Preface

The topic of this book is the classification theorem for compact surfaces. We present the technical tools needed for proving rigorously the classification theorem, give a detailed proof using these tools, and also discuss the history of the theorem and its various "proofs."

We find the classification theorem for compact surfaces quite fascinating because its statement fits very well our intuitive notion of a surface (given that one recognizes that there are non-orientable surfaces as well as orientable surfaces) but a rigorous proof requires a significant amount of work and machinery. Indeed, it took about 60 years until a rigorous proof was finally given by Brahana [4] in 1921. Early versions of the classification theorem were stated by Möbius [13] in 1861 and by Jordan [9] in 1866. Present-day readers will be amused by the "proofs" given by Möbius and Jordan who did not have the required technical tools at their disposal and did not even have the definition of a (topological) surface. More definite versions and "proofs" were given later by von Dyck [6] in 1888 and by Dehn and Heegaard [5] in 1907. One of our goals is to present the history of the proof as complete as possible. A detailed history seems lacking in the literature and should be of interest to anyone interested in topology.

It is our opinion that the classification theorem for compact surfaces provides a natural and wonderful incentive for learning some of the basic tools of algebraic topology, in particular homology groups, a somewhat arduous task without relevant motivations. The reward for such an effort is a thorough understanding of the proof of the classification theorem. Our experience is that self-disciplined and curious students are willing to make such an effort and find it rewarding. It is our hope that our readers will share such feelings.

The classification theorem for compact surfaces is covered in most algebraic topology books. The theorem either appears at the beginning, in which case the presentation is usually rather informal because the machinery needed to give a formal proof has not been introduced yet (as in Massey [12]) or it is given as an application of the machinery, as in Seifert and Threlfall [15], Ahlfors and Sario [1], Munkres [14], and Lee [11] (the proofs in Seifert and Threlfall [15] and Ahlfors and Sario [1] are also very formal). Munkres [14] and Lee [11] give rigorous and essentially complete proofs (except for the fact that surfaces can be triangulated). Munkres' proof appears in Chap. 12 and depends on material on the fundamental group from Chaps. 9 and 11. Lee's proof starts in Chap. 6 and ends in Chap. 10, which depends on Chap. 7 on the fundamental group. These proofs are very nice but we feel that the reader will have a hard time jumping in without having read a significant portion of these books. We make further comparisons between Munkres' and Lee's approach with ours in Chap. 6.

We thought that it would be useful for a wider audience to present a proof of the classification theorem for compact surfaces more leisurely than that of Ahlfors and Sario [1] (or Seifert and Threlfall [15] or Munkres [14] or Lee [11]) but more formal and more complete than other sources such as Massey [12], Amstrong [2], Kinsey [10], Henle [8], Bloch [3], Fulton [7], and Thurston [16]. Such a proof should be accessible to readers who have a certain amount of "mathematical maturity." This definitely includes first-year graduate students but also strongly motivated upper-level undergraduates. Our hope is that after reading our guide, the reader will be well prepared to read and compare other proofs of the theorem on the classification of surfaces, especially in Seifert and Threlfall [15], Ahlfors and Sario [1], Massey [12], Munkres [14], and Lee [11]. It is also our hope that our introductory chapter on homology (Chap. 5) will inspire the reader to undertake a deeper study of homology and cohomology, two fascinating and powerful theories.

We begin with an informal presentation of the theorem, very much as in Massey's excellent book [12]. Then, we develop the technical tools to give a rigorous proof: the definition of a surface in Chap. 2, simplicial complexes and triangulations in Chap. 3, the fundamental group and orientability in Chap. 4, and homology groups in Chap. 5. The proof of the classification theorem for compact surfaces is given in Chap. 6, the main chapter of this book.

In order not to interrupt the main thread of the book (the classification theorem), we felt that it was best to put some of the material in some appendices. For instance, a review of basic topological preliminaries (metric spaces, normed spaces, topological spaces, continuous functions, limits, connected sets, and compact sets) is given in Appendix C. The history of the theorem and its "proofs" are discussed quite extensively in Appendix D. Finally, a proof that every surface can be triangulated is given in Appendix E. Various notes are collected in Appendix F.

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