

Chapter 2

Problem Formulation

In this chapter, we explain some important aspects of beamforming and differential arrays. The problem of a DMA design is formulated while we progress in defining some useful concepts. We start with the definition of the steering vector for a plane wave with the conventional anechoic farfield model. We give the general definition of the beampattern as well as its expression for directional arrays. We then derive the gain in signal-to-noise ratio (SNR), which can be very useful in the evaluation of DMAs under different types of noise. Finally, we discuss the Vandermonde matrix, which always appears, explicitly or implicitly, in the design of DMAs.

2.1 Signal Model

We consider a source signal (plane wave) that propagates in an acoustic environment (anechoic farfield model) at the speed of sound, i.e., $c = 340$ m/s, and impinges on a uniform linear sensor array consisting of M omnidirectional microphones, where the distance between two successive sensors is equal to δ (see Fig. 2.1). The direction of the source signal to the array is parameterized by the angle θ . In this scenario, the corresponding steering vector (of length M) is

$$\begin{aligned} \mathbf{d}(\omega, \cos \theta) &= \left[1 \ e^{-j\omega\delta \cos \theta/c} \ \dots \ e^{-j(M-1)\omega\delta \cos \theta/c} \right]^T \\ &= \left[1 \ \left(e^{-j\omega\tau_0 \cos \theta} \right)^1 \ \dots \ \left(e^{-j\omega\tau_0 \cos \theta} \right)^{M-1} \right]^T, \end{aligned} \quad (2.1)$$

where the superscript T is the transpose operator, $j = \sqrt{-1}$ is the imaginary unit, $\omega = 2\pi f$ is the angular frequency, $f > 0$ is the temporal frequency, and $\tau_0 = \delta/c$ is the delay between two successive sensors at the angle $\theta = 0^\circ$. The acoustic wavelength is $\lambda = c/f$. In DMAs [1], it is always assumed that

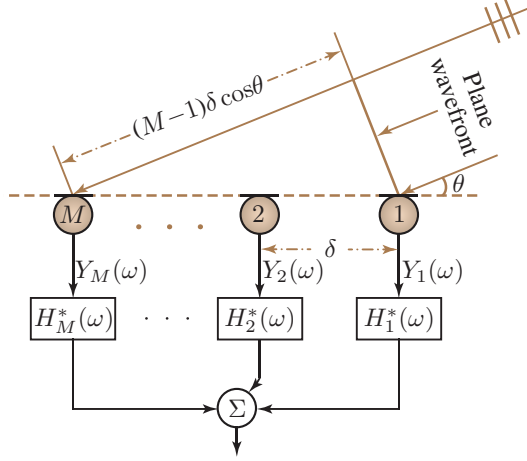


Fig. 2.1 A uniform linear microphone array with processing.

the sensor spacing, δ , is much smaller than the acoustic wavelength, λ , i.e., $\delta \ll \lambda$, implying that

$$\frac{\omega \delta}{c} = \omega \tau_0 \ll 2\pi. \quad (2.2)$$

The condition (2.2) easily holds for small values of δ and at low frequencies but not at high frequencies. With this condition, spatial aliasing, which has the negative effect of creating grating lobes (i.e., copies of the main lobe, which usually points toward the desired signal), is also avoided [2].

We consider fixed directional beamformers¹, like in DMAs, where the main lobe is at the angle $\theta = 0^\circ$ (endfire direction) and the desired signal propagates at the same angle. This position is optimal as will become clearer later. Electronic steering (in the sense that the main lobe can be oriented to any possible direction without affecting the shape of the beampattern) with a uniform linear DMA is not really feasible but we will study some simple possibilities.

As pointed out in [3], there is a fundamental difference between differential arrays and filter-and-sum beamformers. In the latter category, the filters are optimized in such a way that the microphone signals are aligned in order to steer the main lobe in the direction of the desired signal, whereas in the former category the gains are optimized to steer a number of nulls in some specific directions.

The focus of this work is on the design, with small apertures, of beamformers whose beampatterns are very close to the ones obtained with “ideal”

¹ The terms beamformer, beamforming, and beampattern may not be adequate in the context of DMAs but we will still use them for convenience.

DMA. For that, a complex weight, $H_m^*(\omega)$, $m = 1, 2, \dots, M$, is applied at the output of each microphone, where the superscript $*$ denotes complex conjugation. The weighted outputs are then summed together to form the beamformer output as shown in Fig. 2.1. Putting all the gains together in a vector of length M , we get

$$\mathbf{h}(\omega) = [H_1(\omega) H_2(\omega) \cdots H_M(\omega)]^T. \quad (2.3)$$

The objective then is to design such a filter for any directivity pattern of any order. The approach taken here is based on the fundamental observation that for all beampatterns of interest, some constraints must be fulfilled at all frequencies given that the number of microphones is equal to M . In other words, we select M fundamental constraints from a well-defined beampattern of a DMA. For example, in the first-order dipole with two microphones, the two fundamental constraints are a one at the angle 0° and a null at the angle 90° . Since we have two microphones and two constraints, we have a simple linear system of two equations to solve. As a result, the obtained solution is optimal from a mathematical point of view and the derived dipole is the best we can get.

In the next two sections, we discuss some fundamental measures. We are only interested in narrowband measures. The broadband measures can be easily deduced from their respective narrowband counterparts.

2.2 Beampattern

Each beamformer has a pattern of directional sensitivity, i.e., it has different sensitivities from sounds arriving from different directions. The beampattern or directivity pattern describes the sensitivity of the beamformer to a plane wave (source signal) impinging on the array from the direction θ . Mathematically, it is defined as

$$\begin{aligned} \mathcal{B}[\mathbf{h}(\omega), \theta] &= \mathbf{d}^H(\omega, \cos \theta) \mathbf{h}(\omega) \\ &= \sum_{m=1}^M H_m(\omega) e^{j(m-1)\omega\tau_0 \cos \theta}, \end{aligned} \quad (2.4)$$

where the superscript ^{H} is the transpose-conjugate operator.

