Chapter 2
Pinning Control for Complete Synchronization of Complex Dynamical Networks

Abstract This chapter first introduces some complex network models and then gives both local and global stability conditions for complete synchronization of complex dynamical networks. In addition, the concept of virtual control of pinned complex dynamical networks is introduced to illustrate the principle of pinning control. The main reason is explained for why significantly less local controllers are required by specifically pinning the most highly connected nodes in a scale-free network than those required by the randomly pinning scheme, and why there is no significant difference between specifically and randomly pinning schemes for controlling random dynamical networks. Finally, a novel pinning scheme based on ControlRank (CR) is introduced, which is a vertex centrality index exploring the link structure of the directed networks. Moreover, it is shown that pinning the vertex with largest CR is much more effective than pinning the vertex with largest out-degree.

Keywords Complete synchronization · Virtual control · Complex network · ControlRank

2.1 Complex Network Models

In this section, we will introduce three complex network models, that is, ER random network model, BA scale-free network model and a new directed scale-free network model, which will be used in this chapter.

2.1.1 ER Random Network Model

The basic ER random network model is defined as a random graph of labeled N nodes connected by n edges, which are chosen randomly from all the $N(N-1)/2$ possible edges. The network evolution is uniform: Starting with N nodes, every pair of nodes are connected with the same probability p (Fig. 2.1).

The main goal of the random graph theory is to determine in what connection probability p a particular property of a graph will likely arise. For a large N, the

H. Su, X. Wang, *Pinning Control of Complex Networked Systems*,
DOI 10.1007/978-3-642-34578-4_2,

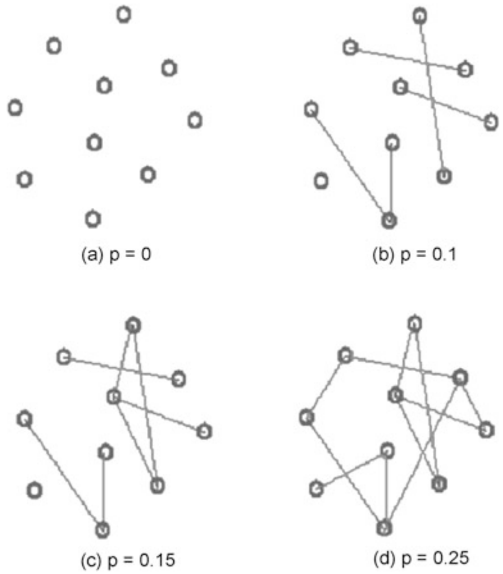

Fig. 2.1 [1] Evolution of an ER random graph. One starts with $N = 10$ isolated nodes in (**a**), and connects every pair of nodes with probability (**b**) $p = 0.1$, (**c**) $p = 0.15$, and (**d**) $p = 0.25$, respectively

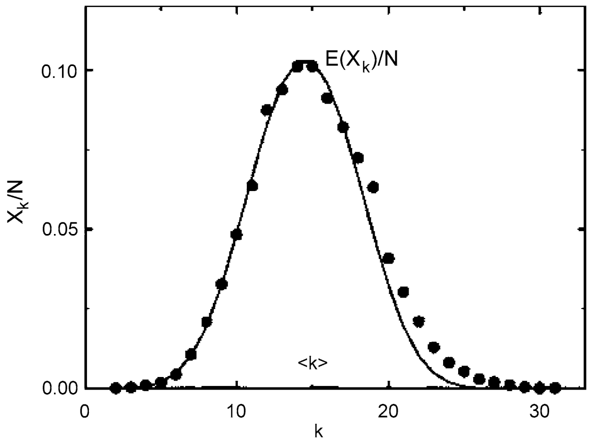

Fig. 2.2 [1] Degree distribution that results from the numerical simulation of an ER random network, which is generated with $N = 10000$ and connection probability $p = 0.0015$. (After Albert and Barabási [2])

ER model generates a homogeneous random network, whose connectivity approximately follows a Poisson distribution described by (Fig. 2.2)

$$P(k) \approx e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!} \tag{2.1}$$

where $\langle k \rangle$, the so-called average degree of the network, is the average of k_i over all i nodes in the network. With this connectivity distribution, nodes in the network are quite uniformly spread out, which is known as a homogeneous feature of the distribution.

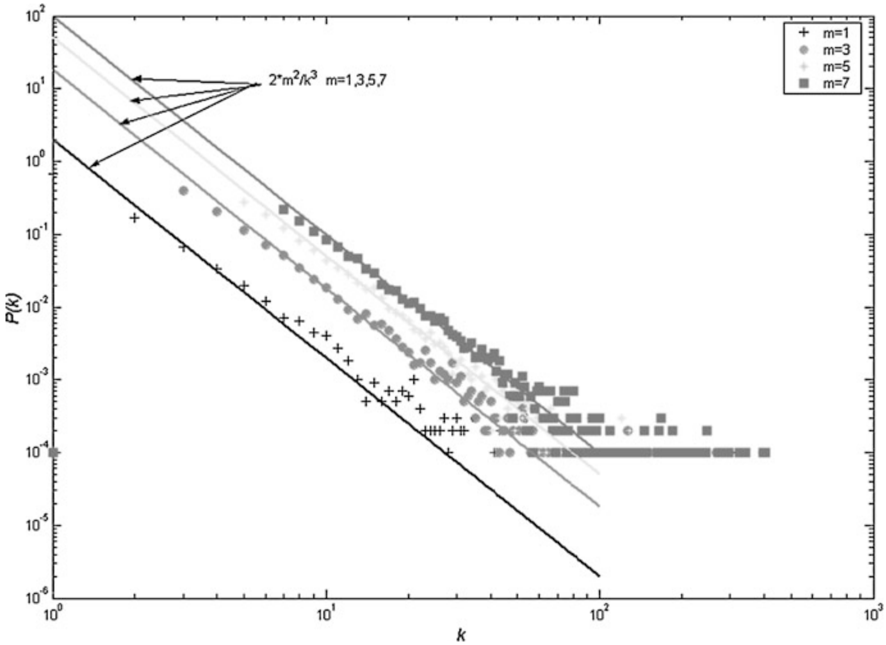

Fig. 2.3 [1] Degree distribution $P(k)$ of a BA scale-free model, with $N = m_0 + t = 10000$ and $m = m_0 = 1, 3, 5, 7$, respectively

2.1.2 BA Scale-Free Network Model

The BA scale-free model is generated as follows [1].

1. *Growth*: Starting with a small number (m_0) of nodes, at every time step, add a new node with $m(\leq m_0)$ edges that link the new node to m different nodes already presented in the network.
2. *Preferential attachment*: When choosing the nodes to which the new node connects, assume that the probability $\Pi(k_i)$ that a new node will be connected to node i depends on the degree k_i of node i, in such way that

$$\Pi(k_i) = \frac{k_i}{\sum_j k_j}. \tag{2.2}$$

After t time steps, we get a network having $N = t + m_0$ nodes and mt edges. This network evolves into a scale-invariant state with the probability that a node has k edges following a power-law distribution $P(k) \sim 2m^2 k^{-\gamma_{BA}}$ with an exponent $\gamma_{BA} = 3$ (Fig. 2.3), where the scaling exponent γ_{BA} is independent of m, i.e., γ_{BA} is scale-invariant, and in this sense the network is said to be scale-free.

It has been suggested that the BA scale-free network model has captured the basic mechanisms, growth, and preferential attachment, responsible for the scale-free feature and "rich gets richer" phenomenon in many real-life complex networks [2–5].

2.1.3 A Directed Scale-Free Network Model

It has been demonstrated that many real complex networks display a scale-free feature [2, 5–7]. Since the BA model describes undirected scale-free networks [7, 8], some researchers have investigated models of directed scale-free networks [9–11].

Based on the Tadic model [11], a new directed scale-free network model was proposed in this subsection. Except for the power law property of its degree distribution, the directed model is also strongly connected, following the rules as follows:

1. *Strongly Connected*. At a unique time t, one vertex i $(i = t)$ is added. Meanwhile, two old vertices n $(n \le t)$ and k $(k \le t)$ are selected connecting i by two directed links $n \to i$ and $i \to k$.
2. *Growth and Rearrangements.* When i is added, $M\ (t)(M(t) > 2)$ new links are distributed between all vertices. The first two links are added using following rule 1, then the other $M(t) - 2$ new links are added, in which $\alpha(M(t) - 2)$ $(0 \le \alpha \le 1)$ new links are between i and the old vertices, whereas the rest $(1 - \alpha)(M(t) - 2)$ are links updated between the old vertices.
3. *Preferential Attachment*. The probability Π_{in} that the new link $n \to i$ going from an old vertex n depends on the in-degree $d_{\text{in}}(n, t)$ of n at t is
$$\Pi_{\text{in}} = \frac{\alpha(M(t) - 2) + d_{\text{in}}(n, t)}{(1 + \alpha)(M(t) - 2) \times t};$$
while the probability Π_{out} that the new link $i \to k$ pointing to an old vertex k depends on the out-degree $d_{\text{out}}(k, t)$ of vertex k at t is
$$\Pi_{\text{out}} = \frac{\alpha(M(t) - 2) + d_{\text{out}}(k, t)}{(1 + \alpha)(M(t) - 2) \times t}.$$
4. *Preferential Update*. Link $n \to k$ is established between two old vertices n and k, which are selected with probabilities Π_{out} and Π_{in} defined above.

As a result, a directed scale-free network is obtained, whose in-degree and out-degree both follow power law distribution (Fig. 2.4(a)–(b)), and the accumulation distribution of ControlRank is neither exponential distribution nor power law distribution, which follows a stretched exponential distribution (Fig. 2.4(c)). A proof of the power law distribution of the in-degree and out-degree can be found in [10, 11].

2.2 Stability Conditions for Complete Synchronization of Complex Dynamical Networks

Before stating the main results of this section, we need the following definitions and lemmas, which are taken from [13–16].

Definition 2.1 A function $\phi : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ is increasing if $(x - y)^{\mathrm{T}}(\phi(x, t) - \phi(y, t)) \ge 0$ for all x, y, t.

