Control of Networks of Coupled Dynamical Systems

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Abstract We study networks of coupled dynamical systems where an external forcing control signal is applied to the network in order to align the state of all the individual systems to the forcing signal. By considering the control signal as the state of a virtual dynamical system, this problem can be studied as a synchronization problem. The main focus of this chapter is to link the effectiveness of such control to various properties of the underlying graph. For instance, we study the relationship between control effectiveness and the network as a function of the number of nodes in the network. For vertex-balanced graphs, if the number of systems receiving control does not grow as fast as the total number of systems, then the strength of the control needed to effect control will be unbounded as the number of vertices grows. In order to achieve control in systems coupled via locally connected graphs, as the number of systems grows, both the control and the coupling among *all* systems need to increase. Furthermore, the algebraic connectivity of the graph is an indicator of how easy it is to control the network. We also show that for the cycle graph, the best way to achieve control is by applying control to systems that are approximately equally spaced apart. In addition, we show that when the number of controlled systems is small, it is beneficial to put the control at vertices with large degrees, whereas when the number of controlled systems is large, it is beneficial to put the control at vertices with small degrees. Finally, we give evidence to show that applying control to minimize the distances between all systems to the set of controlled systems could lead to a more effective control.

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1 Introduction

In recent years, there is much research activity to study synchronization in networks of nonlinear dynamical systems [1–5]. In these studies criteria are derived that ensure all systems are synchronized to the same behavior. An important area of study is how these criteria are related to the topology of the network [6–10]. In these studies there are generally no external forcing; the dynamical systems only interact with each other. A related research area is control in such networks [11–18], where forcing is applied to a subset of dynamical systems in order to bring the entire network to follow a specific trajectory. It was shown that control can be achieved by forcing the behavior of a few systems. In [11] it was shown that applying control to highly connected nodes in a scale-free network facilitates control. In [14] numerical results were presented on control in scale-free networks. In [19] a similar problem in a power flow network is studied from a control theory perspective, but the graph topology is not taken into account. In [13] it was shown that such control is possible if the underlying topology contains a spanning directed tree. In this chapter we enumerate some recent results in this area.

2 Notations and Definitions

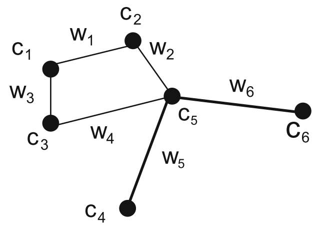
We consider edge-weighted graphs (V, E, W) where each edge $e \in E$ has a weight $0 < w_e \in W$. The Laplacian matrix of a graph is a zero row sum matrix L such that $L_{ij} = -w_{ij}$ where w_{ij} is the weight of the edge (i, j) and $L_{ij} = 0$ for all other $i \neq j$. Thus L_{ii} is the (weighted) degree of vertex i. We also consider VE-weighted (vertex and edge weighted) graphs defined as (V, E, W, C) with a value $c_i \in C$ associated to each vertex i. Such graphs are shown in Fig. 1 where we denote a VE-weighted graph (V, E, W, C) by assigning a label w_i to each edge and adding a label c_i to each vertex i. The Laplacian of a VE-weighted graph is defined as $L' = L + \text{diag}(c_1, c_2, \ldots, c_n)$ where L is the Laplacian matrix of (V, E, W). For a Hermitian matrix A, we ordered its eigenvalues as: $\lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A)$. For matrices A and B, we write $A \succeq B$ if $A - B$ is positive semidefinite.

Definition 1. The interaction graph of an n by n matrix G is the graph $\mathscr{G}$ of n vertices with an edge from vertex i to vertex j with weight G_{ji} if and only if $G_{ji} \neq 0$.

In particular, if one reverses the direction of all the edges in the interaction graph, we obtain the graph of a matrix as usually defined in combinatorial graph theory [20].

A directed tree is a directed graph with n vertices and $n - 1$ edges and a *root* vertex such that there is a directed path from the root vertex to every other vertex. A spanning directed tree of a graph is a subgraph that is a directed tree with the same vertex set as the graph. A spanning directed forest is a set of subgraphs that are directed trees where the union of their vertex sets is the same as the vertex set of the graph.

Fig. 1 VE-weighted graph



For undirected graphs, the eccentricity of a vertex v is $\max_w d(v,w)$. The *graph center* of a graph is defined as the set of vertices of minimum eccentricity.

Definition 2. A function f is V-uniformly decreasing if there is a matrix V such that $(y-z)^{\mathrm{T}}V(f(y,t)-f(z,t)) \le -\mu\|y-z\|^2$ for some $\mu > 0$ and all y,z,t. The function f is V-uniformly increasing if $-f$ is V-uniformly decreasing.

Definition 3. Let B be an irreducible square matrix B with nonpositive off-diagonal elements. The quantities $\beta(B)$ and $\gamma(B)$ are defined as follows. Decompose B uniquely as $B = L + U$, where L is a zero row sum matrix and U is a diagonal matrix. Let w be the unique positive vector such that $w^{\mathrm{T}}L = 0$ and $\max_v w_v = 1$. The vector w exists by Perron–Frobenius theory [21]. Let $W = \mathrm{diag}(w)$. Then $\gamma(B) = \min_{x\neq 0, x\perp \mathbf{1}} \frac{x^{\mathrm{T}}WBx}{x^{\mathrm{T}}\left(W - \frac{ww^{\mathrm{T}}}{\sum_v w_v}\right)x}$ and $\beta(B) = \min_{x\neq 0}\frac{x^{\mathrm{T}}WBx}{x^{\mathrm{T}}Wx}$. We define $\gamma(0) = +\infty$.

Definition 4. Let A be a zero row sums matrix written in Frobenius normal form [20]:

$$A = Q\begin{pmatrix} B_1 & B_{12} & \cdots & B_{1k} \\ & B_2 & \cdots & B_{2k} \\ & & \ddots & \vdots \\ & & & B_k \end{pmatrix} Q^{\mathrm{T}} \tag{1}$$

where Q is a permutation matrix and B_i are irreducible matrices. Then $\eta(A)$ is defined as

$$\eta(A) = \min(\beta(B_1), \beta(B_2), \ldots, \beta(B_{k-1}), \gamma(B_k))$$

Theorem 1. *Let L be the Laplacian matrix of an undirected weighted graph. Then the second smallest eigenvalue* λ_2 *of* L *satisfies* $\lambda_2(L) = \min_{x\neq 0, x\perp\mathbf{1}}\frac{x^{\mathrm{T}}Lx}{x^{\mathrm{T}}x}$.

Proof. First note that L is a zero row sums symmetric matrix. By Gershgorin's circle criterion, all eigenvalues of L are nonnegative, and $\lambda_1 = 0$ with eigenvector $\mathbf{1}$. The conclusion then follows from (4.2.7) of [22, p. 178]. □

Theorem 2 (Weyl's eigenvalue interlacing theorem). *Let* A, B *be Hermitian matrices with eigenvalues* $\lambda_i(A)$, $\lambda_i(B)$ *and* $\lambda_i(A+B)$ *be arranged in increasing order. For each* j,k *such that* $1 \le j,k \le n$ *and* $j+k \ge n+1$ *we have:*

$$\lambda_{j+k-n}(A+B) \le \lambda_j(A) + \lambda_k(B) \tag{2}$$

and for each j, k *such that* $1 \leq j, k \leq n$ *and* $j + k \leq n + 1$ *we have:*

$$\lambda_j(A) + \lambda_k(B) \leq \lambda_{j+k-1}(A + B) \tag{3}$$

Proof. Theorem 4.3.7 in [22, p. 184–185]. □

3 Synchronization in Networks of Dynamical Systems

We consider a network of n coupled dynamical systems whose state equations are written as:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = f(x_i, t) - \alpha \sum_j A_{ij} D(t)(x_i - x_j) \tag{4}$$

where x_i is the state vector of the i-th system and $\alpha > 0$ is a scalar global coupling coefficient. The parameter α can be absorbed into $D(t)$, but we'll find it useful to separate it out. The scalar $A_{ij} \geq 0$, $i \neq j$ denotes the coupling coefficient between the i-th and the j-th system. The total number of systems is denoted by n (i.e., $1 \leq i \leq n$). The matrix $D(t)$ describes the linear coupling between two systems which is the same between any pair of systems. By setting $L_{ij} = -A_{ij}$ for $i \neq j$ and $L_{ii} = \sum_j A_{ij}$, this is rewritten as:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = f(x_i, t) - \alpha \sum_j L_{ij} D(t) x_j \tag{5}$$

Let us assume that $A_{ii} = 0$. Then the matrix $L = \{L_{ij}\}$ is a zero row sum matrix with nonpositive off-diagonal elements. The underlying topology of the network is expressed as the weighted graph such that A and L are its adjacency matrix and Laplacian matrix, respectively.

We say the system in (4) synchronizes (globally) if $\|x_i - x_j\| \to 0$ *as* $t \to \infty$.

Conditions for global and local synchronization have been obtained using a variety of techniques [23–27]. In many cases, the synchronization conditions depend on properties of the matrix L.