The frequency-independent beampattern of an N th-order DMA is well known. It is defined as [4]

$$\mathcal{B}_N(\theta) = \sum_{n=0}^N a_{N,n} \cos^n \theta, \quad (2.5)$$

where $a_{N,n}$, $n = 0, 1, \dots, N$, are real coefficients. The different values of these coefficients determine the different directional patterns of the N th-order DMA. In the direction of the desired signal, i.e., for $\theta = 0^\circ$, the beam pattern must be equal to 1, i.e., $\mathcal{B}_N(0^\circ) = 1$. Therefore, we have

$$\sum_{n=0}^N a_{N,n} = 1. \quad (2.6)$$

As a result, we always choose the first coefficient as

$$a_{N,0} = 1 - \sum_{n=1}^N a_{N,n}. \quad (2.7)$$

It follows from (2.5) that an N th-order DMA has at most N (distinct) nulls. All interesting patterns have at least one null in some direction. Since $\cos \theta$ is an even function, so is $\mathcal{B}_N(\theta)$. Therefore, on a polar plot, $\mathcal{B}_N(\theta)$ is symmetric about the axis $0^\circ - 180^\circ$ and any DMA design can be restricted to this range. Polar patterns are a very convenient way to describe the directional sensitivity of the DMAs.

The directivity factor (see also Section 2.3) of an N th-order DMA, defined as the ratio between the directivity pattern at the endfire direction $\theta = 0^\circ$ and the averaged directivity pattern over the whole space, is² [4], [5], [6]

$$\begin{aligned} \mathcal{G}_N &= \frac{\mathcal{B}_N^2(0^\circ)}{\frac{1}{\pi} \int_0^\pi \mathcal{B}_N^2(\theta) d\theta} \\ &= \frac{\pi}{\int_0^\pi \left(\sum_{n=0}^N a_{N,n} \cos^n \theta \right)^2 d\theta} \end{aligned} \quad (2.8)$$

and what we call the directivity index is

$$\mathcal{D}_N = 10 \log_{10} \mathcal{G}_N. \quad (2.9)$$

We find that the first-order, second-order, and third-order directivity factors are

$$\mathcal{G}_1 = \frac{1}{a_{1,0}^2 + \frac{1}{2}a_{1,1}^2}, \quad (2.10)$$

$$\mathcal{G}_2 = \frac{1}{a_{2,0}^2 + \frac{1}{2}a_{2,1}^2 + \frac{3}{8}a_{2,2}^2 + a_{2,0}a_{2,2}}, \quad (2.11)$$

² This situation corresponds to the cylindrically isotropic noise field.

$$\mathcal{G}_3 = \frac{1}{a_{3,0}^2 + \frac{1}{2}a_{3,1}^2 + \frac{3}{8}a_{3,2}^2 + a_{3,0}a_{3,2} + \frac{5}{16}a_{3,3}^2 + \frac{3}{4}a_{3,1}a_{3,3}}. \quad (2.12)$$

The hypercardioid is the pattern obtained from the maximization of the directivity factor³.

The front-to-back ratio is defined as the ratio of the power of the output of the array to signals propagating from the front-half plane to the output power for signals arriving from the rear-half plane [7]. This ratio, for the cylindrically isotropic noise field, is mathematically defined as [4], [7]

$$\mathcal{F}_N = \frac{\int_0^{\pi/2} \mathcal{B}_N^2(\theta) d\theta}{\int_{\pi/2}^{\pi} \mathcal{B}_N^2(\theta) d\theta}. \quad (2.13)$$

The supercardioid is the pattern obtained from the maximization of the front-to-back ratio⁴ [7].

First-order directivity patterns have the form:

$$\mathcal{B}_1(\theta) = (1 - a_{1,1}) + a_{1,1} \cos \theta \quad (2.14)$$

and the most important ones are as follows.

- Dipole: $a_{1,1} = 1$, null at $\cos \theta = 0$, and $\mathcal{D}_1 = 3$ dB.
- Cardioid: $a_{1,1} = \frac{1}{2}$, null at $\cos \theta = -1$, and $\mathcal{D}_1 = 4.3$ dB.
- Hypercardioid: $a_{1,1} = \frac{2}{3}$, null at $\cos \theta = -1/2$, and $\mathcal{D}_1 = 4.8$ dB.
- Supercardioid: $a_{1,1} = 2 - \sqrt{2}$, null at $\cos \theta = (1 - \sqrt{2})/(2 - \sqrt{2})$, and $\mathcal{D}_1 = 4.6$ dB.

Figure 2.2 shows these different polar patterns. What is exactly shown are the values of the magnitude squared beampattern in dB, i.e., $10 \log_{10} \mathcal{B}_1^2(\theta)$.

Second-order beampatterns are described by the equation:

$$\mathcal{B}_2(\theta) = (1 - a_{2,1} - a_{2,2}) + a_{2,1} \cos \theta + a_{2,2} \cos^2 \theta. \quad (2.15)$$

The second-order dipole has a null at $\cos \theta = 0$ and a one (maximum) at $\cos \theta = -1$. Replacing these values in (2.15), we find that $a_{2,1} = 0$ and $a_{2,2} = 1$. By analogy with the first-order and second-order dipoles, we define the N th-order dipole as

$$\mathcal{B}_{D,N}(\theta) = \cos^N \theta, \quad (2.16)$$

³ Another type of hypercardioid can be obtained by maximizing the directivity factor in the presence of a spherically isotropic noise field. There is not much difference, however, between the two patterns.

⁴ Another type of supercardioid can be obtained by maximizing the front-to-back ratio in the presence of a spherically isotropic noise field. There is not much difference, however, between the two patterns.

