Chapter 1
Introduction

Abstract In this introductory chapter, we describe the contents of our book.

On December 11, 1944, A.N. Kolmogorov gave a lecture "Problems of probability theory" at the meeting of the Moscow Mathematical Society. A few pages containing a summary and a sketch of this lecture were found in his archives. They were published in Kolmogorov (1993). The sixth Kolmogorov problem is as follows.

> 6. Distributions of scalar, vector and tensor functions which are invariant with respect to different groups of transformations. This problem is of interest from the point of view of Statistical Mechanics of Continuous Media, in particular, of Statistical Turbulence Theory. (*Translated from the Russian by V.V. Sazonov*.)

Several results concerning this problem were obtained by one of Kolmogorov's pupils, A.M. Yaglom, and published in Yaglom (1957, 1960, 1961, 1963). Applications to Statistical Mechanics of Continuous Media mentioned by Kolmogorov were presented in the books by Yaglom (1987a,b) and Monin and Yaglom (2007a,b).

A new wave of interest in Kolmogorov's sixth problem occurred at the end of the last millennium. Precise measurements of the cosmic microwave background (CMB) showed its fluctuations. Cosmologists assumed that these fluctuations are generated by a homogeneous and isotropic random field. From the observations, one can define the intensity tensor P of the CMB. In fact, it is a tensor-valued function of position $\mathbf{x}$, time t and photon direction $\mathbf{n}$. Since we measure the intensity tensor here ($\mathbf{x}$ is fixed) and now (t is fixed), P becomes a tensor-valued function on the sphere S^2. More precisely, $\mathsf{P}(\mathbf{n})$ is a random section of some tensor bundle over S^2. Therefore, cosmology requires the theory of random sections of vector and tensor bundles.

The author has been interested in questions connected to the sixth Kolmogorov problem since the second half of the 1970s, when he was a graduate student of Taras Shevchenko National University of Kyiv. At that time, the paper by Yaglom (1957) was refereed at the students' seminar in probability led by Professor M.Ĭ. Yadrenko.

In Chap. 2 we consider spectral expansions of invariant random fields in vector bundles. A scalar, i.e. real-valued (resp. complex-valued) random field $X(t)$ on a parametric space T may be considered as a random section of the one-dimensional trivial bundle $T \times \mathbb{R}$ (resp. $T \times \mathbb{C}$) over T. In order to simplify the exposition, we consider the cases of a scalar random field and a vector random field separately.

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All definitions in Subsection 2.1.1 are either classical or directly generalise classical definitions. This is not the case for Subsection 2.1.2, where we consider random sections of vector bundles. Indeed, let $\xi = (E, \pi, T)$ be a vector bundle, and let $\mathbf{X}(t)$ be a random section of the bundle ξ. Let s and t be two points in the base T with $s \neq t$. Random vectors $\mathbf{X}(s)$ and $\mathbf{X}(t)$ lie in different spaces. The following problem arises: how does one define a mean-square continuous random field in this situation?

To overcome this difficulty, we extend an idea of Kolmogorov formulated by him for the case of a finite-dimensional trivial vector bundle and published by his pupils Rozanov (1958) and Yaglom (1961). We define a scalar random field on the total space E, which we call the field *associated* to the vector random field $\mathbf{X}(t)$. Then, we call $\mathbf{X}(t)$ mean-square continuous if the associated scalar random field is mean-square continuous.

Let G be a topological group acting continuously from the left on the base T. We would like to call a vector random field $\mathbf{X}(t)$ *wide-sense left G-invariant*, if the associated scalar random field is wide-sense left G-invariant under some left-continuous action of G on the total space E. However, in general there exists no natural continuous left action of G on E. In Definition 2.12, we define an action of G on E *associated* to its action on the base space T. Then, we call a vector random field $\mathbf{X}(t)$ wide-sense left G-invariant, if the associated scalar random field is wide-sense left G-invariant under the associated action. For the case of the trivial vector bundle, our definition becomes a classical one.

In Example 2.16, we consider an important example of an associated action: the so-called *homogeneous*, or *equivariant* vector bundles. They are interesting for us for several reasons.

