Chapter 4
The Double Multi-Layer Potential Operator

In this chapter we take on the task of introducing and studying what we call double multi-layer potential operators, associated with arbitrary elliptic, higher-order, homogeneous, constant (complex) matrix-valued coefficients. As a preamble, we first take a look at the nature of fundamental solutions associated with such operators.

4.1 Differential Operators and Fundamental Solutions

Fix an Euclidean space dimension $n \in \mathbb{N}$. For some fixed $M \in \mathbb{N}$ and $m \in \mathbb{N}$, consider a $M \times M$ system of homogeneous differential operators of order $2m$ in $\mathbb{R}^n$ with complex constant coefficients:

$$L = \sum_{|\alpha|=|\beta|=m} \partial^\alpha A_{\alpha\beta} \partial^\beta = \big(L_{jk}\big)_{1\leq j,k\leq M} \tag{4.1}$$

where, for each multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ of length m,

$$A_{\alpha\beta} := \Big(a_{jk}^{\alpha\beta}\Big)_{1\leq j,k\leq M} \in \mathbb{C}^{M\times M} \quad \text{and} \quad L_{jk} := \sum_{|\alpha|=|\beta|=m} \partial^\alpha a_{jk}^{\alpha\beta} \partial^\beta, \tag{4.2}$$

for some coefficients $a_{jk}^{\alpha\beta} \in \mathbb{C}$, $j,k \in \{1,\dots,M\}$. Throughout, we shall use the superscript t to indicate transposition. In particular,

$$L^t := \sum_{|\alpha|=|\beta|=m} \partial^\alpha A_{\beta\alpha}^t \partial^\beta, \tag{4.3}$$

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and we shall call L `symmetric` if $L^t = L$. The complex conjugate of L is the operator $\overline{L}$ defined by the requirement that $\overline{L}u = \overline{L\overline{u}}$ for each smooth function u. Hence,

$$\overline{L} = \overline{L}^{\,t} = \sum_{|\alpha|=|\beta|=m} \partial^\alpha \overline{A_{\beta\alpha}} \partial^\beta. \tag{4.4}$$

The formal adjoint of L is then defined as

$$L^* := \overline{L}^{\,t} = \sum_{|\alpha|=|\beta|=m} \partial^\alpha \overline{A_{\beta\alpha}}^{\,t} \partial^\beta, \tag{4.5}$$

and we shall call the operator L `self-adjoint` if $L^* = L$. Given a tensor coefficient $A = \big(A_{\alpha\beta}\big)_{|\alpha|=|\beta|=m}$ with complex entries, define

$$A^t := \big(A^t_{\beta\alpha}\big)_{|\alpha|=|\beta|=m}, \quad \overline{A} := \big(\overline{A_{\alpha\beta}}\big)_{|\alpha|=|\beta|=m}, \quad A^* := \big(\overline{A_{\beta\alpha}}^{\,t}\big)_{|\alpha|=|\beta|=m}. \tag{4.6}$$

Also, call A `symmetric` if $A = A^t$, and `self-adjoint` if $A = A^*$. In particular,

$$A \ \text{ symmetric} \iff A_{\alpha\beta} = (A_{\beta\alpha})^t, \qquad \forall\, \alpha, \beta \in \mathbb{N}_0^n : |\alpha| = |\beta| = m, \tag{4.7}$$

$$A \ \text{ self-adjoint} \iff A_{\alpha\beta} = (\overline{A_{\beta\alpha}})^t, \qquad \forall\, \alpha, \beta \in \mathbb{N}_0^n : |\alpha| = |\beta| = m. \tag{4.8}$$

Let us temporarily use the notation L_A to indicate that the system L is associated with the tensor coefficient $A = \big(A_{\alpha\beta}\big)_{|\alpha|=|\beta|=m}$ as in (4.1). Then

$$(L_A)^t = L_{A^t}, \quad \overline{L_A} = L_{\overline{A}}, \quad (L_A)^* = L_{A^*}, \tag{4.9}$$

from which it is clear that

$$A \ \text{ symmetric} \implies L_A \text{ symmetric}, \tag{4.10}$$

$$A \ \text{ self-adjoint} \implies L_A \text{ self-adjoint}. \tag{4.11}$$

In general, the converse implications in (4.10) and (4.11) may fail, but

$$\left.\begin{array}{r} \text{if } \ L_A \ \text{ is symmetric and if} \\ \widetilde{A} := \Big(\tfrac{1}{2}\big(A_{\alpha\beta} + A^t_{\beta\alpha}\big)\Big)_{|\alpha|=|\beta|=m} \end{array}\right\} \implies L_A = L_{\widetilde{A}} \ \text{ and } \ \widetilde{A} \ \text{ is symmetric.} \tag{4.12}$$

Likewise,

$$\left.\begin{array}{r}\text{if } L_A \text{ is self-adjoint and if}\\ \widetilde{A} := \left(\tfrac{1}{2}\left(A_{\alpha\beta} + \overline{A_{\beta\alpha}}^{\,t}\right)\right)_{|\alpha|=|\beta|=m}\end{array}\right\} \Longrightarrow L_A = L_{\widetilde{A}} \text{ and } \widetilde{A} \text{ is self-adjoint.} \tag{4.13}$$

We now proceed to define several concepts of ellipticity (listed below in increasing order of strength).

Definition 4.1. The operator L in (4.1) is said to be W-`elliptic` provided that

$$det\left[\Big(\sum_{|\alpha|=|\beta|=m} a^{\alpha\beta}_{jk}\xi^\alpha\xi^\beta\Big)_{1\leq j,k\leq M}\right] \neq 0, \qquad \forall\, \xi \in \mathbb{R}^n \setminus \{0\}. \tag{4.14}$$

Also, the differential operator L in (4.1) is said to be `elliptic` whenever the `Legendre--Hadamard condition` is satisfied. That is, there exists $C > 0$ such that

$$\mathrm{Re}\left(\sum_{|\alpha|=|\beta|=m}\sum_{j,k=1}^{M} a^{\alpha\beta}_{jk}\xi^\alpha\xi^\beta\,\overline{\eta_j}\,\eta_k\right) \geq C|\xi|^{2m}|\eta|^2, \quad \forall\, \xi \in \mathbb{R}^n,\ \forall\, \eta \in \mathbb{C}^M. \tag{4.15}$$

Finally, L is called S-`elliptic` provided there exists $C > 0$ such that

$$\mathrm{Re}\left(\sum_{|\alpha|=|\beta|=m}\sum_{j,k=1}^{M} a^{\alpha\beta}_{jk}\zeta^\alpha_j\overline{\zeta^\beta_k}\right) \geq C\sum_{|\alpha|=m}\sum_{j=1}^{M}\frac{\alpha!}{m!}|\zeta^\alpha_j|^2, \tag{4.16}$$

for all families of numbers $\zeta^\alpha_j \in \mathbb{C},\ |\alpha| = m,\ 1 \leq j \leq M$.

Notice that the Legendre–Hadamard property (4.15) implies W-ellipticity, as introduced in (4.14). Also, (4.15) is equivalent to the strict positivity condition

$$\mathrm{Re}\left(\sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}\xi^\alpha\xi^\beta\right) \geq C|\xi|^{2m}I_{M\times M}, \qquad \forall\, \xi \in \mathbb{R}^n, \tag{4.17}$$

in the sense of $M \times M$ matrices with complex entries (in which case we have $\mathrm{Re}\,A := (A + A^*)/2$). Let us also note that being S-elliptic is a stronger concept than mere ellipticity (in the sense of Legendre–Hadamard). Indeed, since in general the multinomial formula gives

$$\Big(\sum_{i=1}^{n} x_i\Big)^N = \sum_{|\alpha|=N}\frac{|\alpha|!}{\alpha!}X^\alpha, \qquad \forall\, X := (x_1, \dots, x_n) \tag{4.18}$$

we conclude

$$\sum_{|\alpha|=m} \frac{m!}{\alpha!}\,\xi^{2\alpha} = \left(\sum_{k=1}^{n} \xi_k^2\right)^m = |\xi|^{2m}, \qquad \forall \xi \in \mathbb{R}^n, \tag{4.19}$$

so (4.16) reduces to (4.15) when $\zeta_j^\alpha := \xi^\alpha \eta_j$ with $\xi \in \mathbb{R}^n$ and $\eta = (\eta_j)_{1 \le j \le M} \in \mathbb{C}^M$.

Finally, we remark that we will occasionally require only a much weaker condition than (4.16), namely the `semi-positivity condition` to the effect that

$$\mathrm{Re}\left(\sum_{|\alpha|=|\beta|=m} \sum_{j,k=1}^{M} a_{jk}^{\alpha\beta} \zeta_j^\alpha \overline{\zeta_k^\beta}\right) \ge 0, \quad \forall\, \zeta_j^\alpha \in \mathbb{C},\ |\alpha| = m,\ \ 1 \le j \le M. \tag{4.20}$$

We wish to note that given a differential operator $L = L_A$ as in (4.1), corresponding to some tensor coefficient $A = \big(A_{\alpha\beta}\big)_{|\alpha|=|\beta|=m}$, there exist infinitely many other tensor coefficients $B = \big(B_{\alpha\beta}\big)_{|\alpha|=|\beta|=m}$ such that $L = L_B$. In turn, the choice of these B's may affect some of the properties of the operator L discussed above. Concretely, while the quality of being W-elliptic (cf. (4.14)) or satisfying the Legendre–Hadamard condition (4.15) are unaffected by the choice of tensor coefficient B used in the writing of the given operator L, the property of being S-elliptic (cf. (4.16)) and the semi-positivity condition (4.20) may fail for L_B.

A few examples of homogeneous, constant coefficient, higher-order elliptic operators are as follows. First, the polyharmonic operator

$$\Delta^m = \sum_{|\gamma|=m} \frac{m!}{\gamma!}\,\partial^{2\gamma}, \tag{4.21}$$

has been extensively studied in the literature. In the three-dimensional setting, the fourth order operator

$$\partial_1^4 + \partial_2^4 + \partial_3^4 \tag{4.22}$$

has been considered by I. Fredholm who has computed an explicit fundamental solution for it in [48]. More generally, an explicit fundamental solution for the operator

$$\partial_1^4 + \partial_2^4 + \partial_3^4 + c\big(\partial_1^2\partial_2^2 + \partial_1^2\partial_3^2 + \partial_2^2\partial_3^2\big) \tag{4.23}$$

which is elliptic if $c \in (-\frac{1}{2}, \infty)$ has been found in [130] (where elliptic operators of the form $\sum_{j,k=1}^{3} a_{jk}\,\partial_j^2\partial_k^2$ have also been considered).

Moving on, recall that the classical Malgrange–Ehrenpreis theorem asserts that any differential operator of the form $P(\partial) = \sum_{|\alpha|\leq m} a_\alpha \partial^\alpha$, with $a_\alpha \in \mathbb{C}$ not all zero, has a fundamental solution $E \in \mathscr{D}'(\mathbb{R}^n)$ in $\mathbb{R}^n$. In fact, as noted in [100], one may take

$$E(X) := \frac{1}{2\pi i\,\overline{P_m(\eta)}} \int_{z\in\mathbb{C},\,|z|=1} z^m e^{z\langle\eta,X\rangle} \mathscr{F}^{-1}_{\xi\to X}\Big(\frac{\overline{P(i\xi + z\eta)}}{P(i\xi + z\eta)}\Big)\frac{dz}{z}. \tag{4.24}$$

Above, $\mathscr{F}^{-1}_{\xi\to X}$ is the inverse of the Fourier transform, originally defined by the formula $\mathscr{F}_{X\to\xi}\phi(\xi) := \int_{\mathbb{R}^n} e^{-i\langle X,\xi\rangle}\phi(X)\,dX$ for $\phi \in C_c^\infty(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$, then extended by continuity to tempered distributions. Also, $P(\zeta) = \sum_{|\alpha|\leq m} a_\alpha \zeta^\alpha$, $\zeta \in \mathbb{C}^n$, is the characteristic polynomial of $P(\partial)$, and P_m stands for the principal part of P. Finally, $\eta \in \mathbb{C}^n$ is a fixed vector with the property that $P_m(\eta) \neq 0$.

We shall need a result of a somewhat similar nature for systems of differential operators. The theorem below is essentially a collection of results proved in [60, pp. 72–76], [57, p. 169], [114], and [95, p. 104], in various degrees of generality. However, we feel that it is useful to have a unifying statement, accompanied by a fairly self-contained proof, presented here.

Theorem 4.2. *Let L be a homogeneous differential operator L in $\mathbb{R}^n$ of order $2m$, $m \in \mathbb{N}$, with complex matrix-valued constant coefficients as in (4.1) which satisfies (4.14). Then, there exists a matrix of tempered distributions, $E = \big(E_{jk}\big)_{1\leq j,k\leq M}$, such that the following hold:*

(1) For each $1 \leq j, k \leq M$,

$$E_{jk} \in C^\infty(\mathbb{R}^n \setminus \{0\}) \quad \textit{and} \quad E_{jk}(-X) = E_{jk}(X) \quad \forall\, X \in \mathbb{R}^n \setminus \{0\}. \tag{4.25}$$

(2) For each $1 \leq j, k \leq M$,

$$\sum_{r=1}^{M} L^X_{jr}\Big[E_{rk}(X - Y)\Big] = \begin{cases} 0 & \textit{if } j \neq k, \\ \delta_Y(X) & \textit{if } j = k, \end{cases} \tag{4.26}$$

where δ stands for the Dirac delta distribution at 0 and the superscript X denotes the fact that the operator L_{js} in (4.26) is applied in the variable X.

(3) If $1 \leq j, k \leq M$, then

$$E_{jk}(X) = \Phi_{jk}(X) + \Big(\log|X|\Big)P_{jk}(X), \qquad X \in \mathbb{R}^n \setminus \{0\}, \tag{4.27}$$

where $\Phi_{jk} \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is a homogeneous function of degree $2m - n$, and P_{jk} is identically zero when either n is odd, or $n > 2m$, and is a homogeneous polynomial of degree $2m - n$ when $n \leq 2m$. In fact,

$$\Big(P_{jk}(X)\Big)_{1\leq j,k\leq M} = \frac{-1}{(2\pi\, i)^n(2m-n)!}\int\limits_{S^{n-1}} (X\cdot\xi)^{2m-n}\Big(\sum_{|\alpha|=|\beta|=m}\xi^{\alpha+\beta}A_{\alpha\beta}\Big)^{-1}\,d\sigma_\xi \tag{4.28}$$

for $X\in\mathbb{R}^n$.

(4) *For each $\alpha\in\mathbb{N}_0^n$ there exists $C_\alpha>0$ such that*

$$|\partial^\alpha E(X)| \leq \begin{cases} \dfrac{C_\alpha}{|X|^{n-2m+|\alpha|}} & \text{if either } n \text{ is odd, or } n>2m, \text{ or if } |\alpha|>2m-n, \\[2ex] \dfrac{C_\alpha(1+|\log|X||)}{|X|^{n-2m+|\alpha|}} & \text{if } 0\leq|\alpha|\leq 2m-n, \end{cases} \tag{4.29}$$

uniformly for $X\in\mathbb{R}^n\setminus\{0\}$.

(5) *One can assign to each differential operator L as in* (4.1)–(4.14) *a fundamental solution E_L which satisfies (1)-(4) above and, in addition,*

$$\big(E_L\big)^t = E_{L^t},\quad \overline{E_L} = E_{\overline{L}},\quad \big(E_L\big)^* = E_{L^*}. \tag{4.30}$$

(6) *The matrix-valued distribution E has entries which are tempered distributions in $\mathbb{R}^n$. When restricted to $\mathbb{R}^n\setminus\{0\}$, the matrix-valued distribution $\widehat{E}$ (with "hat" denoting the Fourier transform, defined as before) is a C^∞ function and, moreover,*

$$\widehat{E}(\xi) = (-1)^m\Big(\sum_{|\alpha|=|\beta|=m}\xi^{\alpha+\beta}A_{\alpha\beta}\Big)^{-1} \quad \textit{for each}\quad \xi\in\mathbb{R}^n\setminus\{0\}. \tag{4.31}$$

(7) *For each multi-index $\alpha\in\mathbb{N}_0^n$ with $|\alpha|=2m-1$, the function $\partial^\alpha E$ is odd, C^∞ smooth, and homogeneous of degree $n-1$ in $\mathbb{R}^n\setminus\{0\}$.*

Proof. Let $\big(P^{jk}(\xi)\big)_{1\leq j,k\leq M}$ be the inverse of the characteristic matrix

$$L(\xi) := \sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}\xi^{\alpha+\beta},\qquad \xi\in\mathbb{R}^n\setminus\{0\}. \tag{4.32}$$

Following [60, p. 76], if n is odd, define

$$E_{jk}(X) := \frac{1}{4(2\pi\, i)^{n-1}(2m-1)!}\Delta_X^{(n-1)/2}\int\limits_{S^{n-1}}\Big[(X\cdot\xi)^{2m-1}\cdot\mathrm{sign}\,(X\cdot\xi)\Big]P^{jk}(\xi)\,d\sigma_\xi, \tag{4.33}$$

for $X\in\mathbb{R}^n\setminus\{0\}$. Note that this expression is homogeneous of degree $2m-n$. On the other hand, if n is even, set

$$E_{jk}(X) := \frac{-1}{(2\pi\, i)^n (2m)!} \Delta_X^{n/2} \int_{S^{n-1}} \Big[(X \cdot \xi)^{2m} \cdot \log|X \cdot \xi|\Big] P^{jk}(\xi)\, d\sigma_\xi, \quad (4.34)$$

where $X \in \mathbb{R}^n \setminus \{0\}$. It is also clear from (4.33)–(4.34) that E is an even function.

Alternatively, transferring one Laplacian under the integral sign when n is odd, using the fact that $\log\left(\frac{X \cdot \xi}{i}\right) = \log|X \cdot \xi| - \frac{\pi i}{2} \operatorname{sign}(X \cdot \xi)$ and simple parity considerations, it can be checked that

$$E_{jk}(X) = \frac{-1}{(2\pi\, i)^n (2m+q)!} \Delta_X^{(n+q)/2} \int\limits_{S^{n-1}} \Big[(X \cdot \xi)^{2m+q} \cdot \log\Big(\frac{X \cdot \xi}{i}\Big)\Big] P^{jk}(\xi)\, d\sigma_\xi, \quad (4.35)$$

for $X \in \mathbb{R}^n \setminus \{0\}$, where $q := 0$ if n is even and $q := 1$ if n is odd.

As a preamble to checking (4.26), consider $F(t) := t^{2m+q} \log(t/i)$, $t \in \mathbb{R} \setminus \{0\}$, and note that there exists a constant $C_{m,q}$ such that

$$\Big(\frac{d}{dt}\Big)^{2m} F(t) = \frac{(2m+q)!}{q!} t^q \log\frac{t}{i} + C_{m,q}\, t^q. \quad (4.36)$$

Consequently,

$$L_X F(X \cdot \xi) = \Big[\frac{(2m+q)!}{q!} (X \cdot \xi)^q \log\frac{X \cdot \xi}{i} + C_{m,q}\,(X \cdot \xi)^q\Big] L(\xi) \quad (4.37)$$

and, further,

$$\begin{aligned} L_X \Big(-\frac{1}{(2\pi\, i)^n (2m+q)!} \int_{S^{n-1}} \Big[(X \cdot \xi)^{2m+q} \cdot \log\Big(\frac{X \cdot \xi}{i}\Big)\Big] [L(\xi)]^{-1}\, d\sigma_\xi\Big) \\ = \Big(B_{(n+q)/2}(X) + P_q(X)\Big) \cdot I_{M \times M}, \end{aligned} \quad (4.38)$$

where

$$B_{(n+q)/2}(X) := -\frac{1}{(2\pi\, i)^n q!} \int_{S^{n-1}} (X \cdot \xi)^q \cdot \log\Big(\frac{X \cdot \xi}{i}\Big)\, d\sigma_\xi \quad (4.39)$$

is a fundamental solution for $\Delta^{(n+q)/2}$ (cf., e.g., Lemma 4.2 on p. 123 in [122]), $P_q(X)$ is a homogeneous polynomial of degree q, and $I_{M \times M}$ is the identity matrix. Applying $\Delta_X^{(n+q)/2}$ to both sides of (4.38) then gives

$$LE = \delta \cdot I_{M \times M} \quad (4.40)$$

i.e., (4.26) holds.

To show that each component E_{jk} of E belongs to $C^\infty(\mathbb{R}^n \setminus \{0\})$, for every number $\ell \in \{1, \dots, n\}$ and $X \in \mathscr{O}_\ell := \mathbb{R}^n \setminus \{\lambda\, e_\ell : \lambda \leq 0\}$, consider the rotation

$$R_{\ell,X}(\xi) := \xi + 2\xi_\ell \frac{X}{|X|} - \frac{\xi \cdot X + \xi_\ell |X|}{|X|(|X| + x_\ell)}(X + |X|e_\ell), \quad \xi \in \mathbb{R}^n. \tag{4.41}$$

Obviously, $R_{\ell,\lambda X} = R_{\ell,X}$ whenever $X \in \mathscr{O}_\ell$ and $\lambda > 0$, the application

$$\mathscr{O}_\ell \times \mathbb{R}^n \ni (X, \xi) \mapsto R_{\ell,X}(\xi) \in \mathbb{R}^n \tag{4.42}$$

is of class C^∞, and an elementary calculation shows that

$$X \cdot R_{\ell,X}(\xi) = |X|\xi_\ell, \qquad \forall\, X \in \mathscr{O}_\ell, \quad \forall\, \xi \in \mathbb{R}^n. \tag{4.43}$$

Using the rotation invariance of the operation of integrating over S^{n-1}, for each $X \in \mathscr{O}_\ell$ we may then write

$$\begin{aligned}
&\int_{S^{n-1}} \Big[(X \cdot \xi)^{2m+q} \cdot \log\Big(\frac{X \cdot \xi}{i}\Big)\Big] P^{jk}(\xi)\, d\sigma_\xi \\
&\quad = \int_{S^{n-1}} \Big[(X \cdot R_{\ell,X}(\xi))^{2m+q} \cdot \log\Big(\frac{X \cdot R_{\ell,X}(\xi)}{i}\Big)\Big] P^{jk}(R_{\ell,X}(\xi))\, d\sigma_\xi \\
&\quad = \int_{S^{n-1}} \Big[(|X|\xi_\ell)^{2m+q} \cdot \log\Big(\frac{|X|\xi_\ell}{i}\Big)\Big] P^{jk}(R_{\ell,X}(\xi))\, d\sigma_\xi \\
&\quad = |X|^{2m+q} \int_{S^{n-1}} \xi_\ell^{2m+q} \Big\{ \log|X| + \log\Big(\frac{\xi_\ell}{i}\Big)\Big\}\; P^{jk}(R_{\ell,X}(\xi))\, d\sigma_\xi.
\end{aligned} \tag{4.44}$$

From this representation and formula (4.35) it is clear that $E_{jk} \in C^\infty(\mathscr{O}_\ell)$. Since $\mathbb{R}^n \setminus \{0\} = \bigcup_{\ell=1}^n \mathscr{O}_\ell$, the first part of (4.25) follows.