Theorem 3 ([28]). *Consider a network of dynamical systems coupled via a directed graph with state equation:*

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = f(x_i, t) + \sum_j g_{ij} D(t) x_j(t) + u_i(t)$$

The network synchronizes in the sense that $\forall i, j, \lim_{t\to\infty} \|x_i - x_j\| = 0$ *if the following conditions are satisfied:*

1. $G = \{g_{ij}\}$ *is a zero row sums matrix with nonpositive off-diagonal elements.*
2. $\forall i, j, \lim_{t\to\infty} \|u_i - u_j\| = 0$*,*

3. *$f(x,t) + D(t)x$ is V-uniformly decreasing for some symmetric positive definite matrix V,*
4. *$VD(t) \preceq 0$ and is symmetric for all t,*
5. *$\eta(G) \geq 1$.*

4 Control in Networks of Dynamical Systems

Consider the case where in order to control the network in (5), forcing terms are applied to a subset of systems to drive the entire network to follow a prescribed trajectory. In particular we augment (5) by adding linear control terms as follows:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = f(x_i,t) - \alpha \left(\sum_j L_{ij} D(t) x_j + c_i D(t)(x_i - u(t)) \right) \tag{6}$$

where $u(t)$ is the desired target trajectory and $c_i > 0$ if control is applied to the i-th system and $c_i = 0$ otherwise. We define P as the set of systems where such control is applied (i.e., $i \in P \Leftrightarrow c_i > 0$). P is the set of *controlled* systems and the number of controlled systems is denoted as $p = |P|$. We write $C = \mathrm{diag}(c_1, \ldots, c_n)$. When a large control signal is applied with $c_i \to \infty$ for $i \in P$, this implies that $x_i \to u(t)$, i.e. the states of the i-th system is forced to approach the trajectory $u(t)$.

We say that control is achieved in (6) if $x_i \to u(t)$ for all i.

Some questions we are interested in answering are:

- Under what conditions will control be achieved?
- Where shall we apply control to be the most effective?
- How much control needs to be applied to the network in order to achieve control of all the systems?

Assume that $u(t)$ is a trajectory of the individual dynamical system in the network, i.e.

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = f(u(t),t) \tag{7}$$

Then (7) is a virtual system [14] and by setting $x_{n+1}(t) = u(t)$, we obtain a network of $n+1$ systems with state equations

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = f(x_i,t) - \alpha \sum_j \tilde{L}_{ij} D(t) x_j \tag{8}$$

where $\tilde{L}$ is related to L as

$$\tilde{L} = \begin{pmatrix} L_{11}+c_1 & L_{12} & & \dots & L_{1n} & -c_1 \\ L_{21} & L_{22}+c_2 & L_{23} & \dots & L_{2n} & -c_2 \\ \vdots & & & & & \vdots \\ & & & \dots & L_{nn}+c_n & -c_n \\ 0 & & & & & 0 \end{pmatrix}$$

and the *control* problem is reduced to a *synchronization* problem. Control is achieved in (6) under the assumption in (7) if the extended system in (8) synchronizes. We next look at how properties of L and $L + C$ are useful in deriving a criterion for achieving control in (6) in this case.

The matrix $\tilde{L}$ can be written as

$$\begin{pmatrix} L+C & -c \\ 0 & 0 \end{pmatrix}$$

where C is a diagonal matrix with c_i on the diagonal and c is the vector of c_i's. Consider $\tilde{L}$ written in Frobenius normal form, i.e.

$$\tilde{L} = Q \begin{pmatrix} B_1 & B_{12} & \cdots & B_{1q} \\ & B_2 & \cdots & B_{2q} \\ & & \ddots & \vdots \\ & & & B_q \end{pmatrix} Q^{\mathrm{T}} \tag{9}$$

where Q is a permutation matrix and B_i are square irreducible matrices. The Frobenius normal form is not unique, but we pick Q such that $B_q = 0$ is a scalar corresponding to the virtual system.

Then Theorem 3 applied to (8) results in the following condition for achieving control:

Theorem 4. *Control is achieved in (6) if*

1. *L is a zero row sums matrix with nonpositive off-diagonal elements.*
2. *$f(x,t) - D(t)x$ is V-uniformly decreasing for some symmetric positive definite matrix V,*
3. *$VD(t)$ is symmetric positive semidefinite for all t,*
4. *$\beta_{\min} \geq \frac{1}{\alpha}$, where $\beta_{\min} \stackrel{\text{def}}{=} \min_{i<q} \beta(B_i)$.*

In Theorem 4 the inequality $\beta_{\min} \geq \frac{1}{\alpha}$ is a sufficient condition to achieve control. An interpretation of Theorem 4 is that the larger $\beta_{\min}$ is, the easier it is to achieve control. Assume now that $\beta_{\min} = \frac{1}{\alpha}$ is a *sufficient and necessary* condition for achieving control and use $\beta_{\min}\alpha$ as a proxy to denote the *control effectiveness*. In particular, the rest of this chapter will focus on studying $\beta_{\min}$ and how it depends on properties of the graph.

Theorem 5 ([16]). *$\beta_{\min} \geq 0$. Furthermore, $\beta_{\min} > 0$ if and only if there exists a spanning directed forest in the interaction graph of L such that $c_i > 0$ whenever the i-th system is a root of a tree in the forest.*

If $\beta_{\min} = 0$, then for all spanning directed forests there is a tree whose root r satisfies $c_r = 0$. This means that there are systems that do not receive directly or indirectly any external forcing. It follows that in general control cannot be achieved. On the other hand, if $\beta_{\min} > 0$, then under the conditions of Theorem 4 and *a sufficiently large enough α* we can ensure that control is achieved. This can be paraphrased as:

Control can be achieved in a network of dynamical systems if and only if sufficiently strong forcing is applied to roots of trees in a spanning directed forest of the interaction graph of L.

Since the paths from the roots of these trees cover all vertices, this statement shows which systems should receive coupling in order to achieve control, i.e. those vertices which together directly or indirectly influence all other vertices, which is an intuitive conclusion. These vertices can be identified as follows.

To determine the spanning directed forest of the graph of L, we utilize the Frobenius normal form. The Frobenius normal form of L can be chosen to look like

$$L = Q \begin{pmatrix} B_1 & B_{12} & \cdots & & & & B_{1,k+m} \\ & \ddots & & & & & \\ & & B_k & B_{k,k+1} & \cdots & & B_{k,k+m} \\ & & & B_{k+1} & 0 & & 0 \\ & & & & \ddots & & 0 \\ & & & & & & B_{k+m} \end{pmatrix} Q^{\mathrm{T}} \tag{10}$$

where for $1 \leq i \leq k$, there exists at least one $i + 1 \leq j \leq k + m$ such that the submatrix $B_{i,j}$ is nonzero. Such a matrix is called m-reducible [29]. The quantity m denotes the minimum number of trees needed in a spanning directed forest.[1] The matrices B_i, for $k + 1 \leq i \leq k + m$ correspond to the strongly connected components (SCC) of the roots of m trees in a spanning directed forest.

Thus any spanning directed forest must have a root in each of these m strongly connected components. In order to achieve control, forcing must be applied to a node in each of the m strongly connected components. This is illustrated schematically in Fig. 2, where we use the term *residual* vertices to denote vertices in the graph of L that are not in the m strongly connected components. The residual vertices correspond to the indices in B_i, $1 \leq i \leq k$ in (10). Thus there are k SCCs (referred to as RSCC) within the set of residual vertices that can be decomposed by looking at the Frobenius form of the submatrix of L restricted to the residual

[1]The only exception here is the case where L is irreducible, in which case $m = 0$, but there still exist a spanning directed tree in the graph. This case can be treated similar to the case $m = 1$.

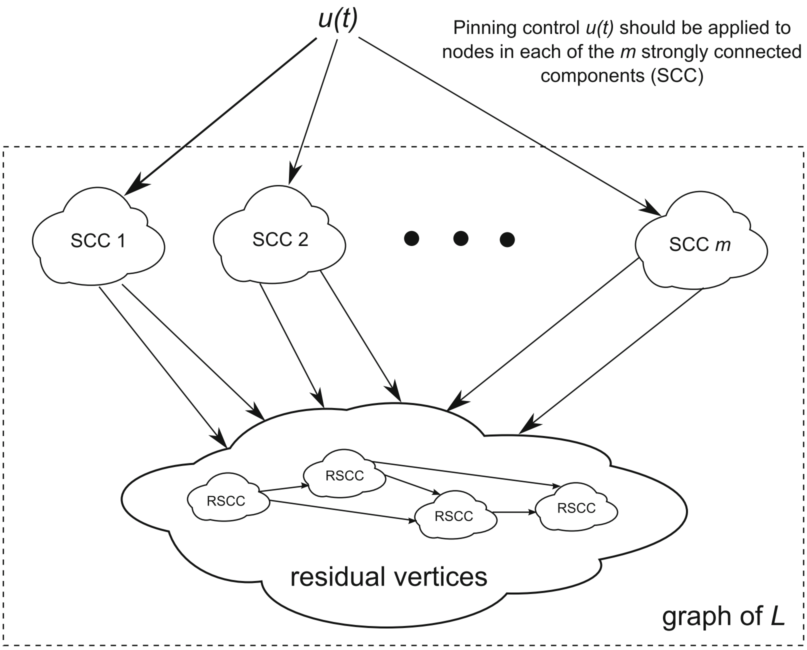


Fig. 2 The m strongly connected components (SCC) influence the residual vertices, but not each other. Forcing term $u(t)$ should be applied to at least one node in each of the m SCCs in order to achieve control. Decomposing the subgraph corresponding to the residual vertices lead to k SCCs (denoted as RSCC) corresponding to the matrices $B_1, \ldots, B_k$ in (10)

vertices. If we collapse these k RSCC, the connections between them form a directed acyclic graph (DAG).