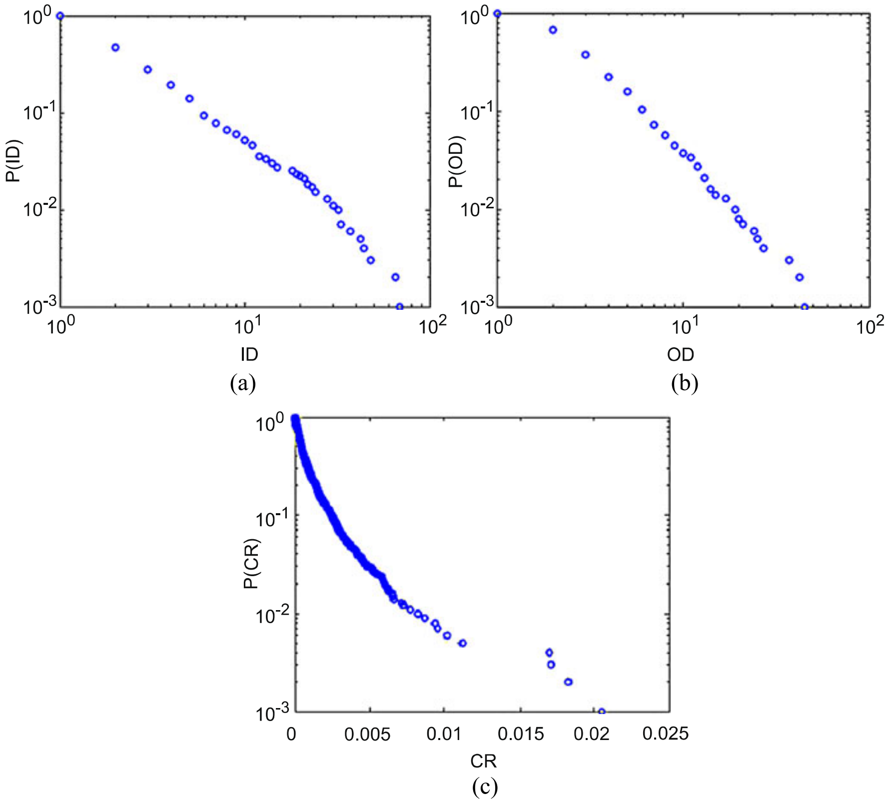

Fig. 2.4 [12] Accumulation distributions of in-degree (**a**), out-degree (**b**), and ControlRank (**c**) for a network of $N = 1000$ vertices with parameters $\alpha = 0.5$, $m_0 = 3$; $M(t) = M = 3$ directed links are added per time unit

Definition 2.2 A function $\phi : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ is uniformly increasing if there exists $c > 0$ such that for all x, y, t

$$(x - y)^{\mathrm{T}}\big(\phi(x,t) - \phi(y,t)\big) \geq c\|x - y\|^2.$$

Definition 2.3 Given a square matrix V, a function $\phi : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ is V-uniformly increasing if $V\phi$ is uniformly increasing.

Definition 2.4 A function $\phi : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ is (V-uniformly) decreasing if $-\phi$ is (V-uniformly) increasing.

Definition 2.5 A function $f(x,t)$ is Lipschitz continuous in x with Lipschitz constant L_C^f if $\|f(x,t) - f(y,t)\| \leq L_C^f \|x - y\|$ for all x, y, t.

Lemma 2.1 *If $f(x,t)$ is V-uniformly decreasing for a symmetric and positive definite matrix V, then $\dot{x} = f(x,t) + \eta(t)$ is asymptotically stable for all $\eta(t)$,* *in the sense that* $\|x - y\| \to 0$ *as* $t \to \infty$ *where* $\dot{x} = f(x,t)$, $x(t_0) = x_0$ *and* $\dot{y} = f(y,t)$, $y(t_0) = y_0$. *Consequently, the zero solution of the error dynamics*

$$\dot{e} = f(x,t) - f(y,t), \quad e(t_0) = x_0 - y_0$$

is asymptotically stable.

Lemma 2.2 *Suppose* $f(x,t)$ *is Lipschitz continuous in* x *with Lipschitz constant* $L_C^f > 0$ *and* V *is symmetric and positive definite. If* $\alpha > (L_C^f)/(\lambda_{\min}(V)) > 0$ *where* $\lambda_{\min}(V)$ *is the smallest eigenvalue of* V, *then* $\dot{x} = f(x,t) - \alpha V$ *is asymptotically stable.*

Definition 2.6 A matrix A is reducible if there exists a permutation matrix P such that PAP^{T} is of the form $\left(\begin{smallmatrix} B & C \\ 0 & D \end{smallmatrix}\right)$, where B and D are square matrices. Matrix A is irreducible if it is not reducible.

Definition 2.7

1. Let M_1 be the class of matrices such that their row i consists of zeros except for one entry α_i and one entry $-\alpha_i$ for some nonzero α_i.
2. The set Wconsists of all matrices with zero row sums, which have only nonpositive off-diagonal elements. The set W_i consists of all irreducible matrices in W.

Lemma 2.3 *If* A *is a symmetric matrix in* W, *then* A *is positive semi-definite and has a zero eigenvalue associated with the eigenvector* $(1, 1, \ldots, 1)$. *Furthermore,* A *can be decomposed as* $A = M^{\mathrm{T}}M$, *where* M *is a matrix in* M_1. *Furthermore, if* A *is irreducible, then the zero eigenvalue has multiplicity* 1.

Definition 2.8 The Kronecker product of matrices A and B is defined as

$$A \otimes B = \begin{pmatrix} a_{11}B & \ldots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \ldots & a_{nm}B \end{pmatrix}$$

where if A is an $n \times m$ matrix and B is a $p \times q$ matrix, then $A \otimes B$ is an $np \times mq$ matrix.

Definition 2.9 Vector $A \otimes f(x_i, t)$ is defined as

$$A \otimes f(x_i,t) = \begin{pmatrix} a_{11}f(x_1,t) + a_{12}f(x_2,t) + \cdots + a_{1m}f(x_m,t) \\ \vdots \\ a_{n1}f(x_1,t) + a_{n2}f(x_2,t) + \cdots + a_{nm}f(x_m,t) \end{pmatrix}$$

where if A is an $n \times m$ matrix and f is a $p \times 1$ function, then $A \otimes f(x_i, t)$ is a $np \times 1$ vector.

Suppose that a complex network consists of N identical linearly and diffusively coupled nodes, with each node being an n-dimensional dynamical system. In network (1.1), letting $a_{ii} = -\sum_{j=1,j\neq i}^{N} a_{ij}$, the coupling matrix $A = (a_{ij}) \in \mathbf{R}^{N\times N}$ represents the coupling configuration of the network, which is assumed in this section as a random network described by the ER model or a scale-free network described by the BA model. Then, the coupling matrix A is symmetric and the matrix $-A$ is in W (Definition 2.7).

Suppose the network is connected in the sense of having no isolated clusters. Then, the symmetric matrix A is irreducible. From Lemma 2.3, we know that zero is an eigenvalue of $-A$ with multiplicity 1, and other eigenvalues of $-A$ are strictly positive.

Suppose that we want to stabilize network (1.1) onto a homogeneous stationary state (1.2). To achieve the goal (1.2), we apply the pinning control strategy on a small fraction δ $(0 < \delta \ll 1)$ of the nodes in network (1.1). Suppose that nodes $i_1, i_2, \ldots, i_l$ are selected, where $l = [\delta N]$ stands for the largest integer smaller than the real number δN. This controlled network can be described as

$$\begin{aligned}
\dot{x}_{ik} &= f(x_{ik}) + \sum_{j=1,j\neq i_k}^{N} c_{i_k j} a_{i_k j} \Gamma(x_j - x_{i_k}) + u_{i_k}, \quad k = 1, 2, \ldots, l, \\
\dot{x}_{ik} &= f(x_{ik}) + \sum_{j=1,j\neq i_k}^{N} c_{i_k j} a_{i_k j} \Gamma(x_j - x_{i_k}), \quad k = l+1, l+2, \ldots, N.
\end{aligned} \tag{2.3}$$

For simplicity, we use the local linear negative feedback control law as follows:

$$u_{i_k} = -c_{i_k i_k} d_{i_k} \Gamma(x_{i_k} - \bar{x}), \quad k = 1, 2, \ldots, l \tag{2.4}$$

where the coupling strength $c_{i_k i_k}$ satisfies $c_{i_k i_k} a_{i_k i_k} + \sum_{j=1,j\neq i_k}^{N} c_{i_k j} a_{i_k j} = 0$, and the feedback gain $d_{i_k} > 0$.

Without loss of generality, we rearrange the order of nodes in the network such that the pinned nodes i_k, $k = 1, 2, \ldots, l$, are the first l nodes in the rearranged network.

Define the following two matrices:

$$\begin{aligned}
D &= \operatorname{diag}(d_1, d_2, \ldots, d_l, 0, \ldots, 0) \in \mathbf{R}^{N\times N}, \\
D' &= \operatorname{diag}(c_{11}d_1, c_{22}d_2, \ldots, c_{ll}d_l, 0, \ldots, 0) \in \mathbf{R}^{N\times N}.
\end{aligned}$$

Substituting (2.4) into (2.3), we can rearrange the controlled network (2.3) and write it by using the Kronecker product (Definition 2.9) as

$$\begin{aligned}
\dot{x} &= f(x) - \big[\big(G + D'\big) \otimes \Gamma\big]x + \big(D' \otimes \Gamma\big)\bar{X} \\
&= I_N \otimes f(x_i) - \big[\big(G + D'\big) \otimes \Gamma\big]x + \big(D' \otimes \Gamma\big)\bar{X}
\end{aligned} \tag{2.5}$$

where $\bar{X} = [\bar{x}^{\mathrm{T}}, \ldots, \bar{x}^{\mathrm{T}}]^{\mathrm{T}}$, and the elements g_{ij} of the symmetric irreducible matrix $G = (g_{ij}) \in \mathbf{R}^{N\times N}$ are defined as

$$g_{ij} = -c_{ij} a_{ij}. \tag{2.6}$$

It is easy to see that G is positive semi-definite, and $G+D'$ is positive definite with the minimal eigenvalue $\lambda_{\min}(G+D')>0$.

2.2.1 Global Stability Conditions

Theorem 2.1 *Suppose that $\varGamma$ is symmetric positive semi-definite. Let T be a matrix such that $f(x)+Tx$ is V-uniformly decreasing for some symmetric positive definite matrix V. The controlled network (2.5) is globally stable about the homogeneous state $\bar{x}$ if there exists a positive definite diagonal matrix U such that the matrix*

$$(U\otimes V)\big[\big(G+D'\big)\otimes\varGamma+I\otimes T\big] \tag{2.7}$$

is positive definite.