Note that (4.44) can be re-written in the form

$$\int_{S^{n-1}} \Big[(X \cdot \xi)^{2m+q} \cdot \log\Big(\frac{X \cdot \xi}{i}\Big)\Big] P^{jk}(\xi)\, d\sigma_\xi = \Psi_{jk}(X) + \Big(\log|X|\Big) Q_{jk}(X) \tag{4.45}$$

where

$$Q_{jk}(X) := \int_{S^{n-1}} (X \cdot \xi)^{2m+q} P^{jk}(\xi)\, d\sigma_\xi, \qquad X \in \mathbb{R}^n, \tag{4.46}$$

is a polynomial of degree $2m + q$ which vanishes (by simple parity considerations) when n is odd, and we have set

$$\Psi_{jk}(X) := |X|^{2m+q} \int_{S^{n-1}} \xi_\ell^{2m+q} \log\Big(\frac{\xi_\ell}{i}\Big) P^{jk}(R_{\ell,X}(\xi))\, d\sigma_\xi, \quad X \in \mathscr{O}_\ell. \tag{4.47}$$

Let us observe that, thanks to (4.45)–(4.47), Ψ_{jk} is actually a smooth, unambiguously defined function in $\bigcup_{\ell=1}^{n} \mathscr{O}_\ell = \mathbb{R}^n \setminus \{0\}$. In particular, the decomposition (4.27) follows from the above discussion in concert with (4.45) and (4.35).

Concerning (4.28), we first observe that

$$P_{jk}(X) = \frac{-1}{(2\pi i)^n (2m+q)!} \Delta_X^{(n+q)/2} \int_{S^{n-1}} (X \cdot \xi)^{2m+q} P^{jk}(\xi)\, d\sigma_\xi, \tag{4.48}$$

as seen from (4.35), (4.27), (4.45)–(4.46), and the identity

$$\Delta_X^K[(X \cdot \xi)^N] = \frac{N!}{(N-2K)!}(X \cdot \xi)^{N-2K} |\xi|^{2K}, \tag{4.49}$$

valid for any $N, K \in \mathbb{N}$ with $N \geq 2K$.

The estimates in (4.29) are a direct consequence of the structure formula (4.27). Finally, (4.33)–(4.34) readily account for the transposition identity (4.30).

The proof of (6) relies on (4.27) which we shall abbreviate as $E = \Phi + \Psi$, where we have set $\Phi(X) := \big(\Phi_{jk}(X)\big)_{j,k}$ and $\Psi(X) := \log|X| \big(P_{jk}(X)\big)_{j,k}$. First, we note that $\Phi \in C^\infty(\mathbb{R}^n \setminus \{0\}) \cap L^1_{loc}(\mathbb{R}^n)$ is a homogeneous function and so, by Theorem 7.1.18 on p. 168 of [57], Φ is a (matrix-valued) tempered distribution in $\mathbb{R}^n$ whose Fourier transform $\widehat{\Phi}$ coincides with a C^∞ function on $\mathbb{R}^n \setminus \{0\}$. As for Ψ, pick a function $\theta \in C_c^\infty(B(0,2))$ such that $\theta \equiv 1$ on $B(0,1)$ and write $\Psi = \Psi_1 + \Psi_2$ with $\Psi_1 := (1-\theta)\Psi \in C^\infty(\mathbb{R}^n)$, a matrix-valued tempered distribution, and $\Psi_2 := \theta\Psi \in L^1_{comp}(\mathbb{R}^n)$. Hence, $\widehat{\Psi}_2 \in C^\infty(\mathbb{R}^n)$. Also, for every multi-index $\beta \in \mathbb{N}_0^n$, the function $X^\beta \partial^\alpha \Psi_1(X)$ belongs to $L^1(\mathbb{R}^n)$ if $\alpha \in \mathbb{N}_0^n$ is such that $|\alpha|$ is large enough. This readily implies that for any $k \in \mathbb{N}$ there exists $\alpha \in \mathbb{N}_0^n$ such that $\widehat{\partial^\alpha \Psi_1} \in C^k(\mathbb{R}^n)$. Thus, the function $\mathbb{R}^n \ni \xi \mapsto \xi^\alpha \widehat{\Psi}_1(\xi)$ belongs to $C^k(\mathbb{R}^n)$ if $|\alpha|$ is large enough, relative to k, forcing $\widehat{\Psi}_1 \in C^\infty(\mathbb{R}^n \setminus \{0\})$.

The above reasoning shows that E is a matrix-valued tempered distribution in $\mathbb{R}^n$ with $\widehat{E}$ a function of class C^∞, when restricted to $\mathbb{R}^n \setminus \{0\}$. Having established this, taking the Fourier transforms of both sides of (4.40) gives $(-1)^m L(\xi)\widehat{E}(\xi) = I_{M\times M}$ in the sense of tempered distributions in $\mathbb{R}^n$. Restricting this identity to $\mathbb{R}^n \setminus \{0\}$ gives (4.31).

Finally, the claim in part (7) is a direct consequence of (1) and (3), and this completes the proof of Theorem 4.2. □

4.2 The Definition of Double Multi-Layer and Non-tangential Maximal Estimates

Let L be a homogeneous differential operator of order $2m$, with constant (possibly matrix-valued) coefficients:

$$L := \sum_{|\alpha|=|\beta|=m} \partial^\alpha A_{\alpha\beta}\, \partial^\beta \tag{4.50}$$

and consider the quadratic form associated with (the representation of) L in (4.50)

$$\mathscr{B}(u,v) := \sum_{|\alpha|=|\beta|=m} \langle A_{\alpha\beta}\, \partial^\beta u, \partial^\alpha v\rangle. \tag{4.51}$$

Also, fix a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ with outward unit normal ν and surface measure σ, as well as two arbitrary functions u, v. which are sufficiently well-behaved on Ω. Starting with $\int_\Omega \mathscr{B}(u,v)dX$ and successively integrating by parts (as to passing on to u all partial derivatives acting on v), we eventually arrive at

$$\sum_{|\alpha|=|\beta|=m} \int_\Omega \langle A_{\alpha\beta}\, \partial^\beta u, \partial^\alpha v\rangle\, dX = (-1)^m \int_\Omega \langle Lu, v\rangle\, dX + \text{ boundary terms.} \tag{4.52}$$

It is precisely the boundary terms which will determine the actual form of the double multi-layer potential operator, which we shall define momentarily. For now, we record the following.

Proposition 4.3. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain with surface measure σ and outward unit normal $\nu = (\nu_1, \dots, \nu_n)$, and suppose that L is as in* (4.50)*. Then*

$$\sum_{\substack{|\alpha|=m\\|\beta|=m}} \int_\Omega \langle A_{\alpha\beta}\, \partial^\beta u, \partial^\alpha v\rangle\, dX = (-1)^m \sum_{|\alpha|=|\beta|=m} \int_\Omega \langle Lu, v\rangle\, dX \tag{4.53}$$

$$- \sum_{|\alpha|=|\beta|=m} \sum_{k=1}^m \sum_{\substack{\gamma+\delta+e_j=\alpha\\ |\gamma|=k-1,\, |\delta|=m-k}} (-1)^k \tfrac{\alpha!(m-k)!(k-1)!}{m!\gamma!\delta!} \int_{\partial\Omega} \langle \nu_j A_{\alpha\beta}\, \partial^{\beta+\gamma} u, \partial^\delta v\rangle\, d\sigma,$$

for any $\mathbb{C}^M$-valued functions u, v which are reasonably behaved in Ω.

Proof. Let us focus on a generic term in the sum in the left hand-side of (4.53), say, corresponding to $\beta \in \mathbb{N}_0^N$ arbitrary and $\alpha = \sum_{k=1}^m e_{j_k}$ where $1 \le j_1, \dots, j_m \le n$ (recall that e_j, $1 \le j \le n$, stands for the vector $(0,\dots,1,\dots,0) \in \mathbb{R}^n$ with the only non-zero component on the j-th position). Then

$$\int_\Omega \langle A_{\alpha\beta}\, \partial^\beta u, \partial^\alpha v\rangle\, dX = (-1)^m \int_\Omega \langle \partial^\alpha A_{\alpha\beta}\, \partial^\beta u, v\rangle\, dX \tag{4.54}$$

$$+ \sum_{k=1}^m (-1)^{k+1} \int_{\partial\Omega} \Big\langle \nu_{j_k} \partial^{(\sum_{r=1}^{k-1} e_{j_r})} A_{\alpha\beta}\, \partial^\beta u, \partial^{(\sum_{r=k+1}^m e_{j_r})} v \Big\rangle.$$

Based on a general identity, to the effect that for any function $G : \mathbb{N}_0^n \to \mathbb{R}$

$$\sum_{1\le j_1,\dots,j_\ell\le n} G\Big(\sum_{r=1}^{\ell} e_{j_r}\Big) = \sum_{\gamma\in\mathbb{N}_0^n:\,|\gamma|=\ell} \frac{\ell!}{\gamma!} G(\gamma), \tag{4.55}$$

and (4.54), we may then write

$$\begin{aligned}
&\sum_{\substack{|\alpha|=m\\|\beta|=m}} \int_\Omega \langle A_{\alpha\beta}\,\partial^\beta u, \partial^\alpha v\rangle\,dX\\
&\quad= \sum_{|\beta|=m}\int_\Omega \sum_{1\le j_1,\dots,j_m\le n} \frac{(\sum_{r=1}^m e_{j_r})!}{m!}\Big\langle A_{(\sum_{r=1}^m e_{j_r})\beta}\partial^\beta u, \partial^{(\sum_{r=1}^m e_{j_r})} v\Big\rangle\,dX\\
&\quad= (-1)^m \sum_{|\beta|=m}\sum_{1\le j_1,\dots,j_m\le n} \frac{(\sum_{r=1}^m e_{j_r})!}{m!}\int_\Omega \Big\langle \partial^{(\sum_{r=1}^m e_{j_r})} A_{(\sum_{r=1}^m e_{j_r})\beta}\partial^\beta u, v\Big\rangle\,dX\\
&\qquad+ \sum_{|\beta|=m}\sum_{1\le j_1,\dots,j_m\le n} \frac{(\sum_{r=1}^m e_{j_r})!}{m!}\sum_{k=1}^m\Big\{(-1)^{k+1}\times\\
&\qquad\qquad\times\int_{\partial\Omega}\Big\langle \nu_{j_k}\partial^{(\sum_{r=1}^{k-1} e_{j_r})} A_{(\sum_{r=1}^m e_{j_r})\beta}\partial^\beta u, \partial^{(\sum_{r=k+1}^m e_{j_r})} v\Big\rangle\,dX\Big\}\\
&\quad= (-1)^m \sum_{|\alpha|=|\beta|=m}\int_\Omega \langle \partial^\alpha A_{\alpha\beta}\,\partial^\beta u, v\rangle\,dX \qquad\qquad (4.56)\\
&\qquad+ \sum_{|\beta|=m}\sum_{k=1}^m\sum_{j=1}^n\sum_{|\gamma|=k-1}\sum_{|\delta|=m-k}\Big\{(-1)^{k+1}\frac{(\gamma+\delta+e_j)!(m-k)!(k-1)!}{m!\gamma!\delta!}\times\\
&\qquad\qquad\times\int_{\partial\Omega}\langle \nu_j A_{(\gamma+\delta+e_j)\beta}\partial^{\beta+\gamma}u, \partial^\delta v\rangle\,d\sigma\Big\}.
\end{aligned}$$

Thus, (4.53) follows. □

The following definition is inspired by the format of the boundary integral in (4.53), in which formally, the function u is replaced by $E(X-\cdot)$, the family of derivatives $(\partial^\delta v)_{|\delta|\le m-1}$ is replaced by the Whitney array $\dot{f} = (f_\delta)_{|\delta|\le m-1}$ (in the process, it helps to keep in mind that $\partial^{\beta+\gamma}[E(X-\cdot)] = (-1)^{m+k-1}(\partial^{\beta+\gamma}E)(X-\cdot)$ for each $\beta\in\mathbb{N}_0$ with $|\beta|=1$ and each $\gamma\in\mathbb{N}_0^n$ with $|\gamma|=k-1$).

Definition 4.4. Let $\Omega\subset\mathbb{R}^n$ be a Lipschitz domain with outward unit normal vector $\nu=(\nu_j)_{1\le j\le n}$ and surface measure σ. Also, let L be a (complex) matrix-valued constant coefficient, homogeneous, W-elliptic differential operator of order $2m$ in $\mathbb{R}^n$ and denote by E a fundamental solution for L in $\mathbb{R}^n$. Then the `double multi-layer potential operator` acting on $\dot{f} = \{f_\delta\}_{|\delta|\leq m-1}$ is given by

$$\dot{\mathscr{D}}\dot{f}(X) := -\sum_{\substack{|\alpha|=m\\|\beta|=m}}\sum_{k=1}^{m}\sum_{\substack{|\delta|=m-k\\|\gamma|=k-1\\\gamma+\delta+e_j=\alpha}}\Big\{\frac{\alpha!(m-k)!(k-1)!}{m!\gamma!\delta!}\times \tag{4.57}$$
$$\times\int_{\partial\Omega}\nu_j(Y)(\partial^{\beta+\gamma}E)(X-Y)A_{\beta\alpha}f_\delta(Y)\,d\sigma(Y)\Big\}$$

for $X \in \mathbb{R}^n \setminus \partial\Omega$.

As a practical matter, $\mathscr{D}\dot{f}(X)$ is obtained by formally replacing, in the boundary integral in (4.53), the function u by $E(X-\cdot)$, the family of derivatives $\{\partial^\delta v\}_{|\delta|\leq m-1}$ by the Whitney array $\dot{f} = \{f_\delta\}_{|\delta|\leq m-1}$, and then multiplying everything by $(-1)^{m-1}$. The reason for which this definition is natural is as follows.

Proposition 4.5. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and consider a (complex) matrix-valued constant coefficient, homogeneous, W-elliptic differential operator L of order $2m$ in $\mathbb{R}^n$. Assume that $\dot{\mathscr{D}}$ is associated with these as in (4.57). Then*

$$L(\dot{\mathscr{D}}\dot{f})(X) = 0 \quad \textit{for each} \quad X \in \mathbb{R}^n \setminus \partial\Omega. \tag{4.58}$$

Also, for each $F \in C_c^\infty(\mathbb{R}^n)$,

$$\sum_{|\alpha|=|\beta|=m}\int_{\Omega}(\partial^\beta E)(X-Y)A_{\beta\alpha}(\partial^\alpha F)(Y)\,dY$$
$$= \begin{cases} F(X) - \dot{\mathscr{D}}(\mathrm{tr}_{m-1}(F))(X) & \textit{if } X \in \Omega,\\ -\dot{\mathscr{D}}(\mathrm{tr}_{m-1}(F))(X) & \textit{if } X \in \mathbb{R}^n \setminus \overline{\Omega}. \end{cases} \tag{4.59}$$

Proof. The identity in (4.58) is clear from definitions. Next, writing (4.53) for L^t in place of L (recall that the superscript t indicates transposition), gives

$$\sum_{\substack{|\alpha|=m\\|\beta|=m}}\int_{\Omega}\langle(A_{\beta\alpha})^t\partial^\beta u, \partial^\alpha v\rangle\,dX = (-1)^m\sum_{|\alpha|=|\beta|=m}\int_{\Omega}\langle L^t u, v\rangle\,dX \tag{4.60}$$
$$-\sum_{|\alpha|=|\beta|=m}\sum_{k=1}^{m}\sum_{\substack{\gamma+\delta+e_j=\alpha\\|\gamma|=k-1,\,|\delta|=m-k}}(-1)^k\frac{\alpha!(m-k)!(k-1)!}{m!\gamma!\delta!}\int_{\partial\Omega}\langle\nu_j(A_{\beta\alpha})^t\partial^{\beta+\gamma}u, \partial^\delta v\rangle\,d\sigma.$$

Specializing this identity to the case when the function $v := F \in C_c^\infty(\mathbb{R}^n)$ and $u := E_{L^t}(X_o - \cdot)\eta$ where $X_o \in \Omega$ is a fixed point and $\eta \in \mathbb{C}^M$ is an arbitrary vector, yields, after some simple algebraic manipulations,

$$\Big\langle \eta\,,\ \sum_{\substack{|\alpha|=m\\|\beta|=m}} \int_\Omega (\partial^\beta E)(X_o - \cdot) A_{\beta\alpha}(\partial^\alpha F)\,dX \Big\rangle \tag{4.61}$$

$$= \Big\langle \eta, F(X_o)\Big\rangle - \Big\langle \eta, \dot{\mathscr{D}}(\mathrm{tr}_{m-1}F)(X_o)\Big\rangle.$$

Above, property (5) in Theorem 4.2 is also used. Since $\eta \in \mathbb{C}^M$ was arbitrary, the version of (4.59) corresponding to $X_o \in \Omega$ is proved. The case when $X_o \in \mathbb{R}^n \setminus \overline{\Omega}$ is treated similarly, completing the proof of the proposition. □

Given $k \in \mathbb{N}_0$, recall that $\mathscr{P}_k$ stands for the space of polynomials of degree $\le k$ in $\mathbb{R}^n$.

Proposition 4.6. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and consider a (complex) matrix-valued constant coefficient, homogeneous, W-elliptic differential operator L of order $2m$ in $\mathbb{R}^n$. Assume that $\dot{\mathscr{D}}$ is associated with these as in (4.57). Then for any $u \in \mathscr{P}_{m-1}$*

$$u = \dot{\mathscr{D}}(\mathrm{tr}_{m-1}u) \quad \text{in } \ \Omega. \tag{4.62}$$

Proof. Fix a function $\psi \in C_c^\infty(\mathbb{R}^n)$ with $\psi \equiv 1$ in a neighborhood of $\overline{\Omega}$. If $u \in \mathscr{P}_{m-1}$, set $F := \psi u \in C_c^\infty(\mathbb{R}^n)$. Since $F \equiv u$ near $\overline{\Omega}$, we have $\partial^\alpha F = 0$ in Ω whenever $|\alpha| = m$, by trivial degree considerations. Then (4.62) is a consequence of the first line in (4.59). □

We now proceed to establish non-tangential maximal function estimates for the double multi-layer operator $\dot{\mathscr{D}}$.

Theorem 4.7. *Given a Lipschitz domain $\Omega \subset \mathbb{R}^n$ and a (complex) matrix-valued constant coefficient, homogeneous, W-elliptic differential operator L of order $2m$ in $\mathbb{R}^n$, consider the double multi-layer potential operator defined in (4.57). Then for each $p \in (1,\infty)$ there exists a finite constant $C = C(\Omega, p, L) > 0$ with the property that*

$$\sum_{j=0}^{m-1} \|\mathscr{N}(\nabla^j \dot{\mathscr{D}}\dot{f})\|_{L^p(\partial\Omega)} \le C\|\dot{f}\|_{\dot{L}^p_{m-1,0}(\partial\Omega)}, \qquad \forall\, \dot{f} \in \dot{L}^p_{m-1,0}(\partial\Omega), \tag{4.63}$$

$$\sum_{j=0}^{m} \|\mathscr{N}(\nabla^j \dot{\mathscr{D}}\dot{f})\|_{L^p(\partial\Omega)} \le C\|\dot{f}\|_{\dot{L}^p_{m-1,1}(\partial\Omega)}, \qquad \forall\, \dot{f} \in \dot{L}^p_{m-1,1}(\partial\Omega). \tag{4.64}$$

Furthermore,

$$\partial^\gamma \dot{\mathscr{D}} \dot{f}\Big\lfloor_{\partial\Omega} \quad \textit{exists for every} \quad \dot{f} \in \dot{L}^p_{m-1,0}(\partial\Omega) \\ \textit{whenever} \ \ \gamma \in \mathbb{N}_0^n \ \ \textit{satisfies} \ \ |\gamma| \leq m-1, \tag{4.65}$$

and

$$\partial^\gamma \dot{\mathscr{D}} \dot{f}\Big\lfloor_{\partial\Omega} \quad \textit{exists for every} \quad \dot{f} \in \dot{L}^p_{m-1,1}(\partial\Omega) \\ \textit{whenever} \ \ \gamma \in \mathbb{N}_0^n \ \ \textit{satisfies} \ \ |\gamma| \leq m. \tag{4.66}$$

Proof. The starting point is the identity

$$\dot{\mathscr{D}}(\dot{f})(X) = \sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n\setminus\overline{\Omega}} (\partial^\beta E)(X-Y) A_{\beta\alpha} (\partial^\alpha F)(Y)\, dY, \quad \forall\, X \in \Omega, \tag{4.67}$$

valid whenever $F \in C_c^\infty(\mathbb{R}^n)$ and $\dot{f} := \mathrm{tr}_{m-1}(F)$. This is readily seen from (4.59) with the roles of Ω and $\mathbb{R}^n \setminus \overline{\Omega}$ reversed (here it is useful to keep in mind that the outward unit normal of the latter domain is $-\nu$).