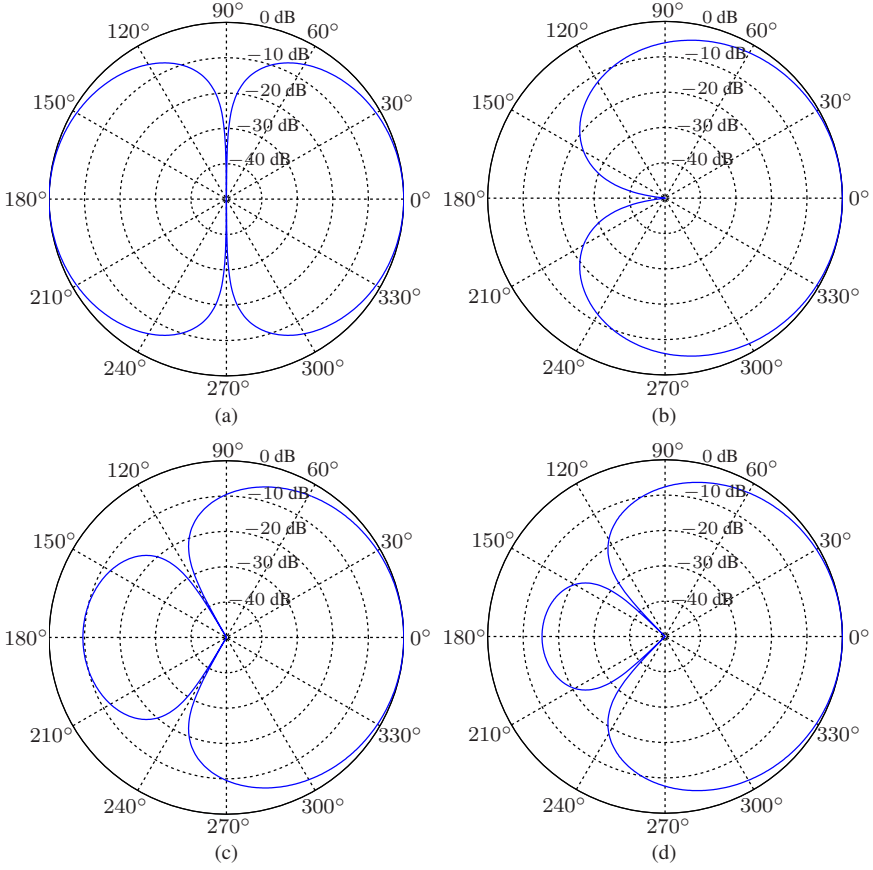


Fig. 2.2 First-order directional patterns: (a) dipole, (b) cardioid, (c) hypercardioid, and (d) supercardioid.

implying that $a_{N,N} = 1$ and $a_{N,N-1} = a_{N,N-2} = \dots = a_{N,0} = 0$. The N th-order dipole has only one (distinct) null (in the range $0^\circ - 180^\circ$) at $\theta = 90^\circ$. The directivity indices of the second-order and third-order dipoles are, respectively, $\mathcal{D}_2 = 4.3$ dB and $\mathcal{D}_3 = 5.1$ dB.

The most well-known second-order cardioid has two nulls; one at $\cos \theta = -1$ and the other one at $\cos \theta = 0$. From these values, we easily deduce from (2.15) that $a_{2,1} = a_{2,2} = \frac{1}{2}$. By analogy with the first-order and second-order cardioids, we define the N th-order cardioid as

$$\mathcal{B}_{C,N}(\theta) = \left(\frac{1}{2} + \frac{1}{2} \cos \theta \right) \cos^{N-1} \theta, \quad (2.17)$$

implying that $a_{N,N} = a_{N,N-1} = \frac{1}{2}$ and $a_{N,N-2} = a_{N,N-3} = \dots = a_{N,0} = 0$. This N th-order cardioid has only two distinct nulls (in the range $0^\circ - 180^\circ$):

one at $\theta = 90^\circ$ and the other one at $\theta = 180^\circ$. The directivity indices of the second-order and third-order cardioids are, respectively, $\mathcal{D}_2 = 6.6$ dB and $\mathcal{D}_3 = 7.6$ dB.

The N th-order hypercardioid and supercardioid are characterized by the fact that they have N distinct nulls in the interval $0^\circ < \theta < 180^\circ$. Hence, their general beampattern is

$$\mathcal{B}_{\text{HS},N}(\theta) = \prod_{n=1}^N [\varsigma_{N,n} + (1 - \varsigma_{N,n}) \cos \theta]. \quad (2.18)$$

Third-order beampatterns have the form

$$\begin{aligned} \mathcal{B}_3(\theta) = & (1 - a_{3,1} - a_{3,2} - a_{3,3}) + a_{3,1} \cos \theta \\ & + a_{3,2} \cos^2 \theta + a_{3,3} \cos^3 \theta. \end{aligned} \quad (2.19)$$

We give the values of $a_{N,n}$ and $\mathcal{D}_N$ for some examples of hypercardioid and supercardioid [4], [6]:

- second-order hypercardioid, $a_{2,1} = \frac{2}{5}$, $a_{2,2} = \frac{4}{5}$, $\mathcal{D}_2 = 7$ dB;
- second-order supercardioid, $a_{2,1} \approx 0.484$, $a_{2,2} \approx 0.413$, $\mathcal{D}_2 = 6.3$ dB;
- third-order hypercardioid, $a_{3,1} = -\frac{4}{7}$, $a_{3,2} = \frac{4}{7}$, $a_{3,3} = \frac{8}{7}$, $\mathcal{D}_3 = 8.4$ dB; and
- third-order supercardioid, $a_{3,1} \approx 0.217$, $a_{3,2} \approx 0.475$, $a_{3,3} \approx 0.286$, $\mathcal{D}_3 = 7.2$ dB.

