

Robust Stability of Differential-Algebraic Equations

Nguyen Huu Du, Vu Hoang Linh, and Volker Mehrmann

Abstract This paper presents a survey of recent results on the robust stability analysis and the distance to instability for linear time-invariant and time-varying differential-algebraic equations (DAEs). Different stability concepts such as exponential and asymptotic stability are studied and their robustness is analyzed under general as well as restricted sets of real or complex perturbations. Formulas for the distances are presented whenever these are available and the continuity of the distances in terms of the data is discussed. Some open problems and challenges are indicated.

Keywords Differential-algebraic equation · Restricted perturbation · Robust stability · Stability radius · Spectrum · Index

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1 Introduction

In many areas of science and engineering one uses mathematical models to simulate, control or optimize a system or process. These mathematical models, however, are typically inexact or contain uncertainties and thus, the following question is of major importance.

How robust is a specific property of a given system described by differential or difference equations under perturbations to the data?

N.H. Du · V.H. Linh (✉)

Faculty of Mathematics, Mechanics and Informatics, Vietnam National University,
334, Nguyen Trai Str., Thanh Xuan, Hanoi, Vietnam
e-mail: linhvh@vnu.edu.vn

N.H. Du

e-mail: dunh@vnu.edu.vn

V. Mehrmann

Institut für Mathematik, MA 4-5, Technische Universität Berlin, 10623 Berlin,
Fed. Rep. Germany
e-mail: mehrmann@math.tu-berlin.de

Here, we say that a certain property of a system is *robust* if it is preserved when an arbitrary (but sufficiently small) perturbation affects the system. An important quantity in this respect is then the distance (measured by an appropriate metric) between the nominal system and the closest perturbed system that does not possess the mentioned property, this is typically called the *radius* of the system property.

In this paper, we deal with robustness and distance problems for differential-algebraic equations (DAEs), with a focus on robust stability and stability radii. Systems of DAEs, which are also called descriptor systems in the control literature, are a very convenient modeling concept in various real-life applications such as mechanical multibody systems, electrical circuit simulation, chemical reactions, semi-discretized partial differential equations, and in general for automatically generated coupled systems, see [12, 39, 47, 63, 68, 84, 85] and the references therein.

DAEs are generalizations of ordinary differential equations (ODEs) in that certain algebraic equations constrain the dynamical behavior. Since the dynamics of DAEs is constrained to a set which often is only given implicitly, many theoretical and numerical difficulties arise, which may lead to a sensitive behavior of the solution of DAEs to perturbation in the data. The difficulties are characterized by fundamental notions for DAEs such as regularity, index, solution subspace, or hidden constraints, which do not arise for ODEs. These properties may be easily lost when the data are subject to arbitrarily small perturbations. As a consequence, usually restrictions to the allowed perturbations have to be made, leading to robustness questions for DAEs that are very different from those for ODEs.

This paper surveys robustness results for linear DAEs with time-invariant or time-varying coefficients of the form

$$E(t)\dot{x}(t) = A(t)x(t) + f(t), \quad (1.1)$$

on the half-line $\mathbb{I} = [0, \infty)$, together with an initial condition

$$x(t_0) = x_0, \quad t_0 \in \mathbb{I}. \quad (1.2)$$

Here we assume that $E, A \in C(\mathbb{I}, \mathbb{K}^{n \times n})$, and $f \in C(\mathbb{I}, \mathbb{K}^n)$ are sufficiently smooth. We use the notation $C(\mathbb{I}, \mathbb{K}^{n \times n})$ to denote the space of continuous functions from $\mathbb{I}$ to $\mathbb{K}^{n \times n}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Linear systems of the form (1.1) arise directly in many applications and via linearization around solution trajectories [22]. They describe the local behavior in the neighborhood of a solution for a general implicit nonlinear system of DAEs

$$F(t, x(t), \dot{x}(t)) = 0, \quad (1.3)$$

the constant coefficient case arising in the case of linearization around stationary solutions.