Consider next $\gamma \in \mathbb{N}_0^n$, $|\gamma| \leq m-1$, $\gamma = \sum_{r=1}^{|\gamma|} e_{j_r}$, with $1 \leq j_1, \dots, j_n \leq n$, and using (4.55), for each $X \in \Omega$ write

$$\partial^\gamma \dot{\mathscr{D}}(\dot{f})(X) = \sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n\setminus\overline{\Omega}} (\partial^{\gamma+\beta} E)(X-Y) A_{\beta\alpha} (\partial^\alpha F)(Y)\, dY \tag{4.68}$$

$$= \sum_{\substack{|\beta|=m \\ 1\leq k_1,\dots,k_m\leq n}} \frac{(\sum_{s=1}^m e_{k_s})!}{m!} \int_{\mathbb{R}^n\setminus\overline{\Omega}} (\partial^{(\sum_{r=1}^{|\gamma|} e_{j_r})+\beta} E)(X-Y) A_{\beta(\sum_{s=1}^m e_{k_s})} \Big(\partial^{(\sum_{s=1}^m e_{k_s})} F\Big)(Y)\, dY.$$

Integrating in (4.68) by parts (to move $\partial^{e_{k_1}}$ from F on to E) and taking into account that the outward unit normal vector to $\mathbb{R}^n \setminus \overline{\Omega}$ is $-\nu$ further leads to

$$\partial^\gamma \dot{\mathscr{D}}(\dot{f})(X) = \sum_{\substack{|\beta|=m \\ 1\leq k_1,\dots,k_m\leq n}} \frac{(\sum_{s=1}^m e_{k_s})!}{m!} \times \tag{4.69}$$

$$\times \Bigg\{ \int_{\mathbb{R}^n\setminus\overline{\Omega}} (\partial^{(\sum_{r=1}^{|\gamma|} e_{j_r})+e_{k_1}+\beta} E)(X-Y) A_{\beta(\sum_{s=1}^m e_{k_s})} \times$$

$$\times \Big(\partial^{(\sum_{s=2}^{m} e_{k_s})} F\Big)(Y)\,dY - \int_{\partial\Omega} \nu_{k_1}(Y)(\partial^{(\sum_{r=1}^{|\gamma|} e_{j_r})+\beta} E)(X-Y) \times$$

$$\times\ A_{\beta(\sum_{s=1}^{m} e_{k_s})}\Big(\partial^{(\sum_{s=2}^{m} e_{k_s})} F\Big)(Y)\,d\sigma(Y)\Bigg\}.$$

We further integrate by parts in the first term in the curly brackets in the right hand-side of (4.69) and move $\partial^{e_{j_1}}$ from E onto F. This gives

$$\partial^\gamma \dot{\mathscr{D}}(\dot{f})(X) = \sum_{\substack{|\beta|=m \\ 1\le k_1,\dots,k_m\le n}} \frac{(\sum_{s=1}^{m} e_{k_s})!}{m!} \times \tag{4.70}$$

$$\times \Bigg\{ \int_{\mathbb{R}^n\setminus\overline{\Omega}} (\partial^{(\sum_{r=2}^{|\gamma|} e_{j_r})+e_{k_1}+\beta} E)(X-Y) \times$$

$$\times A_{\beta(\sum_{s=1}^{m} e_{k_s})}\Big(\partial^{(\sum_{s=2}^{m} e_{k_s})+e_{j_1}} F\Big)(Y)\,dY$$

$$-\int_{\partial\Omega} \nu_{k_1}(Y)(\partial^{(\sum_{r=1}^{|\gamma|} e_{j_r})+\beta} E)(X-Y) \times$$

$$\times A_{\beta(\sum_{s=1}^{m} e_{k_s})}\Big(\partial^{(\sum_{s=2}^{m} e_{k_s})} F\Big)(Y)\,d\sigma(Y)$$

$$+\int_{\partial\Omega} \nu_{j_1}(Y)(\partial^{(\sum_{r=2}^{|\gamma|} e_{j_r})+e_{k_1}+\beta} E)(X-Y) \times$$

$$\times\ A_{\beta(\sum_{s=1}^{m} e_{k_s})}\Big(\partial^{(\sum_{s=2}^{m} e_{k_s})} F\Big)(Y)\,d\sigma(Y)\Bigg\}.$$

Next, recall that $\partial_{\tau_{k_1 j_1}} := \nu_{k_1}\partial^{e_{j_1}} - \nu_{j_1}\partial^{e_{k_1}}$ and notice that the sum of the last two terms in the curly brackets in (4.70) is

$$\int_{\partial\Omega} \partial_{\tau_{k_1 j_1}(Y)}\Big((\partial^{(\sum_{r=2}^{|\gamma|} e_{j_r})+\beta} E)(X-Y)\Big) A_{\beta(\sum_{s=1}^{m} e_{k_s})}\Big(\partial^{(\sum_{s=2}^{m} e_{k_s})} F\Big)(Y)\,d\sigma(Y). \tag{4.71}$$

Therefore,

$$\partial^\gamma \dot{\mathscr{D}}(\dot{f})(X) = \sum_{\substack{|\beta|=m \\ 1\leq k_1,\dots,k_m\leq n}} \frac{(\sum_{s=1}^m e_{k_s})!}{m!} \times \tag{4.72}$$

$$\times \Bigg\{ \int_{\mathbb{R}^n\setminus\overline{\Omega}} (\partial^{(\sum_{r=2}^{|\gamma|} e_{j_r})+e_{k_1}+\beta} E)(X-Y) \times$$

$$\times A_{\beta(\sum_{s=1}^m e_{k_s})} \Big(\partial^{(\sum_{s=2}^m e_{k_s})+e_{j_1}} F\Big)(Y)\, dY$$

$$+ \int_{\partial\Omega} \partial_{\tau_{k_1 j_1}(Y)} \Big((\partial^{(\sum_{r=2}^{|\gamma|} e_{j_r})+\beta} E)(X-Y) \Big) \times$$

$$\times\ A_{\beta(\sum_{s=1}^m e_{k_s})} \Big(\partial^{(\sum_{s=2}^m e_{k_s})} F\Big)(Y)\, d\sigma(Y) \Bigg\}.$$

Starting with the first integral inside the curly brackets in the right-hand side of (4.72), we now repeat the procedure described in (4.68)–(4.72) until we swap $\partial^{(\sum_{r=1}^{|\gamma|} e_{j_r})}$ from E with $\partial^{(\sum_{s=1}^{|\gamma|} e_{k_s})}$ from F. In this fashion, we obtain

$$\partial^\gamma \dot{\mathscr{D}}(\dot{f})(X) = \sum_{\substack{|\beta|=m \\ 1\leq k_1,\dots,k_m\leq n}} \frac{(\sum_{s=1}^m e_{k_s})!}{m!} \times \tag{4.73}$$

$$\times \Big\{ \int_{\mathbb{R}^n\setminus\overline{\Omega}} (\partial^{(\sum_{s=1}^{|\gamma|} e_{k_s})+\beta} E)(X-Y) \times$$

$$\times A_{\beta(\sum_{s=1}^m e_{k_s})} \Big(\partial^{(\sum_{r=1}^{|\gamma|} e_{j_r})+(\sum_{s=|\gamma|+1}^m e_{k_s})} F\Big)(Y)\, dY$$

$$+ \sum_{\ell=1}^{|\gamma|} \int_{\partial\Omega} \partial_{\tau_{k_\ell j_\ell}(Y)} \Big((\partial^{(\sum_{s=1}^{\ell-1} e_{k_s})+(\sum_{r=\ell+1}^{|\gamma|} e_{j_r})+\beta} E)(X-Y) \Big) \times$$

$$\times A_{\beta(\sum_{s=1}^m e_{k_s})} \Big(\partial^{(\sum_{r=1}^{\ell-1} e_{j_r})+(\sum_{s=\ell+1}^m e_{k_s})} F\Big)(Y)\, d\sigma(Y) \Big\}.$$

At this point we use repeated integration by parts in the first integral inside the curly brackets in the right-hand side of (4.73) in order to move the remaining partial derivatives $\partial^{(\sum_{s=|\gamma|+1}^m e_{k_s})}$ from F onto E. Using that $(LE)(X-Y) = 0$ for $X \neq Y$, this gives

$$\partial^\gamma \dot{\mathscr{D}}(\dot{f})(X) = \sum_{\substack{|\beta|=m \\ 1\le k_1,\dots,k_m\le n}} \frac{(\sum_{s=1}^m e_{k_s})!}{m!} \times \tag{4.74}$$

$$\times\Big\{\sum_{\ell=1}^{|\gamma|}\int_{\partial\Omega} \partial_{\tau_{k_\ell j_\ell}(Y)}\Big((\partial^{(\sum_{s=1}^{\ell-1} e_{k_s})+(\sum_{r=\ell+1}^{|\gamma|} e_{j_r})+\beta} E)(X-Y)\Big) \times$$

$$\times A_{\beta(\sum_{s=1}^m e_{k_s})}\Big(\partial^{(\sum_{r=1}^{\ell-1} e_{j_r})+(\sum_{s=\ell+1}^m e_{k_s})} F\Big)(Y)\, d\sigma(Y)$$

$$-\sum_{\ell=|\gamma|+1}^{m}\int_{\partial\Omega} \nu_{k_\ell}(Y)(\partial^{(\sum_{s=1}^{\ell-1} e_{k_s})+\beta} E)(X-Y) A_{\beta(\sum_{s=1}^m e_{k_s})} \times$$

$$\times\Big(\partial^{(\sum_{r=1}^{|\gamma|} e_{j_r})+(\sum_{s=\ell+1}^m e_{k_s})} F\Big)(Y)\, d\sigma(Y)\Big\}.$$

From the identity (4.74), Proposition 3.3 and a limiting argument, we can finally deduce that if $X \in \mathbb{R}^n \setminus \partial\Omega$ and if $\dot{f} = \{f_\delta\}_{|\delta|\le m-1} \in \dot{L}^p_{m-1,0}(\partial\Omega)$, $1 < p < \infty$,

$$\partial^\gamma \dot{\mathscr{D}}(\dot{f})(X) = \sum_{\substack{|\beta|=m \\ 1\le k_1,\dots,k_m\le n}} \frac{(\sum_{s=1}^m e_{k_s})!}{m!} \times \tag{4.75}$$

$$\times\Big\{\sum_{\ell=1}^{|\gamma|}\int_{\partial\Omega} \partial_{\tau_{k_\ell j_\ell}(Y)}\Big((\partial^{(\sum_{s=1}^{\ell-1} e_{k_s})+(\sum_{r=\ell+1}^{|\gamma|} e_{j_r})+\beta} E)(X-Y)\Big) \times$$

$$\times A_{\beta(\sum_{s=1}^m e_{k_s})} f_{(\sum_{r=1}^{\ell-1} e_{j_r})+(\sum_{s=\ell+1}^m e_{k_s})}(Y)\, d\sigma(Y)$$

$$-\sum_{\ell=|\gamma|+1}^{m}\int_{\partial\Omega} \nu_{k_\ell}(Y)(\partial^{(\sum_{s=1}^{\ell-1} e_{k_s})+\beta} E)(X-Y) A_{\beta(\sum_{s=1}^m e_{k_s})} \times$$

$$\times f_{(\sum_{r=1}^{|\gamma|} e_{j_r})+(\sum_{s=\ell+1}^m e_{k_s})}(Y)\, d\sigma(Y)\Big\}.$$

Based on the identity (4.55) we may write

$$\partial^\gamma \dot{\mathscr{D}}(\dot{f})(X) = \sum_{\ell=1}^{|\gamma|} \sum_{\substack{|\beta|=m \\ |\delta|=\ell-1,|\eta|=m-\ell}} \sum_{k=1}^{n} \frac{(\delta+\eta+e_k)!(m-\ell)!(\ell-1)!}{m!\delta!\eta!} \times$$

$$\times\int_{\partial\Omega} \partial_{\tau_{kj_\ell}(Y)}\Big((\partial^{\delta+\omega_\ell+\beta} E)(X-Y)\Big) A_{\beta(\delta+\eta+e_k)} f_{\theta_\ell+\eta}(Y)\, d\sigma(Y)$$

$$- \sum_{\ell=|\gamma|+1}^{m} \sum_{\substack{|\beta|=m \\ |\delta|=\ell-1, |\eta|=m-\ell}} \sum_{k=1}^{n} \tfrac{(\delta+\eta+e_k)!(m-\ell)!(\ell-1)!}{m!\delta!\eta!} \times$$

$$\times \int_{\partial\Omega} \nu_k(Y)(\partial^{\delta+\beta} E)(X-Y) A_{\beta(\delta+\eta+e_k)} f_{\gamma+\eta}(Y)\, d\sigma(Y). \tag{4.76}$$

Above, for each $\ell \in \mathbb{N}$, $1 \leq \ell \leq |\gamma|$, we have set $\theta_\ell := \sum_{r=1}^{\ell-1} e_{j_r}$, $\omega_\ell := \sum_{r=\ell+1}^{|\gamma|} e_{j_r}$, and $\theta_\ell + \omega_\ell + e_{j_\ell} = \gamma$. Summing over all such representations of γ and taking into account multiplicities allows us to conclude the following. For every multi-index $\gamma \in \mathbb{N}_0^n$, $|\gamma| \leq m-1$, $\dot{f} \in \dot{L}^p_{m-1,0}(\partial\Omega)$, $X \in \Omega$, we have the identity

$$\partial^\gamma \dot{\mathscr{D}}(\dot{f})(X) = \sum_{\substack{|\alpha|=m \\ |\beta|=m}} \sum_{\ell=1}^{|\gamma|} \sum_{\substack{\delta+\eta+e_k=\alpha \\ |\delta|=\ell-1, |\eta|=m-\ell}} \Big\{ \sum_{\substack{\theta+\omega+e_j=\gamma \\ |\theta|=\ell-1, |\omega|=|\gamma|-\ell}} C_1(m,\ell,\alpha,\delta,\eta,\gamma,\theta,\omega) \times$$

$$\times \int_{\partial\Omega} \partial_{\tau_{kj}(Y)} \Big((\partial^{\delta+\omega+\beta} E)(X-Y) \Big) A_{\beta\alpha} f_{\theta+\eta}(Y)\, d\sigma(Y) \Big\}$$

$$- \sum_{\substack{|\alpha|=m \\ |\beta|=m}} \sum_{\ell=|\gamma|+1}^{m} \sum_{\substack{\delta+\eta+e_k=\alpha \\ |\delta|=\ell-1, |\eta|=m-\ell}} \Big\{ C_2(m,\ell,\alpha,\delta,\eta) \times \tag{4.77}$$

$$\times \int_{\partial\Omega} \nu_k(Y)(\partial^{\delta+\beta} E)(X-Y) A_{\beta\alpha} f_{\gamma+\eta}(Y)\, d\sigma(Y) \Big\},$$

where

$$C_1(m,\ell,\alpha,\delta,\eta,\gamma,\theta,\omega) := \frac{\alpha!(m-\ell)!(\ell-1)!\gamma!(\ell-1)!(|\gamma|-\ell)!}{m!\,\delta!\,\eta!\,|\gamma|!\,\theta!\,\omega!}, \tag{4.78}$$

$$C_2(m,\ell,\alpha,\delta,\eta) := \frac{\alpha!(m-\ell)!(\ell-1)!}{m!\,\delta!\,\eta!}. \tag{4.79}$$

Now the estimate (4.63) is a simple consequence of (4.77) and (2.527) in Proposition 2.63. Furthermore, the claim in (4.65) also follows from (4.77) and formula (2.530) in Proposition 2.63.

As far as (4.64) is concerned, let us assume that $\dot{f} = \{f_\delta\}_{|\delta|\leq m-1} \in \dot{L}^p_{m-1,1}(\partial\Omega)$ for some $1 < p < \infty$. In this case, thanks to the extra smoothness of $\dot{f}$, we can integrate by parts (cf. Corollary 2.12) one more time in the first integral in the right-hand side of (4.77) and write for each $\gamma \in \mathbb{N}_0^n$ of length $\leq m-1$

$$\begin{aligned}
\partial^\gamma \dot{\mathscr{D}}(\dot{f})(X) = \sum_{\substack{|\alpha|=m\\|\beta|=m}} \sum_{\ell=1}^{|\gamma|} \sum_{\substack{\delta+\eta+e_k=\alpha\\|\delta|=\ell-1,|\eta|=m-\ell}} \sum_{\substack{\theta+\omega+e_j=\gamma\\|\theta|=\ell-1,|\omega|=|\gamma|-\ell}} &\Big\{C_1(m,\ell,\alpha,\delta,\eta,\gamma,\theta,\omega)\times \\
&\times \int_{\partial\Omega} (\partial^{\delta+\omega+\beta}E)(X-Y)A_{\beta\alpha}(\partial_{\tau_{jk}} f_{\theta+\eta})(Y)\,d\sigma(Y)\Big\} \\
- \sum_{\substack{|\alpha|=m\\|\beta|=m}} \sum_{\ell=|\gamma|+1}^{m} \sum_{\substack{\delta+\eta+e_k=\alpha\\|\delta|=\ell-1,|\eta|=m-\ell}} &\Big\{C_2(m,\ell,\alpha,\delta,\eta)\times \qquad (4.80)\\
&\times \int_{\partial\Omega} \nu_k(Y)(\partial^{\delta+\beta}E)(X-Y)A_{\beta\alpha} f_{\gamma+\eta}(Y)\,d\sigma(Y)\Big\},
\end{aligned}$$

at each $X \in \mathbb{R}^n \setminus \partial\Omega$, where the coefficients C_1, C_2 are as in (4.78)–(4.79).

Note that the integrand in the first integral in (4.80) is only weakly singular (since it entails $m-1+|\gamma| \leq 2m-2$ derivatives on E). Hence, terms of this form can absorb yet another partial derivative, say ∂_{x_i}, and still yield integral operators with either weakly singular kernels, or kernels of Calderón–Zygmund type (cf. Proposition 2.63). In particular, their contribution when estimating $\|\mathscr{N}(\partial_i \partial^\gamma \dot{\mathscr{D}} \dot{f})\|_{L^p(\partial\Omega)}$ is bounded by a fixed multiple of $\|\dot{f}\|_{\dot{L}^p_{m-1,1}(\partial\Omega)}$.

As for the terms in the second part of (4.80), it suffices to consider only those which contain $2m-1$ derivatives on E (since the rest are treated as before). Note that this requires $\ell = m$ (hence, $\eta = 0$). After applying ∂_{x_i} to them, at each $X \in \mathbb{R}^n \setminus \partial\Omega$ we obtain

$$\begin{aligned}
&\sum_{\substack{|\alpha|=m\\|\beta|=m}} \sum_{\delta+e_k=\alpha} \frac{\alpha!}{m\,\delta!} \int_{\partial\Omega} \nu_k(Y)\partial_{y_i}[(\partial^{\delta+\beta}E)(X-Y)]A_{\beta\alpha} f_\gamma(Y)\,d\sigma(Y) \\
&\qquad = -\sum_{\substack{|\alpha|=m\\|\beta|=m}} \sum_{\delta+e_k=\alpha} \frac{\alpha!}{m\,\delta!} \int_{\partial\Omega} \partial_{\tau_{ik}(Y)}[(\partial^{\delta+\beta}E)(X-Y)]A_{\beta\alpha} f_\gamma(Y)\,d\sigma(Y) \\
&\qquad\quad + \sum_{\substack{|\alpha|=m\\|\beta|=m}} \sum_{\delta+e_k=\alpha} \frac{\alpha!}{m\,\delta!} \int_{\partial\Omega} \nu_i(Y)\partial_{y_k}[(\partial^{\delta+\beta}E)(X-Y)]A_{\beta\alpha} f_\gamma(Y)\,d\sigma(Y)
\end{aligned}$$

$$= -\sum_{\substack{|\alpha|=m\\|\beta|=m}} \sum_{\delta+e_k=\alpha} \frac{\alpha!}{m\,\delta!} \int_{\partial\Omega} (\partial^{\delta+\beta} E)(X-Y) A_{\beta\alpha} (\partial_{\tau_{ki}} f_\gamma)(Y)\, d\sigma(Y)$$

$$- \sum_{\substack{|\alpha|=m\\|\beta|=m}} \sum_{\delta+e_k=\alpha} \frac{\alpha!}{m\,\delta!} \int_{\partial\Omega} \nu_i(Y) (\partial^{\delta+\beta+e_k} E)(X-Y) A_{\beta\alpha} f_\gamma(Y)\, d\sigma(Y)$$

$$=: A + B. \tag{4.81}$$

Now, the contribution from the terms labeled A is handled as before since, under the assumptions on the indices involved, $(\partial^{\delta+\beta} E)(X-Y)$ is a Calderón–Zygmund kernel. As far as B is concerned, note that $\delta+\beta+e_k = \alpha+\beta$ and that

$$\sum_{\delta+e_k=\alpha} \frac{\alpha!}{m\,\delta!} = \sum_{k=1}^{n} \frac{\alpha_k}{m} = 1, \tag{4.82}$$

since $|\alpha| = m$. Hence,

$$B = -\sum_{\substack{|\alpha|=m\\|\beta|=m}} \int_{\partial\Omega} \nu_i(Y) (\partial^{\alpha+\beta} E)(X-Y) A_{\beta\alpha} f_\gamma(Y)\, d\sigma(Y)$$

$$= -\int_{\partial\Omega} \nu_i(Y) \Big[(L^t E_{L^t})(X-Y) \Big]^t f_\gamma(Y)\, d\sigma(Y)$$

$$= 0. \tag{4.83}$$

In summary, we have shown that for each multi-index $\gamma \in \mathbb{N}_0^n$ of length $\leq m-1$ and each $i \in \{1, \dots, n\}$, there holds

$$\partial_i \partial^\gamma \dot{\mathscr{D}}(\dot{f})(X) = \sum_{\substack{|\alpha|=m\\|\beta|=m}} \sum_{\ell=1}^{|\gamma|} \sum_{\substack{\delta+\eta+e_k=\alpha\\|\delta|=\ell-1,|\eta|=m-\ell}} \sum_{\substack{\theta+\omega+e_j=\gamma\\|\theta|=\ell-1,|\omega|=|\gamma|-\ell}} \Big\{ C_1(m,\ell,\alpha,\delta,\eta,\gamma,\theta,\omega) \times$$

$$\times \int_{\partial\Omega} (\partial^{\delta+\omega+\beta+e_i} E)(X-Y) A_{\beta\alpha} (\partial_{\tau_{jk}} f_{\theta+\eta})(Y)\, d\sigma(Y) \Big\}$$

$$- \sum_{\substack{|\alpha|=m\\|\beta|=m}} \sum_{\ell=|\gamma|+1}^{m-1} \sum_{\substack{\delta+\eta+e_k=\alpha\\|\delta|=\ell-1,|\eta|=m-\ell}} \Big\{ C_2(m,\ell,\alpha,\delta,\eta) \times \tag{4.84}$$

$$\times \int_{\partial\Omega} \nu_k(Y)(\partial^{\delta+\beta+e_i} E)(X-Y) A_{\beta\alpha} f_{\gamma+\eta}(Y)\, d\sigma(Y)\Big\}$$

$$-\sum_{\substack{|\alpha|=m\\ |\beta|=m}} \sum_{\delta+e_k=\alpha} \frac{\alpha!}{m\,\delta!} \int_{\partial\Omega} (\partial^{\delta+\beta} E)(X-Y) A_{\beta\alpha} (\partial_{\tau_{ki}} f_\gamma)(Y)\, d\sigma(Y)$$

at each $X \in \mathbb{R}^n \setminus \partial\Omega$, where the coefficients C_1, C_2 are as in (4.78)–(4.79). This justifies (4.64). To finish the proof of Theorem 4.7 there remains to observe that the claim made in (4.66) follows from (4.84) and formula (2.530) in Proposition 2.63. □

The formulas for the derivatives of the double multi-layer deduced during the course of the proof of Theorem 4.7 are also useful to establish the following smoothing properties of the double multi-layer, measured on the Besov scale.

