2 Modeling the Growth of the World Economy: The Basic Overlapping Generations Model

2.1 Introduction and Motivation

The previous chapter provided a detailed introduction to empirical data concerning the long run growth of the world economy as a whole and to the international economic relations prevailing among nearly 100 nation states. In this chapter we intend to explore within a simple theoretical model the driving forces behind the apparently unbounded growth of the global market economy, and for the moment simply disregard the international relations among countries. In order to go some way towards addressing the frequently expressed fear that globalization has gone too far we begin by envisioning a world economy where globalization has come to an end. In other words, we assume a fully integrated world economy with a single global commodity market and a uniform global labor and capital market. Although the present world economy is still quite far away from achieving full integration dealing with this commonly cited bogey of globalization critics would nevertheless appear worthwhile.

In view of Kaldor's (1961) stylized facts presented in Chap. 1 it is not surprising at all that economic growth attracted the attention of economic theorists in post-WWII period of the 1950s and 1960s. In contrast to the rather pessimistic growth projections of the leading post-Keynesian economists, Harrod (1939) and Domar (1946), the GDP growth rate, especially in countries destroyed in WWII, dramatically exceeded its long run average (of about 2 % p.a.) and remained at the higher level at least for a decade.

As it is well-known, Solow (1956) and Swan (1956) were the first to provide neoclassical growth models of closed economies. This rather optimistic growth models were better akin to the growth reality of the post-war period than the post-Keynesian approaches. However, savings behavior in Solow's and Swan's macroeconomic growth models lacked intertemporal micro-foundations. In order to address this drawback from the perspective of current mainstream growth theory (Acemoglou 2009) our basic growth model of the world economy is based on Diamond's (1965) classic overlapping generations' (OLG) version of neoclassical

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growth theory.[1] This modeling framework enables us to study the relationship between aggregate savings, private capital accumulation and GDP growth within an intertemporal general equilibrium framework. After working through this chapter the reader should be able to address the following questions:

- How can we explain private capital accumulation endogenously on the basis of the rational behavior of all agents in a perfectly competitive market economy?
- Which factors determine the accumulation of capital (investment) and GPD growth, and how do they evolve over time?
- Are there other economic variables determined by the dynamics of capital accumulation?
- Is a world economy with a large savings rate better off than one with a smaller savings rate?

This chapter is organized as follows. In the next section the set-up of the model economy is presented. In Sect. 2.3, the macroeconomic production function and its per capita version are described. The structure of the intertemporal equilibrium is analyzed in Sect. 2.4. The fundamental equation of motion of the intertemporal equilibrium is derived in Sect. 2.5. In Sect. 2.6, the "golden rule" of capital accumulation to achieve maximal consumption per capita is dealt with. Section 2.7 summarizes and concludes.

2.2 The Set-Up of the Model Economy

There are two types of households (= generations) living in the model economy: old households comprise the retired (related symbols are denoted by superscript "2"), and young households represent the "active" labor force and their children (denoted by superscript "1"). Each generation lives for two periods. Consequently, the young generation born at the beginning of period t has to plan for two periods (t, $t+1$), while the planning horizon of retired households consists of one (remaining) period only. In each period, two generations overlap – hence the term *O*ver*l*apping *G*enerations model (or OLG model). The typical length of one period is about 25–30 years. While members of the young households work to gain labor income, members of the old generation simply enjoy their retirement.

For the sake of analytical simplicity, we assume a representative (young) household characterized by a log-linear utility function. Moreover, members of the young generation are assumed to be "workaholics", i.e. they attach no value to leisure. As a consequence, labor time supplied to production firms is completely inelastic to variations in the real wage.[2]

[1] Alternative intertemporal general equilibrium foundations are provided by Ramsey's (1928) infinitely-lived-agent approach which is not dealt with at all in this book.

[2] We make these assumptions to keep the model as simple as possible. They can of course be replaced by more realistic assumptions – e.g. that leisure does have a positive value to households and, thus, labor supply depends on the real wage rate. As e.g. Lopez-Garcia (2008) shows the endogeneity of the labor supply does not alter the main insights concerning growth and public debt.

The utility function of the young generation, born in period t, and at the beginning of period t, is given by:

$$U_t^1 = \ln c_t^1 + \beta \ln c_{t+1}^2, \quad 0 < \beta \leq 1. \tag{2.1}$$

In Eq. 2.1, c_t^1 refers to the per-capita consumption of the young household when working, c_{t+1}^2 denotes the expected per-capita consumption when retired, and β denotes the subjective time discount factor. The time discount factor is a measure of the subjective time preference (consumption today is generally valued more than consumption tomorrow), and specifies the extent to which consumption in the retirement period is valued less than one unit of consumption in the working period.

The available technologies can be described by a macroeconomic production function of the form $Y_t = F(A_t, K_t)$, where Y_t denotes the gross national product (GDP) in period t, A_t stands for the number of productivity-weighted (efficiency) employees and K_t denotes the physical capital stock at the beginning of period t. In the absence of technological progress, the actual number of employees is equal to the productivity-weighted sum of employees. Labor-saving technological progress implies that the same number of workers is producing an ever increasing amount of products. Technological progress thus has the same impact as an increase in workers employed – the number of efficiency workers increases (given a constant number of physical employees). A common specification for the production function is that first introduced by Cobb and Douglas (1934):

$$Y_t = A_t^{1-\alpha} K_t^{\alpha}, \quad 0 < \alpha < 1. \tag{2.2}$$

The technological coefficients α and $(1 - \alpha)$ denote the production elasticity of capital and of efficiency employees, respectively. These coefficients indicate the respective percentage change in output when capital or labor is increased by 1 %.

$0 \leq \delta \leq 1$ denotes the depreciation rate of capital within one period. The capital stock evolves according to the following accumulation equation:

$$K_{t+1} = (1 - \delta)K_t + I_t. \tag{2.3}$$

The labor force L_t (number of young households in generation t) increases by the constant factor $G^L = 1 + g^L > 0$. Hence, the parameter g^L represents the (positive or negative) growth rate of the labor force. The accumulation equation of the labor force has the following form: $L_{t+1} = G^L L_t$.

We assume that the efficiency a_t of employees N_t rises by the constant rate g^τ:

$$a_{t+1} = (1 + g^\tau)a_t = G^\tau a_t, \; a_0 = 1, \tag{2.4}$$

$$A_t = a_t N_t. \tag{2.5}$$

In accordance with the stylized facts of the first chapter, we adopt labor-saving, but not capital-saving technological progress in the basic model.

We want to analyze economic developments of the world economy over time. To do this, we have to make an assumption regarding the formation of expectations of market participants with respect to the evolution of market variables.[3] As in other standard textbooks, we assume that all economic agents have perfect foresight with respect to prices, wages and interest rates, i.e. they expect exactly those prices that induce clearing of all markets in all future periods (i.e. deterministic rational expectations). Expectation formation can be modeled in several ways. We may use, for example, static expectations, adaptive expectations or non-perfect foresight. Such variations, however, are only of relatively minor importance in the growth literature. Under static expectations households presume that wages, interest rates and prices in future periods are identical to those existing today. Adaptive expectations mean that expected future prices depend not only on current prices, but also on past price changes. Given non-perfect foresight, expectations regarding prices in some future periods are realized, but after some future period expectations then become “static”.