Definition 1.1 A function $x : \mathbb{I} \rightarrow \mathbb{R}^n$ is called a *solution* of (1.1) if $x \in C^1(\mathbb{I}, \mathbb{R}^n)$ and x satisfies (1.1) pointwise. It is called a *solution of the initial value problem* (1.1)–(1.2) if x is a solution of (1.1) and satisfies (1.2). An initial condition (1.2) is called *consistent* if the corresponding initial value problem has at least one solution.

We note that, by the tractability index approach [43, 68], the condition on the smoothness of solutions may be relaxed, namely, only a part of x is required to be continuously differentiable.

Recall the following classical stability concepts for ordinary differential equations:

$$\dot{x}(t) = f(t, x(t)), \quad t \in \mathbb{I}, \quad (1.4)$$

with initial condition (1.2), see e.g. [55].

Definition 1.2 A solution $x : t \mapsto x(t; t_0, x_0)$ of (1.4) with initial condition (1.2) is called

1. *stable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that
 - (a) the initial value problem (1.4) with initial condition $x(t_0) = \hat{x}_0$ is solvable on $\mathbb{I}$ for all $\hat{x}_0 \in \mathbb{K}^n$ with $\|\hat{x}_0 - x_0\| < \delta$;
 - (b) the solution $x(t; t_0, \hat{x}_0)$ satisfies $\|x(t; t_0, \hat{x}_0) - x(t; t_0, x_0)\| < \varepsilon$ on $\mathbb{I}$.
2. *asymptotically stable* if it is stable and there exists $\rho > 0$ such that
 - (a) the initial value problem (1.4) with initial condition $x(t_0) = \hat{x}_0$ is solvable on $\mathbb{I}$ for all $\hat{x}_0 \in \mathbb{K}^n$ with $\|\hat{x}_0 - x_0\| < \rho$;
 - (b) the solution $x(t; t_0, \hat{x}_0)$ satisfies $\lim_{t \rightarrow \infty} \|x(t; t_0, \hat{x}_0) - x(t; t_0, x_0)\| = 0$.
3. *exponentially stable* if it is stable and *exponentially attractive*, i.e., if there exist $\delta > 0$, $L > 0$, and $\gamma > 0$ such that
 - (a) the initial value problem (1.4) with initial condition $x(t_0) = \hat{x}_0$ is solvable on $\mathbb{I}$ for all $\hat{x}_0 \in \mathbb{K}^n$ with $\|\hat{x}_0 - x_0\| < \delta$;
 - (b) the solution satisfies the estimate $\|x(t; t_0, \hat{x}_0) - x(t; t_0, x_0)\| < Le^{-\gamma(t-t_0)}$ on $\mathbb{I}$.

If δ does not depend on t_0 , then we say the solution is uniformly (exponentially) stable.

Note that one can transform the ODE (1.4) in such a way that a given solution $x(t; t_0, x_0)$ is mapped to the trivial solution by simply shifting the arguments. When studying the stability of a selected solution, one may therefore assume without loss of generality that the selected solution is the trivial solution, and also that $t_0 = 0$.

In this paper, we restrict the discussion to *regular* DAEs, i.e., we require that (1.1) (or (1.3) locally) have a unique solution for sufficiently smooth E , A , f (F) and appropriately chosen (consistent) initial conditions. One can immediately extend Definition 1.2 verbatim to regular DAEs. However, one has to be careful with the initial conditions and the inhomogeneities, since they are restricted due to the algebraic constraints in the system. This is, in particular, true if one considers the robustness of the stability concepts under perturbations to the system.

The following examples give an illustration for the possible difficulties in the robustness of the stability concepts for DAEs under small perturbations.

Example 1.1 Consider the homogeneous linear time-invariant DAE

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad (1.5)$$

which can be written as $\dot{x}_1 = x_2$, $0 = x_1$ and has only the trivial solution $x_1 = x_2 = 0$.

If we perturb (1.5) by a small ε as

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & \varepsilon \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad (1.6)$$

then solving the second equation of (1.6) for x_2 and substituting into the first equation, we obtain

$$\dot{x}_1 = -(1/\varepsilon)x_1. \quad (1.7)$$

Clearly, if $\varepsilon < 0$, then the perturbed DAE (1.6) is unstable. If $\varepsilon > 0$, then the system is asymptotically stable, but it qualitatively differs from the solution of the original system (1.5). For an arbitrarily prescribed initial value $x_1(0) \neq 0$, the initial value problem for (1.7) has a unique solution. Furthermore, the value of $x_2(0)$ is not required and is uniquely determined by $x_1(0)$. In fact, this small perturbation has changed the index of the DAE (1.5), see Definition 2.2 below.

