned above when considering nonrelativistic scattering theory, the density matrix is convenient when a partially polarized beam is used. The same is true for relativistic theory. For the physical system state ψ_{α} , the density matrix is defined by the expression

$$\rho = \sum_{\alpha} |\psi_{\alpha}\rangle W_{\alpha} \langle \psi_{\alpha}|, \quad (2.297)$$

where W_{α} is the probability of finding the system in this state; therefore, $\sum_{\alpha} W_{\alpha} = 1$.

The probability of finding the system in the region R is determined by the formula

$$w(R) = \text{Tr}(\rho \Pi), \quad (2.298)$$

where Π is the projection operator separating the region R . If the region is an element of the three-dimensional momentum space ($d\vec{f} = df_1 df_2 df_3$), then

$$w(d\vec{f}) = d\vec{f} \text{Tr} \rho_s(f). \quad (2.299)$$

Here, Tr stands for the trace of the matrix $\rho_s(f)$ in spin space and

$$\rho_s(\vec{f}) = \sum_{\alpha} |a_{\alpha}(\vec{f})|^2 [|U_{\alpha}(\vec{f})\rangle W_{\alpha} \langle U_{\alpha}^*(\vec{f})|]. \quad (2.300)$$

The amplitude $a_{\alpha}(\vec{f})$ is related to the wave function in momentum space as

$$\psi_{\alpha}(\vec{f}) = a_{\alpha}(\vec{f}) |U_{\alpha}(\vec{f})\rangle. \quad (2.301)$$

Here, the spinor $U_{\alpha}(\vec{f})$ of the particle moving with the momentum f can be expressed in terms of the spinor of the particle in its R frame by means of the Lorentz transformation:

$$\begin{aligned} U_{\alpha}(f) &= L^{-1}(f) U_{\alpha}(0), \\ w(df) &= (d\vec{f}) |a(\vec{f})|^2 (\gamma)^f = (d\vec{f}) \left[\sum W_{\alpha} |a_{\alpha}(\vec{f})|^2 \right] (\gamma)^f. \end{aligned} \quad (2.302)$$

In this expression, the ratio $(d\vec{f})/(\gamma)^f$ is Lorentz invariant (this is the invariant $d\vec{p}/dE$). Hence, the quantity $|a_{\alpha}(\vec{f}) \gamma^f|^2$ should be Lorentz invariant.

Note that both the density matrix and volume element are not invariant separately. However, if the particle is in a definite energy state, it is possible to make the transformation (sum signs are omitted)

$$\begin{aligned}\rho_s(\vec{f}) &= |a_\alpha(\vec{f})|^2 |U_\alpha(\vec{f})\rangle W_\alpha \langle U_\alpha^*(\vec{f})| \\ &= |a_\alpha(\vec{f})|^2 |U_\alpha(\vec{f})\rangle W_\alpha \langle U_\alpha^*(\vec{f})| |U_\alpha(\vec{f})\rangle (\pm) \langle U_\alpha^+(\vec{f})| \\ &= |\gamma^f a_\alpha(\vec{f})|^2 |U_\alpha(\vec{f})\rangle (\pm) W_\alpha \langle U_\alpha^+(\vec{f})| / (\gamma)^f \equiv \rho_f / (\gamma)^f. \quad (2.303)\end{aligned}$$

The product of the factor $1/(\gamma)^f$ and the volume element in the momentum space $d\vec{f}$ is invariant. The matrix ρ defined by the above expression is covariant, and its matrix elements are calculated by the formula

$$\rho_{ij}(f) = \langle U_i^+(f) | \rho(f) | U_j(f) \rangle. \quad (2.304)$$

The square of the absolute value of the amplitude specifies the probability of finding the system in the spin state α and with the momentum \vec{f} . The mean value of the operator \hat{O} in the initial state with the momentum f is given by the expression

$$\langle \hat{O} \rangle_f \text{Tr} \rho(f) = \text{Tr} [\rho(f) \hat{O}]. \quad (2.305)$$

The mean value of the operator in the final state with the momentum f' is written similarly

$$\langle \hat{O} \rangle_{f'} \text{Tr} \rho(f') = \text{Tr} [\rho(f') \hat{O}]. \quad (2.306)$$

Two density matrices are related by the scattering matrix as

$$\rho'(f') = S(f', t, f) \rho(f) S^+(f', t, f). \quad (2.307)$$

The differential cross section is determined by the expression

$$I = \text{Tr} \rho'(f') / \text{Tr} \rho(f). \quad (2.308)$$

A Hermitian conjugate operator \hat{A}^+ can be defined as: $(AU)^+ = U^+ A^+$, where U is the spinor and $U^+ = U^* \beta$. In application to the S -matrix, we obtain the relation $S^+ = \beta S^* \beta$, where the asterisk in S means complex conjugation.

The application of relativistic formulas (2.307) and (2.308) leads to the differential cross section in the center-of-mass frame, as in the nonrelativistic case. The same conclusion is also valid for polarization. In the case of the spin rotation parameters in the horizontal plane, only an additional kinematic factor appears.

2.10 Relativistic Nucleon–Nucleon Scattering

In the preceding section, we consider the case where both the initial and final states include only one Dirac particle. Now, we discuss the case where both the initial and final states of the reaction include two particles with spin 1/2. Specifically, we analyze nucleon–nucleon elastic scattering.

Each nucleon has its complete set of the spin operators acting in two independent spin spaces. We present these operators (Bjorken and Drell 1964).

In the spin space of the first particle, these are

$$I^{(1)}, \quad \gamma_\mu^{(1)}, \quad \frac{1}{2}\sigma_{\mu\nu}^{(1)}, \quad i\gamma_5^{(1)}\gamma_\mu^{(1)}, \quad \gamma_\mu^{(1)}, \quad \gamma_5^{(1)}, \quad (2.309)$$

in the spin space of the second particle,

$$I^{(2)}, \quad \gamma_\mu^{(2)}, \quad \frac{1}{2}\sigma_{\mu\nu}^{(2)}, \quad i\gamma_5^{(2)}\gamma_\mu^{(2)}, \quad \gamma_\mu^{(2)}, \quad \gamma_5^{(2)}. \quad (2.310)$$

Each of sets (2.309) and (2.310) contains 16 terms. The scattering matrix is composed of the direct product of these terms, i.e., contains $16 \times 16 = 256$ terms, as in the nonrelativistic case. Correspondingly, the number of amplitudes is the same. It is necessary to impose allowable physical conditions in order to reduce this number to the minimum possible number.

