# Chapter 1
# Introduction

In the last 3 decades, there has been significant progress in 3-dimensional topology, due in large part to the application of new techniques from other areas of mathematics and from physics. On the one hand, ideas from geometry have led to geometric decompositions of 3-manifolds and to invariants such as the $A$-polynomial and hyperbolic volume. On the other hand, ideas from quantum physics have led to the development of invariants such as the Jones polynomial and colored Jones polynomials. While ideas generated by these invariants have helped to resolve several problems in knot theory, their relationships to each other, and to classical knot topology, are still poorly understood. Topological quantum field theory predicts that these invariants are in fact tightly related, as does mounting computer evidence. However, at this writing, several outstanding conjectures and open problems have been verified for only a handful of examples.

In this monograph, we initiate a systematic study of relations between quantum knot invariants and geometries of knot complements. We develop the setting and machinery that allows us to establish direct and concrete relations between colored Jones knot polynomials and geometric knot invariants. In several instances, our results provide deeper and more intrinsic explanations for the connections between geometry and quantum topology that have been observed in special cases in the past. In addition, this work leads to some surprising new relations between the two areas, and offers a promising environment for further exploring such connections.

We begin with some history and background on the problems under consideration, then give an overview of the work contained in this manuscript, including some of the results mentioned above.

## 1.1 History and Motivation

W. Thurston's ground-breaking work in the late 1970s established the ubiquity and importance of hyperbolic geometry in three-dimensional topology. In fact, hyperbolic 3-manifolds had been studied since the beginning of the twentieth

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century as a subfield of complex analysis. In the 1960s and 1970s, Andreev [7, 8], Riley [86, 87], and Jørgensen [51] found several families of hyperbolic 3-manifolds with increasingly complex topology. In particular, Riley constructed the first examples of hyperbolic structures on complements of knots in the 3-sphere. In a different direction, Jaco and Shalen [47] and Johannson [48] found a canonical way to decompose a 3-manifold along surfaces of small genus (this is now called the *JSJ decomposition* or *torus decomposition*). In particular, they observed that *simple* 3-manifolds, i.e. ones that do not contain homotopically essential spheres, disks, tori or annuli, have fundamental groups that share similar properties with the groups of hyperbolic 3-manifolds. Thurston's major insight was that the pieces of the JSJ decomposition should admit locally homogeneous geometric structures, and furthermore that the simple pieces should admit complete hyperbolic structures. This insight was formalized in the celebrated *geometrization conjecture*. Thurston proved the conjecture for 3-manifolds with non-empty boundary [93], among others. In 2003, Perelman proved the general conjecture [69, 79, 80].

A special case of Thurston's theorem [93] is that link complements in the 3-sphere satisfy the geometrization conjecture. In particular, the complement of any non-torus, non-satellite knot must admit a complete hyperbolic metric. By Mostow–Prasad rigidity [71, 82], this hyperbolic structure is unique up to isometry. As a result, geometric information about a hyperbolic knot complement, such as its volume, gives topological knot invariants. For arbitrary knots, one can obtain a similar invariant, called the *simplicial volume*, by considering the sum of the volumes of the hyperbolic components in the JSJ decomposition. The simplicial volume is a constant multiple of the Gromov norm of the knot complement [44].

Since the mid-1980s, low-dimensional topology has also been invigorated by ideas from quantum physics, which have led to powerful and subtle invariants. The first major invariant along these lines is the celebrated Jones polynomial, first formulated by Jones in 1985 using operator algebras [49]. Soon after, Kauffman described a direct construction of the polynomial using the combinatorics of link projections [55], and several authors generalized it to links and trivalent graphs [28, 50, 56, 84]. Witten showed that the Jones polynomial of links in the 3-sphere has an interpretation in terms of a $2+1$ dimensional *topological quantum field theory* (TQFT). At the same time, he introduced new invariants for links in arbitrary 3-manifolds, as well as invariants of 3-manifolds [96, 97]. The resulting theory, although defined only at the physical level of rigor, predicted that the Jones-type invariants and their generalizations are intimately connected to geometric structures of 3-manifolds, and particularly to hyperbolic geometry [96, p. 77]. As explained by Atiyah [10], the TQFT proposed by Witten is completely characterized by certain "gluing axioms." In the late 1980s, Reshetikhin and Turaev gave the first mathematically rigorous construction of a TQFT that fit this axiomatic description [85]. Unlike that of [97], which is intrinsically 3-dimensional, the constructions of [85], as well as those of [55, 84], relied on combinatorial descriptions of 3-manifolds and the representation theory of quantum groups. This approach makes it harder to establish connections with the geometry of 3-manifolds.

In the 1990s, Kashaev defined an infinite family of complex valued invariants of links in 3-manifolds, using the combinatorics of triangulations and the quantum dilogarithm function [52]. For links in the 3-sphere, these invariants can also be formulated in terms of tangles and $R$-matrices [53]. Kashaev's invariants are parametrized by the positive integers; there is an invariant for each $n \in \mathbb{N}$. He conjectured that the large-$n$ asymptotics of these invariants determine the volume of hyperbolic knots [54]. Building on these works, H. Murakami and J. Murakami were able to recover Kashaev's invariants as special values of the *colored* Jones polynomials: an infinite family of polynomials, closely related to the Jones polynomial, also parametrized by $n \in \mathbb{N}$ [73]. As a result, Kashaev's original conjecture has been reformulated into the *volume conjecture*, which asserts that the volume of a hyperbolic knot is determined by the large-$n$ asymptotics of the colored Jones polynomials. Furthermore, Murakami and Murakami generalized the conjecture to all knots in $S^3$ by replacing the hyperbolic volume with the simplicial volume [73]. The volume conjecture fits into a more general, conjectural framework relating hyperbolic geometry and quantum topology; for details, see the survey papers [25, 72] and references therein. Despite compelling experimental evidence, the aforementioned conjectures are currently known for only a few examples of hyperbolic knots.