If we add an inhomogeneity to these DAEs, then more essential differences appear.

In Example 1.1 the perturbation is affecting only the coefficient A . The situation is even more complicated if we allow perturbations in the coefficient of $\dot{x}$.

Example 1.2 Consider the well-known singularly perturbed system

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & \varepsilon I_{n_2} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad (1.8)$$

where A_{ij} , $i, j \in \{1, 2\}$, are constant matrices of appropriate sizes, and $\varepsilon > 0$ is a small parameter. Let us assume that A_{22} is invertible. If ε is set to 0, then the leading matrix becomes singular, i.e., we have a DAE, and we can solve the second equation for x_2 , and obtain the so-called *underlying ODE*

$$\dot{x}_1 = (A_{11} - A_{12}A_{22}^{-1}A_{21})x_1.$$

It is well known that for sufficiently small ε , the (asymptotic) stability of (1.8) depends not only on the stability of the so-called *slow subsystem* associated with the underlying ODE, but also on that of the so called *fast subsystem* $\dot{x}_2 = A_{22}x_2$, see [32, 34, 61], associated with the algebraic equation.

In this example, the rank of the leading matrix is changed, when ε moves from zero to a nonzero value. In the case $\varepsilon = 0$, the initial condition must be consistent to ensure the existence of a solution, but obviously this is not required in the case of a nonzero ε . The difficulties increase if A_{22} is singular and/or the leading matrix involves a singular perturbation of a more general structure.

The presented examples already indicate some of the possible difficulties which will be discussed in this paper. We present the analysis of robust exponential and asymptotic stability for linear DAEs with time-invariant or time-varying coefficients. This is a relatively young topic, starting with the work in [20, 82], which generalized results on the distance to instability and the concept of stability radius for ODEs in [50] and [92].

The outline of the paper is as follows. In Sect. 2 we summarize recent results on the robust stability and stability radii for linear time-invariant DAEs. In Sect. 3 we study the robust stability of linear time-varying DAEs. Stability radii, their dependence on the data, and the robustness of stability spectra are analyzed. Some discussions and topics for future research close the paper.

2 Robust Stability of Linear Time-Invariant DAEs

In this section we study homogeneous linear time-invariant DAEs of the form

$$E\dot{x}(t) = Ax(t), \quad t \in \mathbb{I}, \quad (2.1)$$

where E and A are given constant matrices in $\mathbb{K}^{n \times n}$, $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$.

Definition 2.1 A matrix pair (E, A) , $E, A \in \mathbb{K}^{n \times n}$ is called *regular* if there exists $\lambda \in \mathbb{C}$ such that the determinant of $(\lambda E - A)$, denoted by $\det(\lambda E - A)$, is different from zero. Otherwise, if $\det(\lambda E - A) = 0$ for all $\lambda \in \mathbb{C}$, then we say that (E, A) is *singular*.

If (E, A) is regular, then a complex number λ is called a (*generalized finite*) *eigenvalue* of (E, A) if $\det(\lambda E - A) = 0$. The set of all (finite) eigenvalues of (E, A) is called the (*finite*) *spectrum of the pencil* (E, A) and denoted by $\sigma(E, A)$. If E is singular and the pair is regular, then we say that (E, A) has the eigenvalue ∞ .

In the following we only consider regular pairs (E, A) . Such pairs can be transformed to *Weierstraß–Kronecker canonical form*, see [12, 41, 43], i.e., there exist nonsingular matrices $W, T \in \mathbb{C}^{n \times n}$ such that

$$E = W \begin{bmatrix} I_r & 0 \\ 0 & N \end{bmatrix} T^{-1}, \quad A = W \begin{bmatrix} J & 0 \\ 0 & I_{n-r} \end{bmatrix} T^{-1}, \quad (2.2)$$

where I_r, I_{n-r} are identity matrices of indicated size, $J \in \mathbb{C}^{r \times r}$, and $N \in \mathbb{C}^{(n-r) \times (n-r)}$ are matrices in Jordan canonical form and N is nilpotent. If E is invertible, then $r = n$, i.e., the second diagonal block does not occur.