By analogy with the case of pion–nucleon scattering (see the preceding section), but taking into account the presence of four nucleons, we introduce the matrix $S_q(k', t, k)$ through the expression (Stapp 1956)

$$\begin{aligned} S(f', k', t, f, k) &= [\gamma^{(1)}(f', t)\gamma^{(1)}(t)][\gamma^{(2)}(k', t)\gamma^{(2)}(t)]S_q(k', t, k) \\ &\times [\gamma^{(2)}(t)\gamma^{(2)}(k, t)][\gamma^{(1)}(t)\gamma^{(1)}(f, t)]. \end{aligned} \quad (2.311)$$

Here, k and k' are the initial and final relative momenta, respectively. We can introduce the condition of the theory of holes and obtain

$$\begin{aligned} \gamma^{(1)}(t)S_q(k', t, k)\gamma^{(1)}(t) &= S_q(k', t, k), \\ \gamma^{(2)}(t)S_q(k', t, k)\gamma^{(2)}(t) &= S_q(k', t, k). \end{aligned} \quad (2.312)$$

The matrix $S_q(k', t, k)$ can be expanded in the products of 16 spin matrices for particle 1 by 16 spin matrices for particle 2 (see Eq. (2.310)). The main aim is to significantly reduce the number of the resulting 256 terms. To this end, we consider a term appearing from the product of tensor operators:

$$C_{\mu\nu\sigma\rho} \left(\frac{1}{2}\sigma_{\mu\nu}^{(1)} \right) \left(\frac{1}{2}\sigma_{\sigma\rho}^{(2)} \right). \quad (2.313)$$

From the first of expressions (2.312) applied to this term, we obtain

$$t_\mu C_{\mu\nu\sigma\rho} = -t_\mu C_{\nu\mu\sigma\rho} = 0. \quad (2.314)$$

Therefore,

$$C_{\mu\nu\sigma\rho} \left(\frac{1}{2}\sigma_{\mu\nu}^{(1)} \right) = \gamma^{(1)}(t) i\gamma_5^{(1)} \gamma_\lambda^{(1)} C_{\lambda;\sigma\rho}, \quad (2.315)$$

where we take into account that $t_\lambda C_{\lambda\nu\sigma\rho} = 0$. The dependence on $\sigma_{\sigma\rho}^{(2)}$ can be modified to the form

$$C_{\mu\nu\sigma\rho} \left(\frac{1}{2}\sigma_{\mu\nu}^{(1)} \right) \left(\frac{1}{2}\sigma_{\sigma\rho}^{(2)} \right) = C_{\lambda\sigma} \gamma^{(1)}(t) (i\gamma_5^{(1)} \gamma_\lambda^{(1)}) \gamma^{(2)}(t) (i\gamma_5^{(2)} \gamma_\sigma^{(2)}). \quad (2.316)$$

Here, the equality $C_{\lambda\eta}t_\eta = -t_\lambda C_{\lambda\eta}$ is taken into account. Excluding similarly all terms containing $\sigma_{\mu\nu}$, we obtain

$$\begin{aligned}
 S_q(k', t, k) = & \sum_{\pm\pm} (\Lambda^{(1)\pm}(t) \Lambda^{(2)\pm}(t)) \\
 & \cdot [F^{\pm\pm} + G^{(1)\pm\pm} i\gamma_5^{(1)} \gamma^{(1)} \cdot n + G^{(2)\pm\pm} i\gamma_5^{(2)} \\
 & + (C^{\pm\pm} n_\lambda n_\rho + D^{\pm\pm} s_\lambda s_\rho + E^{\pm\pm} d_\lambda d_\rho) \\
 & \cdot (i\gamma_5^{(1)} \gamma_\lambda^{(1)}) (i\gamma_5^{(2)} \gamma_\rho^{(2)})]. \quad (2.317)
 \end{aligned}$$

Here four vectors n , s , and d are defined as follows

$$\begin{aligned}
 n_\lambda & \propto k'_\rho k_\sigma t_\mu \varepsilon_{\rho\sigma\mu\lambda}, \\
 s_\lambda & = N_s [k_\lambda + k'_\lambda - t_\lambda \{t_\rho (k_\rho + k'_\rho)\} (t \cdot t)^{-1}], \\
 d_\lambda & = N_d [k_\lambda - k'_\lambda]. \quad (2.317a)
 \end{aligned}$$

Here N_i ($i = s, d$) are normalization coefficients. The vectors n, s, d, t form an orthogonal set. The vector d retains its sign under time inversion, whereas s changes sign. Therefore all terms containing the products of these two vectors should reduce to zero.

The scattering matrix is written as follows:

$$\begin{aligned}
 S_q(k', t, k) = & F + F^{(1)} \gamma^{(1)}(t) + F^{(2)} \gamma^{(2)}(t) + G_\lambda^{(1)} (i\gamma_5^{(1)} \gamma_\lambda^{(1)}) \\
 & + G_\lambda^{(2)} (i\gamma_5^{(2)} \gamma_\lambda^{(2)}) + C_\lambda^{(1)} \gamma^{(1)}(t) (i\gamma_5^{(1)} \gamma_\lambda^{(1)}) \\
 & + C_\lambda^{(2)} \gamma^{(2)}(t) (i\gamma_5^{(2)} \gamma_\lambda^{(2)}) + G_{\lambda\rho} (i\gamma_5^{(1)} \gamma_\lambda^{(1)}) \cdot (i\gamma_5^{(2)} \gamma_\rho^{(2)}) \\
 & + C_{\lambda\rho} \gamma^{(1)}(t) (i\gamma_5^{(1)} \gamma_\lambda^{(1)}) \gamma^{(2)}(t) (i\gamma_5^{(2)} \gamma_\rho^{(2)}) \\
 & + F^{(3)} \gamma^{(1)}(t) \cdot \gamma^{(2)}(t) + E_\lambda^{(1)} \gamma^{(2)}(t) (i\gamma_5^{(1)} \gamma_\lambda^{(1)}) \\
 & + E_\lambda^{(2)} \gamma^{(1)}(t) (i\gamma_5^{(2)} \gamma_\lambda^{(2)}) \\
 & + D_\lambda^{(1)} \gamma^{(2)}(t) \gamma^{(1)}(t) (i\gamma_5^{(1)} \gamma_\lambda^{(1)}) + D_\lambda^{(2)} \gamma^{(1)}(t) \gamma^{(2)}(t) (i\gamma_5^{(2)} \gamma_\lambda^{(2)}) \\
 & + H_{\lambda\rho}^{(1)} (i\gamma_5^{(1)} \gamma_\lambda^{(1)}) \gamma^{(2)}(t) (i\gamma_5^{(2)} \gamma_\rho^{(2)}) \\
 & + H_{\lambda\rho}^{(2)} (i\gamma_5^{(2)} \gamma_\lambda^{(2)}) \gamma^{(1)}(t) (i\gamma_5^{(1)} \gamma_\rho^{(1)}). \quad (2.318)
 \end{aligned}$$

All parameters appearing in the scattering matrix are functions of the momenta k, k' , and t . These parameters are orthogonal to t with respect to all subscripts. For example $t_\lambda H_{\lambda\rho} = t_\rho H_{\lambda\rho} = 0$ and similar is for the other terms.