Walras’ law (see the mathematical appendix) implies that the goods market clears – regardless of goods prices – when all other markets are in equilibrium. Hence, goods prices can be set equal to 1 for all periods.

$$P_t = 1, \quad t = 1, 2, \ldots \tag{2.6}$$

Finally, natural resources are available for free to producers and consumers (“free gifts of nature”). This assumption implies a high elasticity of substitution between capital and natural resources and the possibility of free disposal. If this (up to the early 1970s quite realistic) assumption is dropped, the interactions between the natural environment and production and consumption has to be modeled explicitly. These interactions are the subject of environmental science and resource economics and will not be discussed in this book (see e.g. Farmer and Bednar-Friedl 2010).

2.3 The Macroeconomic Production Function and Its Per Capita Version

In the basic model of neoclassical growth theory, the technology of the representative firm is depicted by a linear-homogeneous production function with substitutable production factors. It specifies the maximum possible output of the aggregate of all commodities produced in the world economy, Y, for each feasible factor combination. Figure 2.1 illustrates the above mentioned Cobb-Douglas (CD) production function graphically (for $\alpha = 0.3$). In general, homogeneous production functions exhibit the following form:

[3] A more thorough discussion of alternative expectation formation hypotheses in OLG models can be found in De la Croix and Michel (2002, Chap. 1)

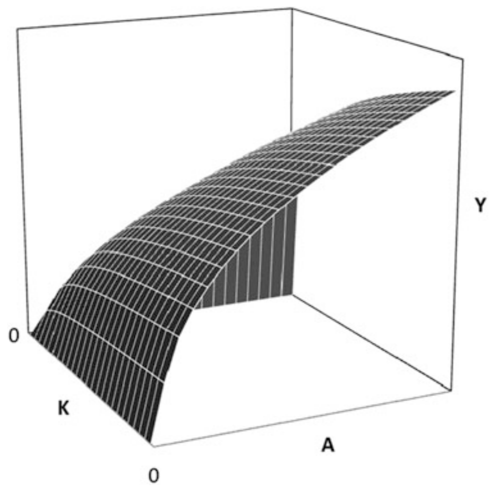


Fig. 2.1 Cobb-Douglas production function

$$Y_t = F(a_t N_t, K_t) \equiv F(A_t, K_t), \text{ where } F(\mu A_t, \mu K_t) = \mu Y_t, \ \mu > 0. \tag{2.7}$$

Homogeneity of degree r implies that if all production inputs are multiplied by an arbitrary (positive) factor μ, the function value changes by the amount μ^r. For linear-homogeneous functions the exponent r is equal to one, i.e. a doubling of all inputs leads to a doubling of production output.

Replacing in Eq. 2.7 μ by $1/A_t$ attributes a new meaning to the production function: it signifies the production per-efficiency employee (= per-efficiency capita product = efficiency-weighted average product).

It is evident from Eq. 2.8 that the per-efficiency capita product, y_t, depends on one variable only, namely the efficiency-weighted capital intensity. The production function Eq. 2.8 is a function of only one variable. The notation used below conforms to the following rule: variables expressing levels are stated using capital letters, per-capita values (and per-efficiency capita values) are depicted using small letters. E.g. $k_t = K_t/A_t$ denotes the capital stock per-efficiency employee and is called the (productivity-weighted) capital intensity.

$$\frac{Y_t}{A_t} \equiv y_t = F\left(1, \frac{K_t}{A_t}\right) \equiv f\left(\frac{K_t}{A_t}\right) = f(k_t) \tag{2.8}$$

The property of substitutability implies that different input combinations can be used to produce the same output. This is in contrast to post-Keynesian models, where production input proportions are fixed (limitational). Under substitutability however, the marginal products of capital and of efficiency-weighted labor can be calculated. How is this done?

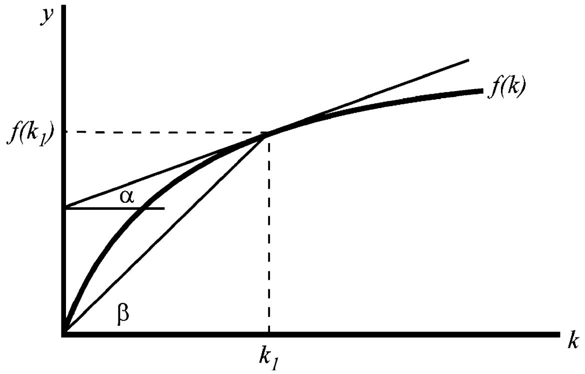


Fig. 2.2 Cobb-Douglas per-capita production function

The marginal products equal the first partial derivatives of the production function $F(A_t, K_t)$ or $f(k_t)$ with respect to A_t and K_t, and with respect to k_t, respectively. Thus, the marginal product of capital can be determined as follows:

$$\frac{\partial F}{\partial K_t} = \frac{\partial [A_t f(K_t/A_t)]}{\partial K_t} = \frac{df}{dk_t} > 0, \quad \frac{\partial^2 F}{\partial K_t \partial k_t} = \frac{d^2 f}{dk_t^2} < 0, \text{ since } \frac{\partial^2 F}{\partial K_t^2} < 0. \tag{2.9}$$

The derivative of the production function with respect to labor yields:

$$\frac{\partial F}{\partial A_t} = \frac{\partial [A_t f(K_t/A_t)]}{\partial A_t} = f(k_t) + A_t \frac{df}{d\,k_t}\left[\frac{-K_t}{(A_t)^2}\right] = f(k_t) - k_t \frac{df}{d\,k_t}, \tag{2.10a}$$

and

$$\frac{\partial^2 F}{\partial A_t \partial k_t} > 0, \text{ since } \frac{\partial^2 F}{\partial A_t \partial K_t} > 0. \tag{2.10b}$$

The Eqs. 2.10a and 2.9 give the marginal products of labor and capital, i.e. the additional output which is due to the input of an additional unit of capital or efficiency-weighted labor. It is striking that for linear-homogeneous production functions both the average and the marginal products are functions of a single variable, namely the capital-labor ratio. As long as this ratio does not change, the per-efficiency capita product and the marginal products do not change.

$$\tan\alpha = \frac{\partial F}{\partial K_1} = \frac{df}{dk_1} \equiv f'(k_1), \quad \tan\beta = \frac{f(k_1)}{k_1} = \frac{Y_1/A_1}{K_1/A_1} = \frac{1}{v_1} \text{ where } v_1 = \frac{K_1}{Y_1} \tag{2.11}$$

Figure 2.2 shows that if the capital intensity is equal to k_1, the per-efficient capita product amounts to $f(k_1)$. Moreover, the tangent of the angle α gives the slope of the production function at this point, i.e. the marginal product of capital in k_1. The tangent of the angle β denotes the (average) productivity of capital. It gives the amount of output per unit of capital and is the reciprocal of the (average) capital coefficient, which indicates the amount of capital required to produce one unit of output.

