Chapter 2
Uniqueness, Stability and Uniform Lipschitz Estimates

Abstract In this chapter, we first prove the uniqueness of solutions to the Dirichlet boundary value problem (1.4) by the sub- and super-solution method. In Sect. 2.2, we use the same method to prove the stability of solutions to the corresponding parabolic initial-boundary value problem. Finally, by the same idea, we prove the uniform Lipschitz estimates for solutions to these two problems, under suitable boundary conditions.

2.1 A Uniqueness Result for the Elliptic System

In this section, we prove the uniqueness of solutions to the following Dirichlet boundary value problem.

$$\begin{cases} \Delta u_i = \kappa u_i \sum_{j\neq i} b_{ij} u_j, & \text{in } \Omega, \\ u_i = \varphi_i, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Here $b_{ij} \geq 0$ are constants, satisfying $b_{ij} = b_{ji}$. φ_i are given nonnegative Lipschitz continuous functions on $\partial\Omega$. We prove the following theorem.

Theorem 2.1.1 *For any $\kappa \geq 0$, there exists a unique solution to the problem* (2.1).

We use the following iteration scheme to prove the uniqueness of solutions for (2.1). First, we know the following harmonic extension is possible:

$$\begin{cases} \Delta u_{i,0} = 0, & \text{in } \Omega, \\ u_{i,0} = \varphi_i, & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

that is, this equation has a unique positive solution $u_{i,0} \in C^2(\Omega) \cap C^0(\bar{\Omega})$ by Theorem 4.3 of [25].

Then the iteration can be defined as:

$$\begin{cases} \Delta u_{i,m+1} = \kappa u_{i,m+1} \sum_{j\neq i} u_{j,m}, & \text{in } \Omega, \\ u_{i,m+1} = \varphi_i & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

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This is a linear equation. It satisfies the maximum principle, so the existence and uniqueness of the solution $u_{i,m+1} \in C^2(\Omega) \cap C^0(\bar{\Omega})$ is clear (cf. Theorem 6.13 in [25]).

Now concerning these $u_{i,m}$, we have the following result.

Proposition 2.1.2 *In* Ω

$$u_{i,0}(x) > u_{i,2}(x) > \cdots > u_{i,2m}(x) > \cdots > u_{i,2m+1}(x) > \cdots > u_{i,3}(x) > u_{i,1}(x).$$

Proof We divide the proof into several claims.

Claim 1 $\forall i, m, u_{i,m} > 0$ in Ω.

Because $\sum_{j \neq i} u_{j,0} > 0$ in Ω, the equation (2.3) satisfies the maximum principle. Because the boundary value $\varphi_i \geq 0$, $u_{i,1} > 0$ in Ω. By induction, we see the claim holds true for all $u_{i,m}$.

Claim 2 $u_{i,1} < u_{i,0}$ in Ω.

From the equation, now we have

$$\begin{cases} \Delta u_{i,1} \geq 0, & \text{in } \Omega, \\ u_{i,1} = u_{i,0}, & \text{on } \partial\Omega, \end{cases}$$

so we get $u_{i,1} < u_{i,0}$ by the comparison principle.

In the following, we assume the conclusion of the proposition is valid until $2m + 1$, that is in Ω

$$u_{i,0} > \cdots > u_{i,2m} > u_{i,2m+1} > u_{i,2m-1} > \cdots > u_{i,1}.$$

Then we have the following.

Claim 3 $u_{i,2m+1} \leq u_{i,2m+2}$.

By (2.3), we have

$$\Delta u_{i,2m+2} \leq \kappa u_{i,2m+2} \sum_{j \neq i} u_{j,2m}. \tag{2.4}$$

$$\Delta u_{i,2m+1} = \kappa u_{i,2m+1} \sum_{j \neq i} u_{j,2m}. \tag{2.5}$$

Because $u_{i,2m+1}$ and $u_{i,2m+2}$ have the same boundary value, comparing (2.4) and (2.5), by the comparison principle again we obtain that $u_{i,2m+1} \leq u_{i,2m+2}$.

Claim 4 $u_{i,2m+2} \leq u_{i,2m}$.

This can be seen by comparing the equations they satisfy:

$$\begin{cases} \Delta u_{i,2m+2} = \kappa u_{i,2m+2} \sum_{j \neq i} u_{j,2m+1}, \\ \Delta u_{i,2m} = \kappa u_{i,2m} \sum_{j \neq i} u_{j,2m-1}. \end{cases}$$

By assumption, we have $u_{j,2m+1} \geq u_{j,2m-1}$, so the claim follows from the comparison principle again.