Definition 2.2 Consider a regular pair (E, A) with $E, A \in \mathbb{K}^{n \times n}$ in Weierstraß–Kronecker form (2.2). If $r < n$ and N has nilpotency index $\nu \in \{1, 2, \dots\}$, i.e., $N^\nu = 0$, $N^i \neq 0$ for $i = 1, 2, \dots, \nu - 1$, then ν is called the *index of the pair* (E, A) and the associated DAE (2.1) and we write $\text{ind}(E, A) = \nu$. If $r = n$ then the DAE has index $\nu = 0$.

The finite spectrum $\sigma(E, A)$ is given by the spectrum of $\sigma(J)$ in the Weierstraß–Kronecker form (2.2) and it is easy to verify that for the *degree of the characteristic polynomial* $\deg \det(\lambda E - A) = \text{rank } E = r$ holds if and only if $\text{ind}(E, A) \leq 1$.

For the regular DAE (2.1) with initial condition (1.2), there always exists a projection matrix $P \in \mathbb{K}^{n \times r}$ so that the projected initial condition $P(x(t_0) - x_0) = 0$ is *consistent*, i.e., the DAE (2.1) with this initial condition has a unique solution, see [12, 43]. Shifting again the solution to the trivial solution $x = 0$, for a DAE of the form (2.1) with regular pair (E, A) , $E, A \in \mathbb{K}^{n \times n}$ we say that this solution is *exponentially stable* if there exist $L > 0$, and $\gamma > 0$ such that the initial value problem

$$E\dot{x} = Ax, \quad P(x(t_0) - \hat{x}_0) = 0,$$

is solvable on $\mathbb{I}$ for all $\hat{x}_0 \in \mathbb{K}^n$, and the solution satisfies the estimate $\|x(t; t_0, \hat{x}_0)\| < Le^{-\gamma(t-t_0)}\|P\hat{x}_0\|$ for all $t \geq t_0$. If the trivial solution is exponentially stable, then we say that (2.1) is *exponentially stable*.

We remark that the property of exponential stability is independent of the choice of projection P . Furthermore, for linear time-invariant systems, the concept of exponential stability is equivalent to that of asymptotic stability and hence we do not have to distinguish these concepts, and discuss only asymptotic stability as is usually done in the literature.

Using the canonical form (2.2), one can easily verify the following statement, see e.g., [20].

Proposition 2.1 *Consider a DAE of the form (2.1) with regular pair (E, A) , $E, A \in \mathbb{K}^{n \times n}$. System (2.1) is asymptotically stable if and only if the pair (E, A) is asymptotically stable, i.e., the finite spectrum satisfies $\sigma(E, A) \subset \mathbb{C}^-$, where $\mathbb{C}^-$ denotes the open left-half complex plane.*

After introducing the basic notation, in the next subsection we discuss the stability radius of a DAE.

2.1 Stability Radii for Linear Time-Invariant DAEs

In this section we study the behavior of the finite spectrum of a regular pair (E, A) under structured perturbations in the matrices E, A . Suppose that the system (2.1) is asymptotically stable and consider a perturbed system

$$(E + B_1 \Delta_1 C_1)\dot{x} = (A + B_2 \Delta_2 C_2)x, \quad (2.3)$$

where $\Delta_i \in \mathbb{K}^{m_i \times q_i}$ ($i = 1, 2$) are perturbations and $B_i \in \mathbb{K}^{n \times m_i}$, $C_i \in \mathbb{K}^{q_i \times n}$ are given matrix pairs that restrict the structure of the perturbations. The matrix pair $(B_1 \Delta_1 C_1, B_2 \Delta_2 C_2)$ is called a *structured perturbation*. For simplicity and for the sake of appropriate perturbation structures, let us consider the case that the restricting matrices satisfy $C_1 = C_2 = C$. Alternatively, the other simplifying case $B_1 = B_2$