2.4 Structure of the Intertemporal Equilibrium

After having presented the basic characteristics of the growth model, we are now able to return to the main question of this chapter: How can we determine the key variables of the model described above, while accounting for all market interactions of economically rational (self-interested) households and firms?

The answer to this question is provided in two steps: First, we use mathematical programming (i.e. constrained optimization) to solve the rational choice problems of households and firms (see the Appendix to this chapter for an introduction to classical optimization). Second, to ensure consistency among the individual optimization solutions, the market clearing conditions in each period need to be invoked. To start with, the rational choice problem of younger households is described first.

2.4.1 Intertemporal Utility Maximization of Younger Households

In line with Diamond (1965) we assume that younger households are not concerned about the welfare of their offspring, i.e. in intergenerational terms, they act egoistic. In other words: they do not leave bequests. Thus, consumption and savings choices in their working period and consumption in their retirement period are made with a view towards maximizing their own lifetime utility. Thus, for all t, the decision problem of households entering the economy in period t reads as follows:

$$\text{Max } U_t^1 = \ln c_t^1 + \beta \ln c_{t+1}^2, \tag{2.12}$$

subject to:

$$c_t^1 + s_t = w_t, \tag{2.13}$$

$$c_{t+1}^2 = (1 + i_{t+1}) s_t \text{ where } 1 + i_{t+1} \equiv q_{t+1} + 1 - \delta. \tag{2.14}$$

The first constraint Eq. 2.13 ensures that per-capita consumption plus per-capita savings of young households equals their income (the wage rate per employee) and based on the second constraint Eq. 2.14 retirement consumption is restricted by the sum of savings made in the working period and interest earned on savings. Active households save by acquiring capital, and the real interest rate is equivalent to the rental price of capital minus depreciation. Since the no-arbitrage condition $i_{t+1} = q_{t+1} - \delta$ holds, the real interest rate (= rate of return on savings) has to be equal to the rental price of capital minus depreciation (= return on investment in physical capital). Obviously, if the real interest rate were smaller (larger) than the capital rental price minus the depreciation rate, then households would just invest in real capital (savings deposits). The relative prices of assets then would change quickly such that respective rates of return once again equate and the no-arbitrage condition is satisfied.

Equations 2.13 and 2.14 can be combined to obtain Eq. 2.15 by calculating s_t from Eq. 2.14 and substituting s_t in Eq. 2.13.[4] This then leaves Eq. 2.15 as the only constraint in the household's utility maximization problem. This is known as the intertemporal budget constraint (i.e. all current and present values of future expenses equal all current and present values of future revenues).

$$c_t^1 + \frac{c_{t+1}^2}{1 + i_{t+1}} = w_t \tag{2.15}$$

The left-hand side of Eq. 2.15 gives the present value of all spending in the two periods of life; the right-hand side the (present value of) total income. If the objective function Eq. 2.12 is maximized subject to the intertemporal budget constraint Eq. 2.15, we obtain the first-order conditions (= FOCs) Eqs. 2.16 and 2.17 for household utility maximization.

$$-\frac{dc_{t+1}^2}{dc_t^1} \equiv \frac{\partial\, U_t^1 / \partial\, c_t^1}{\partial U_t^1 / \partial\, c_{t+1}^2} = \frac{c_{t+1}^2}{\beta c_t^1} = 1 + i_{t+1} \tag{2.16}$$

In the household's optimum, the intertemporal marginal rate of substitution $\left(-dc_{t+1}^2 / dc_t^1\right)$ equals the interest factor.

$$-\frac{dc_{t+1}^2}{dc_t^1} - 1 = i_{t+1} \tag{2.17}$$

Equation 2.17 states that at the optimum the marginal rate of time preference (time-preference rate) is equivalent to tomorrow's real interest rate. What is the rationale behind this result? The left-hand side of Eq. 2.17, the marginal rate of time preference, indicates how much more than one retirement consumption unit the

[4] Here we have to assume that utility maximizing savings per capita are strictly larger than zero. However, this is true since optimal retirement consumption is certainly larger than zero otherwise the marginal utility of retirement consumption would be infinitely large while the price of an additional consumption unit would be finite. This cannot be utility maximizing and thus the optimal retirement consumption must be strictly larger than zero implying, from Eq. 2.14, strictly positive savings.

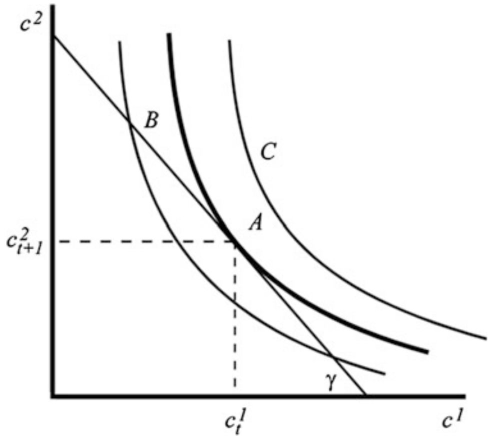


Fig. 2.3 Graphical illustration of the utility maximizing consumption plan

younger household demands for foregoing one working-period consumption unit. On the right-hand side, the real interest rate indicates how much more than one unit the household gets in its retirement period, if it forgoes one unit of consumption in its active period (i.e. if the household saves). Utility maximization implies that the left-hand side in Eq. 2.17 equals the right-hand side.

However, if the right-hand side were larger than the left-hand side, then the household would receive a higher compensation for foregoing current consumption than it demands. The household would then use these surplus earnings to increase its utility. A situation in which the left-hand side of Eq. 2.17 is smaller than the right-hand side can, therefore, never be a utility maximizing situation. The same is true if the left-hand side of Eq. 2.17 is larger than the right-hand side. Maximum utility is thus achieved only if Eq. 2.17 is satisfied.

The optimization condition Eq. 2.16 and the intertemporal budget constraint Eq. 2.15 are sufficient to determine the entire optimal consumption plan of a young household. Figure 2.3 illustrates the decision problem and the optimal consumption plan of the young household. Consumption when young is plotted on the horizontal axis (the abscissa), and the retirement consumption of a young household, which enters the economy in period t, is plotted on the vertical axis (the ordinate). The negatively sloped straight line represents the intertemporal budget constraint. This can be obtained algebraically by solving Eq. 2.15 for c^2_{t+1}: $c^2_{t+1} = (1 + i_{t+1})w_t - (1 + i_{t+1})\, c^1_t$.

If $c^1_t = 0$, we get an intercept of the budget constraint (on the ordinate) of $w_t(1 + i_{t+1})$, and if $c^2_{t+1} = 0$, the intercept (on the abscissa) is w_t. The negative slope of the budget constraint equals $\tan\gamma = 1 + i_{t+1}$. The distance between the intersection of the budget constraint with the abscissa and the utility maximizing consumption point gives the optimal savings s_t of young households.