It is easy to group the terms in expression (2.318). For example, the first two terms can be rewritten in the form

$$F + F^{(1)} \gamma^{(1)}(t) = \sum_{\pm} \frac{1}{2} [1 \pm \gamma^{(1)}(t)] F^\pm, \quad (2.318a)$$

where $F^+ = F + F^{(1)}$, $F^- = F - F^{(1)}$. The other terms can be pairwise grouped similarly. As a result, we arrive at the expression

$$\begin{aligned}
S_q(k', t, k) = \sum_{\pm} \frac{1}{2} [1 \pm \gamma^{(1)}(t)] \cdot [F^{\pm} + G_{\lambda}^{(1)\pm} (i\gamma_5^{(1)} \gamma_{\lambda}^{(1)}) + F^{(2)\pm} \gamma^{(2)}(t) \\
+ G_{\lambda}^{(2)\pm} (i\gamma_5^{(2)} \gamma_{\lambda}^{(2)}) + C_{\lambda}^{(2)\pm} \gamma^{(2)}(t) (i\gamma_5^{(2)} \gamma_{\lambda}^{(2)})] \\
+ [G_{\lambda\rho}^{\pm} (i\gamma_5^{(1)} \gamma_{\lambda}^{(1)}) (i\gamma_5^{(2)} \gamma_{\rho}^{(2)}) \\
+ H_{\lambda\rho}^{(1)\pm} (i\gamma_5^{(1)} \gamma_{\lambda}^{(1)}) \gamma^{(2)}(t) (i\gamma_5^{(2)} \gamma_{\rho}^{(2)}) \\
+ E_{\lambda}^{(1)\pm} \gamma^{(2)}(t) (i\gamma_5^{(1)} \gamma_{\lambda}^{(1)})]. \quad (2.319)
\end{aligned}$$

Grouping similarly the terms with respect to $\gamma^{(2)}(t)$, we obtain

$$\begin{aligned}
S_q(k', t, k) = \sum_{\pm\pm} \left\{ \frac{1}{2} [1 \pm \gamma^{(1)}(t)] \frac{1}{2} [1 \pm \gamma^{(2)}(t)] \right\} \\
\cdot [F^{\pm\pm} + G_{\lambda}^{(1)\pm\pm} (i\gamma_5^{(1)} \gamma_{\lambda}^{(1)}) + G_{\lambda}^{(2)\pm\pm} (i\gamma_5^{(2)} \gamma_{\lambda}^{(2)}) \\
+ G_{\lambda\rho}^{\pm\pm} (i\gamma_5^{(1)} \gamma_{\lambda}^{(1)}) (i\gamma_5^{(2)} \gamma_{\rho}^{(2)})]. \quad (2.320)
\end{aligned}$$

Here, the following orthogonality relations for $G_{\lambda\rho}$ were used:

$$t_{\lambda} G_{\lambda}^{(1)\pm\pm} = t_{\lambda} G_{\lambda}^{(2)\pm\pm} = t_{\lambda} G_{\lambda\rho}^{\pm\pm} = G_{\lambda\rho}^{\pm\pm} t_{\rho} = 0. \quad (2.320a)$$

Since the scattering matrix should be a scalar function, the parameters $G_{\lambda}^{(1)\pm\pm}$ and $G_{\lambda}^{(2)\pm\pm}$ should be pseudovectors. The existing three momenta can provide the single pseudovector

$$n_{\lambda} \propto k'_{\rho} k_{\sigma} t_{\mu} \varepsilon_{\rho\sigma\mu\lambda}. \quad (2.321)$$

Hence, we can write

$$G_{\lambda}^{(1)\pm\pm} = G^{(1)} n_{\lambda} \quad \text{and} \quad G_{\lambda}^{(2)\pm\pm} = G^{(2)} n_{\lambda}. \quad (2.322)$$

To transform the tensor terms $G_{\lambda\rho}^{\pm\pm}$, two normalized vectors s and d are introduced in addition to t and n as follows:

$$s_{\lambda} = N_s \{ k_{\lambda} + k'_{\lambda} - t_{\lambda} [t_{\rho} (k_{\rho} + k'_{\rho})] (t \cdot t)^{-1} \}, \quad d_{\lambda} = N_d (k_{\lambda} - k'_{\lambda}). \quad (2.323)$$

The four vectors t, n, s , and d constitute a set of orthonormalized vectors in which the second-rank tensor $G_{\lambda\rho}^{\pm\pm}$ can be expanded. Relation (2.320a) imposes the constraints on the number of the terms of this tensor. Additional conditions are imposed by the requirement of the invariance of the scattering matrix under space inversion. As a result, we obtain

$$\begin{aligned}
G_{\lambda\rho}^{\pm\pm} = C^{\pm\pm} n_{\lambda} n_{\rho} + D^{\pm\pm} s_{\lambda} s_{\rho} + E^{\pm\pm} d_{\lambda} d_{\rho} \\
+ G'^{\pm\pm} (s_{\lambda} d_{\rho} + d_{\lambda} s_{\rho}) + G^{\pm\pm} (s_{\lambda} d_{\rho} - d_{\lambda} s_{\rho}). \quad (2.324)
\end{aligned}$$

Since the matrix should be invariant under time reversal, two last terms should be zero, because the vector d does not change sign under this operation, whereas the vector s changes sign.

Finally, the relativistic nucleon–nucleon elastic scattering matrix is written in the form

$$\begin{aligned}
S_q(k', t, k) = & \sum_{\pm\pm} [\Lambda^{(1)\pm}(t)] [\Lambda^{(2)\pm}(t)] \\
& \cdot [F^{\pm\pm} + G^{(1)\pm\pm}(i\gamma_5^{(1)}\gamma^{(1)} \cdot n) + G^{(2)\pm\pm}(i\gamma_5^{(2)}\gamma^{(2)} \cdot n)] \\
& + [(C^{\pm\pm}n_\lambda n_\rho + D^{\pm\pm}s_\lambda s_\rho + E^{\pm\pm}d_\lambda d_\rho)(i\gamma_5^{(1)}\gamma_\lambda^{(1)})(i\gamma_5^{(2)}\gamma_\rho^{(2)})].
\end{aligned} \tag{2.325}$$

This is the relativistic formula for nucleon–nucleon scattering in the center-of-mass frame. Several conclusions follow from it. First, the number of free parameters is six, as in the nonrelativistic case. Second, the scattering of antiprotons is described by the same formulas as the scattering of protons. Third, passing to the nonrelativistic case and taking the states only with positive energy, we exactly arrive at Wolfenstein nonrelativistic formulas. As will be shown below, the relativistic case gives only kinematic corrections, which are easily taken into account.