can be treated as well, see [38]. However, the simplification does not fit to the framework of so-called admissible perturbations characterized by Proposition 2.6 given below. Set

$$\Delta = \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_2 \end{bmatrix},$$

and introduce $m = m_1 + m_2$ and $q = q_1 = q_2$. Then we consider the set of destabilizing perturbations

$$\mathcal{V}_{\mathbb{K}}(E, A; B, C) = \{ \Delta \in \mathbb{K}^{m \times q}, (2.3) \text{ is singular or not asymptotically stable} \}$$

and have the following definition.

Definition 2.3 The *structured stability radius* of pair (E, A) subject to structured perturbations as in (2.3) is defined by

$$r_{\mathbb{K}}^{\text{sp}}(E, A; B, C) = \inf \{ \|\Delta\|, \Delta \in \mathcal{V}_{\mathbb{K}}(E, A; B, C) \},$$

where $\|\cdot\|$ is a matrix norm induced by a vector norm. Depending on $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, we talk about the *complex or the real structured stability radius*, respectively.

Note that other properties of pair (E, A) such as the index may still change under a perturbation which is not in $\mathcal{V}_{\mathbb{K}}(E, A; B, C)$. Obviously, we have the estimate

$$r_{\mathbb{C}}^{\text{sp}}(E, A; B, C) \leq r_{\mathbb{R}}^{\text{sp}}(E, A; B, C).$$

To obtain a computable formula for the complex stability radius, let us introduce the matrix functions

$$\mathcal{G}_1(s) = -sC(sE - A)^{-1}B_1, \quad \mathcal{G}_2(s) = C(sE - A)^{-1}B_2, \quad \mathcal{G}(s) = \begin{bmatrix} \mathcal{G}_1(s) & \mathcal{G}_2(s) \end{bmatrix},$$

for $s \in \mathbb{C}, \operatorname{Re} s \geq 0$. Denoting by $i\mathbb{R}$ the imaginary axis of the complex plane, the following result is analogous to that for linear time-invariant ODEs of [51].

Theorem 2.2 Suppose that the matrix pencil (E, A) is regular and asymptotically stable. Then with respect to any matrix norm induced by a vector norm, the complex stability radius of pair (E, A) has the representation

$$r_{\mathbb{C}}^{\text{sp}}(E, A; B, C) = \left\{ \sup_{s \in i\mathbb{R}} \|\mathcal{G}(s)\| \right\}^{-1}. \quad (2.4)$$

Proof The proof can be obtained by using the same techniques as in [33, 37, 38]. $\square$

Note that in [33, 37] the complex structured stability radius is considered with respect to perturbations either in E or in A , i.e., either $B_1 = 0$ or $B_2 = 0$. In [38], formula (2.4) has been proven for perturbations in both E, A . Note further that these

papers discuss both the continuous-time and the discrete-time case. Furthermore, it has been shown in [38] that it is always possible to construct a rank 1 destabilizing perturbation Δ , with a norm that approximates the value of $r_{\mathbb{C}}^{\text{sp}}$ within an arbitrarily prescribed accuracy.

Remark 2.1 The concept of stability radius can be extended to more general sets. Suppose that all the eigenvalues of the unperturbed matrix pencil lie in a prescribed open subset $\mathbb{C}_g$ of the complex plane. Then we want to determine the largest perturbations that the system can tolerate so that its spectrum remains in $\mathbb{C}_g$. In the asymptotic stability analysis of differential equations, the open subset $\mathbb{C}_g$ is chosen to be $\mathbb{C}^-$. As in the other cases, it is trivial to obtain a formula of a $\mathbb{C}_g$ -stability radius analogously to (2.4). In fact, we simply replace $i\mathbb{R}$ by the boundary set of $\mathbb{C}_b = \mathbb{C} \setminus \mathbb{C}_g$. As a consequence of the definition, the strict positivity of a $\mathbb{C}_g$ -stability radius with a relevant subset $\mathbb{C}_g$ implies the continuity of the spectrum with respect to the data.