Theorem 6 ([29]). *Let A be an irreducible matrix with nonnegative row sums and nonpositive off-diagonal elements. Suppose that A is decomposed as $A = L + C$, where L has zero row sums and $C = \mathrm{diag}(c_1, \ldots, c_n)$ is a diagonal matrix. Then*

$$\min_{\lambda} \mathscr{Re}(\lambda(A)) \geq \beta(A)$$

If $C \neq 0$, then

$$\begin{aligned}\beta(A) &\geq \frac{\lambda_2}{\left(\frac{\sqrt{\sum_i (w_i c_i)^2} + \sqrt{\sum_i w_i c_i (w_i c_i + \lambda_2)}}{\sum_i w_i c_i}\right)^2 n + 1} \\ &\geq \frac{\lambda_2}{\left(1 + \sqrt{1 + \frac{\lambda_2}{\sum_i w_i c_i}}\right)^2 n + 1} > 0\end{aligned}$$

where $w = (w_1, \ldots, w_n)$ *is a positive row (eigen)vector such that* $wL = 0$ *and* $\max_i w_i = 1$, $W = \mathrm{diag}(w_1, \ldots, w_n)$ *and* $\lambda_2 = \lambda_2(\frac{1}{2}(WL + L^{\mathrm{T}}W)) > 0$ *is the second smallest eigenvalue of* $\frac{1}{2}(WL + L^{\mathrm{T}}W)$.

Note that $\lambda_2(\frac{1}{2}(WL + L^{\mathrm{T}}W))$ can be considered as the algebraic connectivity of the directed graph of the matrix L [30]. Consider the quantity $\beta_{\min}$ in Theorem 4. In terms of the Frobenius form decomposition of L in (10), $\beta_{\min} = \min_{1 \leq i \leq k+m} \beta(B_i)$. Let us now look at how $\beta(B_i)$ depends on the underlying graph. Each of the matrices B_i in (10) can be written in the form $L_i + D_i$, where L_i is a zero row sum matrix and D_i is diagonal. For $1 \leq i \leq m$, L_{k+i} is the Laplacian matrix of the i-th SCC in Fig. 2 and D_i corresponds to the values of the control parameters c_i. For $1 \leq i \leq k$, L_i corresponds to the Laplacian matrices of the Residual SCCs (RSCC) and D_i corresponds to the weighted coupling from other SCCs. Theorem 6 indicates two ways to increase $\beta(B_i)$. Either by increasing the values in D_i (which are either the control strength c_i or coupling into each of the RSCC) or increasing the algebraic connectivity of the graph corresponding to L_i.

The above discussion suggests that $\beta_{\min}$ depends on the algebraic connectivity of the two sets of SCCs in Fig. 2. For the m SCCs it also depends on the control parameters c_i's that are applied to them, whereas for the k RSCCs in the set of residual vertices it depends on the links from the first set of SCCs and the links between the RSCCs.

The special case when the interaction graph of G contains a spanning directed tree (i.e., $m \leq 1$ and the spanning directed forest has only one tree) was studied in [13].

5 Strongly Connected Directed Graphs

So far, we have given necessary conditions on the topology of the graph to achieve control and show that control should be applied to some node in each SCC in the decomposition in Fig. 2. Next we look at which of the nodes in the SCC is most useful to apply control and how it relates to the topology of the SCC.

Consider the two types of parameters in the coupled network in (6): α and c_i. The parameter α is a global parameter that describes the strength of the coupling between *all* the systems, whereas the parameters c_i (in conjunction with α) describe the strength of the control applied to the i-th system. This is illustrated in Fig. 3. We say that the network is harder to achieve control if a larger c_i or α are needed. Next we study how the topology of the network affects the requirements for these two types of parameters. Let us assume that the graph of L is strongly connected, i.e. the matrix L is irreducible.

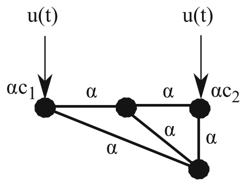


Fig. 3 The parameters α and c_i in (6). The parameter α describes the coupling strength between all systems and c_i describes the strength of the control applied to vertices in P

Theorem 7 ([29]). *Let A be a normal*[2] *square matrix decomposed as $A = L + C$ where L is a real square matrix with zero row sums and nonpositive off-diagonal elements and $C = \mathrm{diag}(c_1, c_2, \ldots, c_n)$ is a nonnegative diagonal matrix. Then*

$$\min_k \mathrm{Re}(\lambda_k(A)) \le \frac{1}{n}\sum_i c_i$$

$$\min_k \mathrm{Re}(\lambda_k(A)) \le \lambda_2\left(\frac{1}{2}\left(L + L^{\mathrm{T}}\right)\right) + c_s$$

where c_s is the second largest c_i.

Corollary 1. *If A is a normal matrix as in Theorem 7, then*

$$\beta_{\min}(A) \le \frac{1}{n}\sum_i c_i$$

Consider the case where the interaction graph of L is vertex-balanced.[3] This implies that L is a normal matrix. In this case L is a normal irreducible matrix with zero row and column sums and $\beta_{\min} = \beta(L + C) = \lambda_1\left(\frac{1}{2}\left(L + L^{\mathrm{T}}\right) + C\right)$.

5.1 Only a Single System Receives Control ($p = 1$)

In this case there is control on only one system, i.e. $c_1 > 0$ and $c_j = 0$ for $j > 1$. This along with Corollary 1 implies that for a fixed coupling parameter c_1, $\beta_{\min} \le \frac{c_1}{n}$ decreases as least as fast as $\frac{1}{n}$ as n increases.

Let us now keep α fixed and sufficiently large. If we also keep c_i fixed, then the above discussion shows that $\beta_{\min} \to 0$ as $n \to \infty$ and thus control cannot be

[2] A matrix is normal if $A^H A = AA^H$.

[3] A directed graph is vertex-balanced if the (weighted) indegree of each vertex is equal to its (weighted) outdegree. Note that undirected graphs are vertex-balanced.

achieved for large n. A way to paraphrase this is that when the underlying topology is a vertex-balanced graph, it takes more effort to achieve control by forcing a single system as the number of systems becomes large. Thus as the number of vertices n grows, the control strength c_1 needs to grow by at least on the order of n in order to keep enforcing control.

For fully connected graphs and random graphs [31, 32], their algebraic connectivities grow on the order or n and Theorem 6 shows that c_1 growing on the order of n is sufficient for control as $\beta(A)$ is bounded away from 0. On the other hand, Theorem 7 shows that $\beta_{\min} \leq c_2 + \lambda_2\left(\frac{1}{2}\left(L + L^{\mathrm{T}}\right)\right) = \lambda_2\left(\frac{1}{2}\left(L + L^{\mathrm{T}}\right)\right)$ and for the nearest neighbor undirected graph, $\lambda_2(L)$ decreases on the order of $\frac{1}{n^2}$ and thus as $n \to \infty$, control is *not* possible regardless of how large c_1 is. In fact, since $\lambda_2\left(\frac{1}{2}\left(L + L^{\mathrm{T}}\right)\right) \to 0$ as $n \to \infty$ for locally connected[4] graphs (which include the nearest neighbor graphs) [30, 33], we have shown two extremes in the ability to apply control. If the graph is random, fully connected or Ramanujan,[5] then a large enough c_1 will achieve control for any n. If the graph is locally connected, control is not possible for large n and a fixed α, even if c_1 is arbitrarily large. In this case α needs to increase as n increases in order to ensure control. In other words, for systems connected via a locally connected network to maintain control for increasing n, it is not sufficient to only increase the control strength, but the coupling among *all* systems also needs to increase. This shows how the topology of the graph can influence the ability to effectively apply control. To illustrate this, Fig. 4 shows the value of $\beta_{\min}$ for various values of c_1 and different size graphs. We see that for a fixed c_1 the value of $\beta_{\min}$ decreases as the number of vertices n increases for both the fully connected graphs and the 1-D nearest neighbor graphs.[6] This shows that a fixed control strength c_1 on a single system will not maintain control as n increases. For fully connected graphs, increasing c_1 will increase $\beta_{\min}$ and allow control to be achieved. However, for nearest neighbor graphs, increasing c_1 does not increase $\beta_{\min}$, and this implies that control cannot be achieved regardless of how large c_1 is.