The three hyperbolas in Fig. 2.3 represent intertemporal indifference curves. Along such curves the lifetime utility U^1_t of generation t is constant. Analytically, these indifference curves are obtained by solving the intertemporal utility function for fixed levels of utility. The further away an indifference curve is from the origin, the higher is lifetime utility. Accordingly, the consumption plan indicated by point A is associated with a higher utility level than consumption plan B, which is also affordable. The negative slope of the intertemporal indifference curve is a consequence of the intertemporal marginal rate of substitution Eq. 2.16. This rate can be analytically derived by totally differentiating the utility function and setting the total differential equal to zero (because the utility level is constant along each indifferent curve).

$$d U_t^1 = \frac{\partial U_t^1}{\partial c_t^1} d c_t^1 + \frac{\partial U_t^1}{\partial c_{t+1}^2} d c_{t+1}^2 = 0 = \frac{1}{c_t^1} d c_t^1 + \frac{\beta}{c_{t+1}^2} d c_{t+1}^2 \tag{2.18}$$

$$-\frac{d c_{t+1}^2}{d c_t^1} = \frac{\partial U_t^1 / \partial c_t^1}{\partial U_t^1 / \partial c_{t+1}^2} = \frac{c_{t+1}^2}{\beta c_t^1} \tag{2.19}$$

Since the intertemporal marginal rate of substitution in Fig. 2.3 corresponds to the negative slope of the intertemporal indifference curve and the interest factor is given by the negative slope of the budget line, the slopes of the intertemporal indifference curve and of the intertemporal budget constraint have to be identical at the optimum – i.e. the intertemporal budget constraint and the indifference curve are at a point of tangency. Point A in Fig. 2.3 represents the tangency point, while point B is a cutting point. Although both consumption plans, A and B, are affordable, the indifference curve associated with the consumption plan A is at a higher utility level. Consumption plan C, which belongs to an even higher indifference curve, is not affordable. Therefore, A gives the optimal consumption plan, i.e. a consumption plan which lies on the indifference curve which is farthest from the origin but still affordable.

By solving Eq. 2.16 for $c_{t+1}^2/(1 + i_{t+1})(= \beta c_t^1)$ and inserting the result into the intertemporal budget constraint Eq. 2.15 we obtain: $c_t^1 + \beta c_t^1 = w_t$. Rearranging yields immediately the optimal (utility maximizing) working-period consumption:

$$c_t^1 = \frac{w_t}{1 + \beta}, t = 1, \ldots. \tag{2.20}$$

Inserting Eq. 2.20 into Eq. 2.13 and solving for s_t yields utility-maximizing savings per capita:

$$s_t^1 = \frac{\beta}{1 + \beta} w_t, \quad t = 1, \ldots. \tag{2.21}$$

Inserting Eq. 2.20 into Eq. 2.16 and solving for c_{t+1}^2 results in the following:

$$c_{t+1}^2 = \frac{\beta(1 + i_{t+1})}{1 + \beta} w_t, \quad t = 1, \ldots. \tag{2.22}$$

Equation 2.20 reveals that current (optimal) consumption depends only on the real wage rate and not on the real interest rate.[5] Equation 2.21 illustrates that the portion of wages not consumed is saved completely. This results from the fact that the public sector is ignored and thus households pay no taxes. Moreover, since current-period consumption is independent of the real interest rate, this is also true for optimal savings. Finally, the amount saved when young (plus interest earned) can be consumed when old Eq. 2.22. As mentioned above savings of retired households (i.e. bequests) are excluded. However, even with the introduction of a bequest motive for old households the following characteristics are still valid.

2.4.2 Old Households

In period 1 the number of retired households equals the number of young households in the previous period, i.e. L_0. Their total consumption in period 1 is identical to their total amount of assets (in real terms) in period 1.

$$L_0 c_1^2 = q_1 K_1 + (1-\delta)K_1 = (1+i_1)K_1 \tag{2.23}$$

These assets include the rental income on capital acquired in the past, plus the market value of the capital stock (after depreciation). Equation 2.23 assumes that the no-arbitrage condition (2.14) applies.

2.4.3 A-Temporal Profit Maximization of Producers

Besides households, the producers of the aggregate commodity also strive to maximize profits in every period t. By assumption, markets are perfectly competitive. To maximize profits, firms have to decide on the number of employees (labor demand, N_t) and on the use of capital services (demand for capital services, K_t^d). The profits, expressed in units of output, are defined as the difference between production output and real factor costs: $\pi_t = F(a_t N_t, K_t^d) - w_t N_t - q_t K_t^d$. In the case of a CD production function, the profit function can be written as $\pi_t = (a_t N_t)^{1-\alpha}(K_t^d)^{\alpha} - w_t N_t - q_t K_t^d$. To determine the profit-maximizing input levels, we set the first partial derivatives of the profit function with respect to N_t and K_t^d equal to zero:

$$\frac{\partial \pi_t}{\partial N_t} = \frac{\partial F(A_t, K_t^d)}{\partial A_t}\frac{\partial A_t}{\partial N_t} - w_t = 0, \tag{2.24}$$

$$\frac{\partial \pi_t}{\partial K_t^d} = \frac{\partial F(A_t, K_t^d)}{\partial K_t^d} - q_t = 0. \tag{2.25}$$

[5] Due to the log-linear intertemporal utility function the substitution effect and the income effect of a change in the real interest rate cancel out.

The first-order condition (2.24) tells us that in each period t firms demand additional workers as long as the physical marginal product is equal to the real (measured in units of output) wage rate. Equation 2.24 is equivalent to:

$$(1-\alpha)\left(k_t^d\right)^{\alpha} a_t = w_t. \tag{2.24a}$$

In the same manner, one arrives at the decision rule for optimal capital input. Firms have to adjust the capital stock such that the marginal product of capital (yield on capital) in each period t is equivalent to the real capital costs. In the case of a CD production function, Eq. 2.25a holds.

$$\alpha\left(k_t^d\right)^{\alpha-1} = q_t \tag{2.25a}$$

With exogenously given technological progress, the number of efficiency employees A_t is a direct consequence of the producer demand for labor.

$$A_t = a_t N_t \tag{2.26}$$

Aggregate production output is determined by the profit maximizing levels of the capital stock and the number of efficiency employees.

$$Y_t = F\left(A_t, K_t^d\right) = A_t^{1-\alpha}\left(K_t^d\right)^{\alpha} \tag{2.27}$$

Finally, linear-homogeneity implies that the aggregate output is distributed across all production factors. Every factor of production is thus paid according to its marginal productivity. Thus, the sum of factor payments corresponds exactly to the production output and there are no surplus profits. Applying Euler's theorem to the aggregate production function Eq. 2.27 implies: $Y_t = (\partial Y_t/\partial N_t)N_t + \left(\partial Y_t/\partial K_t^d\right)K_t^d$. Since through Eqs. 2.24 and 2.25 $(\partial Y_t/\partial N_t) = w_t$ and $\left(\partial Y_t/\partial K_t^d\right) = q_t$, we obtain:

$$Y_t = w_t N_t + q_t K_t^d. \tag{2.28}$$

2.4.4 Market Equilibrium in All Periods

The second step in delineating the structure of the intertemporal equilibrium is to specify the market clearing conditions. In a perfectly competitive market economy no authority or central administration matches or coordinates individual decisions. The coordination of individual decisions results from changes in market prices such that the supply and demand for each good is equal in all markets (market clearing conditions).