On the one hand, they have a natural associated action of some topological group G. Moreover, the above action identifies the vector space fibres over all the points of the base space. Therefore, all random vectors of a random field $\mathbf{X}(t)$ in a homogeneous vector bundle lie in the same space. We prove that for homogeneous vector bundles, our definitions of mean-square continuous field and invariant field are equivalent to the classical ones.

On the other hand, the Hilbert space of the square-integrable sections of a homogeneous vector bundle carries the so-called *induced representation* of the group G. Later on, we use the well-developed theory of induced representations to obtain spectral expansions of invariant random fields in homogeneous vector bundles.

In Example 2.17 we consider two one-dimensional vector bundles over the circle S^1: a trivial bundle (cylinder), and a nontrivial one (Möbius bundle). While there exist a lot of centred mean-square continuous wide-sense left SO(2)-invariant random fields in the trivial bundle, the one and only such field in the Möbius bundle is 0.

We begin our study with the classical case, when the parametric space T is a topological group acting on itself by left translations. The spectral expansion of invariant random fields on a wide class of topological groups was obtained by Yaglom (1961). We formulate his result in our Theorem 2.18.

Examples of using Theorem 2.18 include both classical and new results. In particular, we consider spectral expansions of wide-sense stationary sequences and processes, homogeneous random fields on the lattice $\mathbb{Z}^n$ and on the space $\mathbb{R}^n$, left-homogeneous random fields on low-dimensional Lorentz groups and their covers. The new result here is the description of homogeneous random fields on the group $\mathbb{R}_0^\infty$, which is not locally compact.

Vector invariant random fields in the above situation were also considered by Yaglom (1961). His result is formulated in our Theorem 2.22. We have also corrected some misprints in the original proof. Examples include classical cases of vector homogeneous random fields on commutative locally compact groups with countable base.

The next case is that when the parametric space T is a homogeneous space $T = G/K$, and the group G acts on T by left translations. For simplicity, we consider only the case when K is a massive compact subgroup of the group G. The spectral expansion of the left-invariant random field on T is obtained in Theorem 2.23. This result was also proved by Yaglom (1961).

In Example 2.24 we begin the study of the main objects of this book. We consider the case when T is a two-point homogeneous space. The above case is interesting for the following reasons. On the one hand, it includes classical cases of homogeneous and isotropic random fields on the space $\mathbb{R}^n$ and isotropic random fields on the sphere S^n studied in detail by Yadrenko (1983). On the other hand, the covariance function $R(x, y)$ of such a field depends only on the distance between the points x and y. In other words, R is a function of *one real variable*. This simplifies further studies of various questions connected to such fields.

In Example 2.25 we propose yet another proof of the famous theorem by Schoenberg about the general form of a positive-definite kernel $R(\mathbf{x}, \mathbf{y})$ on the infinite-dimensional separable real Hilbert space $(H, \|\cdot\|)$ that depends only on $\|\mathbf{x} - \mathbf{y}\|$. The spectral expansion of a homogeneous and isotropic field on H is obtained.

In Subsection 2.3.2 we consider an invariant random section $\mathbf{X}(t)$ of a homogeneous vector bundle. Let G be a compact topological group, let K be a compact subgroup, and let W be a unitary representation of K in a finite-dimensional complex Hilbert space H. The unitary representation U of the group G induced by the representation W is realised in the Hilbert space $L^2(G, H)$ of all square-integrable sections of the vector bundle over the homogeneous space $T = G/K$. We introduce a basis in the above space, and find the spectral expansion of the random field $\mathbf{X}(t)$ with respect to the above basis. Note that in the particular case of $G = \mathrm{SO}(3)$, $K = \mathrm{SO}(2)$, $H = \mathbb{C}$, and $W(\varphi) = e^{\mathrm{i}n\varphi}$, where $\varphi \in \mathrm{SO}(2)$ and $n \in \mathbb{Z}$, the elements of the above basis are well known under the name *spin-weighted spherical harmonics*.