Note that this need for increasing c_i (and α) is not true for directed graphs that are not strongly connected (and thus cannot be vertex-balanced). In this case, we need to look at the decomposition in Fig. 2. For instance, if the matrix L is triangular, then $\beta_{\min}$ does not depend on the number of vertices n since all the SCC's are single nodes.

If the graph of L is not strongly connected, then the strongly connected components correspond to the matrices B_i in (10). The value of $\beta_{\min}$ depends on

[4]Roughly speaking, a locally connected graph is a graph where vertices are only connected to neighboring vertices. See Definition 8 in Sect. 8 for a precise definition of a locally connected graph.

[5]Since Ramanujan graphs also have the property that their algebraic connectivities grow on the order of n [34].

[6]These are graphs where the vertices are arranged in a line and are connected only to their nearest neighbors. They are also known as path graphs.

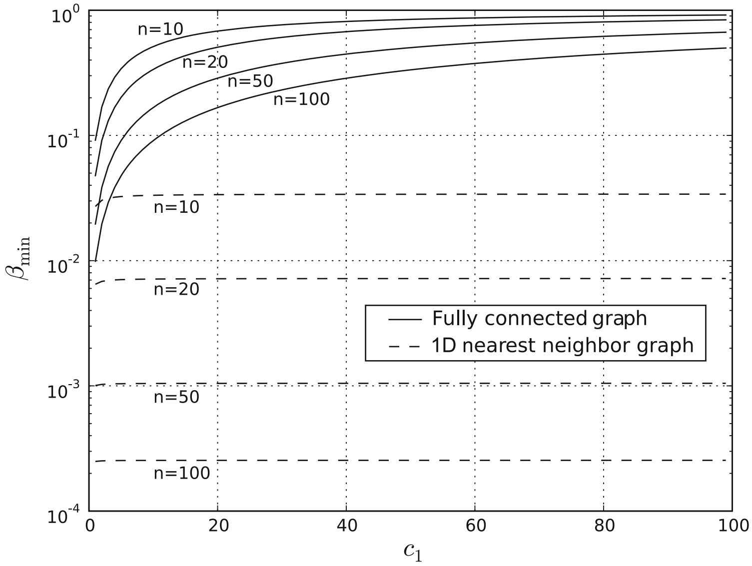


Fig. 4 The value of $\beta_{\min}$ as the control parameter c_1 is varied for fully connected graphs and 1-D nearest neighbor graphs with various number of vertices n

the algebraic connectivity of the strongly connected components, the control matrix C and the amount of coupling among the strongly connected components. In the bound for $\beta(B_i)$, the matrix C defined in Theorem 6 would correspond to control applied to the i-th strongly connected components *and* coupling from other strongly connected components.

6 Undirected Graphs

Consider the case where L is symmetric, i.e. the underlying graph is undirected. In this case $\beta_{\min} = \lambda_{\min}(L + C)$ and we want to study what properties of L and C contributes to maximizing (or minimizing) $\beta_{\min}$. Note that if L is the Laplacian matrix of the graph (V, E, W), then $L + C$ is the Laplacian matrix of the VE-weighted graph (V, E, W, C).

For a given network of systems (6), where the underlying topology is expressed as a weighted graph with Laplacian matrix L, we choose how many systems to apply a control signal to, which systems to apply it to and how large the control gain c_i is. We describe this by specifying the coupling matrix $C = \text{diag}(c_1, \ldots, c_n)$.

What can we say about the matrix C such that control of the network in (6) is achieved? Based on the discussion above, we can attack this problem by looking at conditions for C such that $\lambda_{\min}(L+C) \geq \alpha$. Of particular interest is how it depends on the values of p and P.

The next Lemma establishes the monotonicity of $\lambda_{\min}$, i.e. adding more edges or more control will not decrease $\lambda_{\min}$.

Lemma 1. *Consider two VE-weighted graphs denoted as $\mathcal{G}_1 = (V, E_1, W, C)$ and $\mathcal{G}_2 = (V, E_2, U, F)$ where $C = (c_1, \ldots, c_n)$, $F = (f_1, \ldots, f_n)$, $W = (w_1, \ldots, w_n)$, and $U = (u_1, \ldots, u_n)$. If $c_i \leq f_i$ and $w_i \leq u_i$ for all i, then $\lambda_{\min}(L_1) \leq \lambda_{\min}(L_2)$ where L_1 and L_2 are the Laplacian matrices of the VE-weighted graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively.*

Proof. Note that $L_2' - L_1'$ is symmetric positive semidefinite where L_1' and L_2' are the Laplacian matrices of the weighted graphs (V, E_1, W) and (V, E_2, U), respectively. The result then follows from the fact that L_1' is symmetric and thus by Theorem 1 $\lambda_{\min}(L_1' + C) = \min_{x \neq 0} \frac{x^T(L_1'+C)x}{x^T x} \leq \min_{x \neq 0} \frac{x^T(L_2'+F)x}{x^T x} = \lambda_{\min}(L_2' + F)$. □

6.1 Case $p = n$: Every System Receives Control

Let $c = \min_{c_i \neq 0} c_i$. As $C \succeq cI$, this implies that $\lambda_{\min}(L + C) \geq \lambda_{\min}(L + cI) = \lambda_{\min}(L) + c = c$. Thus λ_m can be made arbitrarily large by choosing c large, i.e. if all systems are controlled, control can be achieved by making the control gains $c_i > 0$ large enough.

6.2 Case $p < n$: Some System does not Receive Control

The scenario is very different from Sec. 6.1 if $c_i = 0$ for some i, i.e. some systems do not receive any control and $p < n$. As we show in Lemma 2, even if the nonzero c_i are arbitrarily large, $\lambda_{\min}(L + C)$ will still remain bounded. The results in this section list various lower and upper bounds for $\lambda_{\min}(L + C)$ that are related to properties of the underlying graph.

Lemma 2.

$$\frac{\lambda_2(L)}{\left(1 + \sqrt{1 + \frac{\lambda_2(L)}{\sum_i c_i}}\right)^2 n + 1} \leq \lambda_{\min}(L + C) \leq \frac{\sum_i c_i}{n} \tag{11}$$

If $p < n$, then

$$\lambda_{\min}(L + C) \leq \lambda_{p+1}(L) \tag{12}$$

Proof. The proof of (11) follows from Theorems 6 and 7. Equation (12) is a consequence of Theorem 2. By setting $j = p + 1$, $k = n - p$ into (2) we get $\lambda_1(L + C) \leq \lambda_{p+1}(L) + \lambda_{n-p}(C)$. Note that C has only p nonzero values on the diagonal, and thus it has $n - p$ zero eigenvalues, i.e. $\lambda_1(C), \ldots, \lambda_{n-p}(C) = 0$. □

Equation (12) suggests that if some system did not receive any control ($p < n$), then control may not be possible if the eigenvalues of L are small, even if the nonzero control gain coefficients c_i are arbitrarily large. This can provide us with guidance on how many systems forcing needs to be applied to. For instance, if $\lambda_{\min}(L + C)$ needs to be larger than a value γ in order to achieve control, where $\gamma \geq \lambda_{p+1}(L)$, then it is necessary to apply control to at least $p + 1$ systems.

One consequence of (11) is that when $\sum_i c_i \to 0$, $\lambda_{\min}(L + C) \to \frac{\sum_i c_i}{n}$.

Definition 5. The *isoperimetric ratio* $r(V')$ of a subset of vertices $V' \subset V$ is the number of edges between V' and $V \backslash V'$ divided by the number of vertices in V'.

Theorem 8. *If* $\emptyset \neq V' \subset V \backslash P$*, then* $\lambda_{\min}(L + C) \leq r(V')$.

Proof. Let v be a vector with $v_i = 1$ for $i \in V'$ and $v_i = 0$ otherwise. It is easy to show that $v^{\mathrm{T}}(L + C)v = v^{\mathrm{T}}Lv$ is equal to the number of edges between V and $V \backslash V'$. It then follows that $\lambda_{\min}(L + C) \leq \frac{v^{\mathrm{T}}(L+C)v}{v^{\mathrm{T}}v} = r(V')$. □

What is interesting to note is that this upper bound on $\lambda_{\min}(L+C)$ is independent of the values of c_i, and thus this upper bound is more useful than $\frac{\sum_i c_i}{n}$ in Lemma 2 when the values of c_i are large.

Corollary 2. *If* $p < n$*, then* $\lambda_{\min}(L + C) \leq p$.

Proof. Let $V' = V \backslash P$. Then $|V'| = n - p$ and the number of edges between V' and $V \backslash V'$ is at most $p(n - p)$. □

We will show in Sect. 7.1.3 that this bound is achieved for the complete graph, when $c \to \infty$.

Corollary 3. *If* $p < n$*, then* $\lambda_{\min}(L + C) \leq \frac{\min\left(\sum_{v \in P} \delta_v, \sum_{v \notin P} \delta_v\right)}{n-p}$*, where* δ_v *is the degree of the vertex* v.