In the basic OLG model there are three markets: the capital market, the labor market and the commodity market. The clearing of these three markets demands:

$$K_t^d = K_t, \forall t, \tag{2.29}$$

$$N_t = L_t, \forall t, \tag{2.30}$$

$$Y_t = L_t c_t^1 + L_{t-1} c_t^2 + K_{t+1} - (1 - \delta) K_t, \forall t. \tag{2.31}$$

Due to Walras' law the sum of nominal (measured in terms of their prices) excess demands (= demand minus supply) on all three markets is equal to zero for all feasible prices. Consequently, if two of the three markets are in equilibrium, the third market must also be in balance. Walras' law is derived for our basic OLG growth model in the mathematical appendix to this chapter.

A pivotal equation for the dynamics of the intertemporal equilibrium is implicitly included in the system of equilibrium conditions (2.29, 2.30 and 2.31). This equation becomes immediately apparent when one considers the equilibrium conditions for period $t = 1$. Equating the left-hand side of Eq. 2.31 and Eq. 2.28 and substituting the budget constraints for both the aggregate consumption of young households (Eq. 2.13 multiplied by L_1 on both sides) and for old households Eq. 2.23 into the right-hand side of Eq. 2.31 yields:

$$w_1 N_1 + q_1 K_1 = w_1 L_1 - L_1 s_1^1 + (1 + i_1) K_1 + K_2 - (1 - \delta) K_1. \tag{2.32}$$

Since $1 + i_1 = q_1 + (1 - \delta)$ and Eqs. 2.29 and 2.30 also apply for $t = 1$, this equation reduces to $K_2 = L_1 s_1$. This can be generalized to:

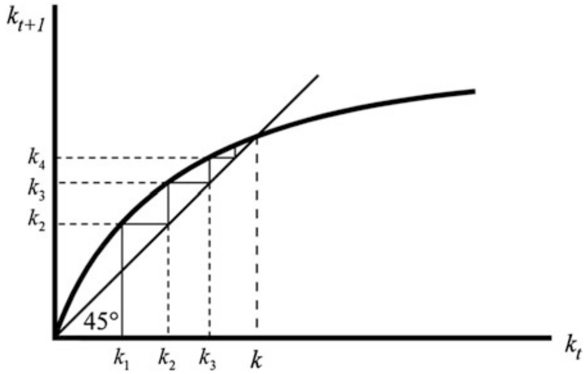
$$K_{t+1} = L_t s_t, \quad t \geq 2. \tag{2.33}$$

In an intertemporal market equilibrium, the optimal aggregate savings of all young households in period t correspond exactly to the optimal aggregate capital stock in $t + 1$. This result becomes immediately apparent when we keep in mind that we have excluded a bequest motive in the basic model. Therefore, in order to consume, old households sell all their assets to the young households of the next generation. The young households save by buying the entire old capital stock plus investing in new capital goods (= gross investment).

2.5 The Fundamental Equation of Motion of the Intertemporal Equilibrium

The next step is to study how the economy evolves over time when in each period households maximize their utility, firms maximize profits, and all markets clear. In order to derive the fundamental equation of motion of the intertemporal equilibrium

Fig. 2.4 The fundamental equation of motion of the basic model



we focus on the accumulation of aggregate capital and on the dynamics of the efficiency-weighted capital intensity (capital-labor ratio). To this end, Eq. 2.33 is divided by $a_t L_t = a_t N_t = A_t$, to obtain:

$$\frac{K_{t+1}}{A_t} = \frac{s_t}{a_t}. \tag{2.34}$$

If the left-hand side is multiplied by $A_{t+1}/A_{t+1} = 1$, we arrive at:

$$\frac{K_{t+1}}{A_{t+1}} \frac{A_{t+1}}{A_t} = k_{t+1} \frac{A_{t+1}}{A_t} = \frac{s_t}{a_t}. \tag{2.34a}$$

Equation 2.34a involves the growth factor of efficiency-weighted labor A_{t+1}/A_t which is equal to the (exogenous) growth factor of labor efficiency times the population growth factor. The latter is called the *natural growth factor*, and it is denoted by G^n. When growth rates are not too large it can be approximated by one plus the natural growth rate g^n:

$$\frac{A_{t+1}}{A_t} = \frac{a_{t+1} L_{t+1}}{a_t L_t} = G^\tau G^L \equiv G^n \approx 1 + g^n. \tag{2.35}$$

Taking Eq. 2.35 into account, we find that Eq. 2.34 is equivalent to:

$$k_{t+1} = \frac{s_t}{G^n a_t}. \tag{2.36}$$

By inserting Eq. 2.24a into Eq. 2.21, we obtain optimal savings per efficiency capita in period t as a function of the capital intensity in the same period:

$$\frac{s_t}{a_t} = \frac{\beta(1-\alpha) k_t^\alpha}{1+\beta}. \tag{2.37}$$

Finally, by inserting Eq. 2.37 into Eq. 2.36 we arrive at the following dynamic equation for k_t:

$$k_{t+1}G^n = \frac{\beta(1-\alpha)k_t^\alpha}{1+\beta}. \tag{2.38}$$

By introducing the aggregate savings rate $\sigma \equiv \beta(1-\alpha)/(1+\beta)$ we obtain the fundamental equation of motion for our basic OLG growth model:

$$k_{t+1} = \frac{\sigma}{G^n}k_t^\alpha, \text{ for } t \geq 1 \text{ and } k_1 = \frac{K_1}{a_1 L_1}. \tag{2.39}$$

Mathematically, the fundamental equation of motion (2.39) is a nonlinear difference equation in k_t (capital per efficiency capita) and determines for each (productivity-weighted) capital intensity k_t the equilibrium (productivity-weighted) capital intensity in the next period k_{t+1}. If the capital intensity of the initial period $t = 1$ is known, the fundamental equation of motion describes the evolution of k_t for all future periods (see Fig. 2.4).

Additionally, the fundamental equation of motion allows us to deduce what determines the absolute change of the capital intensity. Equation 2.39 is equivalent to:

$$k_{t+1} - k_t = (G^n)^{-1}(\sigma k_t^\alpha - G^n k_t). \tag{2.40}$$

Obviously, the capital intensity remains constant if savings per efficiency capita, σk_t^α, are just sufficient to support the additional capital needed for natural growth, $G^n k_t$. This additional capital requirement arises since the accrued and more efficient workers must be equipped with the same capital per efficiency capita as those already employed. If per efficiency capita savings (= per efficiency capita investment) exceed (fall short of) this intensity-sustaining capital requirement, the capital intensity of the next period increases (decreases). Thus, there are two types of investments (= savings): those that are necessary to sustain the current capital intensity and those that increase the current capital intensity. The former are called "capital-widening" investments (savings), the latter "capital-deepening" investments (Müller and Stroebele 1985, 37).