Unlike for the complex stability radius, a general formula for the real stability radius measured by an arbitrary matrix norm is not available. However, if we consider as vector norm the Euclidean norm, then a computable formula has been obtained in [83]. This formula is based on the notion of *real/complex structured singular values*, which, for a given $M \in \mathbb{K}^{p \times m}$, are defined by

$$\mu_{\mathbb{K}}(M) = \left[\inf \{ \sigma_1(\Delta), \Delta \in \mathbb{K}^{m \times p}, \text{ and } \det(I - \Delta M) = 0 \} \right]^{-1},$$

respectively, depending on $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. Here $\sigma_1(\Delta)$ denotes the largest *singular value*, see [42], of the matrix Δ .

Clearly, if M is real, then the complex and the real structured singular values coincide. While for the complex structured singular values, the formula $\mu_{\mathbb{C}}(M) = \sigma_1(M)$ follows trivially, for the real case the formula is more sophisticated.

Proposition 2.3 ([83]) *The real structured singular value of $M \in \mathbb{K}^{p \times m}$ is given by*

$$\mu_{\mathbb{R}}(M) = \inf_{\gamma \in (0,1]} \sigma_2 \begin{bmatrix} \text{Re } M & -\gamma \text{ Im } M \\ \frac{1}{\gamma} \text{ Im } M & \text{Re } M \end{bmatrix},$$

where $\sigma_2(A)$ denotes the second largest singular value of A .

With respect to the Euclidean norm and using a similar argument as in [83], we thus have

$$\begin{aligned} r_{\mathbb{K}}^{\text{sp}}(E, A; B, C) &= \inf \{ \sigma_1(\Delta), \Delta \in \mathcal{V}_{\mathbb{K}}(E, A; B, C) \} \\ &= \inf_{\text{Re } s \geq 0} \inf \{ \sigma_1(\Delta), \Delta \in \mathbb{K}^{m \times q} \text{ and} \\ &\quad \det(s(E + B_1 \Delta_1 C) - (A + B_2 \Delta_2 C)) = 0 \} \end{aligned}$$

$$\begin{aligned}
&= \inf_{\operatorname{Re} s \geq 0} \inf \{ \sigma_1(\Delta), \Delta \in \mathbb{K}^{m \times q} \text{ and} \\
&\quad \det(I + (sE - A)^{-1}(sB_1\Delta_1C - B_2\Delta_2C)) = 0 \} \\
&= \inf_{\operatorname{Re} s \geq 0} \inf \{ \sigma_1(\Delta), \Delta \in \mathbb{K}^{m \times q} \text{ and } \det(I - \Delta\mathcal{G}(s)) = 0 \} \\
&= \left\{ \sup_{\operatorname{Re} s \geq 0} \mu_{\mathbb{K}}(\mathcal{G}(s)) \right\}^{-1},
\end{aligned}$$

and hence we obtain the following theorem.

Theorem 2.4 *Suppose that the matrix pair (E, A) is regular and asymptotically stable. Then the structured stability radii of pair (E, A) , measured in Euclidean norm, are given by*

$$r_{\mathbb{C}}^{\text{sp}}(E, A; B, C) = \left\{ \sup_{\operatorname{Re} s \geq 0} \sigma_1(\mathcal{G}(s)) \right\}^{-1} \quad (2.5)$$

and

$$r_{\mathbb{R}}^{\text{sp}}(E, A; B, C) = \left\{ \sup_{\operatorname{Re} s \geq 0} \inf_{\gamma \in (0, 1]} \sigma_2 \left[\begin{array}{cc} \operatorname{Re} \mathcal{G}(s) & -\gamma \operatorname{Im} \mathcal{G}(s) \\ \frac{1}{\gamma} \operatorname{Im} \mathcal{G}(s) & \operatorname{Re} \mathcal{G}(s) \end{array} \right] \right\}^{-1}, \quad (2.6)$$

respectively.