Figures 2.3 and 2.4 depict the different second-order and third-order directional patterns discussed above.

We are now going to show how the general definition of the beampattern given in (2.4) is very much related to the particular definition of the N th-order directional pattern given in (2.5) for the steering vector defined in (2.1). As a consequence, the dimension (equal to the number microphones) of the vector $\mathbf{d}(\omega, \cos \theta)$ is related to the order N .

Given a function $f(x)$ such that

$$f^{(n)}(x) = \frac{d^n f(x)}{dx^n} \quad (2.20)$$

exists, the MacLaurin's series of $f(x)$ is

$$f(x) = \sum_{n=0}^N \frac{1}{n!} f^{(n)}(0) x^n + R_{N+1}(x), \quad (2.21)$$

where $R_{N+1}(x)$ is some remainder with

$$\lim_{N \rightarrow \infty} R_N(x) = 0. \quad (2.22)$$

We deduce that the MacLaurin's series for the exponential is

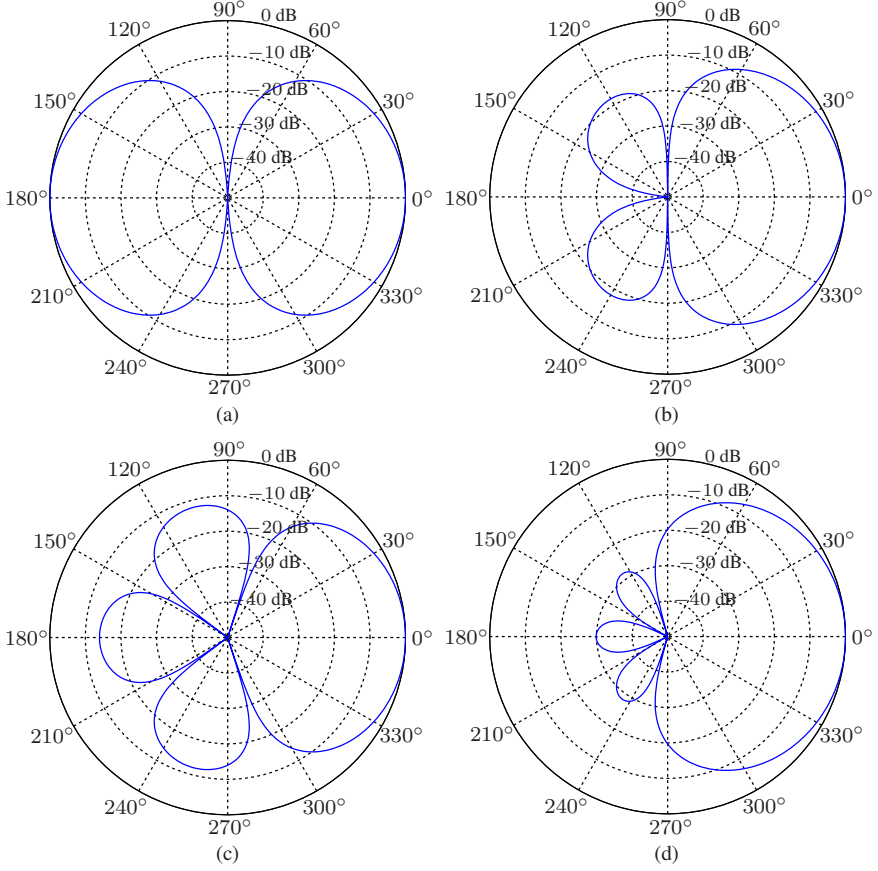


Fig. 2.3 Second-order directional patterns: (a) dipole, (b) cardioid, (c) hypercardioid, and (d) supercardioid.

$$e^x = \sum_{n=0}^N \frac{1}{n!} x^n + R_{N+1}(x). \quad (2.23)$$

Substituting $x = j(m-1)\omega\tau_0 \cos \theta$ in (2.23) and neglecting the remainder, we find that

$$e^{j(m-1)\omega\tau_0 \cos \theta} \approx \sum_{n=0}^N \frac{1}{n!} [j(m-1)\omega\tau_0 \cos \theta]^n. \quad (2.24)$$