Proposition 4.8. *Retain the same context as in Theorem 4.7. Then for each number $p \in (1,\infty)$ there exists a finite constant $C = C(\Omega, L, p) > 0$ such that*

$$\|\dot{\mathscr{D}} \dot{f}\|_{B^{p,p\vee 2}_{m-1+1/p}(\Omega)} \le C\|\dot{f}\|_{\dot{L}^p_{m-1,0}(\partial\Omega)}, \qquad \forall\, \dot{f} \in \dot{L}^p_{m-1,0}(\partial\Omega), \tag{4.85}$$

$$\|\dot{\mathscr{D}} \dot{f}\|_{B^{p,p\vee 2}_{m+1/p}(\Omega)} \le C\|\dot{f}\|_{\dot{L}^p_{m-1,1}(\partial\Omega)}, \qquad \forall\, \dot{f} \in \dot{L}^p_{m-1,1}(\partial\Omega). \tag{4.86}$$

Proof. Estimate (4.85) is a consequence of the (quantitative) lifting result from Proposition 2.24, Proposition 2.68, and formula (4.77). Likewise, estimate (4.86) is implied by Propositions 2.24, 2.68, and formula (4.80). □

Finally, we consider the action of the double multi-layer operator on the Whitney–Hardy spaces $\dot{h}^p_{m-1,1}(\partial\Omega)$.

Theorem 4.9. *Retain the same background assumptions as in Theorem 4.7. Then for any $p \in \big(\frac{n-1}{n}, 1\big]$ there holds*

$$\sum_{j=0}^{m} \|\mathscr{N}(\nabla^j \dot{\mathscr{D}} \dot{f})\|_{L^p(\partial\Omega)} \le C\|\dot{f}\|_{\dot{h}^p_{m-1,1}(\partial\Omega)}, \tag{4.87}$$

for some finite constant $C > 0$ independent of $\dot{f} \in \dot{h}^p_{m-1,1}(\partial\Omega)$.

Proof. This is a consequence of the boundedness of the operator (2.482), the identities (4.80)–(4.83), and (2.528). □

4.3 Carleson Measure Estimates

Fix a bounded Lipschitz domain Ω in $\mathbb{R}^n$, along with a natural number $m \in \mathbb{N}$. As usual, let tr_{m-1} denote the multi-trace operator onto the boundary of Ω. Finally, recall (3.383), (3.384) and Definition 3.49. To proceed, for each $r > 0$ and $X_o \in \partial\Omega$, define

$$\dot{\mathscr{P}}_{m-1}(\partial\Omega) \ni \dot{\omega}_r(\dot{f}) := \text{ the best fit polynomial array}$$
$$\text{to } \dot{f} \in \dot{L}^2_{m-1,0}(\partial\Omega) \text{ on } S_r(X_o), \tag{4.88}$$

i.e., the orthogonal projection of $\dot{f}|_{S_r(X_o)}$ onto $\dot{\mathscr{P}}_{m-1}(S_r(X_o))$ with respect to the natural inner product in $L^2(S_r(X_o)) \oplus \cdots \oplus L^2(S_r(X_o))$, N times, where

$$N := \sum_{k=0}^{m-1} \binom{n+k-1}{n-1} \tag{4.89}$$

is the cardinality of $\{\alpha \in \mathbb{N}_0^n : |\alpha| \leq m-1\}$. In particular, for each $r > 0$,

$$\int_{S_r(X_o)} |\dot{f}(Q) - \dot{\omega}_r(\dot{f})(Q)|^2 \, d\sigma(Q) \tag{4.90}$$
$$= \inf_{\dot{P} \in \dot{\mathscr{P}}_{m-1}(\partial\Omega)} \int_{S_r(X_o)} |\dot{f}(Q) - \dot{P}(Q)|^2 \, d\sigma(Q),$$

and (recall (3.385)), we have

$$\dot{f}^{\#}(X_o) = \sup_{r>0} \left(\fint_{S_r(X_o)} |\dot{f}(Q) - \dot{\omega}_r(\dot{f})(Q)|^2 \, d\sigma(Q) \right)^{1/2}. \tag{4.91}$$

Next, recall that $\dot{\mathscr{P}}_{m-1}(S_r(X_o))$ stands for the space of restrictions of arrays in $\dot{\mathscr{P}}_{m-1}(\partial\Omega)$ to $S_r(X_o)$. Thus, any element $\dot{f}$ in $\dot{\mathscr{P}}_{m-1}(S_r(X_o))$ is of the form $(\mathrm{tr}_{m-1} P)|_{S_r(X_o)}$ for some $P \in \mathscr{P}_{m-1}$. It is important to note that such a polynomial P is unique. Indeed, by linearity, this is an immediate consequence of the fact that

$$P \in \mathscr{P}_{m-1} \text{ and } (\mathrm{tr}_{m-1} P)|_{S_r(X_o)} = 0 \Longrightarrow P \equiv 0 \text{ in } \mathbb{R}^n. \tag{4.92}$$

In turn, this is obvious from the fact that all derivatives of P vanish at, say, X_o. We shall refer to P as the polynomial extension to $\mathbb{R}^n$ of $\dot{f}$. In summary,

$$\mathrm{tr}_{m-1} : \mathscr{P}_{m-1} \longrightarrow \dot{\mathscr{P}}_{m-1}(S_r(X_o)) \text{ is an algebraic isomorphism.} \tag{4.93}$$

In the sequel, we shall frequently identify a generic element $\dot{f}$ from $\dot{\mathscr{P}}_{m-1}(S_r(X_o))$ with its canonical extension to the entire boundary $\partial\Omega$. By definition, the latter is taken to be $\mathrm{tr}_{m-1}P \in \dot{\mathscr{P}}_{m-1}(\partial\Omega)$, where $P \in \mathscr{P}_{m-1}$ is the polynomial extension to $\mathbb{R}^n$ of $\dot{f}$. We emphasize that, by design, the canonical extension of elements from $\dot{\mathscr{P}}_{m-1}(S_r(X_o))$ to elements in $\dot{\mathscr{P}}_{m-1}(\partial\Omega)$ is unique.

The space $\dot{\mathscr{P}}_{m-1}(S_r(X_o))$ has a Hilbert structure when equipped with the inner product naturally inherited from $L^2(S_r(X_o)) \oplus \cdots \oplus L^2(S_r(X_o))$, N times (with N as in (4.89)). Then, given (4.93), the restrictions to $S_r(X_o)$ of the Whitney arrays $\dot{p}_\alpha(X) := \mathrm{tr}_{m-1}[(X - X_o)^\alpha]$, $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m-1$, form a basis in $\dot{\mathscr{P}}_{m-1}(S_r(X_o))$. Applying the Gram–Schmidt process to $\{\dot{p}_\alpha\}_{|\alpha|\leq m-1}$ then yields an orthonormal (relative to $L^2(S_r(X_o)) \oplus \cdots \oplus L^2(S_r(X_o))$) basis for $\dot{\mathscr{P}}_{m-1}(S_r(X_o))$. We denote this basis by $\{\dot{e}_\alpha\}_{|\alpha|\leq m-1}$ and, for each multi-index $\alpha \in \mathbb{N}_0^n$ of length $\leq m-1$, we let $e_\alpha \in \mathscr{P}_{|\alpha|}$ be the polynomial extension to $\mathbb{R}^n$ of $\dot{e}_\alpha$. That is, e_α is a polynomial of degree $|\alpha|$ such that $\dot{e}_\alpha = [\mathrm{tr}_{m-1}e_\alpha]|_{S_r(X_o)}$. Then, for each $\beta \in \mathbb{N}_0^n$, $|\beta| \leq m-1$, it follows that at points in $S_r(X_o)$

$$(\dot{e}_\alpha)_\beta = (\mathrm{tr}_{m-1}e_\alpha)_\beta = \partial^\beta e_\alpha = 0 \quad \text{if} \quad |\beta| > |\alpha|, \tag{4.94}$$

by degree considerations. Also, by carefully keeping track of bounds for the various expressions appearing in the Gram–Schmidt orthonormalization process (as presented on, e.g., p. 120 of [121]) we obtain

$$|(\dot{e}_\alpha)_\beta(X)| \leq \frac{C}{r^{(n-1)/2}}\Big[|X - X_o| + r\Big]^{|\alpha|-|\beta|} \quad \text{if } |\beta| \leq |\alpha|, \quad \forall\, X \in \partial\Omega. \tag{4.95}$$

The point of the above considerations is to facilitate discussing a number of important properties of the best fit polynomial array (cf. (4.88)). To begin with, we agree that $\dot{\omega}_r(\dot{f})$ is always identified with its canonical extension to $\partial\Omega$. Thus,

$$\dot{\omega}_r(\dot{f}) \in \dot{\mathscr{P}}_{m-1}(\partial\Omega), \qquad \forall\, \dot{f} \in \dot{L}^2_{m-1,0}(\partial\Omega). \tag{4.96}$$

Let us also observe that

$$\dot{\omega}_r(\dot{f}) = \dot{f}, \qquad \forall\, \dot{f} \in \dot{\mathscr{P}}_{m-1}(\partial\Omega), \tag{4.97}$$

since, by definition, $\dot{\omega}_r(\dot{f})$ coincides with $\dot{f}|_{S_r(X_o)}$ and any $\dot{f} \in \dot{\mathscr{P}}_{m-1}(\partial\Omega)$ is the canonical extension of $\dot{f}|_{S_r(X_o)}$. As an immediate corollary of (4.96)–(4.97), we note that

$$\dot{\omega}_r(\dot{\omega}_{2r}(\dot{f})) = \dot{\omega}_{2r}(\dot{f}), \qquad \forall\, \dot{f} \in \dot{L}^2_{m-1,0}(\partial\Omega). \tag{4.98}$$

Finally, for each $\dot{f} = \{f_\beta\}_{|\beta|\leq m-1} \in \dot{L}^2_{m-1,0}(\partial\Omega)$, we have

$$\dot{\omega}_r(\dot{f}) = \sum_{|\alpha|,|\beta|\leq m-1} \langle f_\beta, (\dot{e}_\alpha)_\beta\rangle\, \dot{e}_\alpha \quad \text{on } \partial\Omega, \tag{4.99}$$

where $\{\dot{e}_\alpha\}_{|\alpha|\leq m-1}$ is as above and $\langle\cdot,\cdot\rangle$ stands here for the inner product in $L^2(S_r(X_o))$. At points in $S_r(X_o)$ this follows straight from definitions, so the desired result is implied by the uniqueness of the canonical extension to $\partial\Omega$.

Next, we augment the list of algebraic properties of the best fit polynomial array with some useful estimates, contained in the lemma below.

Lemma 4.10. *Consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, and fix $X_o \in \partial\Omega$ and $r > 0$. Then, for each $\dot{f} \in \dot{\mathrm{BMO}}_{m-1}(\partial\Omega)$ and any multi-index $\alpha \in \mathbb{N}_0^n$ of length $\leq m-1$,*

$$\Big(⨍_{S_r(X_o)}\Big|\Big(\dot{f}(Y)-\dot{\omega}_r(\dot{f})(Y)\Big)_\alpha\Big|^2\,d\sigma(Y)\Big)^{1/2} \leq C r^{m-1-|\alpha|}\dot{f}^{\#}(X_o), \tag{4.100}$$

and

$$|(\dot{\omega}_r(\dot{f})-\dot{\omega}_{2r}(\dot{f}))_\alpha(X)| \leq C\,\dot{f}^{\#}(X_o)\Big[|X-X_o|+r\Big]^{m-1-|\alpha|}, \qquad \forall\, X \in \partial\Omega. \tag{4.101}$$

Proof. Fix $r > 0$, $X_o \in \partial\Omega$ and consider the array $\dot{g} = \{g_\alpha\}_{|\alpha|\leq m-1}$ given by

$$\dot{g} := \dot{f} - \dot{\omega}_r(\dot{f}). \tag{4.102}$$

Then, since the best fit polynomial array $\dot{\omega}_r(\dot{f})$ is the projection of $\dot{f}|_{S_r(X_o)}$ onto $\dot{\mathscr{P}}_{m-1}(S_r(X_o))$ with respect to the natural inner product in the space

$$L^2(S_r(X_o)) \oplus \cdots \oplus L^2(S_r(X_o)), \tag{4.103}$$

for each $\alpha \in \mathbb{N}_0^n$ multi-index of length $\leq m-1$ we have

$$\int_{S_r(X_o)}\langle\dot{g}(Y),\dot{P}_\alpha(Y)\rangle\,d\sigma(Y) = \int_{S_r(X_o)}\sum_{|\gamma|\leq m-1} g_\gamma(Y)(\dot{P}_\alpha)_\gamma(Y)\,d\sigma(Y) = 0, \tag{4.104}$$

where

$$P_\alpha(X) := (X-X_o)^\alpha \quad \text{and} \quad \dot{P}_\alpha := \mathrm{tr}_{m-1}P_\alpha \in \dot{\mathscr{P}}_{m-1}(\partial\Omega). \tag{4.105}$$

Note that for each $X \in \mathbb{R}^n$ we have

$$(\dot{P}_\alpha)_\gamma(X) = \begin{cases} 0, & \text{if } \gamma > \alpha, \\ \frac{\alpha!}{(\alpha-\gamma)!}P_{\alpha-\gamma}(X), & \text{if } \gamma \leq \alpha. \end{cases} \tag{4.106}$$

Therefore, from (4.104) and (4.106), we conclude that for each $|\alpha| \leq m-1$ there holds

$$0 = \int_{S_r(X_o)} \langle \dot{g}(Y), \dot{P}_\alpha(Y)\rangle \, d\sigma(Y) = \int_{S_r(X_o)} \sum_{\gamma \leq \alpha} g_\gamma(Y)(\dot{P}_\alpha)_\gamma(Y)\, d\sigma(Y)$$

$$= \int_{S_r(X_o)} \sum_{\gamma \leq \alpha} \frac{\alpha!}{(\alpha-\gamma)!} g_\gamma(Y) P_{\alpha-\gamma}(Y)\, d\sigma(Y). \tag{4.107}$$

In particular, making $\alpha = (0, 0, \cdots, 0)$ in (4.107) implies (recall that the subindex $\emptyset$ denotes the multi-index $(0, 0, \ldots, 0)$)

$$\int_{S_r(X_o)} g_\emptyset(Y)\, d\sigma(Y) = 0, \tag{4.108}$$

and recursively,

$$\alpha! \int_{S_r(X_o)} g_\alpha(Y)\, d\sigma(Y) = -\sum_{\substack{\gamma \leq \alpha \\ |\gamma|<|\alpha|}} \frac{\alpha!}{(\alpha-\gamma)!} \int_{S_r(X_o)} g_\gamma(Y) P_{\alpha-\gamma}(Y)\, d\sigma(Y). \tag{4.109}$$

Using (4.108) and applying Poincaré's inequality we obtain

$$\left(\fint_{S_r(X_o)} g_\emptyset(Y)\, d\sigma(Y) \right)^{1/2} \leq Cr \left(\fint_{S_r(X_o)} |\nabla_{tan} g_\emptyset(Y)|^2 \, d\sigma(Y) \right)^{1/2}. \tag{4.110}$$

In turn, using the compatibility conditions (3.2),

$$\nabla_{tan} g_\alpha = \Big(\sum_{k=1}^n \nu_k \partial_{\tau_{kj}} g_\alpha \Big)_{1 \leq j \leq n} \tag{4.111}$$

$$= \Big(\sum_{k=1}^n \nu_k (\nu_k g_{\alpha+e_j} - \nu_j g_{\alpha+e_k}) \Big)_{1 \leq j \leq n}, \quad \text{if } |\alpha| \leq m-2.$$

Therefore,

$$\left(\fint_{S_r(X_o)} g_\emptyset(Y)\, d\sigma(Y) \right)^{1/2} \leq Cr \sum_{k=1}^n \left(\fint_{S_r(X_o)} |g_{e_k}(Y)|^2 \, d\sigma(Y) \right)^{1/2}$$

$$\leq Cr \dot{f}^{\#}(X_o), \tag{4.112}$$

where the last inequality follows from (4.102) and the definition of the sharp maximal function (3.385). This is the conclusion of our first step. For the second step, fix $\alpha \in \mathbb{N}_0^n$, $|\alpha| = 1$. From (4.109),

$$\int_{S_r(X_o)} g_\alpha(Y)\,d\sigma(Y) = \int_{S_r(X_o)} g_\emptyset(Y) P_\alpha(Y)\,d\sigma(Y), \tag{4.113}$$

and since $|P_\alpha(Y)| \le r^{|\alpha|} = r$ on $S_r(X_o)$, this implies

$$\begin{aligned} \Big|\fint_{S_r(X_o)} g_\alpha(Y)\,d\sigma(Y)\Big| &\le r \fint_{S_r(X_o)} |g_\emptyset(Y)|\,d\sigma(Y) \\ &\le r\Big(\fint_{S_r(X_o)} |g_\emptyset(Y)|^2\,d\sigma(Y)\Big)^{1/2} \\ &\le C r^2 \dot{f}^{\#}(X_o), \end{aligned} \tag{4.114}$$

where the last inequality in (4.114) follows from (4.112). Applying the Poincaré inequality for g_α gives

$$\begin{aligned} \Big(\fint_{S_r(X_o)} |g_\alpha(Y)|^2\,d\sigma(Y)\Big)^{1/2} &\le C r \Big(\fint_{S_r(X_o)} |\nabla_{tan} g_\alpha(Y)|^2\,d\sigma(Y)\Big)^{1/2} \\ &\quad + C\Big|\fint_{S_r(X_o)} g_\alpha(Y)\,d\sigma(Y)\Big|. \end{aligned} \tag{4.115}$$

In turn, using (4.111) yields $|\nabla_{tan}\, g_\alpha|^2 \le \sum_{|\gamma|=2} |g_\gamma|^2$ which, together with (4.114)–(4.115), implies

$$\begin{aligned} \Big(\fint_{S_r(X_o)} |g_\alpha(Y)|^2\,d\sigma(Y)\Big)^{1/2} &\le Cr \sum_{|\gamma|=2} \Big(\fint_{S_r(X_o)} |g_\gamma(Y)|^2\,d\sigma(Y)\Big)^{1/2} \\ &\quad + C\Big|\fint_{S_r(X_o)} g_\alpha(Y)\,d\sigma(Y)\Big| \\ &\le Cr\dot{f}^{\#}(X_o) + Cr^2 \dot{f}^{\#}(X_o) \\ &\le Cr\dot{f}^{\#}(X_o). \end{aligned} \tag{4.116}$$

Utilizing (4.116) instead of the definition of $\dot{f}^{\#}(X_o)$ in the second inequality in (4.112) improves (4.108) to

$$\Big(\fint_{S_r(X_o)} g_\emptyset(Y)\,d\sigma(Y)\Big)^{1/2} \le Cr^2 \dot{f}(X_o), \tag{4.117}$$

The conclusion of our second step is that

$$\left(⨍_{S_r(X_o)} g_\emptyset(Y)\,d\sigma(Y)\right)^{1/2} \le Cr^2 \dot{f}^\#(X_o) \quad \text{and}$$
$$\left(⨍_{S_r(X_o)} |g_\alpha(Y)|^2\,d\sigma(Y)\right)^{1/2} \le C\,r\,\dot{f}^\#(X_o), \quad \text{if } |\alpha| = 1. \tag{4.118}$$

In the third step we fix $\alpha \in \mathbb{N}_0^n$, $|\alpha| = 2$. Then, according to (4.109) we have

$$\int_{S_r(X_o)} g_\alpha(Y)\,d\sigma(Y) = -\int_{S_r(X_o)} g_\emptyset(Y) P_\alpha(Y)\,d\sigma(Y)$$
$$- \sum_{\substack{\gamma\le\alpha \\ |\gamma|=1}} \frac{\alpha!}{(\alpha-\gamma)!} \int_{S_r(X_o)} g_\gamma(Y) P_{\alpha-\gamma}(Y)\,d\sigma(Y). \tag{4.119}$$

Since $|P_{\alpha-\gamma}(Y)| \le r$ and $|P_\alpha(Y)| \le r^2$ for $Y \in S_r(X_o)$, and using (4.118)

$$\left|⨍_{S_r(X_o)} g_\alpha(Y)\,d\sigma(Y)\right| \le r^2 \left(⨍_{S_r(X_o)} g_\emptyset(Y)\,d\sigma(Y)\right)^{1/2}$$
$$+r \sum_{\substack{\gamma\le\alpha \\ |\gamma|=1}} \frac{\alpha!}{(\alpha-\gamma)!} \left(⨍_{S_r(X_o)} |g_\gamma(Y)|^2\,d\sigma(Y)\right)^{1/2}$$
$$\le Cr^4 \dot{f}^\#(X_o) + Cr^2 \dot{f}^\#(X_o)$$
$$\le Cr^2 \dot{f}^\#(X_o). \tag{4.120}$$

However, applying again Poincaré's inequality (4.115) and using (4.111) and (4.120), one has

$$\left(⨍_{S_r(X_o)} |g_\alpha(Y)|^2\,d\sigma(Y)\right)^{1/2} \le C\,r \sum_{|\gamma|=3} \left(⨍_{S_r(X_o)} |g_\gamma(Y)|^2\,d\sigma(Y)\right)^{1/2}$$
$$+C\left|⨍_{S_r(X_o)} g_\alpha(Y)\,d\sigma(Y)\right|$$
$$\le C\,r\,\dot{f}^\#(X_o) + Cr^2 \dot{f}^\#(X_o)$$
$$\le C\,r\,\dot{f}^\#(X_o). \tag{4.121}$$

Recall that $\alpha \in \mathbb{N}_0^n$ is an arbitrary multi-index of length $= 2$. Employing the inequality (4.121) in (4.116) to control

$$\sum_{|\gamma|=2} \left(\fint_{S_r(X_o)} |g_\gamma(Y)|^2 \, d\sigma(Y) \right)^{1/2}, \tag{4.122}$$

allows us to improve (4.116) to

$$\left(\fint_{S_r(X_o)} |g_\delta(Y)|^2 \, d\sigma(Y) \right)^{1/2} \leq C r^2 \dot{f}^\#(X_o) \quad \text{if } |\delta| = 1. \tag{4.123}$$

In turn, using (4.123) in (4.112) gives

$$\left(\fint_{S_r(X_o)} |g_\emptyset(Y)|^2 \, d\sigma(Y) \right)^{1/2} \leq C r^3 \dot{f}^\#(X_o). \tag{4.124}$$

This concludes the third step of our analysis at the end of which we have

$$\left(\fint_{S_r(X_o)} g_\emptyset(Y) \, d\sigma(Y) \right)^{1/2} \leq C r^3 \dot{f}^\#(X_o), \tag{4.125}$$

$$\begin{aligned} \left(\fint_{S_r(X_o)} |g_\gamma(Y)|^2 \, d\sigma(Y) \right)^{1/2} &\leq C r^2 \dot{f}^\#(X_o), \quad \text{if } |\gamma| = 1, \\ \left(\fint_{S_r(X_o)} |g_\alpha(Y)|^2 \, d\sigma(Y) \right)^{1/2} &\leq C r \dot{f}^\#(X_o), \quad \text{if } |\alpha| = 2. \end{aligned} \tag{4.126}$$

Thus, inductively, we obtain (4.100).