In Sect. 2.4 we consider the general case, when G is a compact group acting continuously from the left on a Hausdorff topological space T, and $X(t)$ is a wide-sense mean-square continuous G-invariant random field on T. Under some technical conditions, we obtain the spectral expansion of the field $X(t)$. The above expansion contains the set of uncorrelated centred mean-square continuous random fields $Z_{m\lambda}(s)$ defined on a measurable section S for the action of the group G. In the classical case, when $G = \mathrm{SO}(n)$ acts on $T = \mathbb{R}^n$ by matrix-vector multiplication, we have $S = [0, \infty)$, the corresponding invariant random field is called *isotropic*, and we recover the well-known spectral expansion proved by Yadrenko (1963). In particular, the multiparameter fractional Brownian motion is isotropic. In Example 2.31 we calculate the autocovariance functions of the random processes $Z_{m\ell}(s)$ involved in the spectral expansion of multiparameter fractional Brownian motion.

The vector variant of the previous considerations is represented in Subsection 2.4.2. Let a connected compact Lie group G act continuously from the left on a Hausdorff topological space T. Let λ be an irreducible unitary representation of G. Let $\xi = (E, \pi, T)$ be a vector bundle over T that satisfies the following technical condition. The restriction of the vector bundle ξ to any G-orbit with stationary subgroup K is a homogeneous vector bundle $\eta = (F, \pi, O)$. Moreover, the Hilbert space of square-integrable sections of the bundle η carries the unitary representation of the group G induced by the restriction of λ to K. When the representation λ is trivial, we return to the situation described in the previous paragraph. Under some additional technical conditions we obtain the spectral expansion of an invariant random section $\mathbf{X}(t)$ of the vector bundle ξ. This is the final step in our chain of generalisations.

In the two remaining sections of Chap. 2 we consider two interesting problems. First, we solve the problem formulated by Yaglom (1957) about the spectral expansion of a vector homogeneous and isotropic random field on the space $\mathbb{R}^n$. Our solution involves special numbers, the so-called *Clebsch–Gordan coefficients* of the group $\mathrm{SO}(n)$ (or, more generally, the Clebsch–Gordan coefficients of a compact subgroup G of the group of automorphisms of a commutative locally compact topological group T with countable base). Second, we introduce the so-called *Volterra random fields*. These are isotropic random fields $X(\mathbf{t})$ on the space R^n such that the stochastic processes $Z_{m\ell}(s)$ involved in the spectral expansion of the field $X(\mathbf{t})$ are Volterra processes. For any positive real number R, we find a family of spectral expansions of any Volterra random field. Every expansion of the above family converges in mean-square in the centred ball of radius R in the space $\mathbb{R}^n$. As an example, we calculate the spectral expansion of multiparameter fractional Brownian motion.

We divide all the theory of random fields except their spectral expansions into two areas. The first area does not require any restrictions on the finite-dimensional distributions of the random field under consideration. In other words, we consider the random field as a function with values in the Hilbert space of all centred random vectors with finite expectation of the square of the norm. Such a theory is called L^2 theory and is considered in Chap. 3.

First, we consider the following question. Assume that the random field $X(t)$ is expanded in terms of some random measures Z. How does one restore the measures Z if X is known? In other words, in Sect. 3.1 we prove *inversion formulae* for various spectral expansions obtained in Chap. 2.

The next group of questions of L^2 theory is as follows. Let G be a Lie group that acts on an analytic manifold T. Let L be a linear differential operator on T, and let $\mathscr{A}$ be the set of all centred wide-sense left G-invariant random fields on T. In Sect. 3.2 we consider the following questions.

- Describe the set $\mathscr{A}_L$ of all random fields $X(t)$ such that the field $LX(t)$ may be correctly defined as a result of mean-square differentiation.

- Find sufficient conditions on L for inclusion $L\mathscr{A}_L \subset \mathscr{A}$.
- For each $X \in \mathscr{A}_L$, calculate $\mathsf{E}[X(s)\overline{LX(t)}]$.
- Describe the image $L\mathscr{A}_L$ in terms of the spectral measure of X.