Proof Construct a Lyapunov function $g(\tilde{x})=\frac{1}{2}\tilde{x}^{\mathrm{T}}(U\otimes V)\tilde{x}$, which is radially unbounded with respect to $\tilde{x}$, where $\tilde{x}=x-\bar{x}$. The derivative of $g(\tilde{x})$ along the trajectories of the controlled network (2.5) is

$$\begin{aligned}\dot{g}(\tilde{x})&=\tilde{x}^{\mathrm{T}}(U\otimes V)\dot{\tilde{x}}\\&=\tilde{x}^{\mathrm{T}}(U\otimes V)\big[f(x)-f(\bar{x})+(I\otimes T)\tilde{x}\\&\quad-\big[\big(G+D'\big)\otimes\varGamma+I\otimes T\big]\tilde{x}\big].\end{aligned} \tag{2.8}$$

Hence, $\tilde{x}^{\mathrm{T}}(U\otimes V)[f(x)-f(\bar{x})+(I\otimes T)\tilde{x}]$ in (2.8) is non-positive because of the V-uniformly decreasing property of $f(x)+Tx$. From (2.7), $\tilde{x}^{\mathrm{T}}(U\otimes V)[(G+D')\otimes\varGamma+I\otimes T]\tilde{x}$ is positive. Therefore, this theorem is proved by Lyapunov's direct method. □

Theorem 2.2 *Assume that $f(x)$ is Lipschitz continuous in x with a Lipshitz constant $L_C^f>0$. If $\varGamma$ is symmetric and positive definite, then the controlled dynamical network (2.5) is globally stable about the homogeneous state $\bar{x}$, provided that there exists a constant $\alpha=(L_C^f)/(\lambda_{\min}(\varGamma))>0$ such that*

$$\lambda_{\min}\big(G+D'\big)>\alpha \tag{2.9}$$

where $\lambda_{\min}(\varGamma)$ and $\lambda_{\min}(G+D)$ are the minimal eigenvalues of matrices $\varGamma$ and $G+D$, respectively.

Proof With Lemma 2.2, there exists a constant $\alpha=(L_C^f)/(\lambda_{\min}(\varGamma))$ such that for any $\varepsilon>0$ small enough, the trajectories of $\dot{x}=f(x)-(\alpha+\varepsilon)\varGamma x$ are uniformly decreasing. Denote $\lambda_{\min}(G+D')=\alpha+\delta>\alpha+\varepsilon$ with some $\delta>0$. We can select

$U = I$, $V = I$, and $T = -(\alpha + \varepsilon)\Gamma$, so that the matrix

$$(U \otimes V)\big[\big(G + D'\big) \otimes \Gamma + I \otimes T\big] = \big[G + D' - (\alpha + \varepsilon)I\big] \otimes \Gamma \tag{2.10}$$

is positive definite. Therefore, by Theorem 2.1, the controlled dynamical network (2.5) is globally stable about the homogeneous state $\bar{x}$, and this completes the proof of Theorem 2.2. □

Theorem 2.2 gives a sufficient condition, inequality (2.9), for the existence of the coupling strength matrix $C_{\text{couple}} = (c_{ij})^{N \times N}$ that can stabilize the dynamical network (2.5). By making some further simplifications, we have the following constructive corollary.

Corollary 2.1 *Suppose that $c_{ij} = c$ and $d_i = cd$. The controlled dynamical network (2.5) is globally stable about the homogeneous state $\bar{x}$ if*

$$c > \frac{L_C^f}{\lambda_{\min}(-A + \operatorname{diag}(d, \dots, d, 0, \dots, 0)) \cdot \lambda_{\min}(\Gamma)}. \tag{2.11}$$

Remark 2.1 If we assume that Γ is only a constant 0–1 matrix, then using the Gershgorin theorem we can verify that only $\Gamma = \operatorname{diag}(1, \dots, 1, 0, \dots, 0)_{n \times n}$ can satisfy the condition as a symmetric and positive semi-definite matrix described in Theorem 2.1, and Γ will be I_n for the symmetric and positive definite matrix described in Theorem 2.2. Hence, condition (2.11) in Corollary 2.1 can be simplified as

$$c > \frac{L_C^f}{\lambda_{\min}(-A + \operatorname{diag}(d, \dots, d, 0, \dots, 0))}. \tag{2.12}$$

If the oscillator $\dot{x}_i = f(x_i)$ is chaotic, then the Lipschitz constant L_C^f of function f also means that for two arbitrary, different initial values x_i^1 and x_i^2, the corresponding different trajectories should satisfy $\|f(x_i^1) - f(x_i^2)\| < L_C^f \|x_i^1 - x_i^2\|$ for all time t. Another more precise concept for a chaotic system is the Lyapunov exponent (LE). However, the relationship between the LEs of the coupled network (1.1) and those of uncoupled chaotic nodes is nontrivial, and only in some simplified coupling the relationship can be easily analyzed.

2.2.2 *Local Stability Conditions*

Theorem 2.3 *Consider the controlled network (2.3). Suppose that there exists a constant $\rho > 0$ such that $[Df(\bar{x}) - \rho]$ is a Hurwitz matrix, $c_{ij} = c$, $d_i = cd$, $\Gamma = I_n$.*

Let $\lambda_1 = \lambda_{\min}(-A + \mathrm{diag}(d, \ldots, d, 0, \ldots, 0))$. *If*

$$c\lambda_1 \geq \rho, \tag{2.13}$$

then the homogeneous stationary state $\bar{x}$ *of the controlled network* (2.3) *is locally exponentially stable.*

Proof Because $c_{ij} = c, d_i = cd, \Gamma = I_n$, we can simplify the controlled network (2.5) as

$$\begin{aligned} \dot{x}_i &= f(x_i) + c\sum_{j=1}^{N} a_{ij}x_j - cd(x_i - \bar{x}), \quad i = 1, 2, \ldots, l, \\ \dot{x}_i &= f(x_i) + c\sum_{j=1}^{N} a_{ij}x_j, \quad i = l+1, l+2, \ldots, N. \end{aligned} \tag{2.14}$$

Linearizing the controlled network (2.14) on the homogeneous stationary state $\bar{x}$ leads to

$$\dot{\eta} = \eta\big[Df(\bar{x})\big] - cB\eta \tag{2.15}$$

where $D(f(\bar{x})) \in \mathbf{R}^{n\times n}$ is the Jacobian of f at $\bar{x}$, $\eta = (\eta_1, \eta_2, \ldots, \eta_N)^{\mathrm{T}} \in \mathbf{R}^{Nn}$, with $\eta_i(t) = x_i(t) - \bar{x}, i = 1, 2, \ldots, N$, and $B = -A + \mathrm{diag}(d, \ldots, d, 0, \ldots, 0)$.

Obviously, B is symmetric and positive definite, so all of its eigenvalues are positive, denoted as

$$0 < \lambda_{\min}\big(-A + \mathrm{diag}(d, \ldots, d, 0, \ldots, 0)\big) = \lambda_1 < \lambda_2 < \cdots < \lambda_N \tag{2.16}$$

with their corresponding (generalized) eigenvectors $\Phi = [\phi_1, \phi_2, \ldots, \phi_N] \in \mathbf{R}^{N\times N}$ which satisfy

$$B\phi_k = \lambda_k\phi_k, \quad k = 1, 2, \ldots, N.$$

Similarly, (2.15) can be expanded in the basis Φ into the following linear time-varying system:

$$\dot{v}_k^{\mathrm{T}} = \big[Df(\bar{x}) - c\lambda_k\big]v_k^{\mathrm{T}}, \quad k = 1, 2, \ldots, N \tag{2.17}$$

where $\eta = \Phi v$ and v_k is the kth row of v, which is equivalent to condition (2.13). □

Theorem 2.4 *Assume that the node* $\dot{x}_i = f(x_i)$ *is chaotic for all* $i = 1, 2, \ldots, N$, *with the maximum positive LE* $h_{\max} > 0$. *If* $c_{ij} = c, d_i = cd, \Gamma = I_n$, *then the controlled network* (2.3) *is locally asymptotically stable about the homogeneous state* $\bar{x}$, *provided that*

$$c > \frac{h_{\max}}{\lambda_{\min}(-A + \mathrm{diag}(d, \ldots, d, 0, \ldots, 0))}. \tag{2.18}$$

Proof Recall the concept of TLEs [17–19] for (2.15), denoted by $\mu_i(\lambda_k)$, for each λ_k, $k = 1, 2, \ldots, N$, defined by

$$\mu_i(\lambda_k) = h_i - c\lambda_k, \quad i = 1, 2, \ldots, m. \tag{2.19}$$

The TLEs characterize the behavior of the infinitesimal vectors, and determine the stability of the controlled states. Hence, the local stability of the controlled network (2.14) about the homogeneous state $\bar{x}$ is converted into the negative TLEs of (2.17). Due to the order of $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$, it is obvious that the following condition must be satisfied:

$$\mu_{\max}(\lambda_2) = h_{\max} - c\lambda_1 < 0, \tag{2.20}$$

which is equivalent to condition (2.18). □

2.3 Virtual Control of Pinned Complex Dynamical Networks

Let the coupling strength $c_{ij} = c$ and the feedback gain $d_i = cd$, as was in Corollary 2.1, and further set Γ as a 0–1 matrix. Rearranging the order of the nodes in the controlled dynamical network (2.3), and ordering the pinned nodes $i_k, k = 1, 2, \ldots, l$, as the first l nodes, we have

$$\begin{aligned} \dot{x}_i &= f(x_i) + c\sum_{j=1}^{N} a_{ij}\Gamma x_j - cd\Gamma(x_i - \bar{x}), \quad i = 1, 2, \ldots, l, \\ \dot{x}_i &= f(x_i) + c\sum_{j=1}^{N} a_{ij}\Gamma x_j, \quad i = l+1, l+2, \ldots, N. \end{aligned} \tag{2.21}$$

From the stability conditions (2.11) and (2.12) [or (2.18)], the dynamical network (2.21) will be globally (or locally asymptotically) stable about the homogeneous stationary state $\bar{x}$, if

$$\begin{cases} \Gamma = I_m, \\ c > \frac{C}{\lambda_{\min}(-A+\operatorname{diag}(d,\ldots,d,0,\ldots,0))}. \end{cases} \tag{2.22}$$

Here, C equals the L_C^f in Theorem 2.2 or the $h_{\max}$ in Theorem 2.4.

Owing to the local error-feedback nature of each pinned node, it is guaranteed that, as the feedback control gain limit $d \to \infty$, the states of the controlled nodes can be pinned to the homogeneous target state $\bar{x}$. Hence, the pinning control stability of network (2.21) is converted to that of the following controlled network:

$$\begin{aligned} x_i &= \bar{x}, \quad i = 1, 2, \ldots, l, \\ \dot{x}_i &= f(x_i) + c\sum_{j=l+1}^{N} a_{ij}\Gamma x_j + c\sum_{j=1}^{l} a_{ij}\Gamma\bar{x}, \quad i = l+1, l+2, \ldots, N. \end{aligned} \tag{2.23}$$

Since

$$a_{ii} = \sum_{j=1, j\neq i}^{N} a_{ij} = -\left(\sum_{j=l+1, j\neq i}^{N} a_{ij} + \sum_{j=1}^{l} a_{ij} \right), \quad i = l+1, l+2, \ldots, N, \tag{2.24}$$

network (2.23) can be equivalently rewritten in the form with a virtual control as

$$\begin{aligned} &x_i = \bar{x}, \quad i = 1, 2, \ldots, l, \\ &\dot{x}_i = f(x_i) + c \sum_{j=l+1}^{N} \tilde{b}_{ij} \Gamma x_j + \tilde{u}_i, \quad i = l+1, l+2, \ldots, N, \end{aligned} \tag{2.25}$$

where the virtual control laws are taken as

$$\tilde{u}_i = -c\tilde{d}_i (x_i - \bar{x}), \quad i = l+1, l+2, \ldots, N, \tag{2.26}$$

with the virtual control feedback gains $\tilde{d}_i = \sum_{j=1}^{l} a_{ij}$.