In the center-of-mass frame, the relativistic matrix given by formula (2.325) reduces to the Wolfenstein–Ashkin nonrelativistic matrix (where $I^{(1)}$ and $I^{(2)}$ are the identity matrices in the spaces of particles 1 and 2, respectively):

$$\begin{aligned}
M = & aI^{(1)}I^{(2)} + c(\sigma_n^{(1)} + \sigma_n^{(2)}) + m\sigma_n^{(1)}\sigma_n^{(2)} \\
& + g(\sigma_{\hat{p}}^{(1)}\sigma_{\hat{p}}^{(2)} + \sigma_{\hat{k}}^{(1)}\sigma_{\hat{k}}^{(2)}) + h(\sigma_{\hat{p}}^{(1)}\sigma_{\hat{p}}^{(2)} - \sigma_{\hat{k}}^{(1)}\sigma_{\hat{k}}^{(2)}).
\end{aligned} \tag{2.326}$$

This matrix differs from the matrix used above in the notation: $a = a$, $c = c$, $b = m$, $l = g - h$, and $f = g + h$.

The ordinary method provides the following expressions for the measured quantities in terms of the elements of scattering matrix (2.326):

$$\begin{aligned}
I_0 R = & \frac{1}{2} \operatorname{Re} \left\{ (M_{00} + \sqrt{2} \cot \theta M_{10})(M_{11} + M_{1-1} - M_{ss})^* \right. \\
& \times \cos(\theta - \theta_L) - \frac{\sqrt{2}}{\sin \theta} [(M_{11} + M_{1-1})M_{10}^* - M_{ss}M_{01}^*] \cos \theta_l \Big\}, \\
I_0 A = & -\frac{1}{2} \operatorname{Re} \left\{ (M_{00} + \sqrt{2} \cot \theta M_{10})(M_{11} + M_{1-1} + M_{ss})^* \right. \\
& \times \sin(\theta - \theta_L) - \frac{\sqrt{2}}{\sin \theta} [(M_{11} + M_{1-1})M_{01}^* - M_{ss}M_{10}^*] \sin \theta_l \Big\}, \\
I_0 R' = & \frac{1}{2} \operatorname{Re} \left\{ (M_{00} + \sqrt{2} \cot \theta M_{10})(M_{11} + M_{1-1} + M_{ss})^* \right. \\
& \times \sin(\theta - \theta_L) + \frac{\sqrt{2}}{\sin \theta} [(M_{11} + M_{1-1})M_{10}^* - M_{ss}M_{01}^*] \sin \theta_l \Big\}, \\
I_0 A' = & \frac{1}{2} \operatorname{Re} \left\{ (M_{00} + \sqrt{2} \cot \theta M_{10})(M_{11} + M_{1-1} + M_{ss})^* \right. \\
& \times \cos(\theta - \theta_L) + \frac{\sqrt{2}}{\sin \theta} [(M_{11} + M_{1-1})M_{01}^* - M_{ss}M_{10}^*] \cos \theta_l \Big\},
\end{aligned}$$

$$\begin{aligned}
I_0 R_t &= \frac{1}{2} \operatorname{Re} \left\{ (M_{00} + \sqrt{2} \cot \theta M_{10})(M_{11} + M_{1-1} - M_{ss})^* \right. \\
&\quad \times \cos(\theta' - \theta'_L) + \frac{\sqrt{2}}{\sin \theta} [(M_{11} + M_{1-1})M_{10}^* + M_{ss}M_{01}^*] \cos \theta_l \left. \right\}, \\
I_0 A_t &= -\frac{1}{2} \operatorname{Re} \left\{ (M_{00} + \sqrt{2} \cot \theta M_{10})(M_{11} + M_{1-1} - M_{ss})^* \right. \\
&\quad \times \sin(\theta' - \theta'_L) - \frac{\sqrt{2}}{\sin \theta} [(M_{11} + M_{1-1})M_{01}^* - M_{ss}M_{10}^*] \sin \theta_l \left. \right\}, \\
I_0 R'_t &= \frac{1}{2} \operatorname{Re} \left\{ (M_{00} + \sqrt{2} \cot \theta M_{10})(M_{11} + M_{1-1} - M_{ss})^* \right. \\
&\quad \times \sin(\theta' - \theta'_L) - \frac{\sqrt{2}}{\sin \theta} [(M_{11} + M_{1-1})M_{10}^* + M_{ss}M_{01}^*] \sin \theta_l \left. \right\}, \\
I_0 A'_t &= \frac{1}{2} \operatorname{Re} \left\{ (M_{00} - \sqrt{2} \cot \theta M_{10})(M_{11} + M_{1-1} - M_{ss})^* \right. \\
&\quad \times \cos(\theta' - \theta'_L) - \frac{\sqrt{2}}{\sin \theta} [(M_{11} + M_{1-1})M_{01}^* + M_{ss}M_{10}^*] \cos \theta_l \left. \right\}, \\
I_0 C_{kp} &= \frac{1}{2 \sin \theta} [|M_{01}|^2 - |M_{10}|^2] \cos(\alpha - \alpha') - \frac{1}{4} [|M_{11} + M_{1-1}|^2 - |M_{ss}|^2] \\
&\quad \times \cos(\alpha + \alpha') - \frac{1}{4 \cos \theta} [|M_{11} - M_{1-1}|^2 - |M_{00}|^2] \sin(\alpha - \alpha'), \\
I_0 C_{kk} &= -\frac{1}{2 \sin \theta} [|M_{01}|^2 - |M_{10}|^2] \sin(\alpha - \alpha') + \frac{1}{4} [|M_{11} + M_{1-1}|^2 - |M_{ss}|^2] \\
&\quad \times \cos(\alpha + \alpha') - \frac{1}{4 \cos \theta} [|M_{11} - M_{1-1}|^2 - |M_{00}|^2] \cos(\alpha - \alpha'), \\
I_0 C_{pp} &= -\frac{1}{2 \sin \theta} [|M_{01}|^2 - |M_{10}|^2] \sin(\alpha - \alpha') + \frac{1}{4} [|M_{11} + M_{1-1}|^2 - |M_{ss}|^2] \\
&\quad \times \cos(\alpha + \alpha') - \frac{1}{4 \cos \theta} [|M_{11} - M_{1-1}|^2 - |M_{00}|^2] \cos(\alpha - \alpha').
\end{aligned}$$

Here, θ and θ_l are the scattering angles in the center-of-mass and laboratory frames, respectively, and θ' and θ'_L are the respective angles for the recoil particle.

Under relativistic transformations, the parameters that either are scalars or have only components perpendicular to the reaction plane remain unchanged. These are the following quantities: cross section I ; polarization P ; depolarization tensors of the scattered and recoil particles, D and D_t , respectively; and correlation parameters C_{nn} and A_{nn} .

In the above formalism of the relativistic reaction matrix, the wave functions are represented in the space of angular momenta, where the quantization z axis is fixed.

Jacob and Wick (1959) proposed a relativistic description of reactions using the wave functions quantized along the momentum of the incident and scattered particles in the center-of-mass frame. In this case, the spin projection on the momentum direction has two values, $+1/2$ and $-1/2$, and this projection is called helicity.