Using (2.24) in the general definition of the beam pattern, we obtain

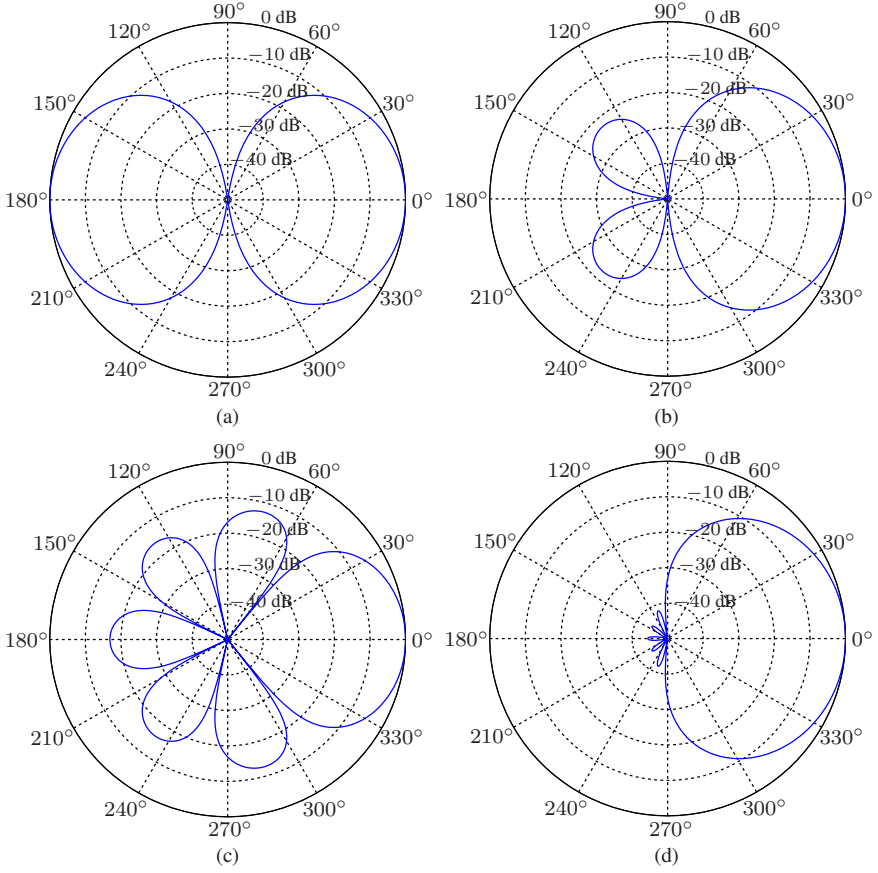


Fig. 2.4 Third-order directional patterns: (a) dipole, (b) cardioid, (c) hypercardioid, and (d) supercardioid.

$$\begin{aligned}
 \mathcal{B}[\mathbf{h}(\omega), \theta] &= \sum_{m=1}^M H_m(\omega) e^{j(m-1)\omega\tau_0 \cos \theta} \\
 &\approx \sum_{m=1}^M H_m(\omega) \sum_{n=0}^N \frac{1}{n!} [j(m-1)\omega\tau_0 \cos \theta]^n \\
 &\approx \sum_{n=0}^N \cos^n \theta \left[\frac{(j\omega\tau_0)^n}{n!} \sum_{m=1}^M (m-1)^n H_m(\omega) \right] \\
 &\approx \sum_{n=0}^N a_{N,n} \cos^n \theta = \mathcal{B}_N(\theta), \tag{2.25}
 \end{aligned}$$

where

$$a_{N,n} \approx \frac{(j\omega\tau_0)^n}{n!} \sum_{m=1}^M (m-1)^n H_m(\omega). \quad (2.26)$$

We observe from (2.25) that as long as $e^{j(m-1)\omega\tau_0 \cos \theta}$ can be approximated by a MacLaurin's series of order N (that is why the microphone spacing should be small), which includes derivatives up to the order N , we can build N th-order differential arrays. We also observe from (2.26) that the gains $H_m(\omega)$, $m = 1, 2, \dots, M$, can be determined given the coefficients $a_{N,n}$, $n = 0, 1, \dots, N$. The least-squares solution ($N+1 > M$) is not appropriate since not only the beampatterns will be highly frequency dependent (it is very hard, if not impossible, to numerically approximate a derivative of order N with a smaller number of points, M) but it is also very hard to have exact nulls in some specific directions and a one at $\theta = 0^\circ$. The minimum-norm solution ($N+1 < M$) is a good choice from both theoretical and practical viewpoints; this concept will be elaborated in Chapter 6. But for all the other chapters, it will always be assumed that the design of an N th-order differential array requires $N+1$ microphones.

2.3 Gain in Signal-to-Noise Ratio (SNR)

The first microphone serves as the reference and we recall that the desired signal comes from the angle $\theta = 0^\circ$. In this case, the m th microphone signal is given by

$$Y_m(\omega) = e^{-j(m-1)\omega\tau_0} X(\omega) + V_m(\omega), \quad m = 1, 2, \dots, M, \quad (2.27)$$

where $X(\omega)$ is the desired signal and $V_m(\omega)$ is the additive noise at the m th microphone. In a vector form, (2.27) becomes

$$\begin{aligned} \mathbf{y}(\omega) &= [Y_1(\omega) \ Y_2(\omega) \ \cdots \ Y_M(\omega)]^T \\ &= \mathbf{d}(\omega, \cos 0^\circ) X(\omega) + \mathbf{v}(\omega), \end{aligned} \quad (2.28)$$

where the noise signal vector, $\mathbf{v}(\omega)$, is defined similarly to $\mathbf{y}(\omega)$.

The beamformer output is simply

$$\begin{aligned} Z(\omega) &= \sum_{m=1}^M H_m^*(\omega) Y_m(\omega) \\ &= \mathbf{h}^H(\omega) \mathbf{y}(\omega) \\ &= \mathbf{h}^H(\omega) \mathbf{d}(\omega, \cos 0^\circ) X(\omega) + \mathbf{h}^H(\omega) \mathbf{v}(\omega), \end{aligned} \quad (2.29)$$

where $Z(\omega)$ is supposed to be the estimate of the desired signal, $X(\omega)$.

We define the input signal-to-noise ratio (SNR) as

$$\text{iSNR}(\omega) = \frac{\phi_X(\omega)}{\phi_{V_1}(\omega)}, \quad (2.30)$$

where $\phi_X(\omega) = E[|X(\omega)|^2]$ and $\phi_{V_1}(\omega) = E[|V_1(\omega)|^2]$ are the variances of $X(\omega)$ and $V_1(\omega)$, respectively.

The output SNR is defined as

$$\begin{aligned} \text{oSNR}[\mathbf{h}(\omega)] &= \phi_X(\omega) \frac{|\mathbf{h}^H(\omega) \mathbf{d}(\omega, \cos 0^\circ)|^2}{\mathbf{h}^H(\omega) \mathbf{\Phi}_{\mathbf{v}}(\omega) \mathbf{h}(\omega)} \\ &= \frac{\phi_X(\omega)}{\phi_{V_1}(\omega)} \cdot \frac{|\mathbf{h}^H(\omega) \mathbf{d}(\omega, \cos 0^\circ)|^2}{\mathbf{h}^H(\omega) \mathbf{\Gamma}_{\mathbf{v}}(\omega) \mathbf{h}(\omega)}, \end{aligned} \quad (2.31)$$

where

$$\mathbf{\Phi}_{\mathbf{v}}(\omega) = E[\mathbf{v}(\omega) \mathbf{v}^H(\omega)] \quad (2.32)$$

and

$$\mathbf{\Gamma}_{\mathbf{v}}(\omega) = \frac{\mathbf{\Phi}_{\mathbf{v}}(\omega)}{\phi_{V_1}(\omega)} \quad (2.33)$$

are the correlation and pseudo-coherence matrices of $\mathbf{v}(\omega)$, respectively.