To give an idea of what is described in Sect. 3.3, consider the following situation. Let $X(t)$ be a centred G-invariant random field on a space T. Let K be a subgroup of the group G, and let C be an orbit of the group K in T. Assume that there exists a left K-invariant measure $\mathrm{d}\mu$ on C. Let $f(t)$ be a continuous function on C with compact support. The Bochner integral

$$Y := \int_C X(t) f(t)\,\mathrm{d}\mu(t)$$

defines a random variable Y, which is a vector of the Hilbert space $H^-_X(C)$—the closed linear span of the vectors $X(t)$, $t \in C$ in the Hilbert space of all random variables with finite variance. Is it possible to express *any* element of the space $H^-_X(C)$ in the form of the above equation?

Let t_0 be a fixed point in C. The random variable $X(t_0)$ should have the following representation:

$$X(t_0) = \int_C X(t)\delta(t - t_0)\,\mathrm{d}\mu(t).$$

In other words, we have to be able to define integrals involving random fields and *distributions* like the Dirac delta.

In Sect. 3.3, we describe the space $L^2_-(C, \mathrm{d}\mu)$ of distributions for which the above integral may be correctly defined as an element of the space $H^-_X(C)$. Moreover, the above integral determines an isometric isomorphism between the spaces $L^2_-(C, \mathrm{d}\mu)$ and $H^-_X(C)$. In our proofs, we use the theory of rigged Hilbert spaces, or *Gel'fand triples*. The above spaces are negative spaces in the triples, therefore our notation includes minus as an index.

The second area lies outside L^2 theory. Here, we impose a simple condition on the finite-dimensional distributions of the random field $X(t)$. Namely, in Chap. 4 we consider *Gaussian* invariant random fields and the properties of their sample paths.

It is well known that the above properties can be studied using the Dudley semimetric ϱ_X. Assume that the covariance function of a centred random field $X(t)$ defined on the metric space (T, ϱ), $R(t_1, t_2) = \mathsf{E}[X(t_1)X(t_2)]$, depends only on the distance $\varrho(t_1, t_2)$. For example, this is true when $T = G/K$ is a two-point homogeneous space and X is a G-invariant random field on T. Then, any two balls of the same radius in the space (T, ϱ_X) are isometric. This simplifies investigations of the properties of sample paths of such fields.

In particular, let $R(\theta)$ be the covariance between $X(s)$ and $X(t)$ when the distance between s and t is equal to θ. To estimate the Dudley integral, we have to estimate

$$\varrho^2_X(s, t) := 2\bigl[R(0) - R(\theta)\bigr]$$

in terms of the spectral measure of the random field $X(t)$. Therefore, we need to prove Abelian and/or Tauberian theorems. In Sect. 4.2, we find the upper estimates of the function $R(\theta)$ in the neighbourhood of 0 for invariant random fields on two-point homogeneous spaces. These estimates are used to find the uniform moduli of continuity of the above fields.

More advanced Abelian and Tauberian theorems in terms of regularly varying functions are considered in Sect. 4.3. The above theorems help us to find more precise moduli of continuity.

In Sect. 4.4 we further investigate one of the series expansions of the multiparameter fractional Brownian motion previously obtained in Example 2.39. We prove that the above expansion is rate optimal.

Finally, in Sect. 4.5 we prove a general functional limit theorem for the multiparameter fractional Brownian motion. The functional law of the iterated logarithm, functional Lévy modulus of continuity and many other results are particular cases.

Chapter 5 is devoted to applications. The idea of an application to approximation theory is as follows. Let $X(t)$ be a centred Gaussian random field on a compact metric space T with continuous sample paths, and let P be the corresponding Gaussian measure in the Banach space $C(T)$ of all continuous functions on T equipped with the sup-norm. Let $\mathscr{L}$ be a measurable linear subset of the linear space $C(T)$. The 0–1 law states that either $\mathsf{P}(\mathscr{L})=0$ or $\mathsf{P}(\mathscr{L})=1$. Moreover, in the latter case the reproducing kernel Hilbert space $\mathscr{H}$ of the measure P is the subset of $\mathscr{L}$.