Denote $\tilde{B} = (\tilde{b}_{ij}) \in \mathbf{R}^{(N-l)\times(N-l)}$ as

$$\begin{aligned} &\tilde{b}_{ij} = a_{ij}, \quad j \neq i, \ j = l+1, \ldots, N, \ i = l+1, l+2, \ldots, N, \\ &\tilde{b}_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij}, \end{aligned} \tag{2.27}$$

which is symmetric. Here, the matrix $-\tilde{B}$ is in W. It should be noted that a_{ij} in (2.27) are the rearranged elements in the original adjacency matrix A.

Given a pinning control strategy, that is, supposing that the nodes $i_1, i_2, \ldots, i_l$ are selected to be under pinning control, and $\tilde{A} \in \mathbf{R}^{(N-l)\times(N-l)}$ is a minor matrix of A with respect to the pinning control scheme, which is obtained by removing the $i_1, i_2, \ldots, i_l$ row–column pairs from A, we have

$$\tilde{A} = \tilde{B} + \mathrm{diag}(\tilde{d}_{l+1}, \tilde{d}_{l+2}, \ldots, \tilde{d}_N). \tag{2.28}$$

Recall that we have let the coupling strength be $c_{ij} = c$ and Γ be a 0–1 matrix.

Corollary 2.2 *If $\tilde{A}$ is irreducible, then the dynamical network (2.25) with the virtual control (2.26) is globally (or locally asymptotically) stable about homogeneous state $\bar{x}$, provided that $\Gamma = I_m$ and*

$$c > \frac{C}{\lambda_{\min}(-\tilde{A})} \tag{2.29}$$

with C being the L_C^f in Theorem 2.2 (or the $h_{\max}$ in Theorem 2.4).

Remark 2.2 If is $\tilde{A}$ reducible, then, following Definition 2.6, $\tilde{A}$ can be blocked into a series of irreducible matrices $\tilde{A}_1, \ldots, \tilde{A}_n$. In this case, the corresponding subnetworks will be globally (or locally asymptotically) stable about homogeneous state $\bar{x}$, provided that

$$c > \frac{C}{\max\{\lambda_{\min}(-\tilde{A}_j) \mid j = 1, 2, \ldots, n\}} \tag{2.30}$$

with C being the L_C^f in Theorem 2.2 (or the $h_{\max}$ in Theorem 2.4).

Clearly, we have the following order of the stable conditions (2.22), (2.29)–(2.30):

$$\begin{aligned} 0 = \lambda_{\min}(-A) &< \lambda_{\min}\big(-A + \mathrm{diag}(d, \ldots, d, 0, \ldots, 0)\big) \\ &< \lim_{d\to\infty} \lambda_{\min}\big(-A + \mathrm{diag}(d, \ldots, d, 0, \ldots, 0)\big) \\ &= \lambda_{\min}(-\tilde{A}) \\ &\le \max\big\{\lambda_{\min}(-\tilde{A}_j) \mid j = 1, 2, \ldots, m\big\}. \end{aligned} \tag{2.31}$$

With Lemma 2.1, it can be verified that the stable condition (2.29) in Corollary 2.2 for network (2.23) is the same as that for synchronizing the following network:

$$\dot{x}_i = f(x_i) + c \sum_{j=l+1}^{N} a_{ij} \Gamma x_j, \quad i = l+1, l+2, \ldots, N. \tag{2.32}$$

Luckily, condition (2.29) is exactly the stable condition for synchronizing network (2.32). Hence, the whole pinned network, controlled by the small amount of local feedback controllers and those virtual controllers, is stabilized onto the homogeneous state $\bar{x}$ via synchronization. This phenomenon will be visually illustrated in the simulation study below.

Pinning strategies mainly include random pinning and specific pinning schemes. In the random pinning scheme, we apply local feedback injections to a fraction δ of randomly selected nodes. While in the specific pinning scheme, we first pin the node of the highest degree, and then continue to select and pin the other nodes in a monotonically decreasing order of degrees.

In a star-like structure of a dynamical network, the kernel node is specifically selected to control for stabilizing the whole network. While in a ring or lattice, only specifically pinning a small amount of nodes is not sufficient to stabilize the whole ring, other factors such as the pinning distribution should also be taken into account. It was proposed in [20] that to stabilize a scale-free dynamical network, the specific pinning scheme requires much fewer local controllers than the random pinning scheme. To study the dynamics spreading in such complex networks in this section, we further perform a simulation-based topological analysis on the virtual control effect in different pinning strategies of complex networks, using the ER random network model and BA scale-free network model as prototypes.

Figure 2.5 is a mini-illustration of the virtual control process in the specific pinning scheme on a network generated by the BA scale-free model with $N = 10$,

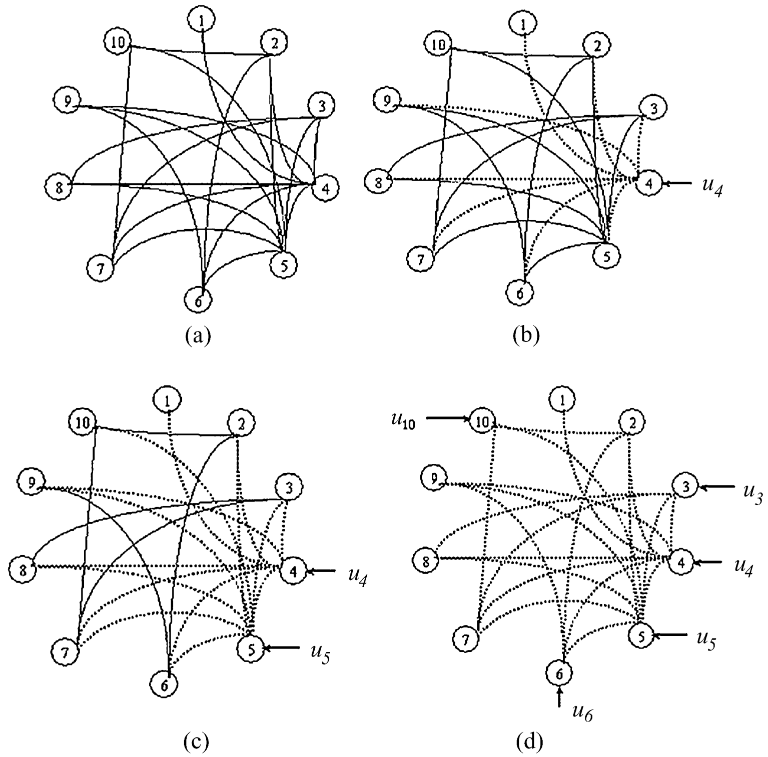

Fig. 2.5 [1] Illustration of virtual control (*dotted line*) in the specific pinning control of a network generated by BA scale-free model with $N = 10$, $m_0 = m = 3$

$m = m_0 = 3$. It can be observed that while the pinned nodes are stabilized onto the homogeneous state $\bar{x}$, their dynamics spread in the network as shown by the dotted lines, which is the virtual control. On the other hand, it can be found in Fig. 2.5(b) that the whole ten-node original network was broken into two parts: An isolated first node, which is only connected with the fourth node, and the remaining 8-node subnetwork. While increasing the pinning fraction of nodes, this subnetwork continues to shrink, and finally reaches the case as shown in Fig. 2.5(d): every unpinned node is 'isolated' and virtually controlled by other pinned and also stabilized nodes.

It can be observed that the virtual control mainly affects the pinned network in two aspects: the percentage of the network that are virtually controlled, and the subnetwork scale that is virtually divided by those pinned nodes.

We have numerically analyzed the effect of virtual control in different pinning schemes for different topological networks. In the simulation, scale-free networks were generated by the BA scale-free model with $N = 3000$ and $m = m_0 = 3$, and random networks were generated by the ER random model containing 3000 nodes with 9000 connections.

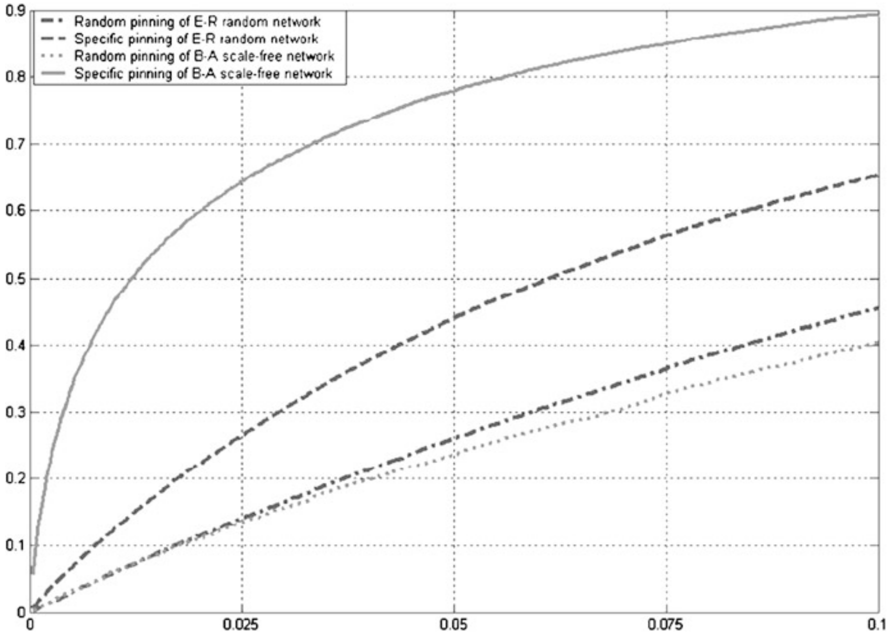

Fig. 2.6 [1] Virtually controlled percentage after pinning a fraction δ of nodes in the networks (10 random groups average output)

Figure 2.6 shows the percentage of those stabilized pinning-controlled nodes in the whole network, after pinning a fraction δ of nodes in an ER random network and in a BA scale-free network. It can be observed that, due to its extremely nonhomogeneous nature, almost 80 % of the nodes have been 'virtually controlled' by only 5 % of specifically pinned nodes in the 3000-node scale-free network. If randomly pinning 5 % of nodes, less than 30 % of nodes could be "virtually controlled." The homogeneous nature of random networks results in the phenomenon that there is no significant difference between the random and specific pinning control schemes for the ER random networks.