Proof. It is clear that the edges between P and $V \backslash P$ are all edges connected to $V \backslash P$ or to P and thus their number is less than or equal to both numbers $\sum_{v \in P} \delta_v$, $\sum_{v \notin P} \delta_v$ and the results follows from Theorem 8. □

Corollary 3 seems to suggest that when p is small (e.g., $p < \frac{n}{2}$) control should be applied to vertices of high degree for more effective control, whereas when p is close to n (e.g., $p > \frac{n}{2}$), the vertices with no control applied ($V \backslash P$) should be vertices of high degree for more effective control. This principle can be illustrated with the star graph of n vertices which has one central vertex v of degree $n - 1$ surrounded by $n - 1$ vertices of degree 1. If $p = 1$, setting $P = \{v\}$ will maximize $\lambda_{\min}(L + C)$ whereas when $p = n - 1$, setting $P = V - \{v\}$ will maximize $\lambda_{\min}(L + C)$ among all configurations.

Corollary 4. *If* $p < n$*, then* $\lambda_{\min}(L + C) \leq \min\left(\frac{p}{n-p}, 1\right) \delta_{\max}$ *where* $\delta_{\max}$ *is the maximal vertex degree of the graph.*

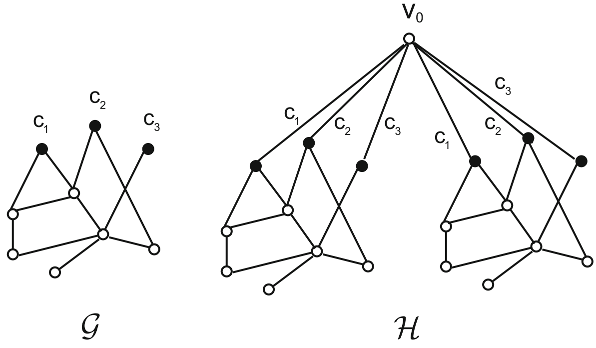


Fig. 5 The weighted graph $\mathscr{H}$ is generated from the VE-weighted graph $\mathscr{G}$ by connecting two copies of $\mathscr{G}$ via an additional vertex v_0. The filled-in vertices of $\mathscr{G}$ indicate vertices where $c_i \neq 0$, i.e. a member of P

The first inequality of Lemma 2 relates $\lambda_{\min}(L+C)$ to the algebraic connectivity λ_2 of the underlying graph with Laplacian matrix L. The next result relates $\lambda_{\min}$ of the Laplacian matrix of an VE-weighted graph[7] to the algebraic connectivity λ_2 of a related weighted graph. For a VE-weighted graph $\mathscr{G}$, construct a graph $\mathscr{H}$ by taking two copies of $\mathscr{G}$ (minus the vertex weights c_i) and adding a new vertex v_0. For each $i \in P$, add an edge of weight c_i from v_0 to vertex v_i of each copy of $\mathscr{G}$ (Fig. 5).

Theorem 9.

$$\lambda_{\min}(L(\mathscr{G})) \geq \lambda_2(L(\mathscr{H}))$$

Proof. Let v be the unit norm vector that minimizes $L(\mathscr{G})$, i.e

$$v^{\mathrm{T}} L(\mathscr{G}) v = \min_{x \neq 0} \frac{x^{\mathrm{T}} L(\mathscr{G}) x}{x^{\mathrm{T}} x} = \lambda_{\min}(L(\mathscr{G}))$$

Let $w^{\mathrm{T}} = (v^{\mathrm{T}}, 0, -v^{\mathrm{T}})$, where v and $-v$ corresponds to the two copies of $\mathscr{G}$ and 0 corresponds to vertex v_0. Since $\sum_i w_i = 0$ and $w^{\mathrm{T}} w = 2$, Theorem 1 shows that $w^{\mathrm{T}} L(\mathscr{H}) w \geq 2\lambda_2(L(\mathscr{H}))$. It is easy to see that $w^{\mathrm{T}} L(\mathscr{H}) w = 2v^{\mathrm{T}} L(\mathscr{G}) v$ and thus $\lambda_{\min}(L(\mathscr{G})) \geq \lambda_2(L(\mathscr{H}))$. □

Corollary 5. *For a VE-weighted graph $\mathscr{G}$ with n vertices and $P \neq \emptyset$,*

$$\lambda_{\min}(L(\mathscr{G})) = \lambda_{\min}(L + C) \geq 2c_m \left(1 - \cos\left(\frac{\pi}{2n+1}\right)\right)$$

where $c_m = \min_{i \in P}\{c_i, 1\}$.

Proof. We only need to prove the case $c_i = 1$ for all $i \in P$, as the other cases are similar. In this case $\mathscr{H}$ is a graph with $2n+1$ vertices. For a graph with n vertices, $\lambda_2(L) \geq 2\left(1 - \cos\left(\frac{\pi}{n}\right)\right)$ (see [35]) and the result follows. □

[7] Recall that the Laplacian matrix of a VE-weighted graph is $L + C$.

The bound in Corollary 5 is tight. For instance, for the path graph P_n with $c_1 = 1$, $c_i = 0$ for $i > 1$, we have $\lambda_{\min}(L + C) = 2\left(1 - \cos\left(\frac{\pi}{2n+1}\right)\right)$ (see, e.g., [36, 37]).

7 Localization of Control

For a fixed $1 \le p < n$ how does the choice of P, i.e. the set of vertices for which c_i is nonzero, affects $\lambda_{\min}(L + C)$? We are interested in the configuration that maximizes or minimizes $\lambda_{\min}(L + C)$.

7.1 The Case of Arbitrarily Large Control where $c_i \to \infty$

In this section we study the case $c_i \to \infty$ as we can more easily find explicit configurations P that maximize or minimize $\lambda_{\min}(L + C)$ for certain classes of graphs. For a fixed set of indices P, let $C(P, c)$ be the diagonal matrix such that $c_i = c$ for $i \in P$ and $c_i = 0$ otherwise. Define $\kappa(P) = \lim_{c\to\infty} \lambda_{\min}(L + C(P, c))$, $\eta_{\max}(p) = \sup_{|P|=p} \kappa(P)$ and $\eta_{\min}(p) = \inf_{|P|=p} \kappa(P)$.[8]

Theorem 10. *If $P \neq \emptyset$, then $\kappa(P) \ge \eta_{\min}(P) \ge \frac{\lambda_2(L)}{4n+1}$.*

Proof. Follows from (11) in Lemma 2. □

Theorem 10 suggests that when we allow the forcing strengths c_i to be large, a network whose underlying graph has a large algebraic connectivity $\lambda_2(L)$ is easier to control.

Lemma 3. *Let L' be the principal submatrix corresponding to the indices $V \backslash P$. Then $\kappa(P) = \lambda_{\min}(L')$.*

Proof. Let v be a unit eigenvector of $L + C(P, c)$ corresponding to $\lambda_{\min}(L + C(P, c))$. For $i \in P$, v_i vanishes as $c \to \infty$. Let w be the subvector of v restricted to $V \backslash P$. Then $w^{\mathrm{T}}(L + C(P, c))w = w^{\mathrm{T}} L' w$ and this also minimizes $w^{\mathrm{T}} L' w$ among all unit vectors w and thus is equal to $\lambda_{\min}(L')$. □

For a fixed p, the configuration that maximizes (minimizes) $\lambda_{\min}(L + C)$ as $c_i \to \infty$ attains $\eta_{\max}(p)$ ($\eta_{\min}$). What can we say about these configurations? The next several subsections study various graphs where such optimal configurations can be explicitly found.

[8] Note that the limit in the definition of κ exists since $\lambda_{\min}(L + C(P, c))$ is a monotonically increasing function of C by Lemma 1 and is bounded for $p < n$ by Corollary 2.

7.1.1 Optimal Configurations of Control Locations: Cycle Graphs

Theorem 11. *For a cycle graph of n vertices and $p < n$,*

$$\eta_{\max}(p) = 2 - 2\cos\left(\frac{\pi}{\lceil \frac{n}{p} \rceil}\right)$$

$$\eta_{\min}(p) = 2 - 2\cos\left(\frac{\pi}{n-p+1}\right)$$

Proof. We show that the configuration P which attains $\eta_{\max}$ and $\eta_{\min}$ is the one which spreads out the most and the least, respectively. L' is block diagonal with the block submatrices of the form:

$$\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & -1 & 2 \end{pmatrix} \tag{13}$$

This is a Toeplitz matrix and its smallest eigenvalue is $2 - 2\cos\left(\frac{\pi}{m+1}\right)$ where m is the order of the matrix. Thus the largest (smallest) value for $\kappa(P)$ is achieved when these submatrices are as small (large) as possible. So to maximize $\kappa(P)$, P should be as dispersed as possible in order to "cut" the cycle graph into as many small pieces as possible. Since $|P| = p$, it will cut the graph into p pieces. If P is placed as evenly around the cycle as possible, then the largest piece is of length $\lceil \frac{n-p}{p} \rceil = \lceil \frac{n}{p} \rceil - 1$. The corresponding Toeplitz matrix has its smallest eigenvalue equal to $\eta_{\max}(p) = 2 - 2\cos\left(\frac{\pi}{\lceil \frac{n}{p} \rceil}\right)$. The submatrix is the largest possible if all elements of P are adjacent on the cycle graph, in which case the submatrix is of order $n - p$. □