We are left with showing (4.101). Let $r > 0$ and $X_o \in \partial\Omega$ be fixed and notice that, by the linearity of $\dot{\omega}_r$, (4.98) and (4.99),

$$\begin{aligned} \dot{\omega}_r(\dot{f}) - \dot{\omega}_{2r}(\dot{f}) &= \dot{\omega}_r(\dot{f} - \dot{\omega}_{2r}(\dot{f})) \\ &= \sum_{|\beta|,|\gamma| \leq m-1} \Big\langle (\dot{f} - \dot{\omega}_{2r}(\dot{f}))_\beta, (\dot{e}_\gamma)_\beta \Big\rangle \dot{e}_\gamma, \end{aligned} \tag{4.127}$$

where the collection $\{\dot{e}_\alpha\}_{|\alpha| \leq m-1}$ is the orthonormal basis in $\dot{\mathscr{P}}_{m-1}(S_r(X_o))$ relative to the Hilbert space $L^2(S_r(X_o)) \oplus \cdots \oplus L^2(S_r(X_o))$, N times, discussed in the preamble of this lemma, and $\langle \cdot, \cdot \rangle$ stands here for the inner product in $L^2(S_r(X_o))$. Then, for every $X \in \partial\Omega$,

$$
\begin{aligned}
&\Big|\Big(\dot\omega_r(\dot f)-\dot\omega_{2r}(\dot f)\Big)_\alpha(X)\Big| \\
&\quad\le \sum_{|\beta|,|\gamma|\le m-1}\Big|\Big\langle(\dot f-\dot\omega_{2r}(\dot f))_\beta,(\dot e_\gamma)_\beta\Big\rangle(\dot e_\gamma)_\alpha(X)\Big| \\
&\quad\le Cr^{\frac{n-1}{2}}\sum_{|\beta|\le|\gamma|\le m-1}\Big(\fint_{S_r(X_o)}\Big|\Big(\dot f-\dot\omega_{2r}(\dot f)\Big)_\beta(Y)\Big|^2\,d\sigma(Y)\Big)^{1/2}\|(\dot e_\gamma)_\alpha\|_{L^\infty(S_r(X_o))} \\
&\quad\le C\sum_{|\beta|\le|\gamma|\le m-1}\Big(\fint_{S_{2r}(X_o)}\Big|\Big(\dot f-\dot\omega_{2r}(\dot f)\Big)_\beta(Y)\Big|^2\,d\sigma(Y)\Big)^{1/2}\Big[|X-X_o|+r\Big]^{|\gamma|-|\alpha|} \\
&\quad\le C\sum_{|\beta|\le|\gamma|\le m-1} r^{m-1-|\beta|}\dot f^\#(X_o)\Big[|X-X_o|+r\Big]^{|\gamma|-|\alpha|}, \qquad (4.128)
\end{aligned}
$$

where the first inequality in (4.128) follows from (4.127), the second is a consequence of Hölder's inequality, the normality of $\{\dot e_\gamma\}_{|\gamma|\le m-1}$ and (4.94), the third is implied by (4.95) and $|S_r(X_o)| \approx |S_{2r}(X_o)|$, while the last follows from (4.100). Then (4.101) is an immediate consequence of (4.128) and the fact that $r\le|X-X_o|+r$ and that $\partial\Omega$ is bounded. □

Next, we present the main result of this section. Before stating it, recall (2.43).

Theorem 4.11. *Consider a W-elliptic homogeneous differential operator L of order $2m$ with (complex) matrix-valued constant coefficients. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and for each $X\in\Omega$ denote by $\rho(X)$ its distance to $\partial\Omega$. Let $\dot{\mathscr{D}}$ be a double multi-layer potential associated with L (as in § 4.2). Then there exists a finite constant $C=C(\Omega,L)>0$ with the property that*

$$
\begin{aligned}
&|\nabla^m\dot{\mathscr{D}}\dot f(X)|^2\rho(X)\,dX \quad \textit{is a Carleson measure on } \Omega\textit{, with} \\
&\textit{Carleson constant} \le C\|\dot f\|^2_{\dot{\mathrm{BMO}}_{m-1}(\partial\Omega)}\ \textit{for each } \dot f\in\dot{\mathrm{BMO}}_{m-1}(\partial\Omega).
\end{aligned}
\qquad (4.129)
$$

Proof. Let $\dot f\in\dot{\mathrm{BMO}}_{m-1}(\partial\Omega)$ and fix $X_o\in\partial\Omega$ and $r>0$. Consider next a function η in $\mathbb{R}^n$ with the following properties

$$\eta\in C_c^\infty(\mathbb{R}^n),\quad 0\le\eta\le1,\quad \eta\equiv1 \text{ on } B_{2r}(X_o),\ \operatorname{supp}\eta\subset B_{4r}(X_o), \qquad (4.130)$$

$$\text{and}\quad |\partial^\alpha\eta|\le\frac{C_\alpha}{r^{|\alpha|}},\qquad \text{for each multi-index } \alpha\in\mathbb{N}_0^n. \qquad (4.131)$$

Also, denote by $\Delta_r(X_o):=\Omega\cap B(X_o,r)$ the Carleson box associated with the surface ball $S_r(X_o)$, and denote by $|\Delta_r(X_o)|$ its n-dimensional Lebesgue measure. Let us agree that $|S_r(X_o)|$ denotes the surface measure of $S_r(X_o)$.

Throughout the proof, we shall abbreviate $\dot{\omega}_r(\dot{f})$, $\dot{\omega}_{2r}(\dot{f})$, etc., as $\dot{\omega}_r$, $\dot{\omega}_{2r}$, etc. Recall (3.9). Writing now $\dot{f} = \eta(\dot{f} - \dot{\omega}_{2r}) + (1-\eta)(\dot{f} - \dot{\omega}_{2r}) + \dot{\omega}_{2r}$ and using the reproducing property (4.62) of $\dot{\mathscr{D}}$ we obtain

$$\nabla^m \dot{\mathscr{D}} \dot{f} = \nabla^m \dot{\mathscr{D}}[\eta(\dot{f} - \dot{\omega}_{2r})] + \nabla^m \dot{\mathscr{D}}[(1-\eta)(\dot{f} - \dot{\omega}_{2r})], \tag{4.132}$$

as $\nabla^m \dot{\mathscr{D}} \dot{\omega}_{2r} = \nabla^m \omega_{2r} = 0$. Consequently, using that $(a+b)^2 \le 2a^2 + 2b^2$, we have,

$$\begin{aligned} \frac{1}{|S_r(X_o)|} \int_{\Delta_r(X_o)} & \rho(X) |\nabla^m \dot{\mathscr{D}} \dot{f}(X)|^2 \, dX \\ & \le \frac{2}{|S_r(X_o)|} \int_{\Delta_r(X_o)} \rho(X) |\nabla^m \dot{\mathscr{D}}[\eta(\dot{f} - \dot{\omega}_{2r})](X)|^2 \, dX \\ & \quad + \frac{2}{|S_r(X_o)|} \int_{\Delta_r(X_o)} \rho(X) |\nabla^m \dot{\mathscr{D}}[(1-\eta)(\dot{f} - \dot{\omega}_{2r})](X)|^2 \, dX \\ & =: I + II. \end{aligned} \tag{4.133}$$

We claim next that there exists a finite constant $C = C(\Omega, L) > 0$ such that the following are satisfied

$$\frac{1}{|S_r(X_o)|} \int_{\Delta_r(X_o)} \rho(X) |\nabla^m \dot{\mathscr{D}}[\eta(\dot{f} - \dot{\omega}_{2r})](X)|^2 \, dX \le C(\dot{f}^{\#}(X_o))^2, \tag{4.134}$$

$$|\nabla^m \dot{\mathscr{D}}[(1-\eta)(\dot{f} - \dot{\omega}_{2r})](X)| \le \frac{C}{r} \|\dot{f}\|_{\dot{\mathrm{BMO}}_{m-1}(\partial\Omega)} \quad \text{if} \quad X \in \Delta_r(X_o). \tag{4.135}$$

Let us assume for the moment that (4.134) and (4.135) hold and continue with the proof of (4.129). Using (3.387), then (4.134) readily implies

$$I \le C \|\dot{f}\|^2_{\dot{\mathrm{BMO}}_{m-1}(\partial\Omega)}. \tag{4.136}$$

Also, based on (4.135), $|S_r(X_o)| \approx r^{n-1}$, $|\Delta_r(X_o)| \approx r^n$, and $|\rho(X)| \le r$ whenever $X \in \Delta_r(X_o)$, we have

$$II \le \frac{C}{r^2} \|\dot{f}\|^2_{\dot{\mathrm{BMO}}_{m-1}(\partial\Omega)} \frac{1}{|S_r(X_o)|} \int_{\Delta_r(X_o)} |\rho(X)| \, dX \le C \|\dot{f}\|^2_{\dot{\mathrm{BMO}}_{m-1}(\partial\Omega)}. \tag{4.137}$$

Then (4.129) follows from (4.133) and (4.136)–(4.137).

We are therefore left with proving (4.134) and (4.135), a task to which we turn now. Starting with (4.134) let us notice that since $|S_r(X_o)| \approx r^{n-1}$,

$$I \leq \frac{C}{r^{n-1}} \int_\Omega \rho(X) |\nabla^m \dot{\mathscr{D}}[\eta(\dot{f} - \dot{\omega}_{2r})(X)]|^2 dX. \tag{4.138}$$

Next, from Proposition 2.67 and the integral identity in (4.77)–(4.79) we may estimate

$$\int_\Omega \rho(X) |\nabla^m \dot{\mathscr{D}}[\eta(\dot{f} - \dot{\omega}_{2r})(X)]|^2 \, dX \leq C \int_{\partial\Omega} |\eta(\dot{f} - \dot{\omega}_{2r})(Q)|^2 \, d\sigma(Q). \tag{4.139}$$

Consequently,

$$\begin{aligned} I &\leq \frac{C}{r^{n-1}} \int_{\partial\Omega} |\eta(\dot{f} - \dot{\omega}_{2r})(Q)|^2 \, d\sigma(Q) \\ &= \frac{C}{r^{n-1}} \int_{S_{4r}(X_o)} |(\dot{f} - \dot{\omega}_{2r})(Q)|^2 \, d\sigma(Q), \end{aligned} \tag{4.140}$$

where the equality above follows from the fact that η is supported in $B_{4r}(X_o)$. Hence,

$$I \leq \frac{C}{r^{n-1}} \int_{S_{4r}(X_o)} |(\dot{f} - \dot{\omega}_{2r})(Q)|^2 \, d\sigma(Q) \leq C(\dot{f}^{\#}(X_o))^2, \tag{4.141}$$

where the last inequality is a consequence of the property (4.90) and the definition of the sharp maximal function $\dot{f}^{\#}(X_o)$. The proof of (4.136) is therefore completed.

Now we focus on proving (4.137) and to this end, let $X \in \Delta_r(X_o)$. Then, since

$$|\nabla^m \dot{\mathscr{D}}[(1-\eta)(\dot{f} - \dot{\omega}_{2r})](X)| = |\nabla(\nabla^{m-1} \dot{\mathscr{D}}[(1-\eta)(\dot{f} - \dot{\omega}_{2r})])(X)|, \tag{4.142}$$

employing (4.77) we obtain

$$\begin{aligned} &|\nabla^m \dot{\mathscr{D}}[(1-\eta)(\dot{f} - \dot{\omega}_{2r})](X)| \\ &\leq C \sum_{\substack{|\gamma|=m-1, |\alpha|=m \\ |\beta|=m, |\delta|=m-1}} \Big| \nabla_X \int_{\partial\Omega} \nu_j(Y) (\partial^{\beta+\gamma} E)(X-Y) A_{\beta\alpha} \times \\ &\qquad \times ((1-\eta)(\dot{f} - \dot{\omega}_{2r}))_\delta(Y) \, d\sigma(Y) \Big|, \end{aligned} \tag{4.143}$$

where $C > 0$ is a universal constant. Using the support properties of η from (4.130), we can replace $\partial\Omega$ by $\partial\Omega \setminus S_{2r}(X_o)$ as the domain of integration in the right hand side of (4.142) since $1 - \eta \equiv 0$ on $S_{2r}(X_o)$. Also, using (4.29),

$$|\nabla^{2m} E(X-Y)| \leq \frac{C}{|X-Y|^n}. \tag{4.144}$$

Therefore, since

$$|(1-\eta)(\dot{f}-\dot{\omega}_{2r})(Y)| \leq C \sum_{|\mu|\leq m-1} |(\dot{f}-\dot{\omega}_{2r})_\mu(Y)| \cdot |\nabla^{m-1-|\mu|}(1-\eta)(Y)|, \tag{4.145}$$

we obtain

$$\begin{aligned} &|\nabla^m \dot{\mathscr{D}}[(1-\eta)(\dot{f}-\dot{\omega}_{2r})](X)| \\ &\quad \leq C \sum_{|\mu|\leq m-1} \int_{\partial\Omega\setminus S_{2r}(X_o)} \frac{|(\dot{f}-\dot{\omega}_{2r})_\mu(Y)|}{|X-Y|^n} |\nabla^{m-1-|\mu|}(1-\eta)(Y)|\, d\sigma(Y). \end{aligned} \tag{4.146}$$

A simple observation is that there exist constants $C_j = C_j(\Omega)$, $j = 1, 2$, such that for $X \in \Delta_r(X_o)$ and $Y \in \partial\Omega \setminus S_{2r}(X_o)$ we have

$$C_1|X-Y| \leq |X_o - Y| \leq C_2|X-Y|. \tag{4.147}$$

Employing (4.147) in (4.146), it follows that

$$\begin{aligned} &|\nabla^m \dot{\mathscr{D}}[(1-\eta)(\dot{f}-\dot{\omega}_{2r})](X)| \\ &\quad \leq C \sum_{|\mu|=m-1} \int_{\partial\Omega\setminus S_{2r}(X_o)} \frac{1}{|X_o-Y|^n} |(\dot{f}-\dot{\omega}_{2r})_\mu(Y)|\, |(1-\eta)(Y)| d\sigma(Y) \\ &\qquad + C \sum_{|\mu|<m-1} \int_{\partial\Omega\setminus S_{2r}(X_o)} \frac{1}{|X_o-Y|^n} |(\dot{f}-\dot{\omega}_{2r})_\mu(Y)|\, |\nabla^{m-1-|\mu|}(1-\eta)(Y)|\, d\sigma(Y) \\ &\quad =: III + IV. \end{aligned} \tag{4.148}$$

For each multi-index $\mu \in \mathbb{N}_0^n$, $|\mu| < m-1$, the function $\nabla^{m-1-|\mu|}(1-\eta)$ is supported in $S_{4r}(X_o) \setminus S_{2r}(X_o)$. This, together with (4.131) imply

$$III \leq C \sum_{|\mu|=m-1} \int_{\partial\Omega\setminus S_{2r}(X_o)} \frac{1}{|X_o-Y|^n} |(\dot{f}-\dot{\omega}_{2r})_\mu(Y)|\, d\sigma(Y), \tag{4.149}$$

$$IV \leq C \sum_{|\mu|<m-1} \frac{1}{r^{m-1-|\mu|}} \int_{S_{4r}(X_o)\setminus S_{2r}(X_o)} \frac{1}{|X_o-Y|^n} |(\dot{f}-\dot{\omega}_{2r})_\mu(Y)|\, d\sigma(Y). \tag{4.150}$$

Using that $|S_{4r}(X_o)| \approx r^{n-1}$ and for each point $Y \in S_{4r}(X_o) \setminus S_{2r}\ (X_o)$ we have $2r \leq |\ X_o - Y| \leq 4r$, (4.150) gives

$$IV \leq C \sum_{|\mu|<m-1} \frac{1}{r^{m-|\mu|}} \cdot \frac{1}{|S_{4r}(X_o)|} \int_{S_{4r}(X_o)} |(\dot{f} - \dot{\omega}_{2r})_\mu(Y)|\, d\sigma(Y). \tag{4.151}$$

$$\leq C \sum_{|\mu|<m-1} \frac{1}{r^{m-|\mu|}} \left(\frac{1}{|S_{4r}(X_o)|} \int_{S_{4r}(X_o)} |(\dot{f} - \dot{\omega}_{2r})_\mu(Y)|\, d\sigma(Y) \right)^{1/2}.$$

Now (4.100) allows us to conclude,

$$IV \leq C \sum_{|\mu|\leq m-1} \frac{1}{r^{m-|\mu|}} \cdot r^{m-1-|\mu|} \dot{f}^{\#}(X_o)$$

$$\leq \frac{C}{r} \dot{f}^{\#}(X_o) \leq \frac{C}{r} \|\dot{f}\|_{\dot{\mathrm{BMO}}_{m-1}(\partial\Omega)}. \tag{4.152}$$

As for *III*, by writing the domain of integration in the right-hand side (4.149) as the union of disjoint annuli $S_{2^{j+1}r}(X_o) \setminus S_{2^j r}(X_o)$, $j \geq 1$, we have

$$III \leq C \sum_{|\mu|=m-1} \sum_{j=1}^{\infty} \int_{S_{2^{j+1}r(X_o)} \setminus S_{2^j r}(X_o)} \frac{1}{(2^j r)^n} |(\dot{f} - \dot{\omega}_{2r})_\mu(Y)|\, d\sigma(Y), \tag{4.153}$$

since, for each $j \geq 1$ and $Y \in S_{2^{j+1}r} \setminus S_{2^j r}$, one has $2^j r \leq |X_o - Y|$. We then write

$$III \leq C \sum_{|\mu|=m-1} \sum_{j=1}^{\infty} \frac{1}{2^j r} \fint_{S_{2^{j+1}r}(X_o)} \Big[|(\dot{f} - \dot{\omega}_{2^{j+1}r})_\mu(Y)|$$

$$+ \sum_{i=1}^{j} |(\dot{\omega}_{2^{i+1}r} - \dot{\omega}_{2^i r})_\mu(Y)| \Big]\, d\sigma(Y)$$

$$\leq C \sum_{|\mu|=m-1} \sum_{j=1}^{\infty} \Big[\frac{1}{2^j r} \Big(\fint_{S_{2^{j+1}r}(X_o)} |(\dot{f} - \dot{\omega}_{2^{j+1}r})_\mu(Y)|^2\, d\sigma(Y) \Big)^{\frac{1}{2}}$$

$$+ \sum_{i=1}^{j} \frac{1}{2^j r} \fint_{S_{2^{j+1}r}(X_o)} |(\dot{\omega}_{2^{i+1}r} - \dot{\omega}_{2^i r})_\mu(Y)|\, d\sigma(Y) \Big]$$

$$\leq C \sum_{|\mu|=m-1} \sum_{j=1}^{\infty} \left(\frac{1}{2^j r} \dot{f}^{\#}(X_o) + \frac{j}{2^j r} \dot{f}^{\#}(X_o) \right)$$

$$\leq \frac{C}{r} \dot{f}^{\#}(X_o) \leq \frac{C}{r} \|\dot{f}\|_{\dot{\mathrm{BMO}}_{m-1}(\partial\Omega)}. \tag{4.154}$$

Above, the first inequality in (4.154) follows from (4.153) by writing $\dot{f} - \dot{\omega}_{2r}$ as a telescoping sum and applying the triangle inequality, while the second one is obtained from Hölder's inequality. The third inequality follows from Lemma 4.10, whereas the last inequality is a consequence of (3.387). Now, (4.154), (4.152) and (4.148) give (4.135) and complete the proof of Theorem 4.11. □

We conclude by establishing the counterpart of Theorem 4.11 in the VMO context. Specifically, we have:

Theorem 4.12. *Retain the setting of Theorem 4.11. Then*

$$\begin{aligned}&|\nabla^m \dot{\mathscr{D}} \dot{f}(X)|^2 \rho(X)\, dX \quad \textit{is a vanishing Carleson} \\ &\textit{measure on } \Omega \textit{ for each } \dot{f} \in \mathrm{V\dot{M}O}_{m-1}(\partial\Omega).\end{aligned} \tag{4.155}$$

Proof. Fix an arbitrary $\dot{f} \in \mathrm{V\dot{M}O}_{m-1}(\partial\Omega)$ along with a threshold $\varepsilon > 0$. Pick a number $s \in (0,1)$ such that $s > \frac{n-1}{2n}$, and choose $\dot{g} \in \dot{B}^{\infty,\infty}_{m-1,s}(\partial\Omega)$ such that $\|\dot{f} - \dot{g}\|_{\mathrm{B\dot{M}O}_{m-1}(\partial\Omega)} < \varepsilon$. That this is possible is ensured by Definition 3.52 and Proposition 3.53. Then, for every $X_o \in \partial\Omega$ and $0 < r < \operatorname{diam}(\Omega)$, we have

$$\begin{aligned} r^{1-n} \int_{\Delta(X_o,r)} |\nabla^m \dot{\mathscr{D}} \dot{f}|^2 \rho\, dX &\leq 2r^{1-n} \int_{\Delta(X_o,r)} |\nabla^m \dot{\mathscr{D}} (\dot{f} - \dot{g})|^2 \rho\, dX \\ &\quad + 2r^{1-n} \int_{\Delta(X_o,r)} |\nabla^m \dot{\mathscr{D}} \dot{g}|^2 \rho\, dX =: I + II. \end{aligned} \tag{4.156}$$

Now, Theorem 4.11 gives that

$$I \leq 2\| \, |\nabla^m \dot{\mathscr{D}} (\dot{f} - \dot{g})|^2 \rho\, dX \, \|_{Car} \leq C \|\dot{f} - \dot{g}\|^2_{\mathrm{B\dot{M}O}_{m-1}(\partial\Omega)} \leq C\varepsilon^2. \tag{4.157}$$

As for term II, anticipating an estimate which will be proved later, we remark that the function $u := |\nabla^m \dot{\mathscr{D}} \dot{g}|^2 \rho$ satisfies $|u| \leq C\rho^{2s-1}$ in Ω, thanks to (4.243), where $C := \|\dot{g}\|^2_{\dot{B}^{\infty,\infty}_{m-1,s}(\partial\Omega)}$ is a finite constant. Consequently, by (2.58), $\mu := u\, dX$ is a vanishing Carleson measure, since $2s - 1 > -1/n$ by our choice of s. In turn, this entails that there exists $R > 0$ such that

$$II \leq \varepsilon \quad \text{if} \quad 0 < r < R, \tag{4.158}$$

uniformly in $X_o \in \partial\Omega$. Altogether, (4.156)–(4.158) show that

$$\lim_{R \to 0^+} \left(\sup_{0<r<R,\, X_o \in \partial\Omega} r^{1-n} \int_{\Delta(X_o,r)} |\nabla^m \dot{\mathscr{D}} \dot{f}|^2 \rho\, dX \right) = 0. \tag{4.159}$$

This proves (4.155). □

4.4 Jump Relations

Throughout this section we define and study the boundary principal-value version of the double multi-layer potential operator $\dot{\mathscr{D}}$ introduced in (4.57).