Denoting the helicities of the initial and final nucleons as λ_1, λ_2 and λ'_1, λ'_2 , respectively, we can write the scattering matrix in the form

$$\langle \lambda'_1 \lambda'_2 | M | \lambda_1 \lambda_2 \rangle. \quad (2.327)$$

These matrix elements are called helicity amplitudes. Let the scattered particle move along the z' axis inclined to the z axis at the angle θ (scattering angle). By analogy with the spinning-top model, matrix element (2.327) can be expanded in terms of the reduced wave functions of the symmetric top, $d_{\mu\mu'}^J$:

$$\langle \lambda'_1 \lambda'_2 | M | \lambda_1 \lambda_2 \rangle = \frac{1}{2ik} \sum_J (2J+1) (\langle \lambda'_1 \lambda'_2 | S(J, E) - 1 | \lambda_1 \lambda_2 \rangle) d_{\mu\mu'}^J(\theta), \quad (2.328)$$

where

$$\mu = \lambda_1 - \lambda_2, \quad \mu' = \lambda'_1 - \lambda'_2. \quad (2.329)$$

As shown above, owing to the invariance of the scattering matrix under space rotation and inversion and time reversal, as well as to the isotopic invariance, five matrix elements are nonzero in the general case. The same requirements applied to the scattering matrix in the helicity representation impose the following conditions: parity conservation

$$\langle \lambda'_1 \lambda'_2 | M | \lambda_1 \lambda_2 \rangle = \langle -\lambda'_1 - \lambda'_2 | M | -\lambda_1 - \lambda_2 \rangle, \quad (2.330)$$

time reversal

$$\langle \lambda'_1 \lambda'_2 | M | \lambda_1 \lambda_2 \rangle = (-1)^{\lambda_1 - \lambda_2 - \lambda'_1 + \lambda'_2} \langle \lambda_1 \lambda_2 | M | \lambda'_1 \lambda'_2 \rangle, \quad (2.331)$$

conservation of the total spin

$$\langle \lambda'_1 \lambda'_2 | M | \lambda_1 \lambda_2 \rangle = \langle \lambda'_2 \lambda'_1 | M | \lambda_2 \lambda_1 \rangle. \quad (2.332)$$

Under these conditions, five matrix elements are also nonzero in the helicity representation, and these elements are denoted as follows:

$$\begin{aligned} \varphi_1 &= \left\langle \frac{1}{2}, \frac{1}{2} \left| M \right| \frac{1}{2}, \frac{1}{2} \right\rangle, & \varphi_2 &= \left\langle \frac{1}{2}, \frac{1}{2} \left| M \right| -\frac{1}{2}, -\frac{1}{2} \right\rangle, \\ \varphi_3 &= \left\langle \frac{1}{2}, -\frac{1}{2} \left| M \right| \frac{1}{2}, -\frac{1}{2} \right\rangle, & & \\ \varphi_4 &= \left\langle \frac{1}{2}, -\frac{1}{2} \left| M \right| -\frac{1}{2}, \frac{1}{2} \right\rangle, & \varphi_5 &= \left\langle \frac{1}{2}, \frac{1}{2} \left| M \right| \frac{1}{2}, -\frac{1}{2} \right\rangle. \end{aligned} \quad (2.333)$$

The amplitudes φ_i , where $i = 1, 2, 3, 4, 5$, are classified according to the physics of the process as the amplitudes without spin flip (φ_1, φ_3), with single spin flip (φ_5), and with double spin flip (φ_2, φ_4).

The relation between the matrix elements in the helicity representation and angular-momentum representation can be obtained as follows (Jacob and Wick 1959). Let the direction of the incident-particle motion be the quantization axis for

the spin wave function. Then, the wave functions of the first and second nucleons in the helicity representation can be written in the form

$$\chi_{1/2}^{(1)} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \quad \chi_{-1/2}^{(1)} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \quad (2.334)$$

in the initial state and

$$\chi_{1/2}^{(1)} = \chi_{-1/2}^{(2)} = \begin{Bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{Bmatrix}, \quad \chi_{-1/2}^{(1)} = \chi_{1/2}^{(2)} = \begin{Bmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{Bmatrix} \quad (2.335)$$

in the final state, where θ is the particle scattering angle in the center-of-mass frame.

Let us rewrite matrix (2.326) as follows:

$$\begin{aligned} M = & a + c(\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)}) \cdot \vec{n} + m(\vec{\sigma}^{(1)} \cdot \vec{n})(\vec{\sigma}^{(2)} \cdot \vec{n}) \\ & + g[(\vec{\sigma}^{(1)} \cdot \vec{P})(\vec{\sigma}^{(2)} \cdot \vec{P}) + (\vec{\sigma}^{(1)} \cdot \vec{K})(\vec{\sigma}^{(2)} \cdot \vec{K})] \\ & + h[(\vec{\sigma}^{(1)} \cdot \vec{P})(\vec{\sigma}^{(2)} \cdot \vec{P}) - (\vec{\sigma}^{(1)} \cdot \vec{K})(\vec{\sigma}^{(2)} \cdot \vec{K})]. \end{aligned} \quad (2.336)$$

The unit vectors \vec{n} , \vec{P} , \vec{K} have the following components:

$$\vec{n}(0, 1, 0), \quad \vec{K}(\cos \theta/2, 0, -\sin \theta/2), \quad \vec{P}(\sin \theta/2, 0, \cos(\theta/2)). \quad (2.337)$$

Therefore, the vector \vec{n} is perpendicular to the scattering plane, whereas the vectors \vec{K} and \vec{P} lie in the scattering plane. Applying matrix (2.336) to wave functions (2.334) and (2.335) and taking into account relations (2.337), we obtain the relation between the amplitudes in the helicity and angular representations. They are given below along with the relations with the amplitudes in the singlet–triplet representation.

2.10.1 Relation Between the Amplitudes in Different Representations

A. In the helicity and angular representations:

$$\begin{aligned} \varphi_1 - \varphi_2 &= a - m - 2g, \\ \varphi_1 + \varphi_2 &= (a + m) \cos \theta + 2ic \sin \theta + 2h, \\ \varphi_3 + \varphi_4 &= a - m + 2g, \\ \varphi_3 - \varphi_4 &= (a + m) \cos \theta + 2ic \sin \theta - 2h, \\ \varphi_5 &= -\frac{1}{2}(a + m) \sin \theta + ic \cos \theta. \end{aligned}$$

