Preface

The purpose of this memoir is to present a theory of asymptotic expansions for functions of two variables, using at the same time functions of one variable and functions of the quotient of these two variables. These *composite asymptotic expansions* (CAsEs for short) are particularly well suited to the description of solutions of singularly perturbed ordinary differential equations. Their use is classical for boundary layers, but less familiar near turning points.

Let us describe the context in a few words. Consider an equation of the form

$$\varepsilon \frac{dy}{dx} = \Phi(x, y, \varepsilon) \tag{1}$$

where Φ is infinitely differentiable, x and y are real or complex variables, and ε is a small parameter, real positive or in a sector of the complex plane. Such an equation might be obtained from an autonomous slow–fast system

$$\begin{aligned} \frac{du}{dt} &= \varepsilon f(u, v, \varepsilon) \\ \frac{dv}{dt} &= g(u, v, \varepsilon) \end{aligned} \tag{2}$$

by elimination of the variable t, i.e. by considering the function $y = v \circ u^{-1}$.

The *slow set* of (1) is the subset $\mathscr{L}$ of $\mathbb{C}^2$ determined by $\Phi(x, y, 0) = 0$. A point (x, y) of $\mathscr{L}$ is called *regular* if $\frac{\partial \Phi}{\partial y}(x, y, 0) \neq 0$; otherwise it is called a *turning point*.

In a neighborhood of a regular point $(\widetilde{x}, \widetilde{y}) \in \mathscr{L}$, one easily sees that (1) has a unique formal solution $\widehat{y} = \sum_{n\geq 0} y_n(x)\varepsilon^n$ and it is known that certain solutions of (1) have $\widehat{y}$ as asymptotic expansion in appropriate domains. The *classical* theory of composite expansions (see e.g. Vasi'leva/Butuzov [59]) is useful to describe the *boundary layer* (also called inner layer) of a solution which takes at $\widetilde{x}$ an initial value close enough to $\widetilde{y}$. For example, in the real framework, if the point $(\widetilde{x}, \widetilde{y})$ is *attracting*, i.e. $\frac{\partial\Phi}{\partial y}(\widetilde{x},\widetilde{y},0) < 0$, then an approximation of the solution $y = y(x,\varepsilon)$ can be given, uniformly on some interval $[\widetilde{x},\widetilde{x}+\delta]$, of the form $\sum_{n\geq 0}\Big(y_n(x) + z_n\big(\frac{x-\widetilde{x}}{\varepsilon}\big)\Big)\varepsilon^n$, containing the formal solution $\widehat{y}$ as well as functions z_n exponentially decreasing at infinity. See also Benoît/El Hamidi/Fruchard [4] for details.

At a turning point (x^*, y^*), the coefficients of the formal solution may have poles or ramified singularities and the classical theory of composite expansions no longer applies. The most common method to obtain an approximation of the solutions is the *matching* of the so-called *inner* and *outer* expansions; in our memoir we would like to propose an alternative.

We present an extensive study of infinite CAsEs of the form

$$\sum_{n\geq 0}\Big(a_n(x) + g_n\big(\tfrac{x-x^*}{\eta}\big)\Big)\eta^n, \tag{3}$$

η a certain root of ε, that are uniformly valid for x in some sector with vertex x^* satisfying $K|\eta| \leq |x-x^*| \leq L$ with some $K, L > 0$, and sometimes even in a full neighborhood of x^* of size $\mathcal{O}(|\eta|)$. Such CAsEs provide outer expansions by re-expanding $g_n\left(\frac{x-x^*}{\eta}\right)$ with respect to η and inner expansions by replacing $x = \eta X$ and re-expanding $a_n(\eta X)$. One advantage of composite expansions is that they provide approximations also in "intermediate" ranges $K|\eta|^\alpha \leq |x| \leq L|\eta|^\beta$, where $0 < \beta \leq \alpha < 1$.