The definition of the gain in SNR is easily derived from the two previous definitions, i.e.,

$$\begin{aligned} \mathcal{G}[\mathbf{h}(\omega)] &= \frac{\text{oSNR}[\mathbf{h}(\omega)]}{\text{iSNR}(\omega)} \\ &= \frac{|\mathbf{h}^H(\omega) \mathbf{d}(\omega, \cos 0^\circ)|^2}{\mathbf{h}^H(\omega) \mathbf{\Gamma}_{\mathbf{v}}(\omega) \mathbf{h}(\omega)}. \end{aligned} \quad (2.34)$$

Assume that the matrix $\mathbf{\Gamma}_{\mathbf{v}}(\omega)$ is nonsingular. In this case, for any two vectors $\mathbf{h}(\omega)$ and $\mathbf{d}(\omega, \cos 0^\circ)$, we have

$$\begin{aligned} |\mathbf{h}^H(\omega) \mathbf{d}(\omega, \cos 0^\circ)|^2 &\leq [\mathbf{h}^H(\omega) \mathbf{\Gamma}_{\mathbf{v}}(\omega) \mathbf{h}(\omega)] \times \\ &\quad [\mathbf{d}^H(\omega, \cos 0^\circ) \mathbf{\Gamma}_{\mathbf{v}}^{-1}(\omega) \mathbf{d}(\omega, \cos 0^\circ)], \end{aligned} \quad (2.35)$$

with equality if and only if $\mathbf{h}(\omega) \propto \mathbf{\Gamma}_{\mathbf{v}}^{-1}(\omega) \mathbf{d}(\omega, \cos 0^\circ)$. Using the inequality (2.35) in (2.34), we deduce an upper bound for the gain:

$$\begin{aligned} \mathcal{G}[\mathbf{h}(\omega)] &\leq \mathbf{d}^H(\omega, \cos 0^\circ) \mathbf{\Gamma}_{\mathbf{v}}^{-1}(\omega) \mathbf{d}(\omega, \cos 0^\circ) \\ &\leq \text{tr}[\mathbf{\Gamma}_{\mathbf{v}}^{-1}(\omega)] \text{tr}[\mathbf{d}(\omega, \cos 0^\circ) \mathbf{d}^H(\omega, \cos 0^\circ)] \\ &\leq M \text{tr}[\mathbf{\Gamma}_{\mathbf{v}}^{-1}(\omega)], \end{aligned} \quad (2.36)$$

where $\text{tr}[\cdot]$ is the trace of a square matrix. We observe how the gain is upper bounded [as long as $\mathbf{\Gamma}_{\mathbf{v}}(\omega)$ is nonsingular] and depends on the number of microphones as well as on the nature of the noise.

In our context, the distortionless constraint is desired, i.e.,

$$\mathbf{h}^H(\omega) \mathbf{d}(\omega, \cos 0^\circ) = 1. \quad (2.37)$$

As a consequence, it is easy to see that the filter:

$$\mathbf{h}_{\max}(\omega) = \frac{\mathbf{\Gamma}_{\mathbf{v}}^{-1}(\omega) \mathbf{d}(\omega, \cos 0^\circ)}{\mathbf{d}^H(\omega, \cos 0^\circ) \mathbf{\Gamma}_{\mathbf{v}}^{-1}(\omega) \mathbf{d}(\omega, \cos 0^\circ)} \quad (2.38)$$

maximizes the gain, which is given by

$$\mathcal{G}_{\max}(\omega) = \mathbf{d}^H(\omega, \cos 0^\circ) \mathbf{\Gamma}_{\mathbf{v}}^{-1}(\omega) \mathbf{d}(\omega, \cos 0^\circ). \quad (2.39)$$

We are interested in three types of noise.

- The temporally and spatially white noise with the same variance at all microphones⁵. In this case, $\mathbf{\Gamma}_{\mathbf{v}}(\omega) = \mathbf{I}_M$, where $\mathbf{I}_M$ is the $M \times M$ identity matrix. Therefore, the white noise gain is

$$\begin{aligned} \mathcal{G}_{\text{wn}}[\mathbf{h}(\omega)] &= \frac{|\mathbf{h}^H(\omega) \mathbf{d}(\omega, \cos 0^\circ)|^2}{\mathbf{h}^H(\omega) \mathbf{h}(\omega)} \\ &= \frac{1}{\mathbf{h}^H(\omega) \mathbf{h}(\omega)}, \end{aligned} \quad (2.40)$$

where in the second line of (2.40), the distortionless constraint is assumed. For

$$\mathbf{h}(\omega) = \frac{\mathbf{d}(\omega, \cos 0^\circ)}{M}, \quad (2.41)$$

we find the maximum possible gain, which is

$$\mathcal{G}_{\text{wn},\max}(\omega) = M. \quad (2.42)$$

In general, the white noise gain of an N th-order DMA is

$$\mathcal{G}_{\text{wn},N}[\mathbf{h}(\omega)] = \frac{1}{\mathbf{h}^H(\omega) \mathbf{h}(\omega)} \leq M. \quad (2.43)$$

We will see how the white noise may be amplified by DMAs, especially at low frequencies.

- The diffuse noise⁶, where

⁵ This noise models well the sensor noise.