A competitive intertemporal equilibrium is completely determined by the above mentioned equilibrium sequence of capital intensities over time. For example, the marginal productivity conditions (2.24a) and (2.25a) immediately determine the period-specific real wage rates and capital rental prices.

$$w_t = a_t(1-\alpha)k_t^\alpha, \quad t = 1, \ldots \tag{2.41}$$

$$q_t = \alpha\, k_t^{\alpha-1}, \quad t = 1, \ldots \tag{2.42}$$

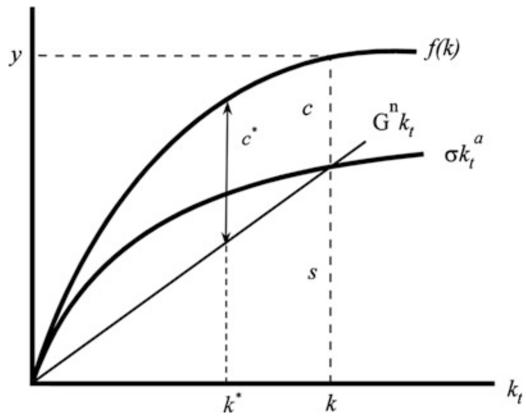


Fig. 2.5 Consumption and savings in the basic OLG model

The optimal consumption levels for young and old households and the optimal savings can be deduced from the Eqs. 2.20, 2.21 and 2.22. Finally, the aggregate output per efficiency capita is a direct result of the production function Eq. 2.8:

$$y_t = f(k_t) = k_t^{\alpha}. \tag{2.43}$$

2.6 Maximal Consumption and the "Golden Rule" of Capital Accumulation

Before closing this chapter it is interesting to explore whether a world economy with a higher savings rate (= higher capital intensity) is always better off than one with a lower savings rate (= lower capital intensity). We begin with the following aggregate accumulation equation:

$$K_{t+1} - K_t = F(K_t, A_t) - C_t - \delta K_t, \tag{2.44}$$

where the depreciation rate is not necessarily equal to one. If we divide both sides by A_t, to arrive at per efficiency capita values, we obtain:

$$k_{t+1}G^n - k_t = f(k_t) - c_t - \delta k_t. \tag{2.45}$$

Variable c_t (with no generation index) denotes total consumption per efficiency employee. Suppose again time-stationary capital intensities, i.e. $k_{t+1} = k_t = k$. Equation 2.45 can then be solved for c:

$$c = f(k) - (g^n + \delta)k. \tag{2.46}$$

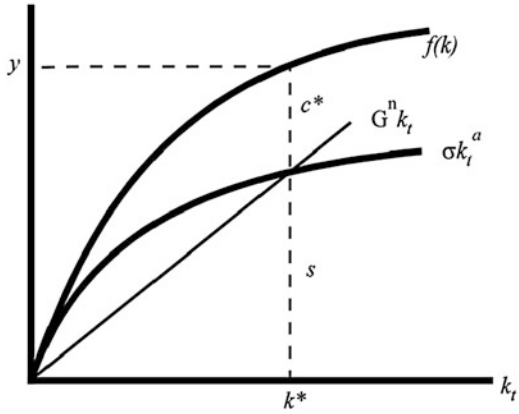


Fig. 2.6 Golden rule savings and consumption

Consumption per efficiency capita is maximized when the savings ratio is such that $(dc/dk)(dk/d\sigma) = 0$ holds. This is equivalent to:

$$\frac{dc}{d\sigma} = \left[\frac{df(k)}{\partial k} - (g^n + \delta)\right]\left(\frac{dk}{d\sigma}\right) = 0. \tag{2.47}$$

It is obvious from Eq. 2.47 that c is maximized only if $f'(k) = (g^n + \delta)$. In the case of a CD production economy the so-called "golden rule" capital intensity k^* is equal to:

$$k^* = \left[\frac{g^n + \delta}{\alpha}\right]^{1/(\alpha - 1)}. \tag{2.48}$$

If, in addition, $\delta = 1$ and Eq. 2.39 is taken into account, then the golden rule capital intensity demands $\sigma = \alpha$, i.e. a savings rate of about 30 % when $\alpha = 0.3$ is assumed.[6]

Figure 2.5 illustrates consumption and savings in the basic model under the assumption of $\delta = 1$. In such a case consumption per efficiency capita is equal to $c = f(k) - G^n k = k^\sigma - \sigma k^\alpha$.

As shown in the figure, consumption is exactly equal to the difference between output per efficiency capita, $f(k)$, and intensity-sustaining savings $G^n k (= \sigma k^\alpha)$. However, as the consumption level c^* associated with capital intensity k^* shows, steady-state $(k_{t+1} = k_t = k)$ intensity k does not maximize consumption per efficiency capita. In order to obtain maximum consumption c^* the savings rate must be

[6] Empirical values for the other model parameters can be found in Auerbach and Kotlikoff (1998, Chaps. 2 and 3).

changed. Hence, we need to search for a savings rate where consumption per efficient capita is maximized. From a purely static perspective, consumption decreases with an increase in the savings rate. But a higher savings rate σ also leads to a higher capital stock in the future and therefore to a greater production capacity and higher potential consumption.

In an intertemporal context we have to weigh short-term consumption losses due to a higher savings rate against the increase in the future capital stock which allows for higher consumption tomorrow. The savings rate which permanently allows for maximum consumption per efficiency capita implies, according to Phelps (1966), the "golden rule" of capital accumulation. This term is borrowed from the "golden rule" of New Testament ethics: "So whatever you wish that men would do to you, do so to them" (Matthew 7, 12). Economically speaking, the "golden rule" consumption level is not only available to currently living generations, but also to all future generations. Graphically, the "golden rule" capital intensity can be found, by maximizing the distance between $f(k)$ and $G^n k$ $(= \sigma k^\alpha)$. Figure 2.6 shows the "golden rule" savings rate and "golden rule" capital intensity leading to long-run maximum consumption.

2.7 Summary and Conclusion

In this chapter the basic OLG growth model of the closed world economy − a log-linear CD version of Diamond's (1965) neoclassical growth model – was introduced and its intertemporal equilibrium dynamics were derived. In contrast to post-Keynesian growth theory our basic OLG growth model rests on solid intertemporal general equilibrium foundations comprising constrained optimization of agents and the clearing of all markets in each model period. Regarding production technology, the linear-homogeneity of the production function and the substitutability of production factors were emphasized. This is in line with neoclassical growth theory. Factor substitutability enables profit-maximizing firms to adapt their capital intensities (capital-labor ratios) to the prevailing relative wage rate.

Another key feature of the basic growth model is the endogeneity of per capita savings. Young households choose savings in order to maximize their life-time utility. In doing so, they also choose optimal (i.e. utility maximizing) consumption when young, and optimal consumption when old. As in the Solow-Swan neoclassical growth model the savings rate is constant, and can be traced back to the time discount factor of younger households. The old households consume their entire wealth (bequests are excluded by definition).

All market participants (young households, old households and producers) interact in competitive markets for capital and labor services and for the produced commodity. Supply and demand in each market are balanced by the perfectly flexible real wage and real interest rate. The first-order conditions (FOCs) for intertemporal utility maxima and period-specific profit maxima in conjunction with market clearing conditions yield the fundamental equation of motion for our basic OLG model of capital accumulation together with the equilibrium dynamics of the efficiency weighted capital intensity. The fundamental equation of motion also allows for the determination of the real wage rate and the real interest rate on the intertemporal equilibrium path.