Definition 4.13. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, and that L is a (complex) matrix-valued constant coefficient, homogeneous, W-elliptic differential operator L of order $2m$ in $\mathbb{R}^n$. For each $\dot{f} \in \dot{L}^p_{m-1,0}(\partial\Omega)$, $1 < p < \infty$, define

$$\dot{K}\dot{f} := \Big\{(\dot{K}\dot{f})_\gamma\Big\}_{|\gamma|\leq m-1} \tag{4.160}$$

where, for each $\gamma \in \mathbb{N}_0^n$ of length $\leq m-1$, we have set

$$\begin{aligned}
(\dot{K}\dot{f})_\gamma(X) := & \sum_{\substack{|\alpha|=m\\|\beta|=m}} \sum_{\ell=1}^{|\gamma|} \sum_{\substack{\delta+\eta+e_k=\alpha\\|\delta|=\ell-1,|\eta|=m-\ell}} \sum_{\substack{\theta+\omega+e_j=\gamma\\|\theta|=\ell-1,|\omega|=|\gamma|-\ell}} \Big\{C_1(m,\ell,\alpha,\delta,\eta,\gamma,\theta,\omega)\times \\
& \times \lim_{\varepsilon\to 0^+} \int\limits_{\substack{Y\in\partial\Omega\\|X-Y|>\varepsilon}} \partial_{\tau_{kj}(Y)}\Big((\partial^{\delta+\omega+\beta}E)(X-Y)\Big)A_{\beta\alpha}f_{\theta+\eta}(Y)\,d\sigma(Y)\Big\} \\
& - \sum_{\substack{|\alpha|=m\\|\beta|=m}} \sum_{\ell=|\gamma|+1}^{m} \sum_{\substack{\delta+\eta+e_k=\alpha\\|\delta|=\ell-1,|\eta|=m-\ell}} \Big\{C_2(m,\ell,\alpha,\delta,\eta)\times \\
& \times \lim_{\varepsilon\to 0^+} \int\limits_{\substack{Y\in\partial\Omega\\|X-Y|>\varepsilon}} \nu_k(Y)(\partial^{\delta+\beta}E)(X-Y)A_{\beta\alpha}f_{\gamma+\eta}(Y)\,d\sigma(Y)\Big\},
\end{aligned} \tag{4.161}$$

for $X \in \partial\Omega$. Above C_1, C_2 are as in (4.78)–(4.79).

To state our next result, recall the non-tangential boundary multi-trace from (3.272). Also, recall the double multi-layer potential operator defined in (4.57), the convention (2.11), and set

$$\dot{\mathscr{D}}^{\pm}\dot{f} := \Big(\dot{\mathscr{D}}\dot{f}\Big)\Big|_{\Omega_\pm}. \tag{4.162}$$

In the sequel we will sometimes abbreviate $\dot{\mathscr{D}}^{+}$ by $\dot{\mathscr{D}}$. Also, I will denote the identity operator.

Theorem 4.14. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and consider a (complex) matrix-valued constant coefficient, homogeneous, W-elliptic differential* *operator L of order $2m$ in $\mathbb{R}^n$. Assume that $\dot{K}$ is associated with these as in* (4.160)–(4.161).

Then for each $\dot{f} \in \dot{L}^p_{m-1,0}(\partial\Omega)$, $1 < p < \infty$, the expression $\dot{K}\dot{f}(X)$ is meaningful at almost every point $X \in \partial\Omega$. Moreover, the operator

$$\dot{K} : \dot{L}^p_{m-1,0}(\partial\Omega) \longrightarrow \dot{L}^p_{m-1,0}(\partial\Omega) \tag{4.163}$$

is well-defined, linear and bounded for each $p \in (1,\infty)$, and the following jump-relation holds

$$\dot{\mathscr{D}}^{\pm}\dot{f}\Big|_{\partial\Omega}^{m-1} = (\pm\tfrac{1}{2}I + \dot{K})\dot{f}, \tag{4.164}$$

for each $\dot{f} \in \dot{L}^p_{m-1,0}(\partial\Omega)$, $1 < p < \infty$.

Furthermore, for each $p \in (1,\infty)$, the operator

$$\dot{K} : \dot{L}^p_{m-1,1}(\partial\Omega) \longrightarrow \dot{L}^p_{m-1,1}(\partial\Omega) \tag{4.165}$$

is also well-defined, linear and bounded. As a corollary,

$$\begin{aligned} \dot{K}^* &: \left(\dot{L}^p_{m-1,0}(\partial\Omega)\right)^* \longrightarrow \left(\dot{L}^p_{m-1,0}(\partial\Omega)\right)^* \\ \dot{K}^* &: \left(\dot{L}^p_{m-1,1}(\partial\Omega)\right)^* \longrightarrow \left(\dot{L}^p_{m-1,1}(\partial\Omega)\right)^* \end{aligned} \tag{4.166}$$

are also well-defined, linear, bounded operators for each $p \in (1,\infty)$.

Proof. The fact that the limits in (4.161) exist at a.e. $X \in \partial\Omega$ is a consequence of Proposition 2.63. This proposition also gives that

$$\sum_{|\gamma|\leq m-1} \|(\dot{K}\dot{f})_\gamma\|_{L^p(\partial\Omega)} \leq C\|\dot{f}\|_{\dot{L}^p_{m-1,0}(\partial\Omega)} \tag{4.167}$$

for some finite constant $C = C(\Omega, L, p) > 0$.

We shall now prove that (4.164) holds which requires that

$$(\dot{\mathscr{D}}^{\pm}\dot{f})_\gamma\Big\lfloor_{\partial\Omega} = \left[\left(\pm\tfrac{1}{2}I + \dot{K}\right)\dot{f}\right]_\gamma, \qquad \forall\, \gamma \in \mathbb{N}_0^n,\ |\gamma| \leq m-1. \tag{4.168}$$

We consider first the case of superscript $+$ in (4.168) and fix for the moment $\gamma \in \mathbb{N}_0^n$ with length $\leq m-1$. Notice that in (4.77), a typical kernel for the first group of integral operators is of the type

$$k(X,Y) = \partial_{\tau_{kj}(Y)}\Big(k_o(X-Y)\Big) \quad \text{where } k_o(Z) := (\partial^{\delta+\omega+\beta}E)(Z)A_{\beta\alpha}, \tag{4.169}$$

where $\alpha, \beta, \delta, \omega \in \mathbb{N}_0^n$, $|\alpha| = |\beta| = m$, $|\delta| = \ell - 1$ and $|\omega| = |\gamma| - \ell$, for some number $\ell \in \{1, \dots, |\gamma|\}$. In particular $|\delta| + |\omega| + |\beta| = m + |\gamma| - 1 \leq 2m - 2$ and, consequently, the kernel $k(X, Y)$ is either weakly singular, or amenable to the Calderón–Zygmund theory reviewed in § 2.8. The integral operators with weakly singular kernels do not jump thus, when restricted to the boundary, they contribute to the left-hand side of (4.168) the boundary operators in the first part of (4.161). In the case when the multi-index $\gamma \in \mathbb{N}_0^n$ satisfies $|\gamma| = m - 1$, we employ the jump relations (2.530) to write

$$\begin{aligned}
\Big(\int_{\partial\Omega} \partial_{\tau_{jk}(Y)}[k_o(\cdot - Y)]g(Y)\,d\sigma(Y)\Big)\Big|_{\partial\Omega}(X) \\
&= \frac{1}{2\sqrt{-1}}[\nu_j\nu_k\widehat{k_o}(\nu) - \nu_k\nu_j\widehat{k_o}(\nu)](X)g(X) \\
&\quad + \lim_{\varepsilon\to 0+} \int\limits_{\substack{Y\in\partial\Omega \\ |X-Y|>\varepsilon}} \partial_{\tau_{jk}(Y)}[k_o(X-Y)]g(Y)\,d\sigma(Y) \\
&= \lim_{\varepsilon\to 0+} \int\limits_{\substack{Y\in\partial\Omega \\ |X-Y|>\varepsilon}} \partial_{\tau_{jk}(Y)}[k_o(X-Y)]g(Y)\,d\sigma(Y). \qquad (4.170)
\end{aligned}$$

for each $g \in L^p(\partial\Omega)$ and almost every $X \in \partial\Omega$. This shows that the restriction to the boundary of the first part in (4.77) gives precisely the first part of (4.161).

Let us now focus on the restriction to the boundary of the second part of (4.77). Notice that whenever $\ell < m$ the kernels in the second part of (4.77) are weakly singular and hence the integral operators with these kernels do not jump. In turn, the restriction to the boundary of these operators give the corresponding terms (i.e., with $\ell < m$) in the second part of (4.161).

Finally, we are left with considering the boundary behavior of the following sum of integral operators (corresponding to the choice $\ell = m$ in the second part of (4.77)):

$$-\sum_{\substack{|\alpha|=m \\ |\beta|=m}} \sum_{\delta+e_k=\alpha} \frac{\alpha!}{m\,\delta!} \int_{\partial\Omega} \nu_k(Y)(\partial^{\delta+\beta}E)(X-Y)A_{\beta\alpha}f_\gamma(Y)\,d\sigma(Y). \qquad (4.171)$$

Since $|\beta| + |\delta| = 2m - 1$, the kernels in (4.171) are Calderón–Zygmund and employing (2.530) the non-tangential limit on $\partial\Omega$ of (4.171) is

$$-\frac{1}{2\sqrt{-1}} \sum_{\substack{|\alpha|=m \\ |\beta|=m}} \sum_{\delta+e_k=\alpha} \frac{\alpha!}{m\,\delta!} \nu_k(X) (\widehat{\partial^{\delta+\beta} E})(\nu(X)) A_{\beta\alpha} f_\gamma(X) \tag{4.172}$$

$$- \lim_{\varepsilon\to 0^+} \sum_{\substack{|\alpha|=m \\ |\beta|=m}} \sum_{\delta+e_k=\alpha} \frac{\alpha!}{m\,\delta!} \int_{\partial\Omega} \nu_k(Y) (\partial^{\delta+\beta} E)(X-Y) A_{\beta\alpha} f_\gamma(Y)\, d\sigma(Y),$$

where as before "hat" denotes the Fourier transform. Based on (4.77) and (4.161), in the light of our previous analysis, in order to prove (4.168) it suffices to show

$$-\frac{1}{2\sqrt{-1}} \sum_{\substack{|\alpha|=m \\ |\beta|=m}} \sum_{\delta+e_k=\alpha} \frac{\alpha!}{m\,\delta!} \nu_k(X) (\widehat{\partial^{\delta+\beta} E})(\nu(X)) A_{\beta\alpha} f_\gamma(X) = \tfrac{1}{2} f_\gamma(X). \tag{4.173}$$

Using that

$$\begin{aligned} (\widehat{\partial^{\delta+\beta} E})(\nu(X)) &= \sqrt{-1}^{|\delta|+|\beta|} \nu(X)^{\delta+\beta} \widehat{E}(\nu(X)) \\ &= (\sqrt{-1})^{2m-1} \nu(X)^{\delta+\beta} \widehat{E}(\nu(X)), \end{aligned} \tag{4.174}$$

we obtain

$$\begin{aligned} &-\tfrac{1}{2\sqrt{-1}} \sum_{\substack{|\alpha|=m \\ |\beta|=m}} \sum_{\delta+e_k=\alpha} \frac{\alpha!}{m\,\delta!} \nu_k(X) (\widehat{\partial^{\delta+\beta} E})(\nu(X)) A_{\beta\alpha} f_\gamma(X) \\ &\quad = \tfrac{1}{2}(-1)^m \sum_{\substack{|\alpha|=m \\ |\beta|=m}} \sum_{\delta+e_k=\alpha} \frac{\alpha!}{m\,\delta!} \nu(X)^{\delta+\beta+e_k} \widehat{E}(\nu(X)) A_{\beta\alpha} f_\gamma(X) \\ &\quad = \tfrac{1}{2}(-1)^m \sum_{\substack{|\alpha|=m \\ |\beta|=m}} \left(\sum_{\delta+e_k=\alpha} \frac{\alpha!}{m\,\delta!} \right) \nu(X)^{\alpha+\beta} \widehat{E}(\nu(X)) A_{\beta\alpha} f_\gamma(X) \\ &\quad = \tfrac{1}{2}(-1)^m \sum_{\substack{|\alpha|=m \\ |\beta|=m}} \nu(X)^{\alpha+\beta} \widehat{E}(\nu(X)) A_{\beta\alpha} f_\gamma(X) \\ &\quad = \tfrac{1}{2}(-1)^m \widehat{E}(\nu(X)) \Big[\sum_{\substack{|\alpha|=m \\ |\beta|=m}} \nu(X)^{\alpha+\beta} A_{\alpha\beta} \Big] f_\gamma(X), \end{aligned} \tag{4.175}$$

where the next-to-last identity follows from (4.82). Finally, (4.31) gives

$$\widehat{E}(\nu(X)) = (-1)^m \Big[\sum_{|\alpha|=|\beta|=m} \nu(X)^{\alpha+\beta} A_{\alpha\beta} \Big]^{-1} \tag{4.176}$$

so the identity (4.173) readily follows. This in turn gives (4.168) and finishes the proof of (4.164).

In order to complete the proof of (4.163) it remains to show that for each Whitney array $\dot{f} \in \dot{L}^p_{m-1,0}(\partial\Omega)$ for some $p \in (1,\infty)$ we have that $\dot{K}\dot{f} \in CC$. This is however a direct consequence of (4.164), the non-tangential maximal function estimate (4.63) and (3.275) in Proposition 3.29.

We turn now to proving that the operator (4.165) is also well-defined and bounded. To this end, fix $p \in (1,\infty)$. Then, for each $\dot{f} \in \dot{L}^p_{m-1,1}(\partial\Omega)$ the jump formula (4.164) holds. Then, the desired properties of the operator (4.165) are a consequence of the non-tangential maximal function estimates (4.64), (3.279) from Proposition 3.30 and (3.265). □

Theorem 4.15. *Retain the same background assumptions as in Theorem 4.14. Then the operator*

$$\dot{K} : \dot{h}^p_{m-1,1}(\partial\Omega) \longrightarrow \dot{h}^p_{m-1,1}(\partial\Omega) \tag{4.177}$$

is well-defined, linear and bounded for each $p \in (\frac{n-1}{n}, \infty)$.

Proof. When $1 < p < \infty$, it follows that $\dot{h}^p_{m-1,1}(\partial\Omega) = \dot{L}^p_{m-1,1}(\partial\Omega)$ and (4.177) follows from (4.165). On the other hand, the embedding $\dot{h}^p_{m-1,1}(\partial\Omega) \hookrightarrow \dot{L}^{p^*}_{m-1,0}(\partial\Omega)$ holds if $\frac{n-1}{n} < p \leq 1$, where p^* is as in (2.433). Hence, in either case, (4.164) holds for $\dot{f} \in \dot{h}^p_{m-1,1}(\partial\Omega)$, so the boundedness of (4.163) is a consequence of this, (4.87), and (3.279) from Proposition 3.30. □

We continue by presenting a structure theorem for the principal-value double multi-layer in the case when the underlying differential operator factors through the Laplacian.

Theorem 4.16. *Retain the same background assumptions as in Theorem 4.14 and, in addition, assume that L is a W-elliptic differential operator which factors as*

$$L = \Delta \tilde{L}, \tag{4.178}$$

where $\tilde{L}$ is a constant coefficient differential operator of order $2m-2$. Then, for each $\gamma \in \mathbb{N}_0^n$, with $|\gamma| \leq m-1$, and $\dot{f} = \{f_\gamma\}_{|\gamma|\leq m-1} \in \dot{L}^p_{m-1,0}(\partial\Omega)$, $1 < p < \infty$, one has

$$(\dot{K}\dot{f})_\gamma(X) = \sum_{|\theta|\le m-1}\sum_{j,k=1}^{n}\lim_{\varepsilon\to 0^+}\int\limits_{\substack{Y\in\partial\Omega\\|X-Y|>\varepsilon}} \partial_{\tau_{jk}(Y)}\Big([p_{\gamma,\theta,j,k}(\partial)E](X-Y)\Big) f_\theta(Y)\,d\sigma(Y)$$
$$+ K_\Delta f_\gamma(X) + \textit{ lower order terms}, \tag{4.179}$$

where each $p_{\gamma,\theta,j,k}(\partial)$ is a differential operator of order $2m-2$.

Proof. Fix $\gamma\in\mathbb{N}_0^n$ such that $|\gamma|\le m-1$. A simple analysis of (4.161) reveals that for (4.179) it suffices to show that

$$-\sum_{\substack{|\alpha|=m\\|\beta|=m}}\sum_{\ell=|\gamma|+1}^{m}\sum_{\substack{\delta+\eta+e_k=\alpha\\|\delta|=\ell-1,|\eta|=m-\ell}} C_2(m,\ell,\alpha,\delta,\eta)\times \tag{4.180}$$
$$\times\lim_{\varepsilon\to 0^+}\int\limits_{\substack{Y\in\partial\Omega\\|X-Y|>\varepsilon}} \nu_k(Y)(\partial^{\delta+\beta}E)(X-Y)A_{\beta\alpha}f_{\gamma+\eta}(Y)\,d\sigma(Y),$$

can be written as in the right-hand side of (4.179) as the remaining terms in (4.161) are already in this form. As for (4.180), we notice that whenever $\ell\ne m$ in (4.180) one obtains a lower order term. Therefore matters are reduced to showing that if

$$I := -\sum_{\substack{|\alpha|=m\\|\beta|=m}}\sum_{\delta+e_k=\alpha}\frac{\alpha_k}{m}\lim_{\varepsilon\to 0^+}\int\limits_{\substack{Y\in\partial\Omega\\|X-Y|>\varepsilon}} \nu_k(Y)(\partial^{\delta+\beta}E)(X-Y)A_{\beta\alpha}f_\gamma(Y)\,d\sigma(Y) \tag{4.181}$$

then

$$I = \sum_{|\theta|\le m-1}\sum_{j,k=1}^{n}\lim_{\varepsilon\to 0^+}\int\limits_{\substack{Y\in\partial\Omega\\|X-Y|>\varepsilon}} \partial_{\tau_{jk}(Y)}\Big([p_{\gamma,\theta,j,k}(\partial)E](X-Y)\Big) f_\theta(Y)\,d\sigma(Y)$$
$$+K_\Delta f_\gamma(X) + \text{ lower order terms }, \tag{4.182}$$

with $p_{\gamma,\theta,j,k}(\partial)$ as in the hypothesis. Above we have used the fact that

$$C_2(m,m,\alpha,\delta,\emptyset) = \frac{\alpha_k}{m}\quad\text{whenever }\ \delta+e_k=\alpha. \tag{4.183}$$

However, based on the definition of I,

$$I = -\sum_{\substack{|\alpha|=m\\|\beta|=m}}\sum_{k=1}^{n}\frac{\alpha_k}{m}\lim_{\varepsilon\to 0^+}\int\limits_{\substack{Y\in\partial\Omega\\|X-Y|>\varepsilon}} \nu_k(Y)(\partial^{\alpha+\beta-e_k}E)(X-Y)A_{\beta\alpha}f_\gamma(Y)\,d\sigma(Y), \tag{4.184}$$

where we make the convention that $\partial^{\alpha+\beta-e_k} E = 0$ if any of the components of the multi-index $\alpha + \beta - e_k$ is negative.