In many cases, the space $\mathscr{H}$ consists of all functions which have some *constructive* property. For example, $\mathscr{H}$ may consist of functions that can be expanded into Fourier series with respect to some exactly determined basis. On the other hand, the linear subset $\mathscr{L}$ consists of all functions which have some *descriptive* property. For example, $\mathscr{L}$ may consist of functions whose modulus of continuity satisfies some conditions. In this situation, the 0–1 law states that some descriptive property of a continuous function on T follows from some constructive property. In other words, the above law is equivalent to a *Bernstein-type theorem* from approximation theory. Several Bernstein-type theorems are proved in Sect. 5.1.

In Sect. 5.2 we consider an application to cosmology. After reading several physical books and papers the author has found a jungle of various choices of coordinates, phase conventions etc. Therefore, Subsection 5.2.1 contains a short introduction to the *deterministic* model of the cosmic microwave background (CMB) for mathematicians. In particular, we discuss different choices of local coordinates in the tangent bundle $\xi=(TS^2,\pi,S^2)$, and fix our choice. We explain both the mathematical and physical sense of the *Stokes parameters* I, Q, U, and V.

The *probabilistic* model of the CMB is discussed in Subsection 5.2.2. We define the set of vector bundles $\xi_s=(E_s,\pi,S^2)$, $s\in\mathbb{Z}$, where the representation of the rotation group $G=\mathrm{SO}(3)$ induced by the representation $W(g_\alpha)=e^{\mathrm{i}s\alpha}$ of the subgroup $K=\mathrm{SO}(2)$ is realised. In particular, the absolute temperature of the CMB, $T(\mathbf{n})$, is a single realisation of a mean-square continuous strict-sense isotropic (i.e., SO(3)-invariant) random field in ξ_0, while the complex polarisation, $(Q\pm\mathrm{i}U)(\mathbf{n})$, is a single realisation of a mean-square continuous strict-sense isotropic random field in $\xi_{\pm 2}$. Because any second-order strict-sense isotropic random field is automatically wide-sense isotropic, Theorem 2.27 immediately gives the spectral expansion of the above random fields. In the case of the absolute temperature, the functions involved in the formulation of the above theorem become familiar *spherical harmonics*, $Y_{\ell m}$, while in the case of the complex polarisation they become *spin-weighted spherical harmonics*, ${}_{\pm 2}Y_{\ell m}$. The expansion coefficients are uncorrelated random variables with finite variance, which do not depend on the index m. In physical terms, the variance as a function of the parameter ℓ is the *power spectrum*.

Following Zaldarriaga and Seljak (1997), we construct the random fields $E(\mathbf{n})$ and $B(\mathbf{n})$. The advantage of these fields is that they are scalar (i.e., live in ξ_0), real-valued, and isotropic. Moreover, only $T(\mathbf{n})$ and $E(\mathbf{n})$ may be correlated, while the two remaining pairs are always uncorrelated.

Our new result is Theorem 5.5. It states that the standard assumption of cosmological theories (the random fields $T(\mathbf{n})$, $E(\mathbf{n})$, and $B(\mathbf{n})$ are jointly isotropic) is equivalent to the assumption that $((Q - \mathrm{i}U)(\mathbf{n}), T(\mathbf{n}), (Q + \mathrm{i}U)(\mathbf{n}))$ is an isotropic random field in $\xi_{-2} \oplus \xi_0 \oplus \xi_2$.

In Sect. 5.3 we consider an application to earthquake engineering. We present an efficient approach for the simulation of homogeneous and partially isotropic random fields based on spectral expansion. It is shown that, by incorporating the partial isotropy of the field into the simulation algorithm, the computational effort required for the simulation is significantly reduced as compared with the case when only the homogeneity of the field is taken into account. The above approach has been applied by Katafygiotis et al. (1999) for simulation of a Gaussian homogeneous and spatially isotropic random field representing the ground motion during a strong earthquake with Clough–Penzien spectrum of the acceleration of apparent propagation of ground motions and Luco–Wong coherency model.

The theory of invariant random fields on spaces with a group action requires good knowledge of various part of mathematics other than Probability and Statistics. In the Appendix (Chap. 6) we discuss differentiable manifolds, vector bundles, Lie groups and Lie algebras, group actions and group representations, special functions, rigged Hilbert spaces, Abelian and Tauberian theorems.

We conclude each chapter and the Appendix (Chap. 6) with bibliographical remarks.

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