Preface

This book is devoted to the study of the main concepts on survival for killed Markov processes and dynamical systems. The first computations on the number of survivals after a long time in branching process was done by Kolmogorov (1938). Later on Yaglom (1947) showed that the limit behavior of sub-critical branching processes conditioned to survival was given by a proper distribution. This pioneer work triggered an important activity in this field.

Quasi-stationary distributions (QSD) capture the long term behavior of a process that will be surely killed when this process is conditioned to survive. A basic and useful property is that the time of killing is exponentially distributed when starting from a QSD, which implies that the rate of survival of a process must be at most exponential in order that QSD can exist.

The study of QSD on finite state irreducible Markov chains started with the pioneering work Darroch and Seneta (1965). For these processes many of the fundamental ideas in the topic can be easily developed. In particular, the Perron–Frobenius theory for finite positive matrices gives all the required information. Thus, the exponential rate of survival is the Perron–Frobenius eigenvalue. The QSD is the Perron–Frobenius normalized eigenmeasure, and the chain of trajectories that are never killed (Q-process) is governed by an h-process where h is the Perron–Frobenius eigenfunction. This relation between a QSD and the kernel of the Q-process is also encountered in many other processes. In the finite case, the fact that the eigenvalues are isolated simplifies the study. Since the dominant eigenvalue is simple, the QSD attracts all the conditioned measures, and is the quasi-limiting distribution, or the Yaglom limit. When the transition kernel is symmetric, the spectral decomposition contains all the data that allow one to understand the survival phenomenon of the chain.

For general countable state Markov chains, this study started in Seneta and Vere-Jones (1996). We give a detailed proof of the characterization of QSDs as a finite mass eigenmeasure due to Pollett, and we give a more general result on the existence of QSDs, which states that if there is no entrance from infinity then exponential survival is a necessary and sufficient condition to have a QSD. We follow the original proof done in Ferrari et al. (1995a). We study also in details the symmetric chains and monotone chains. We also give conditions in order that the exponential decay rate be equal to the Kingman decay parameter of the transition probability.

An important literature has been devoted to time-continuous birth-and-death chains in the last 25 years, many of these works take their basis in the spectral representation given in McGregor (1957a, 1957b). We revisit some of these results. An important problem appears because under exponential survival, the set of QSDs can be a continuum or a singleton. We give the criterion, due to Van Doorn (1991), to identify which case occurs. This is done in terms of the parameters of the chain. When there is a continuum of QSDs one can identify the Yaglom limit, it corresponds to the minimal QSD This result is due to Good (1968). On the other hand, the spectrum can be continuous, and the fact that, in general, there is no spectral gap raises a delicate technical problem. But a useful point is that the bottom of the spectrum is the exponential rate of survival. For birth-and-death chains, we derive the classification of the associated Q-process (as transient, recurrent or positive recurrent) by studying the exponential asymptotic rate function of the survival probability when the chain is restricted to be above some barrier. For some particular birth-and-death chains, we identify explicitly the QSD and other significant parameters and properties.

We present some results for one dimensional diffusions on the half-line killed at the origin. We study the problem of QSD for diffusions that have nice behavior at 0 and at ∞. In this framework, we also study the Q-process and its recurrence classification in relation with spectral properties of the generator of the initial diffusion. We also consider one dimensional diffusions that present singularities at 0 and/or at ∞, and we study for them the problems of existence and properties of QSD. This generalization is motivated by some models in mathematical ecology. These chapters on one dimensional diffusions find their source of inspiration in the work of Mandl (1961). The main tools come from the spectral theory of the Sturm–Liouville operators. The study of QSD in population dynamics is a very active topic nowadays, we include some of the results in the chapters on birth-and-death chains and diffusions.

We also consider the case of dynamical systems (with discrete time evolution) with a trap in the phase space. A QSD in this context deals with trajectories which do not fall into the trap. There are many analogies with the stochastic case concerning the questions, the results and their proofs. The QSDs for dynamical systems have been first studied in the context of expanding systems in Pianigiani and Yorke (1979). We discuss in particular Gibbs QSDs for symbolic systems and absolutely continuous QSDs for repellers. For both cases, we use a similar technique of proof based on quasi-compact operators analogous to those used in the thermodynamic formalism.

One of the main objects in our study is the associated Q-process, that is, the process having trajectories that survive forever, as well as its relations with the QSD and the spectral properties. For a branching process, the Q-process was introduced by Spitzer (unpublished) and in Lamperti and Ney (1968). In this book, we study the Q-process and describe it as an h-process with respect to the eigenfunction h having the same eigenvalue as the minimal QSD. This is done for Markov chains, one-dimensional diffusions and expanding dynamical systems, giving a unified approach. Thus, it appears that the Gibbs invariant measure for topological Markov chains described by Bowen–Ruelle–Sinai can be interpreted as the stationary measure of a Q-process associated to a process that is killed in a region having the Pianigiani–Yorke as its QSD.

The problem of existence of a QSD is in a sense more difficult than the problem of proving the existence of an invariant measure since one has to determine also the rate of decay of the probability of survival. This also implies that the equation for the QSD is nonlinear.

Several techniques have been used in the literature to prove existence and some of them will be explained in details in the present book. Without trying to be exhaustive, one can mention the abstract techniques relying on the Tychonov fixed point theorem (see Tychonov 1935), Krein's lemma (Oikhberg and Troitsky 2005), and the Krein–Rutman Theorem (Dunford and Schwartz 1958). Birkhoff's Theorem on Hilbert's projective metric can also be useful (see Eveson and Nussbaum 1995). Many results are based on spectral techniques, and in particular on various extensions of the Perron–Frobenius Theorem like in the theory of the Ruelle–Perron–Frobenius operator Ruelle (1978), Bowen (1975), the spectral theory of quasi-compact positive operators (see, for example, Ionescu Tulcea and Marinescu 1950 and Nussbaum 1970), and the uniform ergodic theorem (Yosida and Kakutani 1941).

The literature on QSDs is quite large and covers many fundamental and applied domains. This book does not pretend to cover them all. We have not included the developments of QSDs in all directions, for instance, we do not treat the case of branching processes where several seminal results can be found in the book Athreya and Ney (1972) and where there is a lot of active research nowadays. We have selected some of the topics which either have called the attention of specialists, or where the authors of this book have made some contributions.

This book is not intended to be a complete exposition of the theory of QSDs. It mostly reflects the (present) interest of the authors in some parts of this vast field.

In Chap. 2 dealing with general aspects of QSDs, we have mainly gathered some of our own research. We also point out that only Sects. 2.1, 2.2, 2.3 are needed to read independently the other chapters.

Paris, France — Pierre Collet
Santiago, Chile — Servet Martínez
Jaime San Martín

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