Figure 2.7 shows the percentage of the largest cluster in the whole pinning-controlled network, after pinning a fraction δ of nodes in an ER random network and a BA scale-free network. If specifically pinning only 5 % of the "biggest" nodes in a scale-free network, the whole network could be virtually broken into many small clusters. Hence, with the ordered stable conditions (2.31), a smaller coupling strength threshold is required in the specific pinning scheme than that in the random pinning scheme for stabilizing scale-free networks. On the other hand, the whole random network was not dramatically disturbed as compared to the scale-free networks by randomly or specifically pinning a small fraction of nodes.

Hence, for random networks, there is no significant difference in its dynamics spreading when the specific pinning or the random pinning scheme is used. After stabilizing those pinned nodes by these two pinning schemes, the dynamics spread-

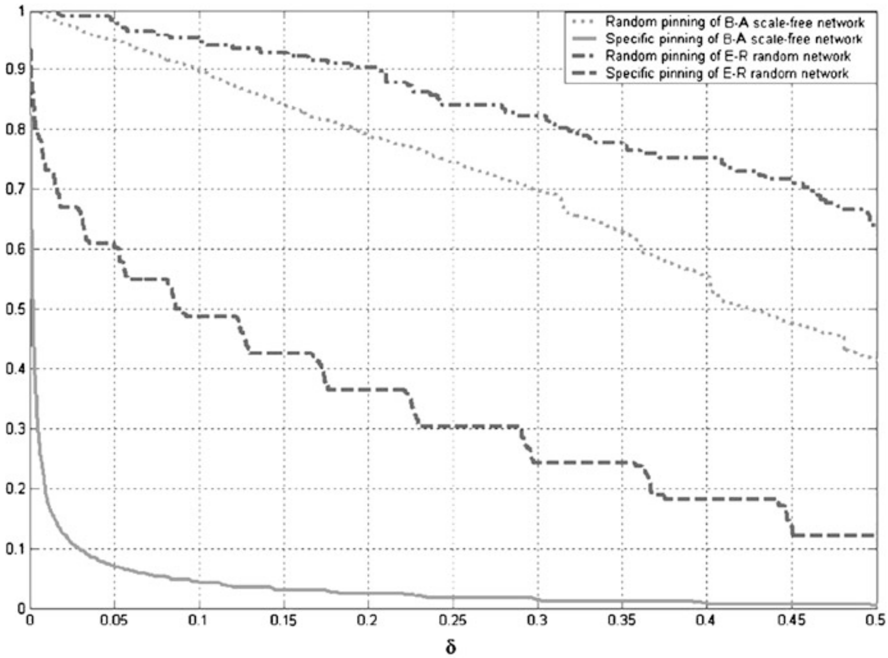

Fig. 2.7 [1] Largest cluster percentage after pinning a fraction δ of nodes in the networks (ten random groups average output)

ing behaviors are almost the same. On the other hand, much fewer local controllers will be required to control a scale-free network by the specific pinning scheme than that by the random pinning scheme. Furthermore, the dynamics of those specifically pinned nodes, which simply are the homogeneous states $\bar{x}$ here, spread in the scale-free networks significantly faster than those of randomly pinned nodes. This phenomenon also reflects the 'robustness and yet fragility' property of scale-free networks.

2.4 Selective Strategies of Pinning Control

2.4.1 Pinning Control Based on Node-Degree

Consider a scale-free dynamical network of the chaotic Chen's oscillators. A single Chen's oscillator is described in the dimensionless form by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} p_1(x_2 - x_1) \\ (p_3 - p_2)x_1 - x_1x_3 + p_3x_2 \\ x_1x_2 - p_2x_3 \end{pmatrix}. \tag{2.33}$$

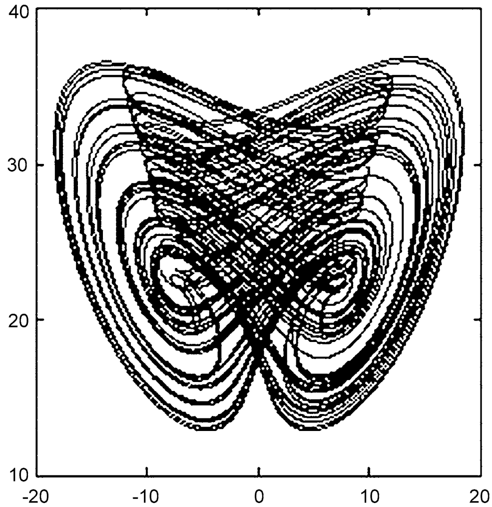

Fig. 2.8 [1] The x_1–x_3 plot of Chen's attractor

If the system parameters are selected as $p_1 = 35$, $p_2 = 3$, $p_3 = 28$, then the single Chen's system (2.33) has a chaotic attractor [21], as shown in Fig. 2.8. In this set of system parameters, one unstable equilibrium point of the oscillator (2.33) is $x^+ = [7.9373, 7.9373, 21]^{\mathrm{T}}$.

Suppose that in each pair two connected Chen's oscillators are linked together through their identical sub-state variables, i.e., $\Gamma = \mathrm{diag}(1, 1, 1)$. The entire dynamical network is described by the following state equations:

$$\begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{pmatrix} = \begin{pmatrix} p_1(x_{i2} - x_{i1}) + c\sum_{j=1}^{N} a_{ij}x_{j1} \\ (p_3 - p_2)x_{i1} - x_{i1}x_{i3} + p_3x_{i2} + c\sum_{j=1}^{N} a_{ij}x_{j2} \\ x_{i1}x_{i2} - p_2x_{i3} + c\sum_{j=1}^{N} a_{ij}x_{j3} \end{pmatrix}, \quad i = 1, 2, \ldots, N. \tag{2.34}$$

We want to stabilize network (2.34) onto the originally unstable equilibrium point $\bar{x} = x^+$ by applying the local linear feedback pinning control to a small fraction δ of nodes. The equations of the controlled network are

$$\begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{pmatrix} = \begin{pmatrix} p_1(x_{i2} - x_{i1}) + c\sum_{j=1}^{N} a_{ij}x_{j1} + u_{i1} \\ (p_3 - p_2)x_{i1} - x_{i1}x_{i3} + p_3x_{i2} + c\sum_{j=1}^{N} a_{ij}x_{j2} + u_{i2} \\ x_{i1}x_{i2} - p_2x_{i3} + c\sum_{j=1}^{N} a_{ij}x_{j3} + u_{i3} \end{pmatrix}, \\ i = 1, 2, \ldots, N, \tag{2.35}$$

where

$$u_{ij} = \begin{cases} -cd(x_{ij} - x^+), & i = i_1, i_2, \ldots, i_l,\ j = 1, 2, 3, \\ 0 & \text{otherwise.} \end{cases}$$

It can be easily checked that the Lipschitz constant for the system is larger than 100, while the maximum positive LE is about 2.01745. In the simulation, we set $C = 2.01745$ and $d = 1000$ according to the stable condition (2.22).

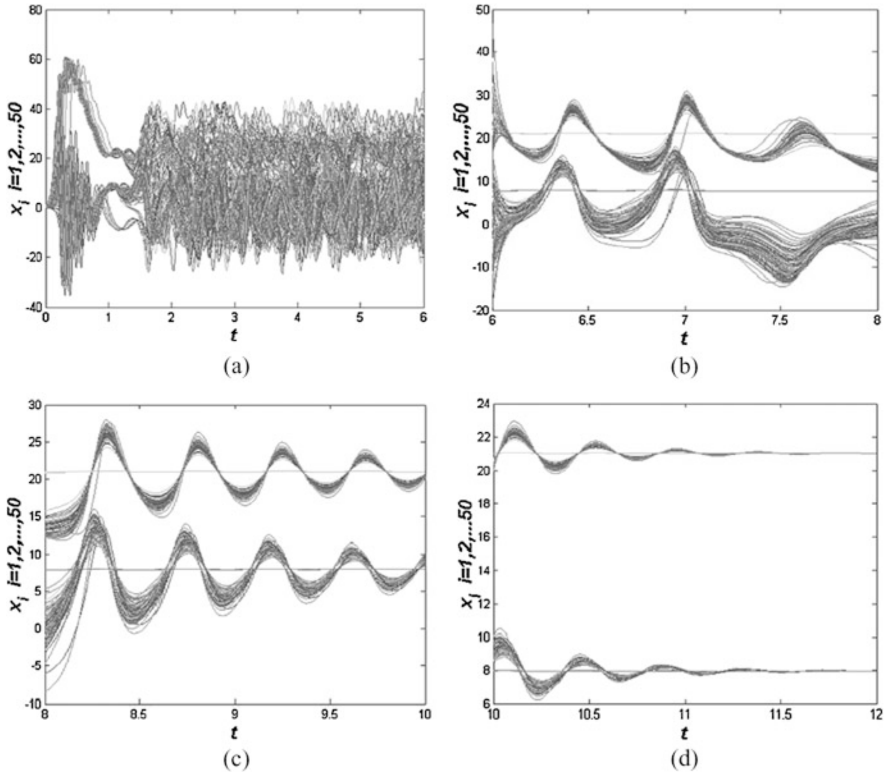

Fig. 2.9 [1] Specifically pinning the biggest node of 19 degrees in a 50-node network generated by the BA scale-free model: (**a**)–(**d**) are stabilizing phases with different coupling strengths; (**a**) $c = 0$, $d = 0$; (**b**) $c = 10$, $d = 1000$; (**c**) $c = 15$, $d = 1000$; (**d**) $c = 20$, $d = 1000$

Figure 2.9 visualizes the process of controlling a 50-node network generated by the BA scale-free model, where only the "biggest" node is pinned, which has 19 degrees, and the corresponding stable threshold of the coupling strength c is 6.2765. Figure 2.10 is the same stabilization process with two "biggest" nodes pinned, which have 19 degrees and 17 degrees, respectively, yielding a smaller threshold for the coupling strength $c = 3.4862$. It can be observed that increasing the pinning fraction δ will accelerate the convergence of the network stabilization, as shown in Fig. 2.11, though we also know that a bigger δ means more controllers are used.