At the same time, a growing body of evidence points to strong relations between the coefficients of the Jones and colored Jones polynomials and the volume of hyperbolic links. One such form of evidence consists of numerical computations, for example those by Champanerkar, Kofman, and Paterson [18]. A second form of evidence consists of theorems proved for several classes of links, for example alternating links by Dasbach and Lin [23]. The authors of this monograph have extended those results to closed 3-braids [34], highly twisted links [32], and certain sums of alternating tangles [33]. The approach in all of these results is somewhat indirect, in that they relate hyperbolic volume to the Jones polynomial by estimating both quantities in terms of the twist number of a link diagram. To mention two examples, for alternating links the result follows from Lackenby's volume estimate in terms of the twist number in any alternating projections [58] and the relation of the twist number to the colored Jones polynomial observed by Dasbach and Lin [23]. For highly twisted links, our argument works as follows. First, we proved an effective version of Gromov and Thurston's $2\pi$-theorem and applied it to estimate the hyperbolic link volume in terms of the twist number of any highly twisted projection. Second, we relied on the combinatorial properties of *Turaev surfaces*, as studied in [21], to relate the twist numbers to the coefficients of Jones polynomials. However, for general links, twist numbers have a highly imperfect relationship to hyperbolic volume [35]. This limits the applicability of these methods to special families of knots and links.

In this monograph, we modify our approach to these problems, focusing on the topology of incompressible surfaces in knot complements and their relations to the colored Jones knot polynomials. Our motivation for the project has been twofold. On the one hand, certain spanning surfaces of knots have been shown to carry information on colored Jones polynomials [21]. On the other hand, essential surfaces also shed light on volumes of manifolds [6] and additional geometry

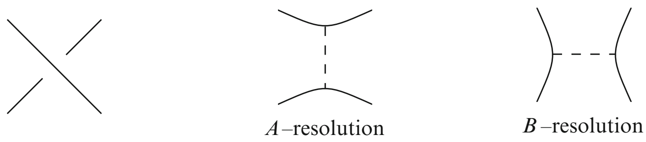


**Fig. 1.1** $A$- and $B$-resolutions at a crossing of $D$

and topology (e.g. [2, 65, 68]). With these ideas in mind, we develop a machine that allows us to establish relationships between colored Jones polynomials and topological/geometric invariants.

For example, under mild diagrammatic hypotheses that arise naturally in the study of Jones-type polynomials, we show that the growth of the degree of the colored Jones polynomials is a *boundary slope* of an essential surface in the knot complement, as predicted by Garoufalidis [42]. Furthermore, certain coefficients of the polynomials measure how far this surface is from being a fiber in the knot complement. Our work leads to direct and detailed relations between hyperbolic geometry and Jones-type polynomials: for certain families of links, coefficients of the Jones and colored Jones polynomials determine the hyperbolic volume to within a factor of 4. Compared to previous arguments, which were all somewhat indirect, the way in which our machine produces volume inequalities gives a clearer and deeper conceptual explanation for why the hyperbolic volume should be related to particular coefficients of the Jones polynomial.

A survey of this monograph, in which the main theorems are illustrated by a running example, is given in [37].

## 1.2 State Graphs, and State Surfaces Far from Fibers

We begin with some terminology and conventions. Throughout this manuscript, $D = D(K)$ will denote a link diagram, in the equatorial 2-sphere of $S^3$. It is worth pointing out two conventions. First, we always assume (without explicit mention) that link diagrams are connected. Second, we abuse notation by referring to the projection 2-sphere using the common term *projection plane*. In particular, $D(K)$ cuts the projection "plane" into compact regions.

Let $D(K)$ be a (connected) diagram of a link $K$, as above, and let $x$ be a crossing of $D$. Associated to $D$ and $x$ are two link diagrams, each with one fewer crossing than $D$, called the *$A$-resolution* and *$B$-resolution* of the crossing.

**Definition 1.1.** A *state* $\sigma$ is a choice of $A$- or $B$-resolution at each crossing of $D$. Resolving every crossing, as in Fig. 1.1, gives rise to a crossing-free diagram $s_\sigma(D)$, which is a collection of disjoint circles in the projection plane. Thus one obtains a *state graph* $\mathbb{G}_\sigma$, whose vertices correspond to circles of $s_\sigma$ and whose edges correspond to former crossings. For a given state $\sigma$, the *reduced state graph* $\mathbb{G}'_\sigma$ is the graph obtained from $\mathbb{G}_\sigma$ by removing all multiple edges between pairs of vertices.

The notion of states on link diagrams was first considered by Kauffman [55] during his construction of the bracket polynomial that provided a new construction and interpretation of the Jones polynomial.

Our primary focus is on the all-$A$ and all-$B$ states. The crossing-free diagram $s_A(D)$ is obtained by applying the $A$-resolution to each crossing of $D$. Its state graph is denoted $\mathbb{G}_A$ or $\mathbb{G}_A(D)$, and its reduced state graph $\mathbb{G}'_A$ or $\mathbb{G}'_A(D)$. Similarly, for the all-$B$ state $s_B(D)$, the state graph is denoted $\mathbb{G}_B$, and the reduced state graph $\mathbb{G}'_B$.

To a state $\sigma$, we associate a *state surface* $S_\sigma$ as follows. The state circles of $\sigma$ bound disjoint disks in the 3-ball below the projection plane; these disks can be connected to one another by half-twisted bands at the crossings. The surface $S_\sigma$ will have $\partial S_\sigma = K$. A special case of this construction is the Seifert surface constructed from the diagram $D(K)$, where the state $\sigma$ is determined by an orientation on $K$.