Finally, we sought for the savings rate and associated capital intensity that maximizes permanent consumption per-efficiency capita. It turns out that higher savings rates are not in general better than lower savings rates. The golden rule for achieving maximum consumption per efficiency capita demands a capital intensity at which the marginal product of capital corresponds exactly to the rate of natural growth plus depreciation rate. If we assume a depreciation rate of one, the savings rate, leading to the "golden rule" capital intensity, must be equal to the production elasticity of capital.

2.8 Exercises

2.8.1. Explain the set-up of the basic OLG model and provide empirically relevant values for basic model parameters such as β, G^n and α. Why is α independent of the length of the model period while β and G^n are not?

2.8.2. Use the CD function Eq. 2.2 to show that the marginal product of capital is always smaller than the average product of capital.

2.8.3. Explain in terms of the marginal rate of substitution and the negative slope of the intertemporal budget constraint why point B in Fig. 2.3 is not utility maximizing.

2.8.4. Show that under the CD production function Eq. 2.2 maximum profits are zero. Which property of general neoclassical production functions implies zero profits?

2.8.5. Why must the younger households finance next-period capital stock even when capital does not depreciate completely during one period?

2.8.6. Verify the derivation of the intertemporal equilibrium dynamics Eq. 2.39 and explain why not the whole savings per efficiency capita cannot be used for capital deepening?

2.8.7. Explain the meaning of the golden rule of capital accumulation and provide a sufficient condition with respect to capital production share such that the savings rate is irrelevant for golden rule capital intensity. (Hint: See Galor and Ryder, 1991)

Appendix

Constrained Optimization

All agents in this chapter aim at optimizing their decisions to reach their goals in the best possible way. However, they are all confronted with various restrictions (constraints) – in some cases they are of a natural or technological nature, in other cases choices are limited due to available income. How can one find the optimum decision in the face of such constraints? The method of mathematical (classical) programming provides a solution. In order to formalize the decision problem we first need to define the following: What are the objectives of the different actors? Which variables are to be included in agent decision making? Which restrictions do they face?

The objectives of the agents can be formalized by use of the objective function, Z. This function assigns a real number to every decision (consisting of a list of n decision variables) made by an agent.

$$Z : \Re^n \to \Re^1 \tag{2.49}$$

We introduced two objective functions in the main text of this chapter: one for households whose goal is to act in such a way that their preferences, represented by a utility function, are met best, and one for firms that try to maximize their profit function.

Concerning the second and third question we know that households can determine consumption quantities and the distribution of consumption over time. We also know that producers can determine the demand for labor as well as for capital. These variables are referred to as decision (choice) variables or instrumental variables. The quantities households can consume depend, among other things, on their income. The production cost of a specific quantity of a good depends, among other things, on the technology used in the production process. Such restrictions are represented in the form of constraints.

Mathematically speaking, the decision problem is to find values for the instrumental variables which maximize the value of the objective function (profit, utility) or minimize it (cost), subject to all constraints. Formally, the optimization problem can be written as one of the following three programs:

$$\text{Max } Z(x) \quad \text{s.t.:} \quad g(x) = b, \quad \text{(classical optimization)} \tag{2.50a}$$

$$\text{Max } Z(x) \quad \text{s.t.:} \quad g(x) \leq b, \quad x \geq 0, \quad \text{(non}-\text{linear optimization)} \tag{2.50b}$$

$$\text{Max } Z(x) = cx \quad \text{s.t.:} \quad Ax \leq b, \quad x \geq 0. \quad \text{(linear optimization)} \tag{2.50c}$$

The objective function Z is a function of n variables, i.e. x is a vector of dimension n (n decision variables). The function $g(x)$ denotes m constraints; b is a column vector of dimension m.

We now turn to classical optimization and try to find a rule which allows us to unveil the optimal decision making of agents. An example of the household objective function $U(x)$ is given by Eq. 2.12; the (only) constraint $g(x)$ by Eq. 2.15.

$$\text{Max } U_t\left(c_t^1, c_{t+1}^2\right) = \ln c_t^1 + \beta \ln c_{t+1}^2 \tag{2.51a}$$

subject to (s.t.):

$$c_t^1 + \frac{c_{t+1}^2}{1 + i_{t+1}} = w_t \tag{2.51b}$$

The two instrumental variables in this optimization problem are c_t^1 and c_{t+1}^2, and are the (only) variables households can determine. Due to the constraint, future consumption can (under certain conditions) be written as a function of current consumption.

$$c_{t+1}^2 = (1 + i_{t+1})(w_t - c_t^1) \tag{2.52a}$$

Or, more generally:

$$c_{t+1}^2 = h(c_t^1), \tag{2.52b}$$

$$\frac{dh}{dc_t^1} = -\frac{\partial g / \partial c_t^1}{\partial g / \partial c_{t+1}^2}. \tag{2.52c}$$

The objective function can also be formulated as a function $\tilde{\Lambda}$ of a single decision variable:

$$\tilde{\Lambda} = \ln\ c_t^1 + \beta \ln\ \left[(1 + i_{t+1})(w_t - c_t^1)\right]. \tag{2.53a}$$

Or, more generally:

$$\tilde{\Lambda} = \tilde{\Lambda}\,(c_t^1, h(c_t^1)). \tag{2.53b}$$

This intermediate step simplifies the search for a value c of the decision variable c_t^1 that maximizes the objective function $\tilde{\Lambda}$ (utility) in our decision problem. Obviously, at a maximum, the following condition has to hold:

$$\tilde{\Lambda}\,(c) \geq \tilde{\Lambda}\,(c + \Delta c). \tag{2.54}$$

If we make use of Taylor's theorem, we can find the maximum of the (modified) objective function Eq. 2.53b. The first-order condition (FOC) of the problem is:

$$\frac{d\tilde{\Lambda}}{dc_t^1} = 0 = \frac{\partial U}{\partial c_t^1} + \frac{\partial U}{\partial h}\frac{dh}{d\,c_t^1}. \tag{2.55}$$

On taking account of Eq. 2.52b, then Eq. 2.55 is equivalent to:

$$\frac{d\tilde{\Lambda}}{dc_t^1} = 0 = \frac{\partial U}{\partial c_t^1} + \frac{\partial U}{\partial h}\left[-\frac{\partial g/\partial c_t^1}{\partial g/\partial c_{t+1}^2}\right] = \frac{\partial U}{\partial c_t^1} + \left[-\frac{\partial U/\partial h}{\partial g/\partial c_{t+1}^2}\right]\frac{\partial g}{\partial c_t^1}. \tag{2.56}$$

We denote the expression in brackets on the right-hand side by λ, so that the maximization problem (2.56) can be written more simply as:

$$\frac{d\tilde{\Lambda}}{dc_t^1} = 0 = \frac{\partial U}{\partial c_t^1} + \lambda\frac{\partial g}{\partial c_t^1}. \tag{2.57}$$