Consider next a generic homogeneous differential operator $\tilde{L}$ of order $2m - 2$,

$$\tilde{L} := \sum_{|\alpha'|=|\beta'|=m-1} \partial^{\alpha'} B_{\alpha'\beta'} \partial^{\beta'}. \tag{4.185}$$

Then, due to (4.178), we have

$$\sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}\, \partial^{\alpha+\beta} = \sum_{|\alpha'|=|\beta'|=m-1} \sum_{j=1}^{n} B_{\alpha'\beta'} \partial^{\alpha'+\beta'+2e_j}. \tag{4.186}$$

A moment's reflection shows then that, for each $\alpha, \beta \in \mathbb{N}_0^n$, with $|\alpha| = |\beta| = m$, we have

$$A_{\alpha\beta} = \sum_{j=1}^{n} B_{(\alpha-e_j)(\beta-e_j)}, \tag{4.187}$$

where $B_{(\alpha-e_j)(\beta-e_j)} = 0$ whenever either $\alpha - e_i$ or $\beta - e_i$ fails to be a multi-index in $\mathbb{N}_0^n$, otherwise they are as in (4.185). Employing (4.187) in (4.184) we obtain

$$I = - \sum_{\substack{|\alpha|=m \\ |\beta|=m}} \sum_{k=1}^{n} \frac{\alpha_k}{m} \lim_{\varepsilon\to 0^+} \int_{\substack{Y\in\partial\Omega \\ |X-Y|>\varepsilon}} \nu_k(Y) \sum_{j=1}^{n} (\partial^{(\alpha-e_j)+(\beta-e_j)+2e_j-e_k} E)(X-Y) \times$$
$$\times B_{(\beta-e_j)(\alpha-e_i)} f_\gamma(Y)\, d\sigma(Y). \tag{4.188}$$

Writing $\nu_k \partial^{e_j} = \partial_{\tau_{kj}} + \nu_j \partial^{e_k}$ we get

$$I = II + III, \tag{4.189}$$

where

$$II := \sum_{\substack{|\alpha|=m \\ |\beta|=m}} \sum_{k=1}^{n} \frac{\alpha_k}{m} \lim_{\varepsilon\to 0^+} \int_{\substack{Y\in\partial\Omega \\ |X-Y|>\varepsilon}} \partial_{\tau_{kj}(Y)} \sum_{j=1}^{n} (\partial^{(\alpha-e_j)+(\beta-e_j)+e_j-e_k} E)(X-Y) \times$$
$$\times B_{(\beta-e_j)(\alpha-e_j)} f_\gamma(Y)\, d\sigma(Y), \tag{4.190}$$

and

$$III := \sum_{\substack{|\alpha|=m \\ |\beta|=m}} \sum_{k=1}^{n} \frac{\alpha_k}{m} \lim_{\varepsilon\to 0^+} \int\limits_{\substack{Y\in\partial\Omega \\ |X-Y|>\varepsilon}} \sum_{j=1}^{n} \nu_j(Y)\partial_Y^{e_j}(\partial^{(\alpha-e_j)+(\beta-e_j)}E)(X-Y)\times$$
$$\times B_{(\beta-e_j)(\alpha-e_j)} f_\gamma(Y)\,d\sigma(Y). \tag{4.191}$$

Observe next that II is of the form

$$\sum_{|\theta|\le m-1} \sum_{j,k=1}^{n} \lim_{\varepsilon\to 0^+} \int\limits_{\substack{Y\in\partial\Omega \\ |X-Y|>\varepsilon}} \partial_{\tau_{jk}(Y)}\Big([p_{\gamma,\theta,j,k}(\partial)E](X-Y)\Big) f_\theta(Y)\,d\sigma(Y). \tag{4.192}$$

As far as III is concerned, notice first that for a fixed $\alpha\in\mathbb{N}_0^n$, $|\alpha|=m$ we have that $\sum_{k=1}^{m}\frac{\alpha_k}{m}=1$. Also, based on (4.185), for each fixed $j\in\{1,\dots,n\}$ we have

$$\sum_{|\alpha|=|\beta|=m} B_{(\alpha-e_j)(\beta-e_j)}\partial^{\alpha+\beta-2e_j} = \tilde{L}. \tag{4.193}$$

Finally, (4.191) and the preceding observations imply

$$III = \sum_{j=1}^{n} \lim_{\varepsilon\to 0^+} \int\limits_{\substack{Y\in\partial\Omega \\ |X-Y|>\varepsilon}} \nu_j(Y)\partial_Y^{e_j}\Big[(\tilde{L}^t E_{L^t})(X-Y)\Big]^t f_\gamma(Y)\,d\sigma(Y), \tag{4.194}$$

and since $\tilde{L}^t E_{L^t} = \Gamma$ (as $L^t = \Delta\tilde{L}^t$), where Γ is the fundamental solution of the Laplacian, (4.194) readily gives that

$$III = K_\Delta f_\gamma(X). \tag{4.195}$$

This finishes the proof of (4.182) and completes the proof of Theorem 4.16. □

Our next theorem addresses the issue of the boundedness of the principal-value double multi-layer on Whitney–BMO spaces. As a preamble, we shall prove the following useful result.

Proposition 4.17. *Let $\Omega\subset\mathbb{R}^n$ be a bounded Lipschitz domain and consider a (complex) matrix-valued constant coefficient, homogeneous, W-elliptic differential operator L of order $2m$ in $\mathbb{R}^n$. In this context, define $\dot{K}$ as in (4.160)–(4.161). Then*

$$\dot{K}(\mathrm{tr}_{m-1}u) = \tfrac{1}{2}\,\mathrm{tr}_{m-1}u \quad on \;\; \partial\Omega, \quad for\, any \;\; u\in\mathscr{P}_{m-1}. \tag{4.196}$$

Proof. This follows by taking the non-tangential limit in (4.62) and using the jump-formula (4.164). □

The action of the principal-value double multi-layer on Whitney–BMO spaces is considered next. The theorem below extends work done in the case when $n = 2$ and $L = \Delta^2$ in [24] and answers the question posed by J. Cohen at the top of page 111 in [24].

Theorem 4.18. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, L is a (complex) matrix-valued constant coefficient, homogeneous, W-elliptic differential operator L of order $2m$ in $\mathbb{R}^n$, and consider $\dot{K}$ as in (4.160)–(4.161). Then*

$$\dot{K} : \mathrm{B\dot{M}O}_{m-1}(\partial\Omega) \longrightarrow \mathrm{B\dot{M}O}_{m-1}(\partial\Omega), \tag{4.197}$$

is well-defined, linear and bounded.

Proof. The first observation is that, given an arbitrary $\dot{f} \in \mathrm{B\dot{M}O}_{m-1}(\partial\Omega)$, due to (4.163),

$$\dot{K}\dot{f} \in \dot{L}^2_{m-1,0}(\partial\Omega). \tag{4.198}$$

Therefore matters reduce to showing

$$(\dot{K}\dot{f})^{\#} \in L^\infty(\partial\Omega). \tag{4.199}$$

To this end, fix $X_o \in \partial\Omega$, $r > 0$ and $\dot{f} \in \mathrm{B\dot{M}O}_{m-1}(\partial\Omega)$. For each $R > 0$, recall the best fit polynomial array $\dot{\omega}_R(\dot{f})$ introduced as in (4.88). We consider next a function η in $\mathbb{R}^n$ as in (4.130). In particular the following hold

$$\eta \in C^\infty_c(\mathbb{R}^n), \quad 0 \leq \eta \leq 1, \quad \eta \equiv 1 \ \text{ on } \ S_{2r}(X_o), \quad \mathrm{supp}\eta \subset S_{4r}(X_o), \tag{4.200}$$

$$|\partial^\alpha \eta| \leq \frac{C_\alpha}{r^{|\alpha|}}, \qquad \text{for each } \alpha \in \mathbb{N}^n_0. \tag{4.201}$$

We write

$$\dot{f} = \eta(\dot{f} - \dot{\omega}_{2r}) + (1-\eta)(\dot{f} - \dot{\omega}_{2r}) + \dot{\omega}_{2r}, \tag{4.202}$$

where for an array $\dot{g}$, the multiplication $\eta\dot{g}$ is in the sense of (3.309). Using the linearity of the operator $\dot{K}$ on $L^2_{m-1,0}(\partial\Omega)$, formulas (4.202) and (4.196) give

$$\begin{aligned}\big(\tfrac{1}{2}I + \dot{K}\big)\dot{f} &= \big(\tfrac{1}{2}I + \dot{K}\big)\Big(\eta(\dot{f} - \dot{\omega}_{2r})\Big)\\ &\quad + \big(\tfrac{1}{2}I + \dot{K}\big)\Big((1-\eta)(\dot{f} - \dot{\omega}_{2r})\Big) + \tfrac{3}{2}\dot{\omega}_{2r}.\end{aligned} \tag{4.203}$$

Therefore

$$
\begin{aligned}
(\tfrac{1}{2}I + \dot{K})\dot{f} - \tfrac{3}{2}\dot{\omega}_{2r} &= (\tfrac{1}{2}I + \dot{K})\Big(\eta(\dot{f} - \dot{\omega}_{2r})\Big) + (\tfrac{1}{2}I + \dot{K})\Big((1-\eta)(\dot{f} - \dot{\omega}_{2r})\Big) \\
&=: I + II. \qquad (4.204)
\end{aligned}
$$

We claim that there exists $C > 0$, independent of X_o, $\dot{f}$ and $r > 0$ such that

$$
\Big(⨍_{S_r(X_o)} |I(Y)|^2\, d\sigma(Y)\Big)^{1/2} \leq C\, \dot{f}^{\#}(X_o), \qquad (4.205)
$$

and that

$$
\begin{aligned}
&\text{there exists } \dot{P} \in \dot{\mathscr{P}}_{m-1}(\partial\Omega) \text{ with the property that} \\
&\Big(⨍_{S_r(X_o)} |II(Y) - \dot{P}(Y)|^2\, d\sigma(Y)\Big)^{1/2} \leq C\, \dot{f}^{\#}(X_o).
\end{aligned} \qquad (4.206)
$$

We start with showing (4.205) and, in this regard, first notice that

$$
\begin{aligned}
\int_{S_r(X_o)} |I(Y)|^2\, d\sigma(Y) &= \int_{S_r(X_o)} \Big|(\tfrac{1}{2}I + \dot{K})\Big(\eta(Y)(\dot{f}(Y) - \dot{\omega}_{2r}(Y))\Big)\Big|^2\, d\sigma(Y) \\
&\leq \int_{\partial\Omega} \Big|(\tfrac{1}{2}I + \dot{K})\Big(\eta(Y)(\dot{f}(Y) - \dot{\omega}_{2r}(Y))\Big)\Big|^2\, d\sigma(Y).
\end{aligned} \qquad (4.207)
$$

Based on Theorem 4.14, the operator $\frac{1}{2}I + \dot{K}$ is bounded on $L^2_{m-1,0}(\partial\Omega)$, and together with (4.207), this implies that there exists a finite constant $C > 0$ such that

$$
\begin{aligned}
&\int_{S_r(X_o)} |I(Y)|^2\, d\sigma(Y) \qquad (4.208) \\
&\quad \leq C \int_{\partial\Omega} |\eta(Y)(\dot{f}(Y) - \dot{\omega}_{2r}(Y))|^2\, d\sigma(Y) \\
&\quad \leq C \sum_{|\alpha| \leq m-1} \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \int_{S_{2r}(X_o)} |\partial^\beta \eta(Y)|^2 \big|\big(\dot{f}(Y) - \dot{\omega}_{2r}(Y)\big)_\gamma\big|^2\, d\sigma(Y).
\end{aligned}
$$

The last identity above follows from (3.309) and the Leibniz rule of differentiation. Employing next (4.201), we have $|\partial^\beta \eta(Y)| \leq Cr^{-|\beta|}$ and hence

$$\int_{S_r(X_o)} |I(Y)|^2\,d\sigma(Y) \le C \sum_{|\alpha|\le m-1}\sum_{\beta+\gamma=\alpha} \frac{1}{r^{2|\beta|}} \int_{S_{2r}(X_o)} |(\dot f(Y) - \dot\omega_{2r}(Y))_\gamma|^2\,d\sigma(Y). \tag{4.209}$$

Due to (4.100) from Lemma 4.10 the above further gives

$$\begin{aligned}\int_{S_r(X_o)} |I(Y)|^2\,d\sigma(Y) &\le C \sum_{|\alpha|\le m-1}\sum_{\beta+\gamma=\alpha} \frac{r^{2(m-1-|\gamma|)}}{r^{2|\beta|}} \Big(\dot f^{\#}(X_o)\Big)^2 \cdot |S_{2r}(X_o)| \\ &\le C \sum_{|\alpha|\le m-1} r^{2(m-1-|\alpha|)} \Big(\dot f^{\#}(X_o)\Big)^2 \cdot |S_r(X_o)|, \end{aligned}\tag{4.210}$$

where in the last inequality in (4.210) we have used that $|\beta| + |\gamma| = |\alpha|$ as well as $|S_r(X_o)| \approx |S_{2r}(X_o)|$. Finally (4.205) now readily follows from (4.210) and the fact that $r^{2(m-1-|\alpha|)} \le C = C(\partial\Omega)$, since $|\alpha| \le m-1$.

We shall focus next on proving (4.206). For each array $\dot g \in L^2_{m-1,0}(\partial\Omega)$, $\dot g = \{g_\gamma\}_{|\gamma|\le m-1}$ and $\alpha \in \mathbb{N}_0^n$, multi-index of length $\le m-1$, we introduce

$$P_\alpha(\dot g; X, X_o) := \sum_{|\beta|\le m-1-|\alpha|} \frac{1}{\beta!} g_{\alpha+\beta}(X_o)(X-X_o)^\beta, \quad X \in \partial\Omega, \tag{4.211}$$

and

$$R_\alpha(\dot g; X, X_o) := g_\alpha(X) - P_\alpha(\dot g; X, X_o), \quad X \in \partial\Omega. \tag{4.212}$$

Going further, we set

$$\begin{aligned}\dot P(\dot g; X, X_o) &:= \Big(P_\alpha(\dot g; X, X_o)\Big)_{|\alpha|\le m-1} \quad \text{and} \\ \dot R(\dot g; X, X_o) &:= \Big(R_\alpha(\dot g; X, X_o)\Big)_{|\alpha|\le m-1},\end{aligned}\tag{4.213}$$

and straightforward algebraic manipulations show that $\dot P(\dot g; X, X_o)$ satisfies the compatibility conditions (3.2). This and degree considerations guarantee that the array $\dot P(\dot g; X, X_o) \in \dot{\mathscr{P}}_{m-1}(\partial\Omega)$, where as before the latter stands for the set of polynomial arrays of degree less than or equal to $m-1$. In turn, based on (4.212), $\dot R(\dot g; X, X_o)$ also satisfies the compatibility conditions (3.2). In the notation we just introduced,

$$\begin{aligned}(\tfrac12 I + \dot K)\Big((1-\eta)(\dot f - \dot\omega_{2r})\Big) &= \dot P\left((\tfrac12 I + \dot K)\big((1-\eta)(\dot f - \dot\omega_{2r})\big); X, X_o\right) \\ &\quad + \dot R\left((\tfrac12 I + \dot K)\big((1-\eta)(\dot f - \dot\omega_{2r})\big); X, X_o\right),\end{aligned}\tag{4.214}$$

and we claim that there exists $C > 0$ (independent of $\dot{f}$, X_o, $r > 0$) such that for each multi-index $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq m-1$, we have

$$\left| R_\alpha \left((\tfrac{1}{2} I + \dot{K}) \big((1-\eta)(\dot{f} - \dot{\omega}_{2r}) \big); X, X_o \right) \right| \leq C \dot{f}^{\#}(X_o). \tag{4.215}$$

The starting point for proving (4.215) is the observation that, based on (4.212) and the jump relations (4.164), for a given $\dot{g} \in L^2_{m-1,0}(\partial\Omega)$ and for almost every point $X \in \partial\Omega$, we have

$$\begin{aligned} & R_\alpha \left((\tfrac{1}{2} I + \dot{K}) \dot{g}; X, X_o \right) \\ & \quad = \mathrm{Tr}\Big[\text{Taylor remainder of order } m-1-|\alpha| \text{ of } \partial^\alpha \dot{\mathscr{D}} \dot{g} \text{ at } X_o \Big](X). \end{aligned} \tag{4.216}$$

Next, if $\dot{g}$ vanishes near X_o, the Taylor formula implies

$$\begin{aligned} & \left| R_\alpha \left((\tfrac{1}{2} I + \dot{K}) \dot{g}; X, X_o \right) \right| \\ & \quad \leq C \sum_{|\gamma| = m - |\alpha|} \int_0^1 |X - X_o|^{m-|\alpha|} \left| \left(\partial^{\gamma+\alpha} \dot{\mathscr{D}} \right) \dot{g}(X_o + t(X - X_o)) \right| dt. \end{aligned} \tag{4.217}$$

It is important to point out that Taylor's formula require that the function $\dot{\mathscr{D}}\dot{g}$ be sufficiently smooth in a neighborhood of X_o. In general, $\dot{\mathscr{D}}\dot{g}$ is not even continuous near X_o, however, it is so if $\dot{g}$ vanishes near X_o. Now, based on (4.57), (4.144) and the support condition on η,

$$\begin{aligned} & \left| \left(\partial^{\gamma+\alpha} \dot{\mathscr{D}} \right) \dot{g}(X_o + t(X - X_o)) \right| \\ & \quad \leq C \int_{\partial\Omega \setminus S_{2r}(X_o)} \frac{1}{|Y - (X_o + t(X - X_o))|^n} \sum_{|\delta| \leq m-1} |\dot{g}_\delta(Y)| \, d\sigma(Y). \end{aligned} \tag{4.218}$$

Furthermore, a simple inspection reveals that there exist $C_j > 0$, $j = 1, 2$, independent of r, X_o and $X \in S_r(X_o)$, such that for each $X \in S_r(X_o)$ the following holds

$$C_1 |Y - (X_o + t(X - X_o))| \leq |X_o - Y| \leq C_2 |Y - (X_o + t(X - X_o))|. \tag{4.219}$$

Collectively, (4.217)–(4.219) further imply

$$\left| R_\alpha \left((\tfrac{1}{2} I + \dot{K}) \dot{g}; X, X_o \right) \right| \leq C \int_{\partial\Omega \setminus S_{2r}(X_o)} \frac{r^{m-|\alpha|}}{|Y - X_o|^n} \sum_{|\delta| \leq m-1} |g_\delta(Y)| \, d\sigma(Y), \tag{4.220}$$

whenever $X \in S_r(X_o)$. However, proceeding as in the proof of Theorem 4.11 when having to estimate the right hand side of (4.137), we have

$$\int_{\partial\Omega\setminus S_{2r}(X_o)} \frac{1}{|Y - X_o|^n} \sum_{|\delta|\leq m-1} |(1-\eta(Y))(\dot{f}(Y)-\dot{\omega}_{2r}(Y))_\delta|\, d\sigma(Y) \leq \frac{C}{r} \dot{f}^{\#}(X_o). \tag{4.221}$$

Above we considered $\dot{g} = (1 - \eta)(\dot{f} - \dot{\omega}_{2r})$ which vanishes in a neighborhood of X_o due to the fact that $\eta \equiv 1$ on $S_{2r}(X_o)$. Then, from (4.220) and (4.221) we deduce

$$\begin{aligned} \Big| R_\alpha \Big((\tfrac{1}{2}I + \dot{K})\big((1-\eta)(\dot{f} - \dot{\omega}_{2r})\big); X, X_o \Big) \Big| &\leq C r^{m-1-|\alpha|} \dot{f}^{\#}(X_o) \\ &\leq C \dot{f}^{\#}(X_o). \end{aligned} \tag{4.222}$$

That (4.206) holds can now be justified by invoking (4.214) and (4.222). Finally, (4.204)–(4.206) give that there exists $\dot{P} \in \dot{\mathscr{P}}_{m-1}(\partial\Omega)$ such that

$$\left(⨍_{S_r(X_o)} \Big| (\tfrac{1}{2}I + \dot{K})\dot{f}(Y) - \dot{P}(Y) \Big|^2 d\sigma(Y) \right)^{1/2} \leq C \dot{f}^{\#}(X_o), \tag{4.223}$$

from which (4.199) readily follows. This finishes the proof of Theorem 4.18. □

4.5 Estimates on Besov, Triebel–Lizorkin, and Weighted Sobolev Spaces

The aim of this section is to study the action of the multi-layer operators $\dot{\mathscr{D}}$ and $\dot{K}$, introduced in Definition 4.4 and Definition 4.13, respectively, on Besov and Triebel–Lizorkin scales in Lipschitz domains.

Theorem 4.19. *Suppose that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, and assume that L is a W-elliptic homogeneous differential operator of order $2m$ with (complex) matrix-valued constant coefficients. Then if the numbers p, q, s satisfy the inequalities $\frac{n-1}{n} < p \leq \infty$, $0 < q \leq \infty$, $(n-1)(\frac{1}{p} - 1)_+ < s < 1$, the operator*

$$\dot{\mathscr{D}} : \dot{B}^{p,q}_{m-1,s}(\partial\Omega) \longrightarrow B^{p,q}_{m-1+s+1/p}(\Omega) \tag{4.224}$$

is well-defined, linear and bounded. Moreover,

$$\dot{\mathscr{D}} : \dot{B}^{p,p}_{m-1,s}(\partial\Omega) \longrightarrow F^{p,q}_{m-1+s+1/p}(\Omega) \tag{4.225}$$

is also well-defined, linear and bounded (with the additional convention that $q = \infty$ if $p = \infty$).

Finally, similar properties hold for $\psi\dot{\mathscr{D}}^-$ *(cf. the convention* (4.162)*), for any cutoff function* $\psi \in C_c^\infty(\mathbb{R}^n)$*.*

Proof. Consider the task of proving that the operator (4.224) is well-defined and bounded when $p, q \neq \infty$. In this case, fix $\dot{f} = \{f_\delta\}_{|\delta|\leq m-1} \in \dot{B}^{p,q}_{m-1,s}(\partial\Omega)$ and note that, by (3.224), it suffices to show that

$$\partial^\alpha \dot{\mathscr{D}} \dot{f} \in B^{p,q}_{-1+s+1/p}(\Omega), \qquad \forall\, \alpha \in \mathbb{N}_0^n, \;\; |\alpha| \leq m, \tag{4.226}$$

with appropriate control of the norm. Let us indicate how this is done in the case when $|\alpha| = m$. Pick $i \in \{1, \dots, n\}$ and write $\alpha = e_i + \gamma$ where γ has length $m-1$. In particular, the discussion in the next couple of paragraphs following (4.80) shows that $\partial^\alpha(\dot{\mathscr{D}}\dot{f})(X)$ can be written in the form

$$\sum_{|\beta|=2m-1} \sum_{|\delta|\leq m-1} \sum_{j,k=1}^{n} \int_{\partial\Omega} (\partial^\beta E)(X-Y) C_{\beta,\delta,j,k} (\partial_{\tau_{jk}} f_\delta)(Y)\, d\sigma(Y), \tag{4.227}$$

where $C_{\beta,\delta,j,k}$ are suitable matrices. Since Proposition 2.58 gives $\partial_{\tau_{jk}} f_\delta \in B^{p,q}_{s-1}(\partial\Omega)$, for every δ, j, k, it suffices to show that the assignment

$$B^{p,q}_{s-1}(\partial\Omega) \ni f \mapsto \nabla R_{\beta'} f \in B^{p,q}_{-1+s+1/p}(\Omega) \tag{4.228}$$

is bounded, where for each $\beta' \in \mathbb{N}_0^n$ of length $2m-2$ we have set

$$R_{\beta'} f(X) := \int_{\partial\Omega} k(X-Y) f(Y)\, d\sigma(Y), \quad X \in \Omega. \tag{4.229}$$

where $k(X)$ is a generic entry in $(\partial^{\beta'} E)(X-Y)$. We shall show that, whenever $|\beta'| = 2m-2$,

$$R_{\beta'} : B^{p,q}_{s-1}(\partial\Omega) \longrightarrow B^{p,q}_{s+1/p}(\Omega) \quad \text{boundedly}, \tag{4.230}$$

from which (4.228) follows.

