

Preface

This work continues Edward Nelson’s programme of devising “radically elementary” approaches to analysis broadly conceived. This research agenda was initiated by Nelson in the mid-seventies through the invention of Internal Set Theory (**IST**) [59] and reached a first climax with the publication of *Radically Elementary Probability Theory*, which appeared in 1987 in the *Annals of Mathematics Studies* monograph series [60].

The objective of Nelson’s 1987 monograph was to make the theory of stochastic processes (including continuous-time processes!) “readily available to anyone who can add, multiply, and reason” (from the preface [60, p. vii]) through an elementary, yet fully rigorous use of infinitesimals and unlimited numbers by invoking a very modest and easily accessible fragment of nonstandard analysis. The core concepts which make this possible are (a) the notion of a finite set with an unlimited number of elements and (b) the notion of a positive infinitesimal number; the point is that the employment of these concepts enables one to treat stochastic continuous-time phenomena as stochastic processes on finite probability spaces with discrete time lines of infinitesimal spacing.

This work extends Nelson’s elementarization even to stochastic analysis, covering topics such as stochastic integration and differentiation (Itô’s formula), change of measure (Girsanov’s theorem), the link between diffusions and semi-elliptic partial differential equations (Dynkin’s formula, Feynman–Kac formula), jump-diffusion processes (Lévy processes) as well as applications of stochastic analysis in financial economics (fundamental theorems of asset pricing), financial engineering (volatility invariance in the Black–Scholes model), and mathematical physics (rigorous definition of the Feynman path integral).

Viewed from an axiomatic perspective, we shall follow Nelson’s example in assuming not just the axioms of conventional mathematics (say, Zermelo–Fraenkel set theory with Choice, **ZFC**) but also some elementary axioms that allow for basic nonstandard analysis; the resulting extension of **ZFC** will be called Minimal Internal Set Theory and is a subsystem of **IST**. Nelson [59] showed through an elaborate set-theoretic argument that **IST** is a conservative extension of **ZFC**; in Appendix A, we shall give a simple proof for the fact that at least a powerful

subsystem of Minimal Internal Set Theory is a conservative extension of **ZFC** and hence consistent relative to **ZFC**. In Appendix B, the relation of Minimal Internal Set Theory to Robinsonian nonstandard analysis is briefly discussed. The remainder of the text, however, requires no acquaintance with model theory or any other part of mathematical logic whatsoever.

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