Figure 2.12 is the counterpart by random pinning. If only randomly selecting two nodes as the same small fraction as in the previous specific pinning scheme, two nodes with 6 degrees and 9 degrees, respectively, are chosen for this round, and the corresponding stable condition threshold for the coupling strength c is 8.0895 with $d = 1000$. However, to achieve the stabilization of the whole network onto the same unstable equilibrium point x^+, a much larger value of the coupling strength c is required in the random pinning scheme than that of the specific pinning scheme, as shown in Fig. 2.12(b)–(c). Increasing the pinning fraction to 10 % (5 nodes)

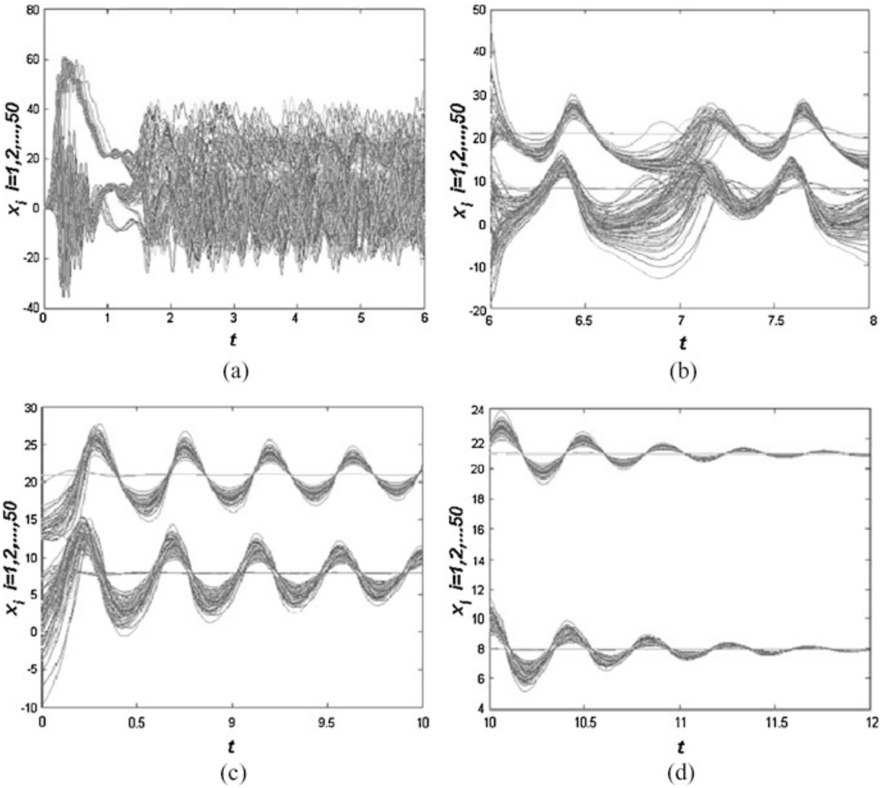

Fig. 2.10 [1] Specifically pinning two biggest nodes of 19 and 17 degrees, respectively, in the same 50-node network as in Fig. 2.9: (**a**)–(**d**) are stabilizing phases with different coupling strengths; (**a**) $c = 5$, $d = 0$; (**b**) $c = 5$, $d = 1000$; (**c**) $c = 8$, $d = 1000$; (**d**) $c = 10$, $d = 1000$

helps stabilize the whole network faster. This time, five nodes having 3, 3, 9, 3, 4, degrees, respectively, are selected for control, resulting in a corresponding threshold 5.4537. Figure 2.12(d) shows the stabilization process with the coupling strength $c = 15$. Hence, for the random pinning scheme, to achieve the same stabilization process as that by the specific pinning scheme, the costs are increasing in both the coupling strength and the pinning fraction. That is to say, to control such a scale-free dynamical network, random pinning control is much more "expensive" than specific pinning control, as compared to Fig. 2.13.

A common feature in this subsection is the synchronization phenomenon in the pinning-controlled dynamical networks, regardless of being stabilized onto the homogeneous state x^+ or not. The oscillation behavior of the smallest node in different pinning-controlled networks is a reflection of the dynamics spreading in scale-free networks, as shown in Figs. 2.11 and 2.13.

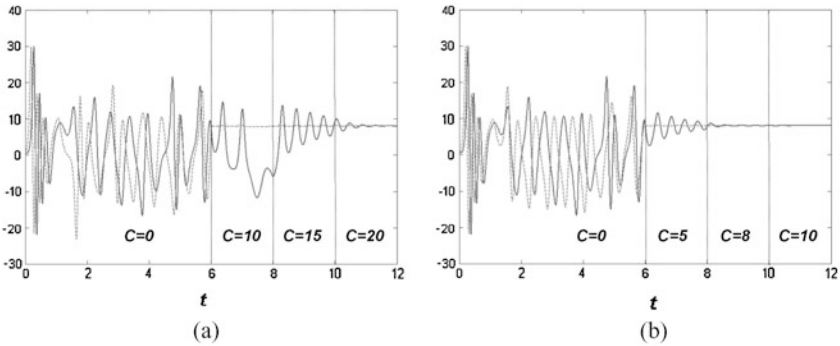

Fig. 2.11 [1] The convergence comparison of the smallest node for different specific pinning fractions. (**a**) Specifically pinning the biggest node: $N = 50$, the biggest node (–) and the smallest node (—). (**b**) Specifically pinning two biggest nodes: $N = 50$, the biggest node (–) and the smallest node (—)

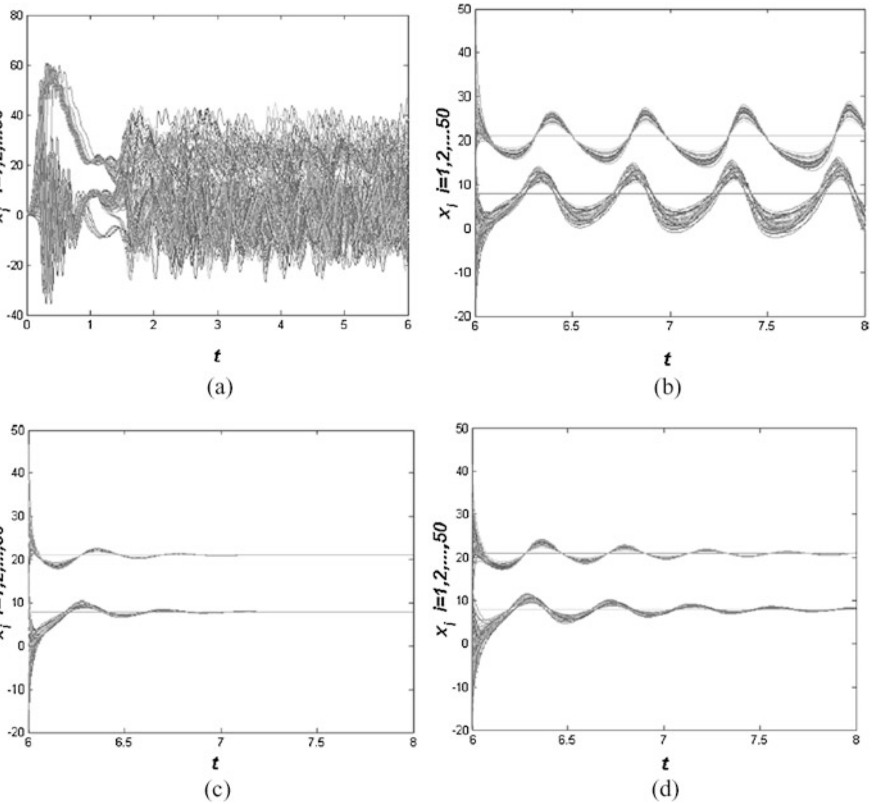

Fig. 2.12 [1] Randomly pinning some nodes in the same 50-node network as in Fig. 2.9. In (**b**) and (**c**), two nodes having 6 degrees and 9 degrees, respectively, are randomly selected for control with feedback gain $d = 1000$ and the coupling strength $c = 15, 30$, respectively. In (**d**), five nodes are randomly selected for the feedback control with gain $d = 1000$ and coupling strength $c = 15$

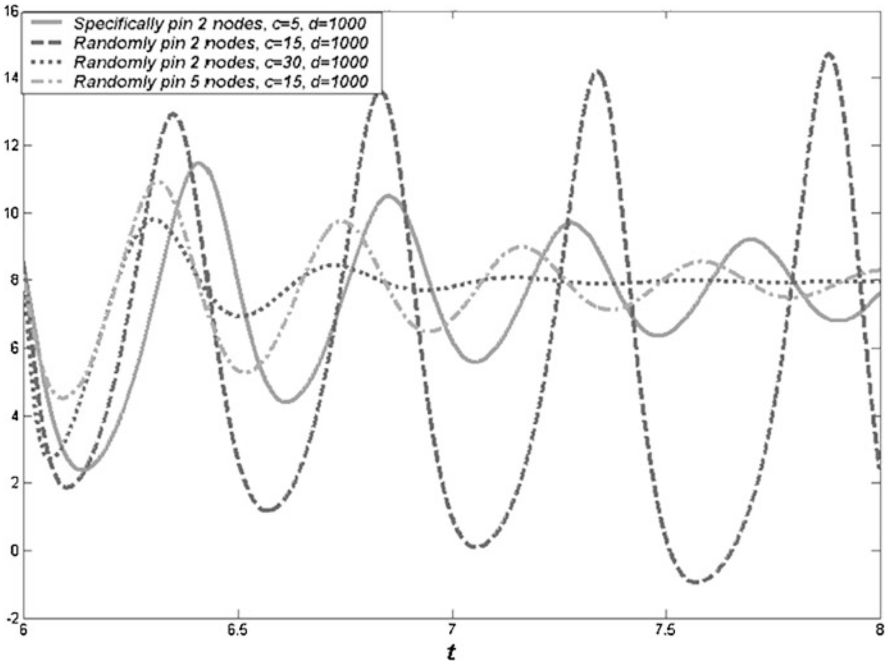

Fig. 2.13 [1] Stabilization process of the smallest node: compassion of the dynamics spreading and controllers' costs by using different pinning schemes

2.4.2 Pinning Control Based on ControlRank

It has been shown for a class of undirected dynamical networks that applying local linear feedback injections to a small fraction of the most highly connected vertices is much more effective than pinning the same number of randomly selected vertices [1, 20, 22]. However, vertex degree as a general index has exploited only little information of the relationships among vertices. Here, the purpose is trying to develop a new effective pinning scheme for directed networks.

In order to efficiently explore the implicative control ability of vertices in dynamical networks, we propose a new vertex centrality index called ControlRank (CR). CR is inspired by PageRank [23, 24], which is used in Google search engine for scoring and ranking web pages from link analysis. CR does not only depend on out-degrees but also makes full use of the link structure of networks. Meanwhile, factors affecting CR are related to the lower bound of pinning stabilizability in dynamical networks [25]. For a class of directed scale-free dynamical networks which are strongly connected, if the vertex of the largest CR is pinned, then the whole graph can be synchronized to its equilibrium better than when using the pinning strategy based on largest out-degree.