Claim 5 $u_{i,2m+3} \geq u_{i,2m+1}$.

This can be seen by comparing the equations they satisfy:

$$\begin{cases} \Delta u_{i,2m+3} = \kappa u_{i,2m+2} \sum_{j \neq i} u_{j,2m+2}, \\ \Delta u_{i,2m+1} = \kappa u_{i,2m+1} \sum_{j \neq i} u_{j,2m}. \end{cases}$$

By Claim 4, we have $u_{j,2m} \geq u_{j,2m+2}$, so the claim follows from the comparison principle again.

Now we know that there exist two family of functions u_i and v_i, such that $\lim_{m\to\infty} u_{j,2m}(x) = u_j(x)$ and $\lim_{m\to\infty} u_{j,2m+1}(x) = v_j(x)$, $\forall x \in \Omega$. Moreover, by standard elliptic estimates, we know this convergence is smooth in Ω and uniformly on $\overline{\Omega}$. So by taking the limit in (2.3), we obtain the following equations:

$$\begin{cases} \Delta u_i = \kappa u_i \sum_{j \neq i} v_j, \\ \Delta v_i = \kappa v_i \sum_{j \neq i} u_j. \end{cases} \tag{2.6}$$

Because $u_{i,2m+1} \leq u_{j,2m}$, by taking limit, we also have

$$v_i \leq u_i. \tag{2.7}$$

Now summing (2.6), we have

$$\begin{cases} \Delta\left(\sum_i u_i\right) = \kappa \sum_i \left(u_i \sum_{j \neq i} v_j\right), \\ \Delta\left(\sum_i v_i\right) = \kappa \sum_i \left(v_i \sum_{j \neq i} u_j\right). \end{cases} \tag{2.8}$$

It is easily seen that

$$\sum_i \left(u_i \sum_{j \neq i} v_j\right) = \sum_i v_i \left(\sum_{j \neq i} u_j\right),$$

so we must have $\sum_i u_i \equiv \sum_i v_i$ because they have the same boundary value. This means, by (2.7), $u_i \equiv v_i \in C^2(\Omega) \cap C^0(\bar{\Omega})$. In particular, they satisfy (2.1). This proves the existence part of Theorem 2.1.1. $\square$

Proposition 2.1.3 *If there exist another positive solution w_i of* (2.1), *we must have $u_i \equiv w_i$.*

Proof We will prove $u_{i,2m} \geq w_i \geq u_{i,2m+1}$, $\forall m$, and then the proposition follows immediately. We divide the proof into several claims.

Claim 1 $w_i \leq u_{i,0}$.

This is because

$$\begin{cases} \Delta w_i \geq 0, & \text{in } \Omega, \\ w_i = u_{i,0}, & \text{on } \partial\Omega. \end{cases}$$

Claim 2 $w_i \geq u_{i,1}$.

This is because

$$\begin{cases} \Delta w_i = \kappa w_i \sum_{j \neq i} w_j, \\ \Delta u_{i,1} = \kappa u_{i,1} \sum_{j \neq i} u_{j,0}. \end{cases}$$

Noting that we have $w_j < u_{j,0}$, so we can apply the comparison principle to get the claim.

In the following, we assume that our claim is valid until $2m + 1$, that is

$$u_{i,2m} \geq w_i \geq u_{i,2m+1}.$$

Then we have the following.

Claim 3 $u_{i,2m+2} \geq w_i$.

This can be seen by comparing the equations they satisfy:

$$\begin{cases} \Delta w_i = \kappa w_i \sum_{j \neq i} w_j, \\ \Delta u_{i,2m+2} = \kappa u_{i,2m+3} \sum_{j \neq i} u_{j,2m+1}. \end{cases}$$

By assumption, we have $u_{j,2m+1} \leq w_j$, so the claim follows from the comparison principle again.

Claim 4 $u_{i,2m+3} \leq w_i$.