When $\sigma$ is the all-$A$ or all-$B$ state, the surfaces $S_\sigma$ hold significance for both geometric topology and quantum topology. The graph $\mathbb{G}_A$ canonically embeds as a spine of the surface $S_A$. On the quantum side, the combinatorics of this embedding can be used to recover the colored Jones polynomials $J^n_K(t)$ [21, 23]. On the geometric side, as we will see below, the combinatorics of $\mathbb{G}_A$ dictates a geometric decomposition of the 3-manifold $M_A$ obtained by cutting the link complement along the surface $S_A$. Because every statement has a $B$-state counterpart (by taking a mirror of the diagram), we will mainly discuss the all-$A$ state for ease of exposition.

**Definition 1.2.** Let $M = S^3 \setminus K$ denote the 3-manifold with torus boundary component(s) obtained by removing a tubular neighborhood of $K$ from $S^3$. Let $S_A$ be the all-$A$ state surface, as above, and let $M \backslash\backslash S_A$ denote the path-metric closure of $M \setminus S_A$. Note that $(S^3 \setminus K)\backslash\backslash S_A$ is homeomorphic to the 3-manifold $S^3\backslash\backslash S_A$ obtained by removing a regular neighborhood of $S_A$ from $S^3$. We will usually write $S^3\backslash\backslash S_A$ for short, and denote this manifold with boundary by $M_A$.

We will refer to $P = \partial M_A \cap \partial M$ as the *parabolic locus* of $M_A$. This parabolic locus consists of annuli. The remaining, non-parabolic boundary $\partial M_A \setminus \partial M$ is the unit normal bundle of $S_A$.

**Definition 1.3.** Let $M$ be an orientable 3-manifold and $S \subset M$ a properly embedded surface. We say that $S$ is *essential* in $M$ if the boundary of a regular neighborhood of $S$, denoted $\widetilde{S}$, is incompressible and boundary-incompressible. If $S$ is orientable, then $\widetilde{S}$ consists of two copies of $S$, and the definition is equivalent to the standard notion of "incompressible and boundary-incompressible." If $S$ is non-orientable, this is equivalent to $\pi_1$-injectivity of $S$, the stronger of two possible senses of incompressibility.

In the setting of Definition 1.2, the surface $S_A$ is often non-orientable. In this case, $S^3\backslash\backslash\widetilde{S_A}$ is the disjoint union of $M_A = S^3\backslash\backslash S_A$ and a twisted $I$-bundle over $S_A$. Since we are interested in the topology of $M_A$, it is appropriate to look at the incompressibility of $\widetilde{S_A}$.

Guided by the combinatorial structure of the state graph $\mathbb{G}_A$, we construct a decomposition of $M_A$ into topological balls. The connectivity properties of $\mathbb{G}_A$ govern the behavior of this decomposition; in particular, if $\mathbb{G}_A$ has no loop edges, we obtain a decomposition of $M_A$ into checkerboard ideal polyhedra with 4-valent vertices (Theorem 3.12). This decomposition generalizes Menasco's decomposition of alternating link complements, which has been used frequently in the literature [64]. As a first application of our machinery, we use normal surface theory with respect to our polyhedral decomposition to give a new proof of the following theorem of Ozawa [76].

**Theorem 3.19 (Ozawa).** *Let $D(K)$ be a diagram of a link $K$. Then the all-$A$ state surface $S_A$ is essential in $S^3 \setminus K$ if and only if $\mathbb{G}_A$ contains no 1-edge loops. Similarly, the surface $S_B$ is essential in $S^3 \setminus K$ if and only if $\mathbb{G}_B$ contains no 1-edge loops.*

Our polyhedral decomposition is designed to provide much more detailed information about the topology and geometry of $M_A = S^3 \backslash\backslash S_A$. In particular, we can characterize exactly when the surface $S_A$ is a fiber of the link complement.

**Theorem 5.11.** *Let $D(K)$ be any link diagram, and let $S_A$ be the spanning surface determined by the all-$A$ state of this diagram. Then the following are equivalent:*

*(1) The reduced graph $\mathbb{G}'_A$ is a tree.*
*(2) $S^3 \setminus K$ fibers over $S^1$, with fiber $S_A$.*
*(3) $M_A = S^3 \backslash\backslash S_A$ is an $I$-bundle over $S_A$.*

It is remarkable to note that the state graph connectivity conditions that ensure incompressibility of the state surfaces first arose in the study of Jones-type knot polynomials. The following definition, formulated by Lickorish and Thistlethwaite [61,92], captures exactly the class of link diagrams whose polynomial invariants are especially well-behaved.

**Definition 1.4.** A link diagram $D(K)$ is called $A$-adequate (resp. $B$-adequate) if $\mathbb{G}_A$ (resp. $\mathbb{G}_B$) has no 1-edge loops. If both conditions hold for a diagram $D(K)$, then $D(K)$ and $K$ are called *adequate*. If $D(K)$ is either $A$- or $B$-adequate, then $D(K)$ and $K$ are called *semi-adequate*. As we will discuss in the next section, the hypothesis of semi-adequacy is rather mild.