In turn, via real interpolation, (4.230) will be a consequence of the fact that

$$R_{\beta'} : B^{p,p}_{s-1}(\partial\Omega) \longrightarrow B^{p,p}_{s+1/p}(\Omega) \quad \text{boundedly}, \tag{4.231}$$

for each multi-index $\beta' \in \mathbb{N}_0^N$ with $|\beta'| = 2m-2$, if $\frac{n-1}{n} < p \leq \infty$, as well as $(n-1)(\frac{1}{p}-1)_+ < s < 1$. Finally, given that $L(R_{\beta'} f) = 0$ in Ω, the claim (4.231) follows from Proposition 2.64 and Theorem 2.41. This finishes the treatment of the case when $|\alpha| = m$. The case when $|\alpha| \leq m-1$ is similar and simpler, completing the proof of (4.224) in the case when $p, q \neq \infty$.

The case when $p \neq \infty$ and $0 < q \leq \infty$ can then be covered from what we have proved so far and real interpolation; cf. (3.314) and (3.317). By the same token, matters can be reduced to considering the case when $p = q = \infty$. To this end, let us fix $\dot{f} = \{f_\eta\}_{|\eta|\leq m-1} \in \dot{B}^{\infty,\infty}_{m-1,s}(\partial\Omega)$ of norm one, and abbreviate (4.4) as

$$\dot{\mathscr{D}}\dot{f}(X) = \sum_{\substack{|\alpha|=|\beta|=m\\ \gamma+\eta+e_j=\alpha}} C^{\alpha,\beta}_{\gamma,\eta,j} \int_{\partial\Omega} \nu_j(Y)(\partial^{\beta+\gamma}E)(X-Y)A_{\beta\alpha} f_\eta(Y)\,d\sigma(Y), \quad X \in \Omega, \tag{4.232}$$

where the $C^{\alpha,\beta}_{\gamma,\eta,j}$'s are real constants. For each multi-index $\gamma \in \mathbb{N}_0^n$ of length $\leq m-1$, consider next, in analogy with (3.78) and (3.73),

$$P_\gamma(X,Y) := \sum_{|\delta|\leq m-1-|\gamma|} \frac{1}{\delta!} f_{\delta+\gamma}(Y)(X-Y)^\delta, \quad X \in \mathbb{R}^n, \ \ Y \in \partial\Omega, \tag{4.233}$$

$$R_\gamma(X,Y) := f_\gamma(X) - P_\gamma(X,Y), \qquad X, Y \in \partial\Omega. \tag{4.234}$$

For ease of reference, for each $Y \in \partial\Omega$ let us also set

$$\dot{P}(\cdot,Y) := \{P_\gamma(\cdot,Y)\}_{|\gamma|\leq m-1}, \qquad \dot{R}(\cdot,Y) := \{R_\gamma(\cdot,Y)\}_{|\gamma|\leq m-1}. \tag{4.235}$$

A direct calculation shows that $\dot{P}(\cdot,Y) \in CC$ for every $Y \in \partial\Omega$ (cf. (3.81), or the discussion on p. 177 in [119]). Since $f_\gamma(X) = P_\gamma(X,Z) + R_\gamma(X,Z)$ if $|\gamma| \leq m-1$ and $X, Z \in \partial\Omega$, we have

$$\dot{f} = \dot{P}(\cdot,Z) + \dot{R}(\cdot,Z), \quad \forall\, Z \in \partial\Omega. \tag{4.236}$$

In particular, $\dot{R}(\cdot,Y) \in CC$ for every $Y \in \partial\Omega$. Furthermore, since $\dot{P}(\cdot,Z) =$ has polynomial components of degree $\leq m-1$, it follows from (4.236) and the reproducing property (4.62) that

$$\begin{aligned} \dot{\mathscr{D}}\dot{f}(X) &= \dot{\mathscr{D}}(\dot{P}(\cdot,Z))(X) + \dot{\mathscr{D}}(\dot{R}(\cdot,Z))(X) \\ &= P_{(0,\dots,0)}(X,Z) + \dot{\mathscr{D}}(\dot{R}(\cdot,Z))(X) \end{aligned} \tag{4.237}$$

for every $X \in \Omega$ and $Z \in \partial\Omega$. Consequently, for each multi-index $\theta \in \mathbb{N}_0^n$ with $|\theta| = m$, we arrive at the identity

$$\begin{aligned} \partial^\theta \dot{\mathscr{D}}\dot{f}(X) &= \partial^\theta \dot{\mathscr{D}}(\dot{R}(\cdot,Z))(X) \\ &= \sum_{\substack{|\alpha|=|\beta|=m\\ \gamma+\eta+e_j=\alpha}} C^{\alpha,\beta}_{\gamma,\eta,j} \int_{\partial\Omega} \nu_j(Y)(\partial^{\beta+\gamma+\theta}E)(X-Y)A_{\beta\alpha} R_\eta(Y,Z)\,d\sigma(Y), \end{aligned} \tag{4.238}$$

valid for each $X \in \Omega$ and $Z \in \partial\Omega$. Next, given a point $X \in \Omega$, we specialize (4.238) by choosing $Z := \pi(X)$, where $\pi : \Omega \to \partial\Omega$ is a mapping chosen with the property that $\operatorname{dist}(X, \partial\Omega) \approx |X - \pi(X)|$, uniformly for $X \in \Omega$. In turn, for each $\theta \in \mathbb{N}_0^n$ with $|\theta| = m$, this and (4.29) entail

$$|\partial^\theta \dot{\mathscr{D}} \dot{f}(X)| \leq C \sum_{|\eta| \leq m-1} \int_{\partial\Omega} \frac{1}{|X - Y|^{n-1+m-|\eta|}} |R_\eta(Y, \pi(X))| \, d\sigma(Y), \tag{4.239}$$

for every $X \in \Omega$. We may therefore estimate

$$\begin{aligned}
\rho^{1-s}(X)|\partial^\theta \dot{\mathscr{D}} \dot{f}(X)| &\leq C \sum_{|\eta| \leq m-1} \int_{\partial\Omega} \frac{\rho(X)^{1-s}}{|X - Y|^{n-1+m-|\eta|}} |R_\eta(Y, \pi(X))| \, d\sigma(Y) \\
&\leq C \sum_{|\eta| \leq m-1} \int_{\partial\Omega} \frac{\rho(X)^{1-s}}{|X - Y|^{n-1+m-|\eta|}|Y - \pi(X)|^{m-1-|\eta|+s}} \, d\sigma(Y) \\
&= C \sum_{|\eta| \leq m-1} \int_{|Y-\pi(X)|>c\rho(X)} \cdots + C \sum_{|\eta| \leq m-1} \int_{|Y-\pi(X)|<c\rho(X)} \cdots \\
&=: I_1 + I_2,
\end{aligned} \tag{4.240}$$

where the second inequality above utilizes (3.75). Localizing and passing from graph to local (Euclidean) coordinates, we see that

$$\begin{aligned}
I_1 &\leq C \sum_{|\eta| \leq m-1} \int\limits_{\substack{|y-x|>ct \\ y \in \mathbb{R}^{n-1}}} \frac{t^{1-s}}{[t + |x - y|]^{n-1+m-|\eta|}} \cdot |x - y|^{m-1-|\eta|+s} \, dy \\
&\leq C \sum_{|\eta| \leq m-1} \frac{t^{1-s}}{t^{n-1+m-|\eta|}} \cdot t^{m-1-|\eta|+s} \cdot t^{n-1} \cdot \int\limits_{\substack{|z|\geq c \\ z \in \mathbb{R}^{n-1}}} \frac{|z|^{m-1-|\eta|+s}}{[1 + |z|]^{n-1+m-|\eta|}} \, dz \leq C,
\end{aligned} \tag{4.241}$$

where in the second step we have changed variables twice (first from $y - x$ to y, then from y to tz). In a similar fashion,

$$\begin{aligned}
I_2 &\leq C \sum_{|\eta| \leq m-1} \int\limits_{|y-x|<ct} \frac{t^{-s} \cdot |y - x|}{|x - y|^{n-1+m-|\eta|}} |x - y|^{m-1-|\eta|+s} \, dy \\
&\leq C \sum_{|\eta| \leq m-1} \frac{t^{-s}}{t^{n-1-s}} \cdot t^{n-1} \cdot \int\limits_{\substack{|z|\leq c \\ z \in \mathbb{R}^{n-1}}} \frac{dz}{|z|^{n-1-s}} \leq C.
\end{aligned} \tag{4.242}$$

Altogether, (4.240)–(4.242) show that there exists $C = C(\Omega, L, s)$ such that

$$\sup_{X\in\Omega}\Big[\rho(X)^{1-s}|\nabla^m \dot{\mathscr{D}}\dot{f}(X)|\Big] \le C\|\dot{f}\|_{\dot{B}^{\infty,\infty}_{m-1,s}(\partial\Omega)}. \tag{4.243}$$

Next, we remark that for each $0 < s < 1$,

$$\dot{B}^{\infty,\infty}_{m-1,s}(\partial\Omega) \hookrightarrow \dot{B}^{p,p}_{m-1,r}(\partial\Omega) \xrightarrow{\nabla^j \dot{\mathscr{D}}} B^{p,p}_{m-1-j+r+1/p}(\Omega) \hookrightarrow L^\infty(\Omega), \tag{4.244}$$

whenever $j \in \{1,\dots,m-1\}$, $0 < r < s$ and $\frac{n-1}{r} < p < \infty$ Indeed, the first inclusion in (4.244) is a consequence of Proposition 2.51, the boundedness of the second arrow is based (4.224), and the last inclusion is a standard embedding result. Consequently, there exists $C > 0$ such that for each $\dot{f} \in \dot{B}^{\infty,\infty}_{m-1,s}(\partial\Omega)$,

$$\sup_{X\in\Omega}\Big[\sum_{j=0}^{m-1}|\nabla^j \dot{\mathscr{D}}\dot{f}(X)|\Big] \le C\|\dot{f}\|_{\dot{B}^{\infty,\infty}_{m-1,s}(\partial\Omega)}. \tag{4.245}$$

Together, estimates (4.243), (4.245) and (2.227) give that $\nabla^{m-1}\dot{\mathscr{D}}\dot{f} \in C^s(\Omega)$. Hence $\dot{\mathscr{D}}\dot{f} \in C^{m-1+s}(\Omega)$ which, by (2.185), gives $\dot{\mathscr{D}}\dot{f} \in B^{\infty,\infty}_{m-1+s}(\Omega)$, as desired.

Moving on, (4.225) is a consequence of (4.224) with $p = q$, the fact that the diagonal of the Besov scale coincides with the diagonal of the Triebel–Lizorkin scale, and (2.265) from Theorem 2.41.

Consider now the last claim in the statement of Theorem 4.19. Given a function $\psi \in C^\infty_c(\mathbb{R}^n)$, pick $R > 0$ large enough so that $B_R(0)$ contains $\bar{\Omega} \cup \operatorname{supp}\psi$. Set now $D_R := B_R(0) \setminus \overline{\Omega}$ and denote by $\dot{\mathscr{D}}_R$ the double multi-layer associated with the bounded Lipschitz domain $D_R \subset \mathbb{R}^n$. Finally, let

$$\iota_R : \dot{B}^{p,q}_{m-1,s}(\partial\Omega) \to \dot{B}^{p,q}_{m-1,s}(\partial\Omega) \oplus \dot{B}^{p,q}_{m-1,s}(\partial B_R(0)) \equiv \dot{B}^{p,q}_{m-1,s}(\partial D_R),$$

$$\pi_R : \dot{B}^{p,q}_{m-1,s}(\partial D_R) \equiv \dot{B}^{p,q}_{m-1,s}(\partial\Omega) \oplus \dot{B}^{p,q}_{m-1,s}(\partial B_R(0)) \to \dot{B}^{p,q}_{m-1,s}(\partial\Omega), \tag{4.246}$$

act according to $\iota_R(\dot{f}) := (\dot{f}, \dot{0})$ and $\pi_R(\dot{f}, \dot{g}) := \dot{f}$, for every $\dot{f}$.

Based on definitions, it is straightforward to check that, when acting on Whitney–Besov spaces,

$$\psi\dot{\mathscr{D}}^- = -(\psi\dot{\mathscr{D}}_R)\circ\iota_R \quad \text{in } D_R. \tag{4.247}$$

With this in hand, the desired conclusions about $\psi\dot{\mathscr{D}}^-$ follow from what we have established in the first part of the proof. □

In the following result we establish mapping properties for the double multi-layer acting on Whitney–Besov spaces and taking values in the weighted Sobolev spaces introduced in § 2.5.

Theorem 4.20. *Retain the same background hypotheses as in Theorem 4.19. In addition, assume that $1 < p < \infty$, $0 < s < 1$, and set $a := 1 - s - 1/p \in (-1/p, 1 - 1/p)$. Then the operator*

$$\dot{\mathscr{D}} : \dot{B}^{p,q}_{m-1,s}(\partial\Omega) \longrightarrow W^{m,p}_{a}(\Omega) \tag{4.248}$$

is well-defined, linear and bounded. Moreover, a similar boundedness result holds for $\psi\dot{\mathscr{D}}^-$ (cf. the convention (4.162)), for any cutoff function $\psi \in C^\infty_c(\mathbb{R}^n)$.

Proof. This is a consequence of Proposition 2.64 (used with $k = 1$) and the fact that for each $\dot{f} = \{f_\delta\}_{|\delta|\leq m-1} \in \dot{B}^{p,q}_{m-1,s}(\partial\Omega)$ and each $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \leq m$ we may represent $\partial^\alpha \dot{\mathscr{D}} \dot{f}$ as a finite linear combination of terms of the form (4.227). □

Our next result deals with the mapping properties of the boundary multiple layer $\dot{K}$ and its relation with $\dot{\mathscr{D}}$, when considered on Whitney–Besov spaces. Recall (4.162).

Theorem 4.21. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, and assume that L is a W-elliptic homogeneous differential operator of order $2m$ with (complex) matrix-valued constant coefficients.*

Then the boundary layer potential operator $\dot{K}$, originally introduced in Definition 4.13, extends to a bounded mapping

$$\dot{K} : \dot{B}^{p,q}_{m-1,s}(\partial\Omega) \longrightarrow \dot{B}^{p,q}_{m-1,s}(\partial\Omega) \tag{4.249}$$

whenever $\frac{n-1}{n} < p \leq \infty$, $(n-1)\big(1/p - 1\big)_+ < s < 1$, and $0 < q \leq \infty$. In addition, with the same assumptions on p, q and s,

$$\mathrm{tr}_{m-1} \circ \dot{\mathscr{D}}^{\pm} = \pm\tfrac{1}{2} I + \dot{K} \quad \text{in} \quad \dot{B}^{p,q}_{m-1,s}(\partial\Omega). \tag{4.250}$$

Proof. Consider first the case when $1 < p = q < \infty$, and $0 < s < 1$. The fact that the operator (4.249) is bounded in this situation follows by interpolating by the real method (cf. (3.319) in Theorem 3.39 between (4.163) and (4.165) in Theorem 4.14). Furthermore, the jump formula (4.250) holds since the two operators involved are bounded on $\dot{B}^{p,p}_{m-1,s}(\partial\Omega)$ and coincide on $\dot{L}^p_{m-1,1}(\partial\Omega)$ which, under the current assumptions on the indices, is densely embedded into $\dot{B}^{p,p}_{m-1,s}(\partial\Omega)$, thanks to (3.118). Once (4.249)–(4.250) have been dealt with in the range of indices $1 < p = q < \infty$, $0 < s < 1$, their validity can be further extended to the case when $\frac{n-1}{n} < p = q < \infty$ and $(n-1)(1/p - 1)_+ < s < 1$ by arguing as follows. First, for each such indices p, s, it is possible to find indices $p^* \in (1, \infty)$ and $s^* \in (0, 1)$ with the property that $\dot{B}^{p,p}_{m-1,s}(\partial\Omega) \hookrightarrow \dot{B}^{p^*,p^*}_{m-1,s^*}(\partial\Omega)$ in a bounded fashion. As a result, for each $\dot{f} \in \dot{B}^{p,p}_{m-1,s}(\partial\Omega)$ we may write

$$\left\|(\pm\tfrac{1}{2}I + \dot{K})\dot{f}\right\|_{\dot{B}^{p,p}_{m-1,s}(\partial\Omega)} = \left\|\mathrm{tr}_{m-1}\dot{\mathscr{D}}^{\pm}\dot{f}\right\|_{\dot{B}^{p,p}_{m-1,s}(\partial\Omega)} \leq C\|\dot{f}\|_{\dot{B}^{p,p}_{m-1,s}(\partial\Omega)}, \tag{4.251}$$

where $C > 0$ is a constant that does not depend on $\dot{f}$. The identity in (4.251) follows from the fact that $\dot{f} \in \dot{B}^{p^*,q^*}_{m-1,s^*}(\partial\Omega)$ while the inequality follows from the fact that $\dot{f}$ belongs to $\dot{B}^{p,q}_{m-1,s}(\partial\Omega)$, Theorems 4.19 and 3.9.

Going further, the range $\frac{n-1}{n} < p = q < \infty$ and $(n-1)\big(\frac{1}{p} - 1\big)_+ < s < 1$, where (4.249)–(4.250) are valid, can be extended to $\frac{n-1}{n} < p < \infty$, $0 < q \leq \infty$ and $(n-1)\big(\frac{1}{p}-1\big)_+ < s < 1$, by invoking Theorem 3.39. In fact, by employing the same interpolation result, there only remains to deal with the case when $p = q = \infty$ and $0 < s < 1$.

However, if $\dot{f} \in \dot{B}^{\infty,\infty}_{m-1,s}(\partial\Omega)$ then $\dot{f} \in \dot{L}^{p}_{m-1,0}(\partial\Omega)$ for any $p \in (1, \infty)$. Consequently,

$$(\pm\tfrac{1}{2}I + \dot{K})\dot{f} = \dot{\mathscr{D}}^{\pm}\dot{f}\Big|^{m-1}_{\partial\Omega} = \mathrm{tr}_{m-1}\dot{\mathscr{D}}^{\pm}\dot{f}, \tag{4.252}$$

by (4.164), the fact that $\dot{\mathscr{D}} : \dot{B}^{\infty,\infty}_{m-1,s}(\partial\Omega) \to C^{m-1+s}(\bar{\Omega})$ boundedly (plus a similar statement for Ω_-), and (3.102) with $p = q = \infty$. This concludes the proof of the theorem. □

For further reference, let us also record here the following useful result.

Corollary 4.22. *In the same setting as in Theorem 4.21, the adjoint of the boundary layer potential operator $\dot{K}$ extends to a bounded mapping*

$$\dot{K}^* : \Big(\dot{B}^{p,q}_{m-1,s}(\partial\Omega)\Big)^* \longrightarrow \Big(\dot{B}^{p,q}_{m-1,s}(\partial\Omega)\Big)^*, \tag{4.253}$$

whenever $1 \leq p, q \leq \infty$ *and* $0 < s < 1$.

Proof. This is an immediate consequence of Theorem 4.21 and duality. □

For later use, let us point out that, by taking boundary traces in (4.247) and duality, we obtain

$$\iota_R \circ \dot{K} = -\dot{K}_R \circ \iota_R \quad \text{and} \quad \dot{K}^* \circ \iota_R^* = -\iota_R^* \circ \dot{K}_R^*, \tag{4.254}$$

where $\dot{K}_R$ is the boundary-to-boundary version of the double multi-layer integral operator for the domain $D_R = B_R(0) \setminus \overline{\Omega}$.

Our next result deals with the mapping properties of the operator $\dot{K}$ on the space $\mathrm{V\dot{M}O}_{m-1}(\partial\Omega)$. Specifically we have

Theorem 4.23. *Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, and that L is a W-elliptic homogeneous differential operator of order $2m$ with (complex) matrix-valued constant coefficients. Then the boundary layer potential operator $\dot{K}$, originally introduced in Definition 4.13, induces a mapping*

$$\dot{K} : \mathrm{V\dot{M}O}_{m-1}(\partial\Omega) \longrightarrow \mathrm{V\dot{M}O}_{m-1}(\partial\Omega) \tag{4.255}$$

which is well-defined, linear and bounded.

Proof. This is an immediate consequence of Theorems 4.21 and 4.18 and the definition of the space $\mathrm{V\dot{M}O}_{m-1}(\partial\Omega)$ given in (3.413). □

The last result in this section deals with the limiting cases $s = 0$ and $s = 1$ in (4.224), (4.225). To state it, recall that $p \vee 2 = \max\{p, 2\}$.

Theorem 4.24. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, and suppose that L is a W-elliptic homogeneous differential operator of order $2m$ with (complex) matrix-valued constant coefficients. Then the operators*

$$\dot{\mathscr{D}} : \dot{L}^p_{m-1,s}(\partial\Omega) \longrightarrow B^{p,p\vee 2}_{m-1+s+1/p}(\Omega), \qquad 1 < p < \infty, \tag{4.256}$$

$$\dot{\mathscr{D}} : \dot{L}^p_{m-1,s}(\partial\Omega) \longrightarrow F^{p,q}_{m-1+s+1/p}(\Omega), \quad 2 \leq p < \infty, \;\; 0 < q \leq \infty, \tag{4.257}$$

are well-defined, linear and bounded if either $s = 0$, or $s = 1$.

Proof. Consider first (4.256) in which case, by Proposition 2.24, it suffices to show that $\nabla^{m-1+s}\dot{\mathscr{D}}\dot{f} \in B^{p,p\vee 2}_{1/p}(\Omega)$ for each $\dot{f} \in \dot{L}^p_{m-1,s}(\partial\Omega)$ (assuming that $s \in \{0, 1\}$). This, however, is a consequence of (4.80)–(4.83) and Proposition 2.68.

Next, the claim about (4.257) is a consequence of what we have just proved, the fact that $B^{p,p\vee 2}_{m-1+s+1/p}(\Omega) \hookrightarrow F^{p,p}_{m-1+s+1/p}(\Omega)$ if $2 \leq p < \infty$, and the last part in Theorem 2.41. □

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