B. In the angular and helicity representations (inverse to case A):

$$\begin{aligned}
 a &= \frac{1}{4}[(\varphi_1 - \varphi_2 + \varphi_3 + \varphi_4) + (\varphi_1 + \varphi_2 + \varphi_3 - \varphi_4) \cos \theta - 4 \sin \theta \varphi_5], \\
 ic &= \frac{1}{4}[(\varphi_1 + \varphi_2 + \varphi_3 - \varphi_4) \sin \theta + 4 \cos \theta \varphi_5], \\
 m &= \frac{1}{4}[(-\varphi_1 + \varphi_2 - \varphi_3 - \varphi_4) + (\varphi_1 + \varphi_2 + \varphi_3 - \varphi_4) \cos \theta - 4 \sin \theta \varphi_5], \\
 g &= \frac{1}{4}(-\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4), \\
 h &= \frac{1}{4}(-\varphi_1 - \varphi_2 + \varphi_3 - \varphi_4).
 \end{aligned}$$

C. In the helicity and singlet–triplet representations:

$$\begin{aligned}
 \varphi_1 - \varphi_2 &= M_{ss}, \\
 \varphi_1 + \varphi_2 &= \cos \theta M_{00} - \sqrt{2} \sin \theta M_{10}, \\
 \varphi_3 + \varphi_4 &= M_{11} + M_{1-1}, \\
 \varphi_1 - \varphi_2 &= \cos \theta M_{11} + \sin \theta M_{01} - \cos \theta M_{1-1}, \\
 \varphi_5 &= -\frac{1}{2} \sin \theta M_{11} + \frac{1}{\sqrt{2}} \cos \theta M_{01} - \frac{1}{2} \sin \theta M_{1-1} \\
 &= -\frac{1}{2} \sin \theta M_{00} - \frac{1}{\sqrt{2}} \cos \theta M_{10}.
 \end{aligned}$$

The helicity amplitudes can be expressed in terms of the phase shifts. To this end, the matrix elements in helicity space should be transformed to the matrix elements in singlet–triplet space that are already expressed in terms of the phases.

D. The matrix elements in the helicity representation can be expressed in terms of the elements in the singlet–triplet representation as follows:

$$\langle \lambda'_1 \lambda'_2 | M | \lambda_1 \lambda_2 \rangle = \sum \langle \lambda'_1 \lambda'_2 | S m_s \rangle \langle S m_s | M | S m'_s \rangle \langle S m'_s | \lambda_1 \lambda_2 \rangle. \quad (2.338)$$

Since the elements of the matrix M , as well as the Clebsch–Gordan coefficients, are known, the helicity amplitudes can be found from this relation (see “Relation between the amplitudes in different representations”).

2.11 Isospin T , C and G Parities

2.11.1 Isotopic Invariance

There are many experimental evidences that the properties of the proton and neutron in nuclear interaction are very similar. It can be assumed that they are the components of the same object called the nucleon. First, their masses are very close: the

proton and neutron masses are 938.27 and 939.57 MeV, respectively; hence, the mass difference is as small as 1.3 MeV. This small difference ($\approx 0.15\%$) is assumingly attributed to the electromagnetic interaction. Second, they strongly interact with each other and are constituents of nuclei. Third, it is known that, excluding the relatively weak electromagnetic interaction in the pp system, this system is almost equivalent to the system of two neutrons nn . This equality of the interactions between two protons and two neutrons is called *charge symmetry*. In particular, additional experimental evidence of charge symmetry follows from nucleon–nucleon scattering experiments. It is shown that the pp and nn elastic scattering processes are identical, excluding the Coulomb interaction. The same data are also obtained in pion–nucleon and kaon–nucleon elastic scattering processes. In addition, the binding energies (B), energy levels, and other properties of the mirror nuclei in which the protons and neutrons are interchanged are very close to each other. The closeness of the binding energies of two mirror nuclei is exemplified as follows:

$$\text{H}^3 = (nnp) \rightarrow B = 8.192 \text{ MeV}, \quad \text{He}^3 = (ppn) \rightarrow B = 7.728 \text{ MeV}. \quad (2.339)$$

Here, the nucleon compositions of the tritium and helium-3 nuclei are given in the parentheses. The difference between the binding energies of these nuclei is as small as $\Delta B \approx 0.5$ MeV and can be attributed to the energy of the Coulomb repulsion between two protons in the helium-3 nucleus:

$$V_C(R) = \frac{1}{2}Z(Z-1)\frac{6e^2}{5R}, \quad R \approx 1.45 \cdot 10^{-13} \text{ cm}. \quad (2.340)$$

The closeness of the energy levels of the mirror nuclei can be illustrated by the example

$$\begin{aligned} \text{B}^{11} &= (6n5p) \rightarrow E = (1.98; 2.14; 4.46; 5.03; 6.76) \text{ MeV}, \\ \text{C}^{11} &= (6p5n) \rightarrow E = (-; 1.85; 4.23; 4.77; 6.40) \text{ MeV}. \end{aligned} \quad (2.341)$$

Here, the numbers of protons and neutrons in a given nucleus are presented in the first parentheses and the level energies of this nucleus are indicated in the second parentheses. The observed similarity of the energy levels of these mirror nuclei is most simply explained by the hypothesis of the *charge symmetry* between the proton and neutron. The detailed discussion of these problems can be found in Schiff (1968).

It is substantial that both the proton and neutron have a half-integer spin and are governed by the same statistics (Dirac–Fermi statistics). This means that the system of two protons or two neutrons is described by the wave function $\psi(\vec{r}_1, \vec{s}_1; \vec{r}_2, \vec{s}_2)$ (where \vec{r} and \vec{s} are the radius-vector and spin of the particles, respectively) anti-symmetric under the simultaneous permutation of the coordinates and spins of the particles. However, the experimentally observed charge symmetry is only one of the manifestations of a deeper similarity of the proton and neutron. This new type of symmetry was introduced by Heisenberg (1932) and was called isotopic invariance. Its main meaning is that the forces between the pp , np , and nn pairs are identical and the proton and neutron are two components of the nucleon. Let us illustrate this hypothesis by an example from low-energy scattering. The interaction of the

proton with the proton or the neutron at low energies is characterized by two parameters: the scattering length a and effective scattering radius r_0 . They determine the scattering phase δ in the S -state through the relation

$$k \cot \delta = -\frac{1}{a} + \frac{1}{2} r_0 k^2, \quad (2.342)$$

where k is the wavenumber. The following parameters are experimentally determined (Nishijima 1964):

$$\begin{aligned} np : \quad & {}^3S_1, \quad r_{0t} = (1.704 \pm 0.028) \cdot 10^{-13} \text{ cm}, \\ & a_t = (5.39 \pm 0.03) \cdot 10^{-13} \text{ cm}; \\ & {}^1S_1, \quad r_{0S} = (2.670 \pm 0.023) \cdot 10^{-13} \text{ cm}, \\ & a_S = (-23.74 \pm 0.09) \cdot 10^{-13} \text{ cm}; \\ pp : \quad & {}^1S_1, \quad r_{0S} = (2.77) \cdot 10^{-13} \text{ cm}, \quad a_S = (-17.77) \cdot 10^{-13} \text{ cm}. \end{aligned} \quad (2.343)$$

The parameters a and r_0 of the np - and pp -scattering processes in the same 1S_1 states are in qualitative agreement. The quantitative difference can be ignored, because this difference can be eliminated by changing the potential-well depth by 3 %.