Building on Theorem 5.11, we start with an $A$-adequate diagram $D$ and strive to understand the geometric and topological complexity of $S^3 \backslash\backslash S_A$. In Chap. 2, we will see that the 3-manifold $M_A = S^3 \backslash\backslash S_A$ is in fact a handlebody, and thus atoroidal. The annulus version of the JSJ decomposition theory [47, 48] provides a way to cut $M_A$ along annuli (disjoint from the parabolic locus) into three types of pieces: $I$-bundles over sub-surfaces of $S_A$, Seifert fibered spaces, and the *guts*, which is the portion that admits a hyperbolic metric with totally geodesic boundary. The Seifert fibered components are solid tori. Thus $\chi(\text{guts}(M_A)) = 0$ precisely when $\text{guts}(M_A) = \emptyset$ and $M_A$ is a union of $I$-bundles and solid tori. In this case, $M_A$ is called a *book of $I$-bundles* and $S_A$ is called a *fibroid* [20]. The guts are the complex, interesting pieces of the geometric decomposition of $M_A$. Because hyperbolic surfaces, and guts, have negative Euler characteristic, it is convenient to work with the following definition.

**Definition 1.5.** Let $Y$ be a compact cell complex, whose connected components are $Y_1, \ldots, Y_n$. Then the Euler characteristic of $Y$ can be split into positive and negative parts:

$$\chi_+(Y) = \sum_{i=1}^{n} \max\{\chi(Y_i), 0\}, \qquad \chi_-(Y) = \sum_{i=1}^{n} \max\{-\chi(Y_i), 0\}.$$

It follows immediately that $\chi(Y) = \chi_+(Y) - \chi_-(Y)$. This notation is borrowed from the Thurston norm [94]. By convention, when $Y = \emptyset$, the above sums have no terms, hence $\chi_+(\emptyset) = \chi_-(\emptyset) = 0$.

The negative Euler characteristic $\chi_-(\mathrm{guts}(M_A))$ serves as a useful measurement of how far $S_A$ is from being a fiber or a fibroid in $S^3 \setminus K$. In fact, $\chi_-(\mathrm{guts})$ is a key measurement of complexity in Agol's virtual fibering criterion [5], which is needed in the proof of the virtual fibering conjecture for hyperbolic 3-manifolds [4]. The Euler characteristic of guts also has a direct connection to hyperbolic geometry. Agol, Storm, and Thurston have shown that for any essential surface $S$ in a hyperbolic 3-manifold $M$, a constant times $\chi_-(\mathrm{guts}(M))$ gives a lower bound for $\mathrm{vol}(M)$ [6]. This is applied below, in Sect. 1.5. On the other hand, the Euler characteristic $\chi(\mathbb{G}'_A)$ of the reduced graph $\mathbb{G}'_A$ first arose in the study of Jones-type polynomials [23,90], and in fact expresses one of their coefficients. This is explored in Sect. 1.4.

One of our main results is a diagrammatic formula for the guts of state surfaces for all $A$-adequate diagrams. In relating guts to reduced state graphs, it provides a bridge between hyperbolic geometry and quantum topology.

**Theorem 5.14.** *Let $D(K)$ be an $A$-adequate diagram, and let $S_A$ be the essential spanning surface determined by this diagram. Then*

$$\chi_-(\mathrm{guts}(S^3 \backslash\backslash S_A)) = \chi_-(\mathbb{G}'_A) - ||E_c||,$$

*where $||E_c|| \geq 0$ is a diagrammatic quantity defined in Definition 5.9.*

In many cases, the correction term $||E_c||$ vanishes. For example, this happens for alternating links [58], as well as for most Montesinos links. See Theorem 8.6, stated on p. 15 and Corollary 5.19 on p. 88. In each of these cases, Theorem 5.14 says that a geometric quantity, $\chi_-(\mathrm{guts}(M_A))$, is equal to $\chi_-(\mathbb{G}'_A)$, which, as shown in [23], expresses a coefficient of the Jones polynomial.

## 1.3 Which Links are Semi-adequate?

We will be considering semi-adequate links throughout this manuscript. (After taking a mirror if necessary, such a link is $A$-adequate.) Before we continue with the description of our results, it is worth making some remarks about the class of semi-adequate links. It turns out that the class is very broad, and that the condition that a knot be semi-adequate seems to be rather mild. For example, with the exception of two 11-crossing knots that we will discuss below, and a handful of 12-crossings knots, all knots with at most 12 crossings are semi-adequate. Furthermore, every minimal crossing diagram for each of these semi-adequate knots is semi-adequate [90, 92]. Thus, apart from a few exceptions, our results in this monograph apply directly to the diagrams in the knot tables up to 12 crossings. The situation is similar with the larger tabulated knots: Stoimenow has computed that among the 253,293 prime knots with 15 crossings tabulated in [46], at least 249,649 are semi-adequate [91].

Several well-studied families of links are semi-adequate. These include alternating links, positive or negative closed braids, all closed 3-braids, all Montesinos links, and planar cables of all of the above. We refer the reader to [61, 91, 92] for more discussion and examples.

Nevertheless, there exist knots and links that are not semi-adequate. Before discussing examples, we recall that the Jones polynomial can be used to detect semi-adequacy. Indeed, the last coefficient of an $A$-adequate link must be $\pm 1$. Similarly, the first coefficient of an $B$-adequate link must be $\pm 1$ [92]. With the notation of Knotinfo [17], the knot $K = 11n_{95}$ has Jones polynomial equal to $J_K(t) = 2t^2 - 3t^3 + 5t^4 - 6t^5 + 6t^6 - 5t^7 + 4t^8 - 2t^9$. Hence, $K$ is not semi-adequate; this is the first such knot in the knot tables. An infinite family of non semi-adequate knots, detected by the extreme coefficients of their Jones polynomial, can be obtained by [63, Theorem 5]. However, as we discuss below, the extreme coefficients of the Jones polynomial are not a complete obstruction to semi-adequacy.