7.1.2 Optimal Configurations of Control Locations: Path Graphs

Theorem 12. *For a path graph of n vertices and $p < n$,*

$$\eta_{\max}(p) = 2 - 2\cos\left(\frac{\pi}{\lceil \frac{n}{p} \rceil}\right)$$

$$\eta_{\min}(p) = 2 - 2\cos\left(\frac{\pi}{2(n-p)+1}\right)$$

Proof. Similar to Theorem 11, the configuration that attains $\eta_{\max}$ and $\eta_{\min}$ is the one which spreads out the most and the least, respectively, with some edge effects in this case. The proof is similar to that of Theorem 11, except that in this case L' is block diagonal with blocks of the form (13) and of the form (perhaps after a simultaneous row and column permutation)

$$\begin{pmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & -1 & 2 \end{pmatrix} \tag{14}$$

For P being internal vertices of the path graph (i.e., those vertices with degree 2), this splits L' into $p+1$ blocks with $p-1$ blocks of the form (13) and 2 blocks of the form (14). For matrices of the form (14) the smallest eigenvalue is $2-2\cos\left(\frac{\pi}{2m+1}\right)$ (see [36, 37]), i.e. the same as a matrix of the form (13) of order $2m$. Thus the optimal splitting into submatrices is such that the blocks of the form (14) are about half the size as the blocks of the form (13). This means that η_{max} is obtained for a configuration that splits it into $p-1$ blocks of the form (13) of order $\approx \frac{n-p}{p}$ and 2 blocks of the form (14) of order $\approx \frac{n-p}{2p}$. As for $\eta_{\min}$ the biggest block is created when all vertices of P is on one side of the graph, resulting in a single block of the form (14) of size $n-p$. □

It is interesting to note that $\eta_{\max}(p)$ is the same for path graphs and cycle graphs, suggesting that the path graph of n vertices is as easy to control as the cycle graph of n vertices.

If a vertex has low eccentricity, then it has a shorter distance to other vertices. So it is reasonable to suspect that control applied to these vertices are most effective. Thus it is reasonable to suspect that placing P inside the graph center will maximize $\kappa(P)$. Computer experiments show that $\eta_{\max}(1)$ is attained for a configuration such that the vertex in P minimizes the distance to $V\backslash P$ [9] for all graphs of 7 vertices or less. However, for graphs with 8 vertices, there is a graph where the $\eta_{\max}$-maximizing configuration P is not in the graph center.

7.1.3 Optimal Configurations of Control Locations: Complete Graphs

For the complete graph, it is clear that only the cardinality p of the set P and not P itself affects $\lambda_{\min}(L+C)$. The principal submatrix corresponding to $V\backslash P$ is $L^K_{n-p}+pI$ where L^K_{n-p} is the Laplacian matrix of the complete graph of $n-p$ vertices and thus $\eta_{\max}(p)=\eta_{\min}(p)=\kappa(P)=\lambda_{\min}(L^K_{n-p}+pI)=p$ for $p<n$.

[9] That is, the $\eta_{\max}$-maximizing set $P=\{i\}$ is a subset of the graph center.

7.1.4 Optimal Configurations of Control Locations: Type I Trees

Next we find graphs for which we can explicit determine P that maximizes $\kappa(P)$ for $p = 1$.

Theorem 13. *Let Y be the Laplacian eigenvector corresponding to λ_2. If there exists a cut vertex x such that $Y(x) = 0$, then $\eta_{\max}(1) = \kappa(\{x\}) = \lambda_2(L)$.*

Proof. Follows from [38, Corollary 7] and Lemma 2. □

Definition 6 ([39]). A tree is called a *Type I* tree if an eigenvector Y corresponding to λ_2 contains a zero element.

Corollary 6. *If the graph is a Type I tree, then by setting P to be any internal vertex x (i.e., has degree > 1) with $Y(x) = 0$ for some eigenvector Y corresponding to λ_2, we obtain $\eta_{\max}(1) = \lambda_2$.*

7.2 *Localization of Control Sites Given a Finite Control Budget*

Let us now consider the problem of determining which vertices to apply control, i.e. determining the set P, given a finite control budget of the form $\sum_i c_i \leq \gamma$. We like to determine where and how much control should be applied in order to produce more effective control by maximizing $\beta_{\min}$. As before, there is a big difference between the case $p = n$ and the case $p < n$. If the graph is vertex-balanced and control can be applied to every system, a configuration of optimal control that maximizes $\beta_{\min}$ is simply setting $c_i = \frac{\gamma}{n}$ for all i. This is shown in the following result.

Theorem 14. *Let G be the Laplacian matrix of a vertex-balanced graph. Then $\beta_{\min} \leq \frac{\sum_i c_i}{n}$. Furthermore, $\beta_{\min} = \frac{\sum_i c_i}{n}$ if $c_i = c$ for all i.*

Proof. First note that $\beta_{\min} \leq \frac{\sum_i c_i}{n}$ by Corollary 1. Next consider the case where $c_i = c = \frac{\sum_i c_i}{n}$ for all i. Then $C = cI$ and thus $\beta_{\min} = \lambda_1(\frac{1}{2}(G + G^{\mathrm{T}}) + C) = \lambda_1(\frac{1}{2}(G + G^{\mathrm{T}})) + c = c$ where we have used the fact that $G + G^{\mathrm{T}}$ is a singular matrix. □

Thus the upper bound $\frac{\sum_i c_i}{n}$ on $\beta_{\min}$ in Theorem 14 is achieved when all systems are applied control ($p = n$). When $p < n$, this upper bound is also approached for the complete graph when $n \to \infty$. In particular, for the complete graph of n vertices with Laplacian matrix L, $\lambda_2(L) = n$ and Lemma 2 shows (see also [29]) that

$$\frac{1}{\frac{n}{\sum_i c_i} + 2 + 2\sqrt{1 + \frac{n}{\sum_i c_i}} + 2 + \frac{1}{n}} \leq \beta_{\min} \leq \frac{\sum_i c_i}{n}$$

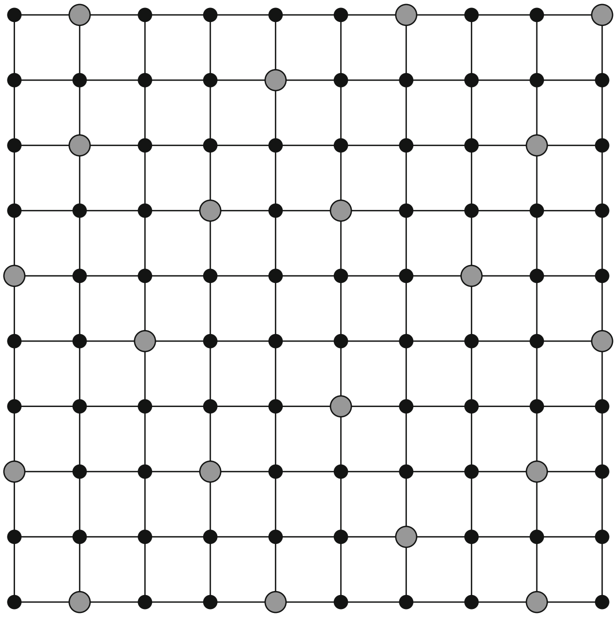


Fig. 6 Control configuration where the value of $\beta_{\min}$ is large. The locations where control is applied are shown larger and in *gray* $\beta_{\min} = 0.7687$

This implies that if $\sum_i c_i$ is bounded for all n, then $\lambda_{\min}(L+C) \to \frac{\sum_i c_i}{n}$ as $n \to \infty$ for the sequence of complete graphs.

But in general the situation is different if we can only apply control to a small number of vertices ($p < n$).

Consider the 2D grid graph of n vertices with Laplacian matrix G_n (Fig. 13). Let us first assume that all the nonzero control strengths are equal, i.e. if $c_i > 0$, then $c_i = c$. Where should the control be applied to maximize $\beta_{\min}$? We performed the following simple experiments to study this question. First p vertices are randomly chosen where control is applied. Then for each vertex with control, its control is moved to another location that increases $\beta_{\min}$. This operation is reiterated until no such move will increase $\beta_{\min}$. We show the resulting configuration of control in Fig. 6. We use $n = 100$, $p = 20$, $\alpha = 1$ and $c = 100$. The locations where control is applied are shown larger and in gray. We see that control is applied to vertices whose locations are spread out in the graph.

The value of $\beta_{\min} = 0.7687$ is significantly lower than the upper bound of $\frac{\sum_i c_i}{n} = 20$. Corollary 4 gives us a better upper bound of 1 which is independent of the value of c_i. In particular the upper bound in Theorem 8 is relatively close to $\beta_{\min}$ as the isoperimetric ratio $r(V \backslash P) = \frac{70}{80} = 0.875$ for this configuration of the set P.