These experimental facts led to the notion of the nucleon isospin τ and to the discovery of isotopic invariance in strong interactions. The isospin is a vector in isotopic space and has the same properties as the Pauli matrices. Isospin space is not a real space, but only a mathematical notion. Correspondingly, the nature of isospin is unknown.

The isotopic invariance of strong interactions can be formulated as invariance under “rotations in isospin space.” The isospin operator $T(T_1, T_2, T_3)$ and isospinor ψ are written in the explicit form

$$T_1 = \frac{1}{2} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \quad T_2 = \frac{1}{2} \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}, \quad T_3 = \frac{1}{2} \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}, \quad \psi = \begin{vmatrix} \psi_1 \\ \psi_2 \end{vmatrix}. \quad (2.344)$$

Let us consider the rotation by angle π about the x_2 axis in this space. The isospinor is transformed as:

$$\psi \rightarrow e^{i\pi T_2} \psi = e^{i\frac{\pi}{2} \tau_2} \psi = \left(\cos \frac{\pi}{2} + i \tau_2 \sin \frac{\pi}{2} \right) \psi = i \tau_2 \psi. \quad (2.345)$$

Here, $T_i = 1/2 \tau_i$, where $i = 1, 2, 3$; T is the nucleon isospin equal $1/2$ with the eigenvalues $\pm 1/2$; and τ is the analog of the Pauli operator in isotopic space. It is usually accepted that the eigenvalue $+1/2$ corresponds to the proton and is described by the function ψ_1 , whereas $-1/2$ corresponds to the neutron and is described by the function ψ_2 in Eq. (2.344). In particular, for the isospinors corresponding to two components of the isodoublet (to the proton and neutron), we have

$$|p\rangle = \begin{vmatrix} 1 \\ 0 \end{vmatrix} \rightarrow \begin{vmatrix} 0 \\ -1 \end{vmatrix} = |n\rangle, \quad |n\rangle = \begin{vmatrix} 0 \\ 1 \end{vmatrix} \rightarrow \begin{vmatrix} 1 \\ 0 \end{vmatrix} = |p\rangle, \quad (2.346)$$

which correspond to the transformations

$$\begin{aligned}\tau_+ \psi &= \frac{1}{2}(\tau_1 + i\tau_2)\psi = \begin{vmatrix} 0 & 0 \\ -1 & 0 \end{vmatrix} \psi, & n \rightarrow p, \\ \tau_- \psi &= \frac{1}{2}(\tau_1 - i\tau_2)\psi = \begin{vmatrix} 0 & 0 \\ -1 & 0 \end{vmatrix} \psi, & p \rightarrow -n,\end{aligned}\quad (2.347)$$

where p and n mean the charge states $|p\rangle$ and $|n\rangle$ of the nucleon. Similar transformations for the antinucleon have the form

$$\bar{p} \rightarrow -\bar{n}, \quad \bar{n} \rightarrow \bar{p}. \quad (2.348)$$

As seen, strong interactions are charge independent or, more widely, isotopically invariant.

2.11.2 Charge Conjugation

The charge conjugation operation is defined only in relativistic theory. Let us represent the ψ operator in the form of the expansion

$$\psi = \sum \frac{1}{\sqrt{2\varepsilon}} (a_p e^{-i(\omega t - \vec{p} \cdot \vec{r})} + b_p e^{i(\omega t - \vec{p} \cdot \vec{r})}), \quad (2.349)$$

where a_p and b_p are the annihilation and creation operators for a particle with the momentum p , respectively. The charge conjugation operation reduces to the change of the particles to the antiparticles and vice versa; i.e.,

$$C : \quad a_p \rightarrow b_p, \quad b_p \rightarrow a_p. \quad (2.350)$$

The action of operation (2.350) on operator (2.349) provides the charge conjugate operator ψ^C ; it is easy to see that

$$\psi^C(t, \vec{r}) = \psi^+(t, \vec{r}). \quad (2.351)$$

This equality expresses the property of the charge symmetry of the particles and antiparticles. According to relation (2.349), the operator C changes the particle to the antiparticle that is not identical to the particle. As a result, this operator has no eigenfunctions and eigenvalues. For this reason, the charge conjugation operation does not generally lead to new physical consequences. However, there is exclusion. In the application to a system where the number of particles coincides with the number of antiparticles, the operator C has eigenfunctions and eigenvalues. The following transformations are illustrative:

$$C|\Lambda\rangle = |\bar{\Lambda}\rangle, \quad C|n\rangle = |\bar{n}\rangle, \quad C|p\rangle = -|\bar{p}\rangle. \quad (2.352)$$

The first relation is obvious, because the Λ hyperon is the isotopic singlet. To prove the last two relations, we write the wave function of a pair of nucleons and

antinucleons and the charge conjugation matrix in the explicit form (Lifshitz and Pitaevskii 1973)

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} p \\ n \\ \bar{n} \\ \bar{p} \end{pmatrix} \quad (2.353)$$

and

$$C = \begin{pmatrix} 0 & -i\tau_2 \\ i\tau_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \\ 0 & +1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \quad (2.354)$$

The action of operator (2.354) on wave function (2.353) gives

$$\psi^C = \begin{pmatrix} \psi_1^C \\ \psi_2^C \end{pmatrix} = C\psi = \begin{pmatrix} 0 & -i\tau_2 \\ i\tau_2 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} -i\tau_2\psi_2 \\ i\tau_2\psi_1 \end{pmatrix}. \quad (2.355)$$

We obtain two matrix equations of the second rank, which are solved separately:

$$\psi_1^C = \begin{pmatrix} C(p) \\ C(n) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \bar{n} \\ \bar{p} \end{pmatrix} = \begin{pmatrix} -\bar{p} \\ \bar{n} \end{pmatrix}. \quad (2.356)$$

Therefore,

$$C(p) = -\bar{p}, \quad C(n) = \bar{n}. \quad (2.357)$$

We write and solve the second matrix equation:

$$\psi_2^C = \begin{pmatrix} C(\bar{n}) \\ C(\bar{p}) \end{pmatrix} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} p \\ n \end{pmatrix} = \begin{pmatrix} n \\ -p \end{pmatrix}. \quad (2.358)$$

Hence (Pilkuhn 1979),

$$C(\bar{n}) = n, \quad C(\bar{p}) = -p. \quad (2.359)$$

Since $C^2 = 1$, it is easy to verify that results (2.358) and (2.359) coincide.

2.11.3 *G Transformation*

The simultaneous application of two conservation laws leads to new selection rules, which do not follow from any individual law (Lee and Yang 1956).