Thistlethwaite [92] showed that certain coefficients of the 2-variable Kauffman polynomial [56] provide the obstruction to semi-adequacy. Building on Thistlethwaite's results, Stoimenow obtained a set of semi-adequacy criteria and applied them to several knots whose adequacy could not be determined by the Jones polynomial. For example, he showed that the knot $K' = 11n_{118}$ is not semi-adequate. Note that in this case, the last coefficient of the Jones polynomial, $J_{K'}(t) = 2t^2 - 2t^3 + 3t^4 - 4t^5 + 4t^6 - 3t^7 + 2t^8 - t^9$, is $-1$.

Ozawa has considered link diagrams and Kauffman states $\sigma$ that are adequate (meaning $\mathbb{G}_\sigma$ has no 1-edge loops) and homogeneous (meaning $\mathbb{G}_\sigma$ contains a set of cut vertices that decompose it into a collection of all-$A$ and all-$B$ state graphs) [76]. See Definition 2.22 for more details. Semi-adequate diagrams clearly have this property, but the class of [76] is broader. As an example, consider the 12-crossing knot $K'' = 12n_{0706}$. This is not semi-adequate since both the extreme coefficients of the Jones polynomial are equal to 2. Indeed $J_{K''}(t) = 2t^{-4} - 4t^{-3} + 6t^{-2} - 8t^{-1} + 9 - 8t + 6t^2 - 4t^3 + 2t^4$. However $K''$ can be written as a 5-string braid that is homogeneous in the sense of Cromwell [19]. Thus the Seifert state of this closed braid diagram is homogeneous and adequate.

Ozawa proved that the state surface $S_\sigma$ corresponding to a $\sigma$-adequate, $\sigma$-homogeneous diagram is always essential in $S^3 \setminus K$. In [29], Futer gave a direct proof of a slightly weaker version of Theorem 5.11, and also generalized it to $\sigma$-adequate, $\sigma$-homogeneous link diagrams. It turns out that many properties of the polyhedral decompositions that we develop below, as well as a number of results proved using the polyhedral decomposition, also extend to all adequate, homogeneous states. See Sects. 2.4, 3.4, 4.5, 5.6 where, in particular, we obtain analogues of Theorems 3.19, 5.11 and 5.14 in this generalized setting. Our study of the geometry of such links is continued in [31].

## 1.4 Essential Surfaces and Colored Jones Polynomials

The Jones and colored Jones polynomials have many known connections to the state graphs of diagrams. To specify notation, let

$$J_K^n(t) = \alpha_n t^{m_n} + \beta_n t^{m_n - 1} + \ldots + \beta'_n t^{r_n + 1} + \alpha'_n t^{r_n},$$

denote the $n$-th *colored Jones polynomial* of a link $K$. Recall that $J_K^2(t)$ is the usual Jones polynomial. Consider the sequences

$$js_K := \left\{ \frac{4m_n}{n^2} \; : \; n > 0 \right\} \quad \text{and} \quad js_K^* := \left\{ \frac{4r_n}{n^2} \; : \; n > 0 \right\}.$$

Garoufalidis' slope conjecture predicts that for each knot $K$, every cluster point (i.e., every limit of a subsequence) of $js_K$ or $js_K^*$ is a *boundary slope* of $K$ [42], i.e. a fraction $p/q$ such that the homology class $p\mu + q\lambda$ occurs as the boundary of an essential surface in $S^3 \setminus K$.

For a given diagram $D(K)$, there is a lower bound for $r_n$ in terms of data about the state graph $\mathbb{G}_A(D)$, and this bound is sharp when $D(K)$ is $A$-adequate. Similarly, there is an upper bound on $m_n$ in terms of $\mathbb{G}_B$ that is realized when $D(K)$ is $B$-adequate [60]. In [36], building on these properties and using Theorem 3.19, we relate the extreme degree of $J_K^n(t)$ to the boundary slope of $S_A$, as predicted by the slope conjecture.

**Theorem 1.6 ([36]).** *Let $D(K)$ be an $A$-adequate diagram of a knot $K$ and let $b(S_A) \in \mathbb{Z}$ denote the boundary slope of the essential surface $S_A$. Then*

$$\lim_{n\to\infty} \frac{4r_n}{n^2} = b(S_A),$$

*where $r_n$ is the lowest degree of $J_K^n(t)$.*

Similarly, if $D(K)$ is a $B$-adequate diagram of a knot $K$, let $b(S_B) \in \mathbb{Z}$ denote the boundary slope of the essential surface $S_B$. Then

$$\lim_{n\to\infty} \frac{4m_n}{n^2} = b(S_B),$$

where $m_n$ is the highest degree of $J_K^n(t)$.

Work of Garoufalidis and Le [41, 43] implies that each coefficient of $J_K^n(t)$ satisfies linear recursive relations in $n$. For adequate links, these relations manifest themselves in a very strong form: Dasbach and Lin showed that if $K$ is $A$-adequate, then the absolute values $|\beta_n'|$ and $|\alpha_n'|$ are independent of $n > 1$ [23]. In fact, $|\alpha_n'| = 1$ and $|\beta_n'| = 1 - \chi(\mathbb{G}_A')$, where $\mathbb{G}_A'$ is the reduced graph. Similarly, if $D$ is $B$-adequate, then $|\alpha_n| = 1$ and $|\beta_n| = 1 - \chi(\mathbb{G}_B')$. Thus we can define the *stable values*

$$\beta_K' := \left|\beta_n'\right| = 1 - \chi(\mathbb{G}_A'), \qquad \text{and} \qquad \beta_K := |\beta_n| = 1 - \chi(\mathbb{G}_B').$$

The main results of this monograph explore the idea that the stable coefficient $\beta_K'$ does an excellent job of measuring the geometric and topological complexity of the manifold $M_A = S^3 \backslash\backslash S_A$. (Similarly, $\beta_K$ measures the complexity of $M_B = S^3 \backslash\backslash M_B$.) For instance, it follows from Theorem 5.11 that $\beta_K'$ is exactly the obstruction to $S_A$ being a fiber.