Composite expansions and even more composite approximations are not new. In the context of matched asymptotic expansions, they are often found as the sum of inner and outer expansions, less the terms common to both see e.g. Kevorkian/Cole [35], p.13, or Skinner [54], p.4. We propose to work directly with composite expansions using the many properties and theorems we present and compare with inner and outer expansions in a second step. More remarks concerning the relations with previous work will be given in Chap. 7.

There is a vast literature on singular perturbation methods for ordinary differential equations. The presentation of another book on this subject should be well motivated. We hope to shed some new light on composite expansions with the following features.

- We work with *infinite* asymptotic expansions, discuss their algebraic and analytic properties, and use them to obtain new results.
- The expansions are mainly presented in the complex domain. Their regions of validity are modified sectors with vertex at a turning point.
- We also present a Gevrey theory of CAsEs—so far the Gevrey theory was confined to classical uniform asymptotic expansions where the coefficients depend upon one or more variables different from the expansion variable. In the literature on singularly perturbed ordinary differential equations the Gevrey theory was mainly used in full neighborhoods of a turning point

(e.g. Canalis-Durand/Ramis/Schäfke/Sibuya [9]) whereas here we have (quasi-) sectors with vertex at a turning point.
- We exhibit three applications, in two of which new results are obtained, thanks to the Gevrey theory of CASEs.

The plan of the memoir is as follows. In the first chapter, we present simple examples of linear differential equations to convince the reader that composite asymptotic expansions are natural, even unavoidable, near turning points.

The second chapter presents the definition of CASEs and their compatibility with algebraic and analytic operations and discusses the relation with the method of matched asymptotic expansions. The third chapter contains the Gevrey version of the theory of CASEs. The relation with exponentially small terms is an important feature.

Chapter 4 states and proves our most important theorem on the existence of Gevrey CASEs. It essentially states, for a family of holomorphic functions of (x,η) that are bounded on a so-called consistent good covering of a punctured neighborhood of $(0,0)$, that each of them has a Gevrey CASE provided their differences are exponentially small. We emphasize that here, unlike in the classical Gevrey theory, *two* kinds of exponential smallness appear: the differences in the x-plane are $\mathcal{O}\left(e^{-c|x|^p/|\eta|^p}\right)$ and hence only small away from the origin whereas the differences in the η-plane are $\mathcal{O}\left(e^{-c/|\eta|^p}\right)$ as usual.

Chapter 5 applies the theory of CASEs to singularly perturbed nonlinear ordinary differential equations of first-order near turning points. We present two methods for obtaining CASEs. The first is direct, following the classical lines: study formal solutions of the form (3), then prove the existence of an analytic solution having this formal solution as a CASE. The second method is more indirect and uses the main result of Chap. 4, embedding a single solution in a family of solutions on a good covering of a neighborhood of the turning point and showing that their differences satisfy exponential estimates. A reader familiar with differential equations in the real domain might find this method too "complex," but it is, to our knowledge, the simplest one in the context of singular perturbation.

We first treat a type of equations allowing analytic continuation of the solutions to a small neighborhood of the turning point and then consider more general equations. The most powerful result in this section is Theorem 5.17.

In the sixth chapter, we apply the previous results and use the general properties of CASEs for several well-known problems of singular perturbation: canards solutions near multiple turning points, non-smooth canards, and Ackerberg–O'Malley resonance. The linear differential equation of second order appearing in the third problem is reduced to a first-order Riccati equation to which our results are applied. For the first and third problems, we obtain new results, and for the second problem, we can add Gevrey properties and Gevrey CASEs to the known results.

The memoir focuses on ordinary differential equations and only considers first-order nonlinear and second-order linear equations, but we are convinced that CASEs can be very useful for equations of higher order and for other types of functional equations, especially difference equations.

We have included a few exercises for the benefit of the reader—in most of them, the results of the corresponding section are applied to examples.

Acknowledgements The work of the second author was supported in part by grants of the French National Research Agency (ref. ANR-10-BLAN 0102 and ANR-11-BS01-0009).

http://www.springer.com/978-3-642-34034-5

Composite Asymptotic Expansions
Fruchard, A.; Schafke, R.
2013, X, 161 p. 21 illus., Softcover
ISBN: 978-3-642-34034-5