This can be seen by comparing the equations they satisfy:

$$\begin{cases} \Delta w_i = \kappa w_i \sum_{j \neq i} w_j, \\ \Delta u_{i,2m+3} = \kappa u_{i,2m+3} \sum_{j \neq i} u_{j,2m+2}. \end{cases}$$

By Claim 3, we have $u_{j,2m+2} \geq w_j$, so the claim follows from the comparison principle again. □

Remark 2.1.4 From our proof, we know that the uniqueness result still holds for equations of more general form:

$$\begin{cases} \Delta u_i = u_i \sum_{j \neq i} b_{ij}(x) u_j, & \text{in } \Omega \\ u_i = \varphi_i & \text{on } \partial\Omega, \end{cases}$$

where $b_{ij}(x)$ are positive (and smooth enough) functions defined in $\overline{\Omega}$, which satisfy $b_{ij} \equiv b_{ji}$.

2.2 Asymptotics in the Parabolic Case

The method in the previous section can also be used to prove the stability of solutions to the following parabolic initial-boundary value problem.

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - \Delta u_i = -\kappa u_i \sum_{j \neq i} b_{ij} u_j, & \text{in } \Omega \times (0, +\infty), \\ u_i = \varphi_i, & \text{on } \partial\Omega \times (0, +\infty), \\ u_i = \phi_i, & \text{on } \Omega \times \{0\}. \end{cases} \tag{2.9}$$

Here $b_{ij} > 0$ and φ_i are those given in Theorem 2.1.1. ϕ_i are given nonnegative Lipschitz continuous functions in Ω, such that $\phi_i = \varphi_i$ on $\partial\Omega$. We prove the following theorem.

Theorem 2.2.1 *For any $\kappa \geq 0$, there exists a unique global solution U of* (2.9). *As $t \to +\infty$, $U(t)$ converges to the solution of* (2.1) *in $C(\overline{\Omega})$.*

Proof Let us consider the iteration scheme analogous to (2.3). First, we consider

$$\begin{cases} \dfrac{\partial u_{i,0}}{\partial t} - \Delta u_{i,0} = 0, & \text{in } \Omega \times (0, +\infty), \\ u_{i,0} = \varphi_i & \text{on } \partial\Omega \times (0, +\infty), \\ u_{i,0} = \phi_i & \text{on } \Omega \times \{0\}. \end{cases}$$

We know this equation has a unique positive solution $u_{i,0}(x,t)$. We also have

$$\lim_{t\to+\infty} u_{i,0}(x,t) = u_{i,0}(x),$$

where the convergence is (for example), in the space of $C^0(\overline{\Omega})$ and $u_{i,0}(x)$ is the solution of (2.2). In fact, we can prove that

$$\int_\Omega \left|\frac{\partial u_{i,0}}{\partial t}\right|^2 dx \le C_1 e^{-C_2 t}$$

for some positive constants C_1 and C_2.

Now the iteration can be defined as:

$$\begin{cases} \dfrac{\partial u_{i,m+1}}{\partial t} - \Delta u_{i,m+1} = -\kappa u_{i,m+1} \sum_{j\ne i} u_{j,m}, & \text{in } \Omega \times (0,+\infty), \\ u_{i,m+1} = \varphi_i & \text{on } \partial\Omega \times (0,+\infty), \\ u_{i,m+1} = \phi_i & \text{on } \Omega \times \{0\}. \end{cases} \tag{2.10}$$

This is just a linear parabolic equation, and there exists a unique global solution $u_{i,m+1}(x,t)$. Differentiating (2.10) in time t, we get

$$\frac{\partial}{\partial t}\frac{\partial u_{i,m+1}}{\partial t} - \Delta \frac{\partial u_{i,m+1}}{\partial t} = -\kappa \frac{\partial u_{i,m+1}}{\partial t} \sum_{j\ne i} u_{j,m} - \kappa u_{i,m+1} \sum_{j\ne i} \frac{\partial u_{j,m}}{\partial t}. \tag{2.11}$$

By the induction assumption and maximum principle, we know there exist constants C_m', $C_{m,1}$ and $C_{m,2}$ such that for $t > 1$,

$$\sum_{j\ne i} u_{j,m+1} \le C_m', \tag{2.12}$$

$$\int_\Omega \left|\frac{\partial u_{i,m}}{\partial t}\right|^2 dx \le C_{m,1} e^{-C_{m,2} t}. \tag{2.13}$$

Multiplying (2.11) by $\frac{\partial u_{i,m+1}}{\partial t}$, with the help of (2.12), we get (note that we have the boundary condition $\frac{\partial u_{i,m+1}}{\partial t} = 0$ on $\partial\Omega$)