The joint application of the isotopic transformation T and charge conjugation C is described by the product of both operators and is denoted as G :

$$G = Ce^{i\pi T_3}. \quad (2.360)$$

Since $p \leftrightarrow -\bar{p}$ and $n \leftrightarrow \bar{n}$ under charge conjugation, operator (2.360) provides (Lifshitz and Pitaevskii 1973)

$$G: \quad p \rightarrow -\bar{n}, \quad n \rightarrow \bar{p}, \quad \bar{p} \rightarrow -n, \quad \bar{n} \rightarrow p. \quad (2.361)$$

The operator G commutes with the operators of all three isospin components T_1 , T_2 , and T_3 . This is directly verified by writing the explicit expressions for the operators in the form of four-row matrices transforming the nucleon and antinucleon states. Let us represent these states in the form of the column

$$\begin{pmatrix} p \\ n \\ \bar{n} \\ \bar{p} \end{pmatrix}$$

and generalize isospin to this case:

$$\begin{aligned} T_1 &= \frac{1}{2} \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_1 \end{pmatrix}, & T_2 &= \frac{1}{2} \begin{pmatrix} \tau_2 & 0 \\ 0 & \tau_2 \end{pmatrix}, & T_3 &= \frac{1}{2} \begin{pmatrix} \tau_3 & 0 \\ 0 & \tau_3 \end{pmatrix}, \\ C &= \begin{pmatrix} 0 & -i\tau_2 \\ i\tau_2 & 0 \end{pmatrix}, & G &= \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \end{aligned} \quad (2.362)$$

Here, 0 and I are two-row matrices.

If the operation G transforms a particle (or a system of particles) to itself, the notion of the G parity appears: the state can remain unchanged or change sign. For this, the baryon number and hypercharge Y ($Y = B + S$, where B is the baryon number and S is strangeness) of the particles should be zero. Indeed, charge conjugation (transition from the particles to the antiparticles) changes the signs of the electric charge Z and hypercharge. Rotation in isospace changes Z , but does not change Y and B . Therefore, the joint application of both transformations changes the numbers Y and B if they are nonzero.

An important property of the G parity is that it is the same for all components of the same isomultiplet. This follows from the commutativity of the operator G with all components of T and, therefore, with all rotations in isospace.

At $Y = 0$, we have $Z = T_3$; therefore, T_3 and thereby T are integers. The isomultiplet with an integer T value is described by a symmetric isospinor of the even rank $2T$, which is equivalent to the irreducible isotensor of the rank T . One of the components of such isomultiplet is a neutral particle ($T_3 = 0$). It corresponds to the isotensor ψ_{ih} with the nonzero component ψ_{33} . The rotation by the angle π about the x_2 axis leads to the multiplication of this isotensor by $(-1)^T$. The G parity of a neutral particle with the charge parity C is given by the expression

$$G = C(-1)^T. \quad (2.363)$$

According to the above consideration, the G parity of all components of the isomultiplet is thus defined.

For example, let us consider the pion isotriplet ($T = 1$). The charge parity of the π^0 meson is $C = +1$. This follows from the fact that the π^0 meson decays into an even number of particles, namely, into two charge-odd particles (photons). Therefore, the G parity of the pions is $G = -1$. In particular, it follows from this that strong interactions can transfer the system of pions to another system of pions only without change of the parity of the number of particles.

Table 2.3 G and C parities of mesons and baryons

	π^+	π^0	π^-	η	K^+	K^0	\bar{K}^0	K^-
G	$-\pi^+$	$-\pi^0$	$-\pi^-$	η	\bar{K}^0	K^-	$-K^+$	$-K^0$
C	$-\pi^-$	π^0	$-\pi^+$	η	$-K^-$	\bar{K}^0	K^0	$-K^+$
	Σ^+	Σ^0	Σ^-	Λ	p	n	\bar{n}	\bar{p}
G	$-\bar{\Sigma}^+$	$-\bar{\Sigma}^0$	$-\bar{\Sigma}^-$	$\bar{\Lambda}$	\bar{n}	\bar{p}	$-p$	$-n$
C	$-\bar{\Sigma}^-$	$\bar{\Sigma}^0$	$-\bar{\Sigma}^+$	$\bar{\Lambda}$	$-\bar{p}$	\bar{n}	n	$-p$

The η meson is an isosinglet ($T = 0$), and its charge parity is $C = +1$, because the η meson, as well as the π^0 meson, decays into two photons. Therefore, the η meson has the positive G parity ($G = +1$). Hence, strong interactions cannot lead to the $\eta \rightarrow 3\pi$ decay.

It is desirable to extend the notion of the G parity to other single-particle states. The state of the particle m of the charge multiplet with the momentum p , helicity λ , and third isospin component T_3 is described by the function $|m, T_3, p, \lambda\rangle$, and the state of the corresponding antiparticle, by the function $|\bar{m}, T_3, p, \lambda\rangle$. Since the operation G changes the particle to the antiparticle and does not change the other variables, we can write (Pilkun 1979):

$$G|m, T_3, p, \lambda\rangle = \eta_G|\bar{m}, T_3, p, \lambda\rangle. \quad (2.364)$$

Here, η_G is independent of T_3 ; the function of the state of the antiparticle is transformed in isotopic space similarly to the function of the state of the particle. As a result, for the multiplet of the particles, we have the relation

$$\eta_G = \eta_G(-1)^{T+T_3} \quad (2.365)$$

or in the other form

$$\eta_G = \eta_G(-1)^{T+T_3}. \quad (2.366)$$

For the π^0 meson, $C = +1$, $T = 1$, and $T_3 = 0$; therefore, $G = -1$. At the same time, although the parity of the η meson is $C = +1$, but $G = 1$ because $T = 0$ and $T_3 = 0$. This agrees with the above results.

Formula (2.366) can be considered as the expansion of formula (2.363) to the charged components of the multiplet, which should have the same G parity as the truly neutral component of the multiplet. For the pion isotriplet, $C|\pi^\pm\rangle = G|\pi^\pm\rangle = -|\pi^\mp\rangle$. Let us consider the G and C parities for the hyperons and K mesons. It is substantial that a baryon can emit (virtually) one pion and hyperon can emit one K meson with the satisfaction of the selection rules in the isospin, hypercharge, and baryon number. From the $\Sigma \rightarrow \pi \Lambda$ decay, it follows that $\eta_G(\Sigma) = \eta_G(\pi)\eta_G(\Lambda) = \eta_G(\pi)$, because the G parity of the Λ hyperon is positive. From the $N \rightarrow K \Lambda$ decay, it follows that $\eta_G(K) = \eta_G(N)\eta_G(\Lambda) = \eta_G(N)$. From the $\Xi \rightarrow K \Lambda$ decay, it follows that $\eta_G(\Xi) = \eta_G(K) = \eta_G(N)$. The results thus obtained are presented in Table 2.3.

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<http://www.springer.com/978-3-642-32162-7>

Introduction to Polarization Physics

Nurushev, S.B.; Runtso, M.F.; Strikhanov, M.N.

2013, XVIII, 430 p. 112 illus., Softcover

ISBN: 978-3-642-32162-7