⁶ This situation corresponds to the spherically isotropic noise field.

$$\begin{aligned}
[\mathbf{\Gamma}_{\mathbf{v}}(\omega)]_{ij} &= [\mathbf{\Gamma}_{\text{dn}}(\omega)]_{ij} = \frac{\sin[\omega(j-i)\tau_0]}{\omega(j-i)\tau_0} \\
&= \text{sinc}[\omega(j-i)\tau_0].
\end{aligned} \tag{2.44}$$

In this scenario, the gain in SNR, $\mathcal{G}_{\text{dn}}[\mathbf{h}(\omega)]$, is called the directivity factor and the directivity index is simply defined as [2], [4]

$$\mathcal{D}[\mathbf{h}(\omega)] = 10 \log_{10} \mathcal{G}_{\text{dn}}[\mathbf{h}(\omega)]. \tag{2.45}$$

With diffuse noise, the filter $\mathbf{h}(\omega)$ is often found by maximizing the directivity factor. As a result, the optimal filter is given by (2.38).

- The noise comes from a point source at the angle θ_n . In this case, the pseudo-coherence matrix is

$$\mathbf{\Gamma}_{\mathbf{v}}(\omega) = \mathbf{d}(\omega, \cos \theta_n) \mathbf{d}^H(\omega, \cos \theta_n), \tag{2.46}$$

where

$$\mathbf{d}(\omega, \cos \theta_n) = \left[1 \ e^{-j\omega\tau_0 \cos \theta_n} \ \dots \ e^{-j(M-1)\omega\tau_0 \cos \theta_n} \right]^T \tag{2.47}$$

is the steering vector of the noise source. We observe from (2.46) that the pseudo-coherence matrix is singular. In fact, this is the only possibility where the gain in SNR, $\mathcal{G}_{\text{ns}}[\mathbf{h}(\omega)]$, is not upper bounded and can go to infinity. We deduce that this gain is

$$\begin{aligned}
\mathcal{G}_{\text{ns}}[\mathbf{h}(\omega)] &= \frac{|\mathbf{h}^H(\omega) \mathbf{d}(\omega, \cos 0^\circ)|^2}{|\mathbf{h}^H(\omega) \mathbf{d}(\omega, \cos \theta_n)|^2} \\
&= \frac{1}{|\mathbf{h}^H(\omega) \mathbf{d}(\omega, \cos \theta_n)|^2}.
\end{aligned} \tag{2.48}$$

When the noise and desired signals come from the same direction, i.e., when $\theta_n = 0^\circ$, then there is no possible gain, i.e., $\mathcal{G}_{\text{ns}}[\mathbf{h}(\omega)] = 1$, $\forall \mathbf{h}(\omega)$. We also deduce the gain of an N th-order DMA:

$$\mathcal{G}_{\text{ns},N}(\theta_n) = \frac{1}{|\mathcal{B}_N(\theta_n)|^2}. \tag{2.49}$$

Figures 2.5, 2.6, and 2.7 depict this gain, as a function of the direction of the noise, for the different first-order, second-order, and third-order patterns (dipole, cardioid, hypercardioid, and supercardioid).

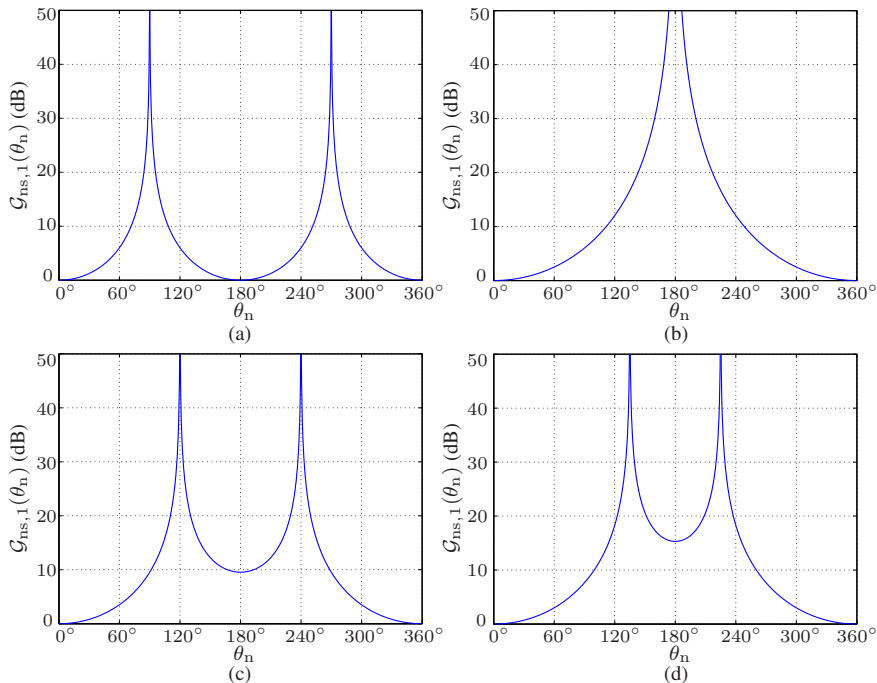


Fig. 2.5 Gain in SNR as a function of the direction (θ_n) of the point noise source for the first-order DMA: (a) dipole, (b) cardioid, (c) hypercardioid, and (d) supercardioid.

2.4 Vandermonde Matrix

Given the definition of the steering vector and combining steering vectors for different angles in a matrix, we obtain the Vandermonde structure. Therefore, it is extremely useful to exploit the structure of this matrix.