This is the solution to the household's decision problem. However, a simpler route is provided by a function that leads us directly to condition (2.57). This is:

$$\Lambda(c_t^1, c_{t+1}^2, \lambda) = U(c_t^1, c_{t+1}^2) + \lambda(w - g(c_t^1, c_{t+1}^2)). \tag{2.58}$$

This function is called the Lagrangian function and the variable λ the Lagrangian multiplier. After calculating the first derivative with respect to the two instrumental variables, the first-order (necessary) conditions for the solution of the optimization problem follows. Thus, differentiating Eq. 2.58 with respect to λ results directly in the constraint. The Lagrangian function of young households has the following form:

$$\Lambda(c_t^1, c_{t+1}^2, \lambda) = \ln c_t^1 + \beta \ln c_{t+1}^2 + \lambda\left[w - c_t^1 - \frac{c_{t+1}^2}{1 + i_{t+1}}\right]. \tag{2.59}$$

The first-order conditions (FOCs) are:

$$\frac{\partial \Lambda}{\partial c_t^1} = \frac{1}{c_t^1} - \lambda = 0, \tag{2.60a}$$

$$\frac{\partial \Lambda}{\partial c_{t+1}^2} = \beta\frac{1}{c_{t+1}^2} - \frac{1}{1 + i_{t+1}}\lambda = 0, \tag{2.60b}$$

$$\frac{\partial \Lambda}{\partial \lambda} = w_t - c_t^1 - \frac{c_{t+1}^2}{1 + i_{t+1}} = 0. \tag{2.60c}$$

If we solve condition (2.60a) for variable λ and substitute the solution into Eq. 2.60b then, assuming the constraint Eq. 2.60c is also taken into account, we can determine the optimal consumption in period t Eq. 2.20, the optimal consumption in period $t+1$ Eq. 2.22 and the optimal savings per capita Eq. 2.21.

To ensure that Eqs. 2.20, 2.21 and 2.22 constitute a maximum (and not a minimum), we have to check the second-order conditions:

$$\frac{\partial^2 \Lambda}{\partial \left(c_t^1\right)^2} = -\frac{1}{\left(c_t^1\right)^2} < 0, \tag{2.61a}$$

$$\frac{\partial^2 \Lambda}{\partial \left(c_{t+1}^2\right)^2} = -\beta \frac{1}{\left(c_{t+1}^2\right)^2} < 0. \tag{2.62b}$$

Both conditions are negative, satisfying the second-order conditions for a strict (local) maximum.

One last and very important question remains: What is the meaning of the Lagrange multiplier in this optimization problem?

The Lagrange multiplier reflects the sensitivity of the value of the objective function with respect to a marginal change in the constants b (cf. Eq. 2.50a) of the constraints. In the optimization problem of young households the Lagrange multiplier is equal to:

$$\lambda_t = \frac{\partial U_t}{\partial w_t}. \tag{2.63}$$

It indicates the amount by which the optimum value of the utility function increases when disposable income rises by one unit.

Walras' Law

Finally, we want to show that our basic growth model satisfies Walras' law. We therefore note the budget constraints of all economic agents for any period t and express all values in monetary units (and not in terms of output units as is done in the main text). In addition, we multiply all per-capita values by the number of corresponding number of individuals. Moreover, we indicate what savings of young households are used for, i.e. to buy investment goods and old capital at the reproduction price P_t. Thus, we have:

$$L_t s_t P_t = P_t I_t + P_t (1-\delta) K_t. \tag{2.64}$$

This equality implies that the budget constraint of young households can be rewritten as follows:

$$P_t L_t c_t^1 + P_t I_t + P_t(1-\delta)K_t = W_t L_t, \tag{2.65}$$

while the aggregate budget constraint of old households reads as follows:

$$P_t L_{t-1} c_t^2 = Q_t K_t + P_t(1-\delta)K_t. \tag{2.66}$$

The linear-homogeneity of the production function implies that at a maximum profits are zero:

$$\Pi_t = 0 = P_t Y_t - W_t N_t - Q_t K_t^d. \tag{2.67}$$

Adding the left-hand sides and the right-hand sides of Eqs. 2.65 and 2.66 yields:

$$P_t\left[L_t c_t^1 + L_{t-1} c_t^2\right] = W_t L_t + Q_t K_t^d - P_t I_t. \tag{2.68}$$

Clearing of the labor and capital market ($N_t = L_t$ and $K_t^d = K_t$) implies:

$$P_t\left[L_t c_t^1 + L_{t-1} c_t^2\right] + P_t I_t = W_t N_t + Q_t K_t = P_t Y_t. \tag{2.69}$$

Since $I_t = K_{t+1} - (1-\delta)K_t$ holds, Eq. 2.69 becomes:

$$P_t\left[L_t c_t^1 + L_{t-1} c_t^2 + K_{t+1} - (1-\delta)K_t - Y_t\right] = 0. \tag{2.70}$$

Since $P_t > 0$, the sum of the terms in square brackets in Eq. 2.70 must be zero.

Thus, we have shown that the product market clears once the labor and the capital markets clear. Equation 2.31 is thus an identity, not a constraint. Thus, we cannot determine the price level in this economy; it has therefore to be set exogenously (e.g. – and as we have assumed here – it can be set equal to one).

References

Acemoglou, D. (2009). *Introduction to modern economic growth*. Princeton: Princeton University Press.

Auerbach, A. J., & Kotlikoff, L. J. (1998). *Macroeconomics: An integrated approach* (2nd ed.). Cambridge, MA: The MIT Press.

Bednar-Friedl, B., & Farmer, K. (2010). *Intertemporal resource economics: An introduction to the overlapping generations approach*. Berlin: Springer.

Cobb, C. W., & Douglas, P. H. (1934). *The theory of wages*. New York: Macmillan.

De la Croix, D., & Michel, P. (2002). *A theory of economic growth: Dynamics and policy in overlapping generations*. Cambridge: Cambridge University Press.

Diamond, P. (1965). National debt in a neoclassical growth model. *American Economic Review, 55*, 1126–1150.

Domar, E. D. (1946). Capital expansion, rate of growth and employment. *Econometrica, 14*, 137–147.

Galor, O., & Ryder, H. (1991). Dynamic efficiency of steady-state equilibria in an overlapping generations model with productive capital. *Economics Letters, 35*(4), 385–390.

Harrod, R. F. (1939). An essay in dynamic theory. *Economic Journal, 49*, 14–33.

Kaldor, N. (1961). Capital accumulation and economic growth. In F. A. Lutz & D. C. Hague (Eds.), *The theory of capital*. London: Macmillan.

Lopez-Garcia, M. A. (2008). On the role of public debt in an OLG model with endogenous labor supply. *Journal of Macroeconomics, 30*(3), 1323–1328.

Müller, K. W., & Stroebele, W. (1985). *Wachstumstheorie*. München: Oldenburg.

Phelps, E. S. (1966). *Golden rules of economic growth*. New York: Norton.

Ramsey, F. (1928). A mathematical theory of saving. *Economic Journal, 38*, 543–559.

Solow, R. M. (1956). A contribution to the theory of economic growth. *Quartely Journal of Economics, 70*, 65–94.

Swan, T. W. (1956). Economic growth and capital accumulation. *Economic Record, 32*, 334–361.

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