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\left|\frac{\partial u_{i,m+1}}{\partial t}\right|^2 + \int_\Omega \left|\nabla \frac{\partial u_{i,m+1}}{\partial t}\right|^2 \le \kappa C_m' \int_\Omega \sum_{j\ne i}\left|\frac{\partial u_{j,m}}{\partial t}\right|\left|\frac{\partial u_{i,m+1}}{\partial t}\right|.$$

Using Cauchy inequality, we get

$$\frac{d}{dt}\int_{\Omega}\frac{1}{2}\left|\frac{\partial u_{i,m+1}}{\partial t}\right|^2+\int_{\Omega}\left|\nabla\frac{\partial u_{i,m+1}}{\partial t}\right|^2$$
$$\leq \kappa C'_m\left(\int_{\Omega}\sum_{j\neq i}\left|\frac{\partial u_{j,m}}{\partial t}\right|^2\right)^{\frac{1}{2}}\left(\int_{\Omega}\left|\frac{\partial u_{i,m+1}}{\partial t}\right|^2\right)^{\frac{1}{2}}.$$

By (2.13) and the Poincare inequality, we get

$$\int_{\Omega}\left|\frac{\partial u_{i,m+1}}{\partial t}\right|^2 dx\leq C_{m+1,1}e^{-C_{m+1,2}t},$$

for some positive constants $C_{m+1,1}$ and $C_{m+1,2}$.

By standard parabolic estimate, this also imply

$$\sup_{\Omega}\left|\frac{\partial u_{i,m+1}}{\partial t}\right|\leq C_{m+1,1}e^{-C_{m+1.2}t},$$

for another two constants $C_{m+1,1}$ and $C_{m+1,2}$. This implies

$$\lim_{t\to+\infty}u_{i,m+1}(x,t)=u_{i,m+1}(x),$$

where $u_{i,m+1}(x)$ is the solution of (2.3). Furthermore, the convergence can be taken (for example) in the space of $C^0(\overline{\Omega})$.

The same method of Sect. 2.2 gives, in $\Omega\times(0,+\infty)$

$$u_{i,0}>\cdots>u_{i,2m}>u_{i,2m+2}>\cdots>u_i>\cdots>u_{i,2m+1}>u_{i,2m-1}>\cdots>u_{i,1}.$$

Now our Theorem 2.2.1 can be easily seen. In fact, $\forall\varepsilon>0$, there exists a m, such that

$$\max_{\overline{\Omega}}\left|u_{i,2m}(x)-u_i(x)\right|<\varepsilon$$

and

$$\max_{\overline{\Omega}}\left|u_{i,2m+1}(x)-u_i(x)\right|<\varepsilon.$$

We also have that there exists a $T>0$, depending on m only, such that, $\forall t>T$,

$$\max_{\overline{\Omega}}\left|u_{i,2m}(x,t)-u_{i,2m}(x)\right|<\varepsilon,$$

and

$$\max_{\overline{\Omega}}\left|u_{i,2m+1}(x,t)-u_{i,2m+1}(x)\right|<\varepsilon.$$

Combing these together, we get $\forall t > T$,

$$\max_{\overline{\Omega}} \left| u_i(x,t) - u_i(x) \right| < 4\varepsilon.$$

This implies that $u_i(x,t)$ converges to the solution $u_i(x)$ of (2.1) as $t \longrightarrow +\infty$, uniformly on $\overline{\Omega}$. (If the boundary values are sufficiently smooth, the convergence in Theorem 2.2.1 can be improved to be smooth enough.) □

2.3 A Uniform Lipschitz Estimate

Finally, by the same idea as in the previous sections, we prove the uniform Lipschitz estimates for solutions to the above two problems (2.9) and (2.2).

Theorem 2.3.1 *There exists a constant $C > 0$ independent of κ, such that for any $\kappa \geq 0$ and solution $(u_{i,\kappa})$ of (2.1), we have*

$$\sup_{\Omega} |\nabla u_{i,\kappa}| \leq C.$$

Theorem 2.3.2 *There exists a constant $C > 0$ independent of κ, such that for any $\kappa \geq 0$ and solution $(u_{i,\kappa})$ of (2.9), we have*

$$\sup_{\Omega \times [0,+\infty)} Lip(u_{i,\kappa}) \leq C.$$

We will only treat the parabolic case. The elliptic case is similar.