A Vandermonde matrix of size $M \times M$ has the form:

$$\mathbf{V}_M = \begin{bmatrix} 1 & v_1 & v_1^2 & \cdots & v_1^{M-1} \\ 1 & v_2 & v_2^2 & \cdots & v_2^{M-1} \\ 1 & v_3 & v_3^2 & \cdots & v_3^{M-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & v_M & v_M^2 & \cdots & v_M^{M-1} \end{bmatrix}. \quad (2.50)$$

It can be shown that the determinant of $\mathbf{V}_M$ is

$$\det(\mathbf{V}_M) = \prod_{j>i} (v_j - v_i). \quad (2.51)$$

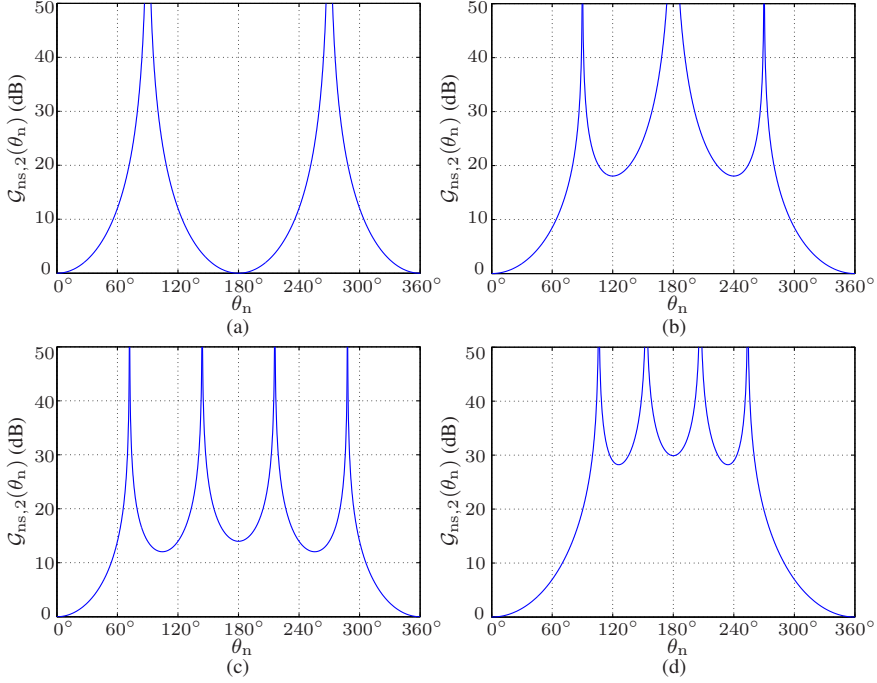


Fig. 2.6 Gain in SNR as a function of the direction (θ_n) of the point noise source for the second-order DMA: (a) dipole, (b) cardioid, (c) hypercardioid, and (d) supercardioid.

As a consequence, as long as the values of v_m are all distinct, the matrix $\mathbf{V}_M$ is nonsingular.

It will be important to have a closed-form expression of the inverse of the Vandermonde matrix. For that, we will use the decomposition proposed in [8]:

$$\mathbf{V}_M^{-1} = \mathbf{U}_M \mathbf{L}_M, \quad (2.52)$$

where $\mathbf{U}_M$ and $\mathbf{L}_M$ are upper and lower triangular matrices, respectively.

The elements l_{ij} of $\mathbf{L}_M$ are given by the relations:

$$l_{ij} = \begin{cases} 0, & i < j \\ 1, & i = j = 1 \\ \prod_{p=1, p \neq j}^i \frac{1}{v_j - v_p}, & \text{otherwise} \end{cases}. \quad (2.53)$$

It is proved in [8] that the elements u_{ij} of $\mathbf{U}_M$ are given by the definition:

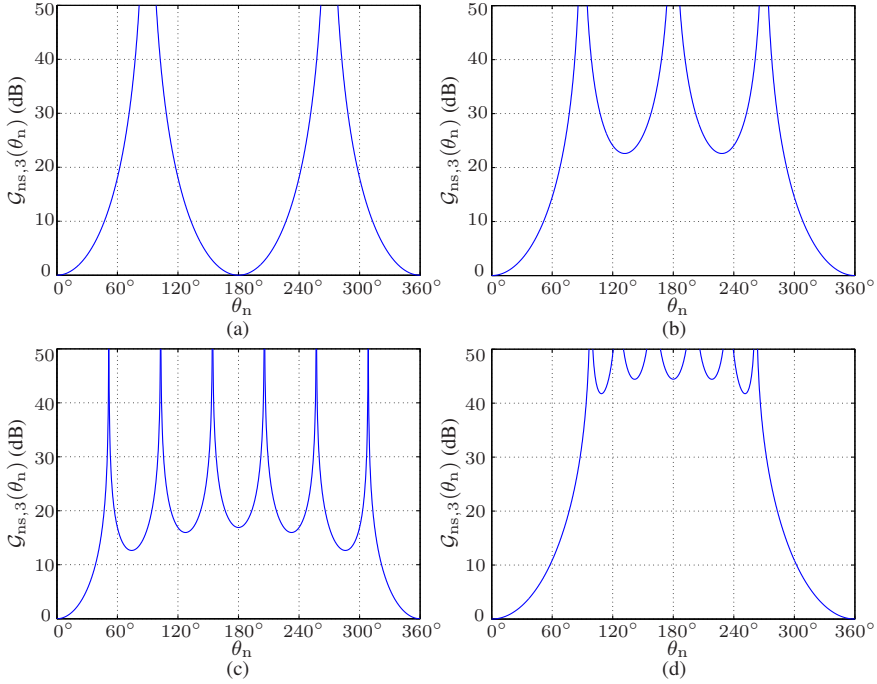


Fig. 2.7 Gain in SNR as a function of the direction (θ_n) of the point noise source for the third-order DMA: (a) dipole, (b) cardioid, (c) hypercardioid, and (d) supercardioid.

$$u_{ij} = \begin{cases} 1, & j = i \\ 0, & j = 1 \\ u_{i-1,j-1} - u_{i,j-1}v_{j-1}, & \text{otherwise} \end{cases}, \quad (2.54)$$

where

$$u_{0j} = 0. \quad (2.55)$$

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