**Corollary 9.16.** *For an $A$-adequate link $K$, the following are equivalent:*

*(1)* $\beta_K' = 0$.
*(2) For* every *$A$-adequate diagram of $D(K)$, $S^3 \setminus K$ fibers over $S^1$ with fiber the corresponding state surface $S_A = S_A(D)$.*
*(3) For* some *$A$-adequate diagram $D(K)$, $M_A = S^3 \backslash\backslash S_A$ is an $I$-bundle over $S_A(D)$.*

Similarly, $\left|\beta_K'\right| = 1$ precisely when $S_A$ is a fibroid of a particular type.

**Theorem 9.18.** *For an $A$-adequate link $K$, the following are equivalent:*

*(1)* $\beta_K' = 1$.
*(2) For* every *$A$-adequate diagram of $K$, the corresponding 3-manifold $M_A$ is a book of $I$-bundles, with $\chi(M_A) = \chi(\mathbb{G}_A) - \chi(\mathbb{G}_A')$, and is not a trivial $I$-bundle over the state surface $S_A$.*
*(3) For* some *$A$-adequate diagram of $K$, the corresponding 3-manifold $M_A$ is a book of $I$-bundles, with $\chi(M_A) = \chi(\mathbb{G}_A) - \chi(\mathbb{G}_A')$.*

In general, the geometric decomposition of $M_A$ contains some non-trivial hyperbolic pieces, namely guts. In this case, $\left|\beta_K'\right|$ measures the complexity of the guts together with certain complicated parts of the maximal $I$-bundle of $M_A$. To state our result we need the following definition.

**Definition 1.7.** A link diagram $D$ is called *prime* if any simple closed curve that meets the diagram transversely in two points bounds a region of the projection plane without any crossings.

Two crossings in $D$ are defined to be *twist equivalent* if there is a simple closed curve in the projection plane that meets $D$ at exactly those two crossings. The diagram is called *twist reduced* if every equivalence class of crossings is a *twist region* (a chain of crossings between two strands of $K$). The number of equivalence classes is denoted $t(D)$, the *twist number* of $D$.

**Theorem 9.20.** *Suppose $K$ is an $A$-adequate link whose stable colored Jones coefficient is $\beta'_K \neq 0$. Then, for every $A$-adequate diagram $D(K)$,*

$$\chi_-(\text{guts}(M_A)) + ||E_c|| \;=\; \left|\beta'_K\right| - 1,$$

*where as above $||E_c|| \;\geq\; 0$ is the diagrammatic quantity of Definition 5.9. Furthermore, if $D$ is prime and every 2-edge loop in $\mathbb{G}_A$ has edges belonging to the same twist region, then $||E_c|| = 0$ and*

$$\chi_-(\text{guts}(M_A)) \;=\; \left|\beta'_K\right| - 1.$$

To briefly discuss the meaning of the correction term $||E_c||$, recall that the non-hyperbolic components of the JSJ decomposition of $M_A$ are $I$-bundles and solid tori. In Chap. 4, we show that the $I$-bundle components with negative Euler characteristic are spanned by *essential product disks (EPDs)*: properly embedded essential disks in $M_A$ whose boundary meets the parabolic locus twice. These disks come in two types: those corresponding to (strings of) complementary regions of $\mathbb{G}_A$ with just two sides, and certain "complicated" ones, which we call *complex*. (See Definition 5.2 on p. 74.) The minimal number of complex EPDs in a spanning set is denoted $||E_c||$; this is exactly the correction term of Theorems 5.14 and 9.20.

It is an open question whether *every* $A$-adequate link admits a diagram for which $||E_c|| = 0$: see Question 10.2 on p. 156. For instance, Lackenby showed that this is the case for prime alternating links [58]. By Theorem 9.20, $||E_c|| = 0$ when every 2-edge loop of $\mathbb{G}_A$ has edges belonging to the same twist region. This is also the case for most Montesinos links (the reader is referred to Chap. 8 for the terminology).

**Corollary 9.21.** *Suppose $K$ is a Montesinos link with a reduced admissible diagram $D(K)$ that contains at least three tangles of positive slope. Then*

$$\chi_-(\text{guts}(M_A)) \;=\; \left|\beta'_K\right| - 1.$$

*Similarly, if $D(K)$ contains at least three tangles of negative slope, then*

$$\chi_-(\text{guts}(M_B)) \;=\; |\beta_K| - 1.$$

When $||E_c|| \;=\; 0$, Theorem 9.20 offers striking evidence that coefficients of the colored Jones polynomials measure something quite geometric: when $\left|\beta'_K\right|$ is large, the link complement $S^3 \setminus K$ contains essential spanning surfaces that are correspondingly far from being a fiber. Whereas the Alexander polynomial and its generalization in Heegaard Floer homology are known to have many connections to the geometric topology of spanning surfaces of a knot [75, 77, 78], the geometric meaning of Jones-type polynomials has traditionally been a mystery. Theorems 9.16, 9.18, and 9.20 establish some of the first detailed connections between surface topology and the Jones polynomial.

## 1.5 Volume Bounds from Topology and Combinatorics

Recall that by the work of Agol, Storm, and Thurston [6], any computation of, or lower bound on, $\chi_{-}(\text{guts})$ of an essential surface $S \subset S^3 \setminus K$ leads to a proportional lower bound on $\text{vol}(S^3 \setminus K)$. For instance, Lackenby's diagrammatic lower bound on the volumes of alternating knots and links came as a result of computing the guts of checkerboard surfaces [58]. However, computing $\chi_{-}(\text{guts})$ has typically been quite hard: apart from alternating knots and links, there are very few infinite families of manifolds for which there are known computations of the guts of essential surface [3, 57].