Suppose that we want to stabilize network (1.1) onto a homogeneous stationary state (1.2). Assume the coupling strengths in network (1.1) are the same, $c_{ij} = c$ and $d_i = cd$.

Synchronization can be achieved by pinning only if the network is strongly connected and the coupling strength is sufficiently large [26, 27]. Recently, rigorous mathematical proof demonstrates that such a stabilization goal can be achieved by pinning even a single vertex [28], and the bounds of the stabilizability are analytically estimated as well [25].

The Laplacian matrix of a directed network is defined as $L = ID - \bar{A}$, where $ID = \mathrm{diag}(ID'_1, ID'_2, \ldots, ID'_N)$ is the diagonal matrix of vertex in-degrees and $\bar{A}$ is the adjacency matrix of the directed network. Clearly, L is a zero row sums matrix with non-positive off-diagonal elements and thus $e = [1, \ldots, 1]^{\mathrm{T}}$ is a right-eigenvector of L corresponding to its largest eigenvalue zero. Suppose that the vertices $i_1, i_2, \ldots, i_l$ are selected as pinned vertices, and let $\tilde{L} \in \mathbf{R}^{(N-l)\times(N-l)}$ be the minor matrix of L with respect to the pinning strategies, which is obtained by removing the $i_1, i_2, \ldots, i_l$ row–column pairs of L.

The network is synchronizable if [25]

$$c > \frac{\rho}{\mathrm{Re}\,\lambda_1(\tilde{L})}, \tag{2.36}$$

here, ρ is a positive constant which guarantees a construct matrix to be an M-matrix for proof of the criterion. $\mathrm{Re}\,\lambda_1(\tilde{L})$ is the smallest real part of the eigenvalues of $\tilde{L}$. In criterion (2.36), $\mathrm{Re}\,\lambda_1(\tilde{L})$ is required to be a positive real number if and only if all vertices in the underlying network can be accessed by a certain vertex [27]. The lower bound of $\mathrm{Re}\,\lambda_1(\tilde{L})$ can be directly obtained [25] from Schur Complement [29] and the results in [30]:

$$\mathrm{Re}\,\lambda_1(\tilde{L}) \geq \frac{1}{2}\left\{2\left[k + \frac{1}{2}(2k)^{-d}\right]\right\}^{-d} \tag{2.37}$$

where k is the largest out-degree of pinned vertices $i_1, i_2, \ldots, i_l$, and d is the largest distance from the pinned vertices set to the unpinned set. Hence, a good pinning strategy should take into account the two factors of bound estimation, the out-degree in the pinned vertex set and the largest distance from the pinned to unpinned vertices.

CR is a vertex centrality index mining link structure of the directed networks, which is defined as follows

$$CR_i = \sum_{i \to j} \frac{CR_j}{ID_j}, \quad i = 1, 2, \ldots, N, \tag{2.38}$$

where CR_i is the CR of vertex i, and ID_j is the in-degree of vertex j. Denoting $CR = (CR_1, CR_2, \ldots, CR_N)$, we have the matrix form of (2.38) as follows:

$$CR = CR \times ID^{-1} \times \bar{A} \tag{2.39}$$

where each row of $ID^{-1} \times \bar{A}$ sums to 1 in a strongly connected network.

As a comparison, the PageRank (PR) vector used in Google is defined as [31, 32]:

$$PR = A \times OD^{-1} \times PR, \tag{2.40}$$

here, the column vector $PR = (PR_1, PR_2, \ldots, PR_N)^{\mathrm{T}}$, PR_i stands for the PR of vertex i, and OD is the diagonal matrix of the vertex out-degrees and each column of $\bar{A} \times OD^{-1}$ sums to 1 in a strongly connected network. Comparing (2.39) with (2.40), we can see a duality between CR and PR vectors: the CR vector of a graph is just the PR vector of the reversal graph, which is obtained by reversing direction of all links in the original graph. Just as PR is the right dominant eigenvector of $\bar{A} \times OD^{-1}$ associated with its largest eigenvalue 1, CR is the left dominant eigenvector of $ID^{-1} \times \bar{A}$ associated with its largest eigenvalue 1.

Suppose the network is strongly connected, then $\bar{A}$ is primitive and irreducible. Meanwhile, $ID^{-1} \times \bar{A}$ is a row normalized stochastic matrix. As a result, $ID^{-1} \times \bar{A}$ has a unique dominant eigenvector, CR, which can be obtained by power iteration from any initial $CR^{(0)}$:

$$CR^{(k)} = CR^{(k-1)} \times ID^{-1} \times \bar{A}. \tag{2.41}$$

Obtain from link analysis of the whole graph, CR contains much more implicative topology information than out-degree and quantitatively measures vertex controllability in the network. On the one hand, CR is a vector of the vertices hub scores in PageRank and obtained by In-link normalization method [32]. As discussed in [32], CR has an obvious positive correlation with out-degrees. On the other hand, CR of a vertex is related to the largest distance from it to all the other vertices. At each iteration in (2.41), vertex j will contribute $CR_j \times \frac{1}{ID_j}$ to each in-degree neighbors, which is in direct ratio to $\frac{1}{ID_j}$. Then if there is a path of length two from vertex p to j through vertex i, j will contribute $CR_j \times \frac{1}{ID_i} \times \frac{1}{ID_j}$ to p indirectly through i in direct ratio to $\frac{1}{ID_i} \times \frac{1}{ID_j}$. Hence, the contribution of CR from j to p attenuates with increment of the length of path from p to j.

In sum, CR depends not only on the out-degree of the vertex, but also on the largest distance to other vertices from it, as illustrated in Fig. 2.14:

In the directed network as shown in Fig. 2.14, vertex 4 has the highest CR, which is of the largest out-degree, and can access the vertex of largest distance through a path of the smallest length.

Remarkably, the lower bound of pinning stability criterion (2.37) is specified by k and d, the largest out-degree of the pinned vertices and the largest distance from pinned vertices to those unpinned. Since CR will be affected by both factors, an intuitive consideration is that pinning a vertex of the largest CR will be a better strategy than pinning a vertex of the largest out-degree [22].

We illustrate the efficiency of the novel pinning strategy using the chaotic Lorenz oscillator [33] as a dynamical vertex in the directed scale-free network. Suppose that two coupled Lorenz oscillators are linked together through the first state variable, i.e., $\Gamma = \mathrm{diag}(1, 0, 0)$. State equations of the network are described by

$$\begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{pmatrix} = \begin{pmatrix} -ax_{i1} + ax_{i2} + c\sum_{j=1}^{N} a_{ij}\Gamma x_{j1} \\ rx_{i1} - x_{i2} - x_{i1}x_{i3} \\ x_{i1}x_{i2} - bx_{i3} \end{pmatrix}, \quad i = 1, 2, \ldots, N, \tag{2.42}$$

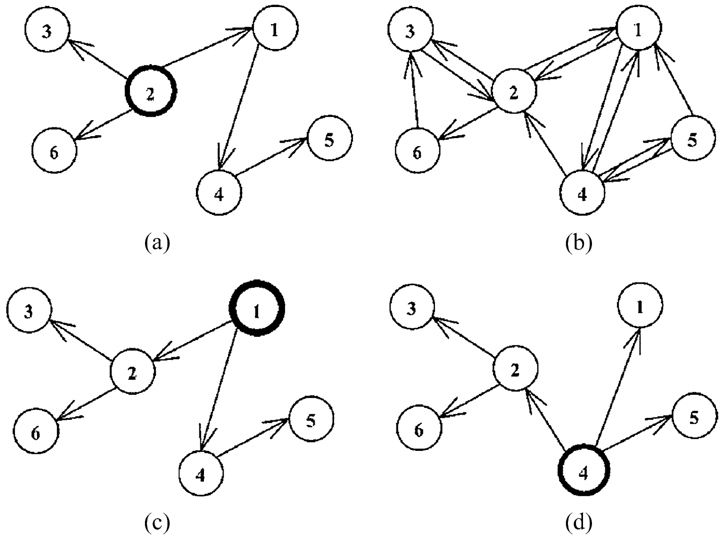

Fig. 2.14 [12] The original directed network is shown in (**a**); vertices 1, 2, and 4 are considered in (**b**), (**c**), and (**d**). For (**b**), vertex 1 with $OD_1 = 2$, $d_1 = 2$, $CR_1 = 0.2182$; for (**c**), vertex 2 with $OD_2 = 3$, $d_2 = 3$, $CR_2 = 0.1091$; for (**d**), vertex 4 with $OD_4 = 3$, $d_4 = 2$, $CR_4 = 0.3636$. OD_i and d_i, respectively, stand for the out-degree of vertex i and the largest distance from i to the other vertices

in which $a = 10$, $b = 8$, $r = 28$. For these parameters, an isolated oscillator has three unstable equilibriums as follows:

$$x^{\pm} = [\pm 8.4853 \quad \pm 8.4853 \quad 27]^{\mathrm{T}}, \qquad x^0 = [0 \quad 0 \quad 0]^{\mathrm{T}}.$$

Suppose that we want to stabilize network (2.42) onto the homogeneous state x^-. The state equations of the pinned scale-free network are

$$\begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{pmatrix} = \begin{pmatrix} -ax_{i1} + ax_{i2} + c\sum_{j=1}^{N} a_{ij}\Gamma x_{j1} + u_i \\ rx_{i1} - x_{i2} - x_{i1}x_{i3} \\ x_{i1}x_{i2} - bx_{i3} \end{pmatrix}, \quad i = 1, 2, \ldots, N, \tag{2.43}$$

where

$$u_i = \begin{cases} cd(x_1^- - x_{i1}), & i = i_1, i_2, \ldots, i_l, \\ 0, & \text{otherwise}, \end{cases} \tag{2.44}$$

with the coupling strength c and feedback gain d.