We repeated the same experiment, but now to minimize $\beta_{\min}$. The result is shown in Fig. 7. We see now that the control is applied to vertices whose locations are

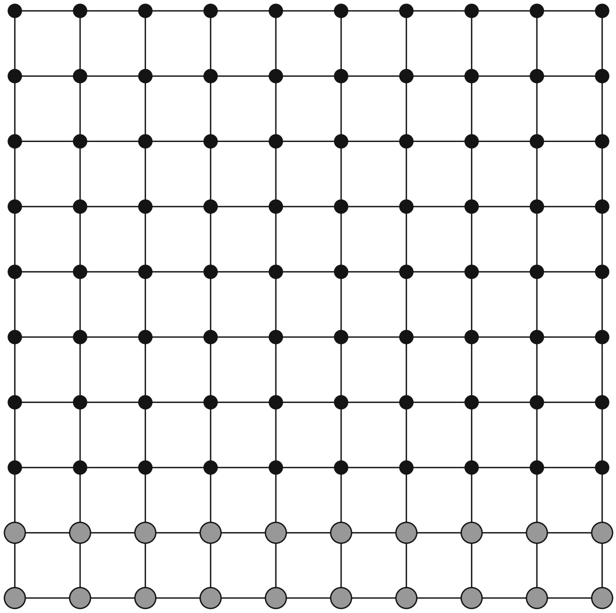


Fig. 7 Control configuration where the value of $\beta_{\min}$ is small. The locations where control is applied are shown larger and in *gray* $\beta_{\min} = 0.034$

close to each other. The difference in $\beta_{\min}$ between these two configuration is more than 20-fold. This experiment suggests that it is more beneficial to apply control at locations which are spread out. One definition of spread out vertices is that the (graph–theoretical) distance between controlled vertices in the graph should not be small. For the grid graph in Fig. 13, the graph–theoretical distance is equivalent to the l_1 distance on the plane.

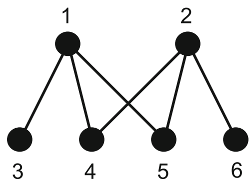
We believe that it is more beneficial to applied control at locations such that the control can easily reach every other system. We consider two quantities that can be used to describe this precisely.

Definition 7. Recall that V denote the set of vertices of the graph and $P \subset V$ the subset of vertices where control is applied.

- Let $d(v, p)$ denote the distance between vertices v and p, i.e. the length of the shortest path between v and p.
- $d(v, P) = \min_{p \in P} d(v, p)$.
- $D_P = \max_{v \in V} d(v, P)$.
- $D_P^a = \frac{\sum_{v \in V \setminus P} d(v, P)}{|V \setminus P|}$

D_P describes the maximal distance between vertices in P and any other vertex,whereas D_P^a describes the average distance between P and the other vertices.

Fig. 8 Graph of 6 vertices. Applying control at vertices 1 or 2 maximizes $\beta_{\min}$. Applying control at vertices 4 or 5 minimizes D_P



The quantity D_P can be used to derive a lower bound on $\beta_{\min}$. In [29] it was shown that if $D_P < \infty$, then

$$\beta_{\min} \geq \frac{c}{2}\left(2\left(r + \frac{c}{2}(2r)^{-D_P}\right)\right)^{-D_P} > 0$$

where r is the maximal vertex degree among vertices of the graph that do not receive control and $c = \min_{i \in P} c_i$. This suggests that P should be chosen to minimize D_P. However, locations for P that minimize D_P do not necessarily maximize $\beta_{\min}$. For instance, for the graph in Fig. 8, assuming a single control ($p = 1$) of strength $c = 10$, applying the control at vertices 4 or 5 minimizes D_P whereas applying the control at vertices 1 or 2 maximizes $\beta_{\min}$.

Let us see whether $\beta_{\min}$ is maximized for a control configuration that minimizes D_P^a.

Statement 1: Under the constraint that $|P| = p$, $\beta_{\min}$ is maximized for a set P such that D_P^a is minimized.

Note that Statement 1 talks about *a* set P since there are in general many sets P which minimize D_P^a, and the statement states that one of them will maximize $\beta_{\min}$.

To provide supporting evidence for Statement 1, we performed the following experiments; 20,000 random sets of 20 control locations are chosen on the grid graph (Fig. 13) and $\beta_{\min}$ and D_P^a are computed. The results are shown in Fig. 9. It is clear that $\beta_{\min}$ tend to be larger for smaller D_P^a.

Next we performed the same experiment on a random graph (Fig. 10) with 100 vertices and 500 edges and a single control location ($p = 1$). Again we see an inverse relationship between $\beta_{\min}$ and D_P^a.

This relationship is again evident when we repeated the experiment with a path graph and parameters $n = 100$, $p = 5$, $c = 100$ (Fig. 11).

Alas, Statement 1 is false in general. For the case $p = 1$, $c = 10$, computer experiments show that Statement 1 is true for all graphs with 6 or less vertices, but there are counterexamples in graphs with 7 vertices.

So far, we have assume that all nonzero c_i are equal. An interesting area of research is to solve the general problem of finite control budget, to find the matrix C that maximizes $\lambda_{\min}(L+C)$ under the constraints that $\sum_i c_i = \gamma$ for some constant γ and the number of nonzero c_i is p. As mentioned before $\lambda_{\min}(L + C) \leq \frac{\gamma}{n}$ by Lemma 2. If p is not fixed, then the answer is clear: set $p = n$ and $c_i = \frac{\gamma}{n}$ for all i. In this case $\lambda_{\min}(L+C) = \lambda_{\min}(L) + \frac{\gamma}{n} = \frac{\gamma}{n}$. For $p < n$, the open question is how to allocate and assign c_i in order to maximize (or minimize) $\lambda_{\min}(L + C)$.

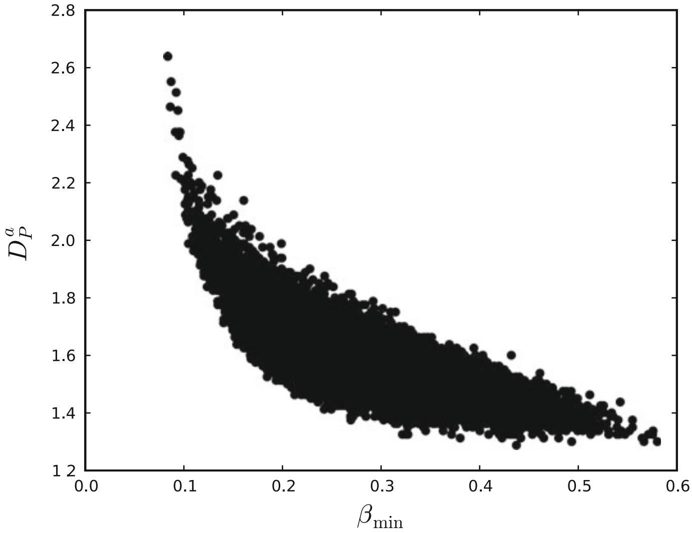


Fig. 9 $\beta_{\min}$ versus D_P^a for 20, 000 random sets of control locations on the grid graph with $n = 100$, $p = 20$, $c = 100$

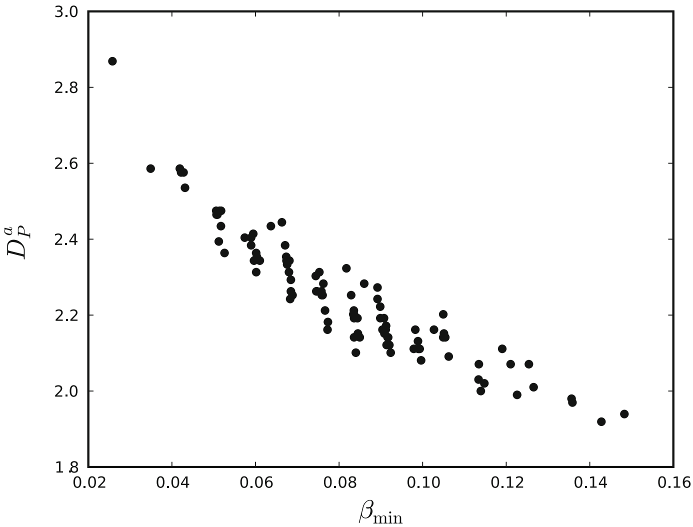


Fig. 10 $\beta_{\min}$ versus D_P^a for various control locations on a random graph with $n = 100$, $p = 1$, and $c = 100$

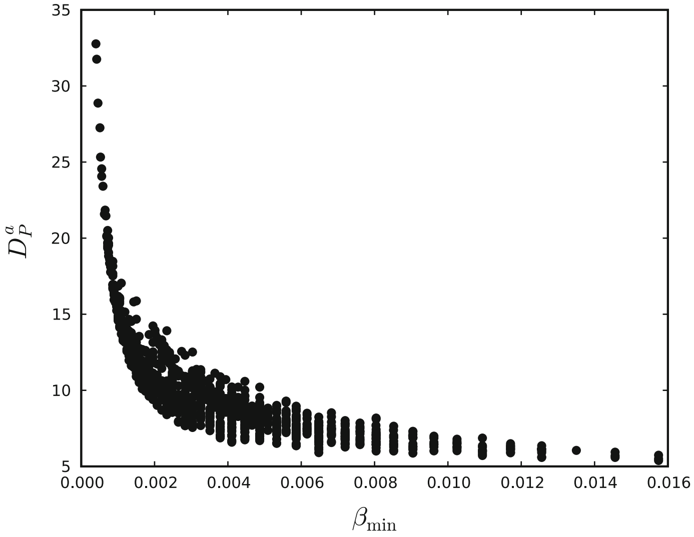


Fig. 11 $\beta_{\min}$ versus D_P^a for $1,000$ random sets of control locations on the path graph with 100 vertices and $p = 5$

8 Asymptotic Behavior for the Case $p < n$ When $n \to \infty$

Earlier in Sect. 5.1 we talked about the asymptotic behavior of a series of graphs for $p = 1$, when only one system is receiving control. Let us now extend this analysis to multiple control sites ($p > 1$).