The results of this manuscript greatly expand the list of manifolds for which such computations exist. In Chap. 9, we combine [6] with our results in Theorems 5.14 and 9.20, as well as some of their specializations, to give lower bounds on hyperbolic volume for all $A$-adequate knots and links. See Theorem 9.3 on p. 140 for the most general result along these lines.

We also focus on two well-studied families of links: namely, positive braids and Montesinos links. For these families, we are able to compute or estimate the quantity $\chi_{-}(\text{guts}(M_A))$ in terms of much simpler diagrammatic data. As a consequence, we obtain tight, two-sided estimates on the volumes of knots and links in terms of the twist number $t(D)$ (see Definition 1.7).

**Theorem 9.7.** *Let $D(K)$ be a diagram of a hyperbolic link $K$, obtained as the closure of a positive braid with at least three crossings in each twist region. Then*

$$\frac{2v_8}{3}\, t(D) \;\leq\; \text{vol}(S^3 \setminus K) \;<\; 10v_3(t(D)-1),$$

*where $v_3 = 1.0149\ldots$ is the volume of a regular ideal tetrahedron and $v_8 = 3.6638\ldots$ is the volume of a regular ideal octahedron.*

Observe that the multiplicative constants in the upper and lower bounds differ by a rather small factor of about 4.155. For Montesinos links, we obtain similarly tight two-sided volume bounds.

**Theorem 9.12.** *Let $K \subset S^3$ be a Montesinos link with a reduced Montesinos diagram $D(K)$. Suppose that $D(K)$ contains at least three positive tangles and at least three negative tangles. Then $K$ is a hyperbolic link, satisfying*

$$\frac{v_8}{4}\,(t(D) - \#K) \;\leq\; \text{vol}(S^3 \setminus K) \;<\; 2v_8\, t(D),$$

*where $v_8 = 3.6638\ldots$ is the volume of a regular ideal octahedron and $\#K$ is the number of link components of $K$. The upper bound on volume is sharp.*

We also relate the volumes of these links to quantum invariants. Recall that the volume conjecture of Kashaev and Murakami–Murakami [54, 73] states that all hyperbolic knots satisfy

$$2\pi \lim_{n\to\infty} \frac{\log\left|J_K^n(e^{2\pi i/n})\right|}{n} = \mathrm{vol}(S^3 \setminus K).$$

If this volume conjecture is true, it would imply for large $n$ a relation between the volume of a knot $K$ and coefficients of $J_K^n(t)$. For example, for $n \gg 0$ one would have $\mathrm{vol}(S^3 \setminus K) < C\,||J_K^n||$, where $||J_K^n||$ denotes the $L^1$-norm of the coefficients of $J_K^n(t)$, and $C$ is an appropriate constant. In recent years, a series of articles by Dasbach and Lin, as well as the authors, has established such relations for several classes of knots [24, 32–34]. In fact, in all known cases, the upper bounds on volume are paired with similar lower bounds. However, in all of the past results, showing that coefficients of $J_K^n(t)$ bound volume below required two steps: first, showing that Jones coefficients give a lower bound on twist number $t(D)$, and then showing that twist number gives a lower bound on volume. Each of these two steps is known to fail outside special families of knots [34,35], and their combination produces an indirect argument in which the constants are far from sharp.

By contrast, our results in this manuscript bound volume below in terms of a topological quantity, $\chi_-(\mathrm{guts})$, that is directly related to colored Jones coefficients. As a consequence, we obtain much sharper lower bounds on volume, along with an intrinsic and satisfactory conceptual explanation for why these lower bounds exist. See Sect. 9.4 in Chap. 9 for more discussion.

Our techniques also imply similar results for additional classes of knots. For instance, Theorems 9.7 and 9.12 have the following corollaries.

**Corollary 9.22.** *Suppose that a hyperbolic link $K$ is the closure of a positive braid with at least three crossings in each twist region. Then*

$$v_8\left(\left|\beta'_K\right| - 1\right) \ \leq\ \mathrm{vol}(S^3 \setminus K) \ <\ 15v_3\left(\left|\beta'_K\right| - 1\right) - 10v_3,$$

*where $v_3 = 1.0149\ldots$ is the volume of a regular ideal tetrahedron and $v_8 = 3.6638\ldots$ is the volume of a regular ideal octahedron.*

**Corollary 9.23.** *Let $K \subset S^3$ be a Montesinos link with a reduced Montesinos diagram $D(K)$. Suppose that $D(K)$ contains at least three positive tangles and at least three negative tangles. Then $K$ is a hyperbolic link, satisfying*

$$v_8\left(\max\{|\beta_K|, \left|\beta'_K\right|\} - 1\right) \ \leq\ \mathrm{vol}(S^3 \setminus K) \ <\ 4v_8\left(|\beta_K| + \left|\beta'_K\right| - 2\right) + 2v_8\,(\#K),$$

*where $\#K$ is the number of link components of $K$.*

## 1.6 Organization

We now give a brief guide to the organization of this monograph.

In Chap. 2, we begin with a connected link diagram $D(K)$, and explain how to construct the state graph $\mathbb{G}_A$ and the state surface $S_A$. Guided by the structure of $\mathbb{G}_A$, we will cut the 3-manifold $M_A = S^3\backslash\backslash S_A$ along a collection of disks into several topological balls. We obtain a collection of *lower* balls that are in one-to-one correspondence with the alternating tangles in $D(K)$ and a single *upper* 3-ball. The boundary of each ball admits a checkerboard coloring into white and shaded regions that we call faces. In the last section of the chapter we discuss the generalization of the decomposition to $\sigma$-homogeneous and $\sigma$-adequate diagrams.