We need an additional assumption on the initial-boundary values here. Let Φ_i be the solution of

$$\begin{cases} \dfrac{\partial \Phi_i}{\partial t} - \Delta \Phi_i = 0, & \text{in } \Omega \times (0,+\infty), \\ \Phi_i = \varphi_i, & \text{on } \partial\Omega \times (0,+\infty), \\ \Phi_i = \phi_i, & \text{on } \Omega \times \{0\}. \end{cases} \tag{2.14}$$

We assume that Φ_i are Lipschitz continuous on the closure of $\Omega \times (0,+\infty)$. Note that by comparison principle, we have (see [11] for the proof in the elliptic case)

$$\begin{cases} \Phi_i \geq u_{i,\kappa}, \\ \Phi_i - \sum_{j \neq i} \Phi_j \leq u_{i,\kappa} - \sum_{j \neq i} u_{j,\kappa}. \end{cases} \tag{2.15}$$

First differentiating (2.9) in a space direction e we obtain an equation for $D_e u := e \cdot \nabla u$:

$$\left(\frac{\partial}{\partial t} - \Delta \right) D_e u_{i,\kappa} = -\kappa D_e u_{i,\kappa} \sum_{j \neq i} u_{j,\kappa} - \kappa u_{i,\kappa} \sum_{j \neq i} D_e u_{j,\kappa}.$$

Now using the Kato inequality for smooth functions ϕ

$$\big|\nabla|\phi|\big| = |\nabla\phi| \quad \text{a.e.}, \ \big|\Delta|\phi|\big| \geq |\Delta\phi|,$$

we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)|D_e u_{i,\kappa}| \leq -\kappa|D_e u_{i,\kappa}|\sum_{j\neq i} u_{j,\kappa} + \kappa u_{i,\kappa}\sum_{j\neq i}|D_e u_{j,\kappa}|.$$

Summing these in i, we get

$$\left(\frac{\partial}{\partial t} - \Delta\right)\sum_i |D_e u_{i,\kappa}| \leq 0.$$

By the assumption on Φ_i and (2.15), we have

$$\sup_{\partial\Omega\times(0,+\infty)}\left|\frac{\partial u_{i,\kappa}}{\partial\nu}\right| \leq C,$$

for all i, where ν is the outward unit normal vector and C is independent of κ. With the assumption of Lipschitz continuity of the boundary values on $\partial\Omega \times (0, +\infty)$, we in fact have

$$\sup_{\partial\Omega\times(0,+\infty)} |\nabla u_{i,\kappa}| \leq C,$$

with a constant C independent of κ again. Next, we also have at $t = 0$, $u_{i,\kappa} = \phi_i$, so

$$\sup_{\Omega\times\{0\}} |\nabla u_{i,\kappa}| = \sup_{\Omega} |\nabla\phi_i|.$$

Now the maximum principle implies a global uniform bound:

$$\sup_{\Omega\times[0,+\infty)} |\nabla u_{i,\kappa}| \leq C.$$

Then by a standard method we can get the uniform Lipschitz bound with respect to the parabolic distance.

Remark 2.3.3 Without the boundary regularity, we can still get an interior uniform bound. Multiplying the equation by $u_{i,\kappa}$ and integrating by parts, we can get a L^2 bound for any $T > 0$

$$\sum_i \int_T^{T+1}\int_\Omega |\nabla u_{i,\kappa}|^2 \leq C,$$

with C independent of κ and T. Then we can use the mean value property for subcaloric (or subharmonic function) to give a uniform upper bound of $|\nabla u_{i,\kappa}|$.

Remark 2.3.4 If we consider the original Lotka–Volterra system

$$\frac{\partial u_i}{\partial t} - \Delta u_i = a_i u_i - u_i^2 - \kappa u_i \sum_{j \neq i} u_j,$$

with homogeneous Dirichlet boundary condition, the above results still hold. In fact, we only need to prove a boundary gradient estimate, which can be guaranteed by the following argument: if we define v_i to be the solution of

$$\frac{\partial v_i}{\partial t} - \Delta v_i = a_i v_i - v_i^2,$$

with the same initial value, then by the maximum principle we have for each κ

$$u_{i,\kappa} \leq v_i,$$

which, together with the boundary condition, implies

$$\left| \frac{\partial u_{i,\kappa}}{\partial \nu} \right| \leq \left| \frac{\partial v_i}{\partial \nu} \right|,$$

where ν is the unit outward normal vector to $\partial \Omega$; using the boundary condition once again we get on the boundary

$$|\nabla u_{i,\kappa}| \leq |\nabla v_i|,$$

where the right hand side is independent of κ.

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