Figure 2.15 visualizes the process of controlling a directed scale-free network of 100 vertices, where only one vertex is pinned with the given coupling strength $c = 70$ and feedback gain $d = 1000$. Figure 2.15(a) is the stabilization process with one randomly selected vertex pinned, and Figs. 2.15(b)–(c) are with one vertex of largest out-degree and largest CR, respectively. It can be observed that the specific pinning (Figs. 2.15(b) and 2.15(c)) stabilizes the whole network much faster than the random pinning strategy (Fig. 2.15(a)). Furthermore, considering the control performance

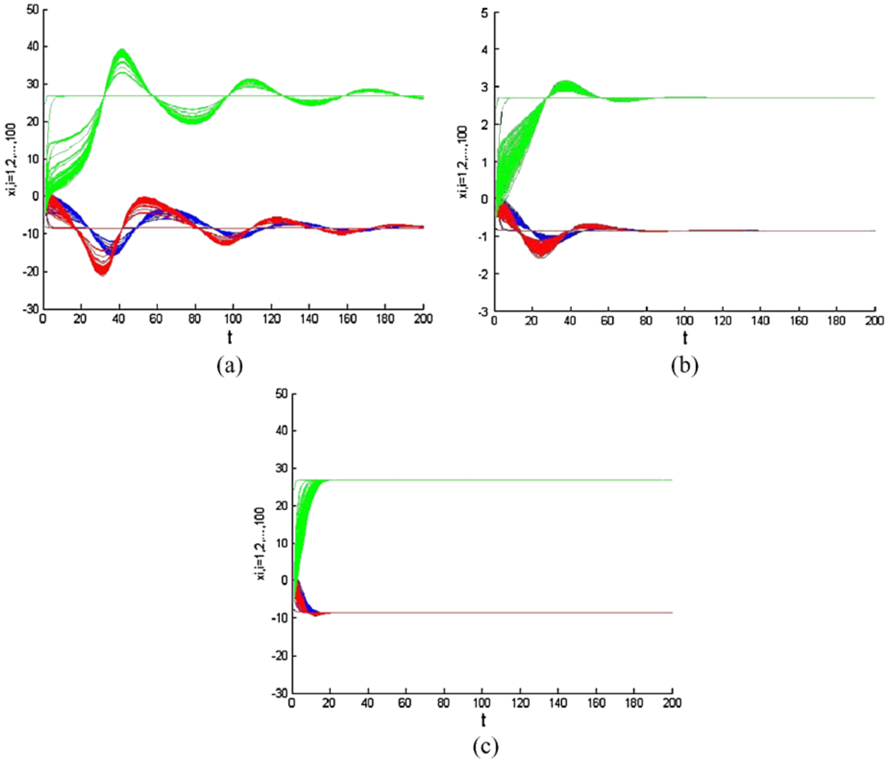

Fig. 2.15 [12] Pinning one vertex in the same 100-vertex network with the same coupling strength $c = 70$ and feedback gain $d = 1000$: (**a**) random pinning; (**b**) pinning one vertex of largest out-degree; (**c**) pinning one vertex of largest CR

indices like rising time, peak time, and overshoot, Figs. 2.15(b) and 2.15(c) imply that CR-based pinning strategy yields better performance than that of the degree-based pinning strategy.

In the following simulation, we take $N = 1000$ and $m_0 = 3$, $M(t) = M = 3$, namely, the original network contains 1000 vertices with about 3000 links. We use variance $\mathrm{var} = \frac{\sum_{i=1}^{N}(x_{i1}(t) - \bar{x}_{i1})^2}{N}$ to measure synchronization capacity, as shown in Fig. 2.16, which indicates that only the pinning strategy based on the largest CR can stabilize the given network. Therefore, the new pinning strategy is more effective even in large-scale networks.

2.5 Conclusions and Notes

In this chapter, we have derived some conditions for stabilizing a complex dynamical network onto its homogeneous state by only pinning a small fraction of the

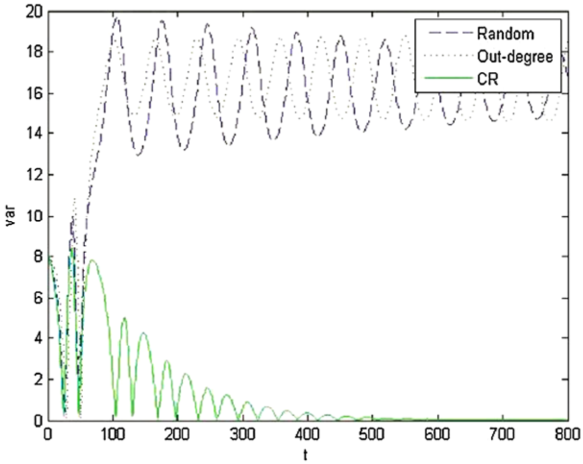

Fig. 2.16 [12] Convergence comparison of the variance for different pinning strategies in the same directed scale-free network of 1000 Lorenz oscillators: coupling strength $c = 150$, feedback gain $d = 1000$. The *blue dashed line* is for the random pinning strategy, the *red dotted line* is for the specific strategy pinning vertex of the largest out-degree, and the *green solid line* is for the specific strategy pinning vertex of the largest CR (Color figure online)

network nodes using local feedback controllers. The placement of the local controllers is affected by the topology of the network. Based on the concept of virtual control, the pinned nodes virtually control other dynamical nodes through the connections among them. Furthermore, the whole network is virtually broken into parts of subnetworks, when those pinned nodes have been stabilized onto the homogeneous state by their local feedback controllers. Owing to the nonhomogeneous nature of scale-free networks, it is much easier to stabilize a scale-free network by specifically placing the local controllers on the big nodes with high degrees than by randomly placing these local controllers into the network. In contrast, to stabilize homogeneous networks such as random graphs, randomly pinning a fraction of network nodes will have no significant difference from specifically pinning those relatively big nodes. Since many real-life complex networks have scale-free features, maintaining the stability of their big nodes is the key to keep the stability of the whole network. On the other hand, for homogeneous networks, randomly stabilizing a fraction of nodes may help control the dynamics of the whole network effectively. Finally, we proposed a new pinning control strategy based on a new vertex centrality index ControlRank for directed dynamical networks. The CR vector is a dual form of the PR vector used in Google search engine for ranking vertices. An important future research topic is how to generalize the CR-based pinning strategy to directed networks which are not strongly connected. Moreover, the relationship between CR and the stabilizability criterion needs rigorous theoretic analysis.

References

1. Li X, Wang X, Chen G (2004) Pinning a complex dynamical network to its equilibrium. IEEE Trans Circuits Syst I, Fundam Theory Appl 51:2074–2087
2. Albert R, Barabási AL (2002) Statistical mechanics of complex networks. Rev Mod Phys 74:47–97

3. Dorogovstsev SN, Mendes JFF (2002) Evolution of networks. Adv Phys 51:1079–1187
4. Li X, Chen G (2003) A local-world evolving network model. Physica A 328:274–286
5. Newman MEJ (2003) The structure and function of complex networks. SIAM Rev 45:167–256
6. Albert R, Jeong H, Barabási AL (1999) Diameter of the World Wide Web. Nature 401:130–131
7. Barabási AL, Albert R (1999) Emergence of scaling in random networks. Science 286:509–512
8. Barabási AL, Albert R, Jeong H (2000) Scale-free characteristics of random networks: the topology of the World-Wide Web. Physica A 281:69–77
9. Bollobas B, Borgs C, Chayes J, Riordan O (2003) Directed scale-free graph. In: Proc of the 14th ACM-SIAM symposium on discrete algorithms, pp 132–139
10. Dorogovtsev SN, Mendes JFF, Samukhin AN (2000) Structure of growing networks with preferential linking. Phys Rev Lett 85:4633–4636
11. Tadic B (2011) Dynamics of directed graphs: the World-Wide Web. Physica A 293:273–284
12. Lu Y, Wang X (2008) Pinning control of directed dynamical networks based on ControlRank. Int J Comput Math 85:1279–1286
13. Wu CW (2002) Synchronization in coupled chaotic circuits and systems. World Scientific, Singapore
14. Wu CW, Chua LO (1995) Synchronization in an array of linearly coupled dynamical systems. IEEE Trans Circuits Syst I, Fundam Theory Appl 42:430–447
15. Wu CW, Chua LO (1995) Application of Kronecker products to the analysis of systems with uniform linear coupling. IEEE Trans Circuits Syst I, Fundam Theory Appl 42:775–778
16. Chua LO, Green DN (1976) A qualitative analysis of the behavior of dynamic nonlinear networks: stability of autonomous networks. IEEE Trans Circuits Syst 23:355–379
17. Ding MZ, Yang YM (1997) Stability of synchronous chaos and on–off intermittency in coupled map lattices. Phys Rev E 56:4009–4016
18. Li X, Chen G (2003) Synchronization and desynchronization of complex dynamical networks: an engineering viewpoint. IEEE Trans Circuits Syst I, Fundam Theory Appl 50:1381–1390
19. Rangarajan G, Ding MZ (2002) Stability of synchronized chaos in coupled dynamical systems. Phys Lett A 296:204–209
20. Wang X, Chen G (2002) Pinning control of scale-free dynamical networks. Physica A 310:521–531
21. Chen G, Ueta T (1999) Yet another chaotic attractor. Int J Bifurc Chaos 9:1465–1466
22. Xiang L, Liu Z, Chen Z, Chen F, Yuan Z (2007) Pinning control of complex dynamical networks with general topology. Physica A 379:298–306
23. Brin S, Page L (1998) The anatomy of a large-scale hypertextual web search engine. In: Proceedings of the seventh international World Wide Web conference, pp 14–18
24. Brin S, Page L (1999) The pagerank citation ranking: bringing order to the web. Technical Report, Computer Science Department, Stanford University, 1999–0120
25. Lu W, Li X, Rong Z (2010) Global stabilization of complex directed networks with a local pinning algorithm. Automatica 46:116–121
26. Wu CW (2005) Synchronization in networks of nonlinear dynamical systems coupled via a directed graph. Nonlinearity 18:1057–1064
27. Wu CW (2005) On Rayleigh–Ritz ratios of a generalized Laplacian matrix of directed graphs. Linear Algebra Appl 402:207–227
28. Chen T, Liu X, Lu W (2007) Pinning complex networks by a single controller. IEEE Trans Circuits Syst I, Fundam Theory Appl 54:1317–1326
29. Boyd S, Ghaoui L, Feron E, Balakrishnan V (1994) Linear matrix inequalities in system and control theory. Studies in applied mathematics, vol 15. Society for Industrial and Applied Mathematics (SIAM), Philadelphia
30. Wu CW (2005) On bounds of extremal eigenvalues of irreducible and m-reducible matrices. Linear Algebra Appl 402:29–45

31. Langville AN, Meyer CD (2006) Google's PageRank and beyond: the science of search engine rankings. Princeton University Press, Princeton
32. Ding C, He X, Husbands P, Zha H, Simon H (2002) PageRank, HITS and a unified framework for link analysis. In: Proceedings of the 25th annual international ACM SIGIR conference on research and development in information retrieval, pp 249–253
33. Ramirez JA (1994) Nonlinear feedback for controlling the Lorenz equation. Phys Rev E 50:2339–2342

http://www.springer.com/978-3-642-34577-7

Pinning Control of Complex Networked Systems
Synchronization, Consensus and Flocking of Networked
Systems via Pinning
Su, H.; Xiaofan, W.
2013, XI, 163 p., Hardcover
ISBN: 978-3-642-34577-7