In Chap. 3, we show that if $D(K)$ is $A$-adequate, each of these balls is a checkerboard colored ideal polyhedron with 4-valent vertices. This amounts to showing that the shaded faces on each of the 3-balls are simply-connected (Theorem 3.12). Furthermore, we show that the ideal polyhedra do not contain normal bigons (Proposition 3.18), which quickly implies Theorem 3.19. In the last section of the chapter, we generalize these results to homogeneous and adequate states.

In Chap. 4, we prove a structural result about the geometric decomposition of $M_A$. As already mentioned, the JSJ decomposition yields three kinds of pieces: $I$-bundles, solid tori, and the guts, which admit a hyperbolic metric with totally geodesic boundary. Let $B$ be an $I$-bundle in the characteristic submanifold of $M_A$. We say that a finite collection of disjoint essential product disks (EPDs) $\{D_1, \dots, D_n\}$ *spans* $B$ if $B \setminus (D_1 \cup \dots \cup D_n)$ is a finite collection of prisms (which are $I$-bundles over a polygon) and solid tori (which are $I$-bundles over an annulus or Möbius band). We prove the following.

**Theorem 4.4.** *Let $B$ be a component of the characteristic submanifold of $M_A$ which is not a solid torus. Then $B$ is spanned by a collection of essential product disks (EPDs) $D_1, \dots, D_n$, with the property that each $D_i$ is embedded in a single polyhedron in the polyhedral decomposition of $M_A$.*

Like all results from the early chapters, Theorem 4.4 generalizes to $\sigma$-adequate and $\sigma$-homogeneous diagrams. See Sect. 4.5 for details.

In Chap. 5, we calculate the number of EPDs required to span the $I$-bundle of $M_A$. We do this by explicitly constructing a suitable spanning set of disks (Lemmas 5.6 and 5.8). The EPDs in the spanning set that lie in the *lower* polyhedra of the decompositions are well understood; they are in one-to-one correspondence with 2-edge loops in the state graph $\mathbb{G}_A$. The EPDs in the spanning set that lie in the upper polyhedron are *complex*; they are not parabolically compressible to EPDs in the lower polyhedra. The construction of this spanning set leads to a proof of Theorem 5.14. The spanning set of Chap. 5 also makes it straightforward to detect when the manifold $M_A$ is an $I$-bundle, leading to a proof of Theorem 5.11.

The main tool used in Chaps. 3–5 is normal surface theory. In fact, our results about normal surfaces in the polyhedral decomposition of $M_A$ can likely be used to attack other topological problems about $A$-adequate links: see Sect. 10.2 in Chap. 10 for variations on this theme.

The results of Chap. 5 reduce the problem of computing the Euler characteristic of the guts of $M_A$ to counting how many complex EPDs are required to span the $I$-bundle of the *upper* polyhedron. In Chap. 6, we restrict attention to prime diagrams and address the problem of how to recognize such EPDs from the structure of the all-$A$ state graph $\mathbb{G}_A$. Our main result there is Theorem 6.4, which describes the basic building blocks for such EPDs. Roughly speaking, each of these building blocks maps onto to a 2-edge loop of $\mathbb{G}_A$.

In Chap. 7, we restrict attention to $A$-adequate diagrams $D(K)$ for which the polyhedral decomposition includes no non-prime arcs or switches (see Definition 2.18 on p. 27). In this case, one can simplify the statement of Theorem 5.14 and give an easier combinatorial estimate for the guts of $M_A$. To state our result, let $b_A$ denote the number of bigons in twist regions of the diagram such that a loop tracing the boundary of this bigon belongs to the $B$-resolution of $D$. (The $A$-resolution of these twist regions is *short* in Fig. 5.4 on p. 86.) Then, define $m_A = \chi(\mathbb{G}_A) - \chi(\mathbb{G}'_A) - b_A$. We prove the following estimate.

**Theorem 7.2.** *Let $D(K)$ be a prime, $A$-adequate diagram, and let $S_A$ be the essential spanning surface determined by this diagram. Suppose that the polyhedral decomposition of $M_A = S^3 \backslash\backslash S_A$ includes no non-prime arcs. Then*

$$\chi_-(\mathbb{G}'_A) - 8m_A \;\leq\; \chi_-(\mathrm{guts}(M_A)) \;\leq\; \chi_-(\mathbb{G}'_A),$$

*where the lower bound is an equality if and only if $m_A = 0$.*

In Chap. 8, we study the polyhedral decompositions of Montesinos links. The main result is the following.

**Theorem 8.6.** *Suppose $K$ is a Montesinos link with a reduced admissible diagram $D(K)$ that contains at least three tangles of positive slope. Then*

$$\chi_-(\mathrm{guts}(M_A)) = \chi_-(\mathbb{G}'_A).$$

*Similarly, if $D(K)$ contains at least three tangles of negative slope, then*

$$\chi_-(\mathrm{guts}(M_B)) = \chi_-(\mathbb{G}'_B).$$

The arguments in Chaps. 6–8 require a detailed and fairly technical analysis of the combinatorial structure of the polyhedral decomposition; we call this analysis *tentacle chasing*. In addition, Chaps. 7 and 8 depend heavily on Theorem 6.4 in Chap. 6.

In Chap. 9, we give the applications to volume estimates and relations with the colored Jones polynomials that were discussed earlier in this introduction. The results in this chapter do not use Chap. 7 at all, and do not directly reference Chap. 6 or the arguments of Chap. 8. Thus, having the statement of Theorem 8.6 at hand, a reader who is eager to see the aforementioned applications may proceed to Chap. 9 immediately after Chap. 5.

In Chap. 10, we state several open questions and problems that have emerged from this work, and discuss potential applications of the methods that we have developed.

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