A consequence of Corollary 5 is that in order to satisfy the control condition $\beta_{\min}\alpha \geq 1$ in undirected graphs, it is sufficient that α grows as $o(n^2)$. On the other hand, recall that $\beta_{\min} \leq \frac{1}{n}\sum_i c_i$. This implies:

If c_i, α are bounded and p grows slower than n, then control is not achievable as $n \to \infty$.

This is illustrated in Fig. 12 where we have computed $\beta_{\min}$ for fully connected graphs where $c_i = 1$, $\alpha = 1$ and $p = \lceil\sqrt{n}\rceil$. This means that if the number of systems where control is applied is small compared with the total number of systems, then the applied control (expressed as αc_i) needs to be large. However, this is not sufficient if the network is locally connected. In particular, we show that for locally connected networks, if p grows slower than n, then control is not possible for a bounded α, regardless of how large the parameters c_i are.

Definition 8 ([30,33]). A locally connected network is defined as a network where the nodes are located on a integer lattice $\mathbb{Z}^d$ and are connected by an edge only if they are at most a distance r apart. The parameters d and r are assumed to be fixed.

It is clear that a subgraph of a locally connected network is also locally connected. An example of a locally connected network for $d = 2$, $r = 1$ is the grid graph shown in Fig. 13.

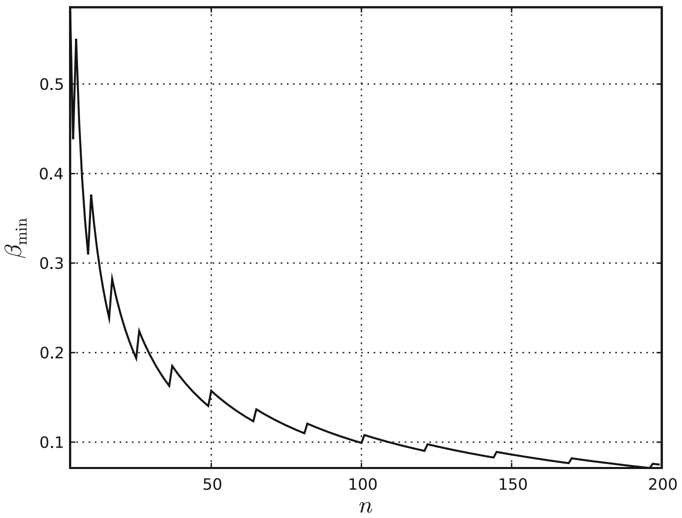


Fig. 12 The value of $\beta_{\min}$ as the number of vertices n is varied for a fully connected graph. The number of systems with control applied is $p = \lceil\sqrt{n}\rceil$. The discontinuity is caused by the discontinuity of $\lceil\cdot\rceil$

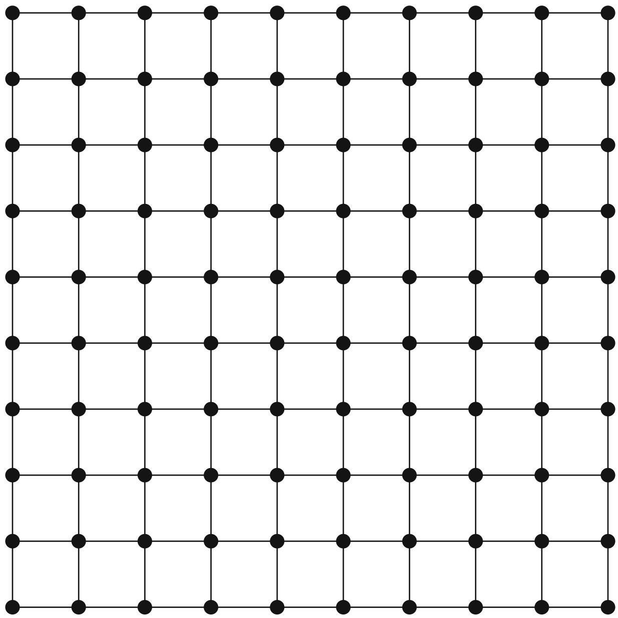


Fig. 13 A locally connected grid graph

Let us assume that α is fixed and that the underlying graph is a locally connected graph. First consider the case where the underlying graph is of the following form: the vertices are arranged in a circle and are connected by an edge if and only if they are less than or equal to r vertices apart. Let us denote this graph as $\mathscr{G}_r$ with Laplacian matrix G_r. For $r = 1$, this is the cycle graph. The eigenvalues of G_r are given by:

$$\lambda_i = 2\left(r - \sum_{l=1}^{r} \cos\left(\frac{2\pi i l}{n}\right)\right), \quad i = 0, \ldots, n-1$$

It is not hard to show that for $p \in o(n) < n$, the smallest $p+1$ eigenvalues of G all converges to 0 as $n \to \infty$. From Lemma 2 it follows that $\lambda_1(G_d + C) \leq \lambda_{p+1}(G_d)$. This implies that $\beta_{\min} \to 0$ as $n \to \infty$.

Next, consider a general locally connected graph with parameters r and d and Laplacian matrix G. It is easy to see that it is a subgraph of a locally connected graph that can be decomposed as the strong product of d graphs of the form $\mathscr{G}_r$. Since the eigenvalues of this graph can be derived from sums and products of eigenvalues of multiple $\mathscr{G}_r$ [40], it is also true that for $p \in o(n)$, the smallest $p+1$ eigenvalues of $G \to 0$ as $n \to \infty$. The same argument as above shows that $\beta_{\min} \to 0$ as $n \to \infty$ in this case as well.

Thus we have shown the following:

For a fixed parameter α, and a locally connected network of n dynamical systems with control applied to p systems, control is not possible as $n \to \infty$ if p grows slower than n.

This is illustrated in Fig. 14, where we show how $\beta_{\min}$ changes as $n \to \infty$. For each n, the graph is a cycle graph of n vertices. We choose $c_i = 100n$, $\alpha = 1$ and $p = \lceil \sqrt{n} \rceil$. We see that $\beta_{\min} \to 0$ as $n \to \infty$.

The above analysis is also valid if the graph is not undirected, but vertex-balanced, i.e. the indegree of each vertex is equal to its outdegree. In this case, the analysis is applied to the symmetric zero row sums matrix $\frac{1}{2}(G + G^{\mathrm{T}})$.

9 The Case of Converging Control Signals

Consider the case where the control signal $u(t)$ applied to the individual systems is not identical, i.e.

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = f(x_i, t) + \sum_j \alpha G_{ij} D(t) x_j + \alpha c_i (x_i - u_i(t)) \tag{15}$$

If $\lim_{t\to\infty} \|u_i - u_j\| \to 0$, i.e. the control signals approach each other asymptotically, then we can still apply the results in [28] and the control results in this chapter will still hold.

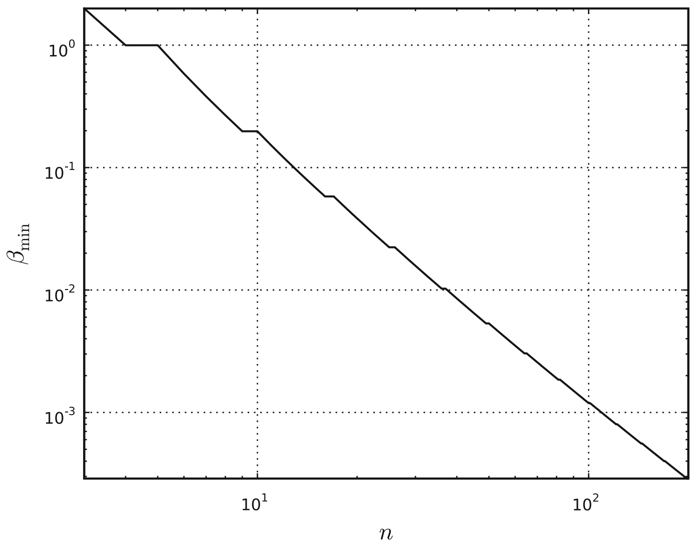


Fig. 14 The value of $\beta_{\min}$ as the number of vertices is varied for a cycle graph

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