For the case $\mathbb{K} = \mathbb{C}$, due to the maximum principle, it suffices to take the supremum over the imaginary axis instead of the right-half complex plane. The same does not hold for the case $\mathbb{K} = \mathbb{R}$, see Example 2.1 below. Further, unlike the case of ODEs, here one cannot replace sup by max since the supremum may be attained only at infinity. It is important to note that the presented results on the structured stability radii do not reflect the fact that an eigenvalue at ∞ may become finite or conversely a finite eigenvalue may move to ∞ , i.e., it may happen that the index or the number of finite eigenvalues of the pair (E, A) changes or that the pair becomes singular.

In the case of inhomogeneous systems, particularly an increase of the index may lead to a loss of solvability of the equation due to inconsistent initial values or a lack of smoothness for the inhomogeneity. As we have demonstrated in Examples 1.1 and 1.2, this can even happen with infinitesimally small perturbations. Furthermore, while the stability radii of an ODE are always strictly positive, those of a DAE may be zero. To see this, considering a pair in canonical form (2.2), we have

$$(sE - A)^{-1} = T \begin{bmatrix} (sI_r - J)^{-1} & 0 \\ 0 & -\sum_{i=0}^{k-1} (sN)^i \end{bmatrix} W^{-1}.$$

Obviously, if $N \neq 0$, then $\|\mathcal{G}_1(s)\|$ and $\|\mathcal{G}_2(s)\|$ may tend to ∞ as $|s| \rightarrow \infty$, which implies that $r_{\mathbb{C}}^{\text{sp}} = 0$. Hence, the perturbations in (2.3) must be further restricted so that the stability radii are strictly positive.

Partitioning the restriction matrices C, B (after transformation to Weierstraß–Kronecker form) as

$$CT = [C_1 \quad C_2], \quad W^{-1}B_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}, \quad W^{-1}B_2 = \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix}, \quad (2.7)$$

according to the structure of (2.2), it is easy to see that if $\text{ind}(E, A) = 1$ then

$$\sup_{s \in i\mathbb{R}} \|\mathcal{G}_1(s)\| < \infty \quad \text{if and only if} \quad C_2 B_{12} = 0$$

and if $\text{ind}(E, A) > 1$ then

$$\sup_{s \in i\mathbb{R}} \|\mathcal{G}_2(s)\| < \infty \quad \text{if and only if} \quad C_2 N^i B_{12} = 0 \quad \text{for } i = 0, 1, \dots, k-1, \quad \text{and}$$

$$C_2 N^i B_{22} = 0 \quad \text{for } i = 1, 2, \dots, k-1.$$

These observations are summarized in the following result.

Proposition 2.5 *Consider a regular pair (E, A) and the associated DAE of the form (2.1). If $\text{ind}(E, A) = 1$, then the structured stability radii of (2.1) are strictly positive if and only if $C_2 B_{12} = 0$. If $\text{ind}(E, A) > 1$, then the structured stability radii of (2.1) are strictly positive if and only if $C_2 N^i B_{12} = 0$ for $i = 0, 1, \dots, k-1$, and $C_2 N^i B_{22} = 0$ for $i = 1, 2, \dots, k-1$, where the transformed structure matrices are defined by (2.7).*

Moreover, if C_2 is of full rank, then $r_{\mathbb{K}}^{\text{sp}}(E, A, B, C) > 0$ if and only if $B_{12} = 0$ for the case $\text{ind}(E, A) = 1$ and $B_{12} = 0, N B_{22} = 0$ for the case $\text{ind}(E, A) > 1$.

According to the characterizations given in Proposition 2.5 and for the sake of simplicity, now the perturbations are further restricted by choosing

$$B_1 = W \begin{bmatrix} B_{11} \\ 0 \end{bmatrix}, \quad (2.8)$$

if $\text{ind}(E, A) = 1$ and

$$B_1 = W \begin{bmatrix} B_{11} \\ 0 \end{bmatrix}, \quad B_2 = W \begin{bmatrix} B_{21} \\ 0 \end{bmatrix} \quad (2.9)$$

if $\text{ind}(E, A) > 1$.

Definition 2.4 ([20]) A structured perturbation as in (2.3) is called *admissible* if it does not alter the nilpotency structure of the Weierstraß–Kronecker form (2.2) of (E, A) , i.e., the nilpotent matrix N and the corresponding left invariant subspace associated with the eigenvalue ∞ are preserved.

In the case that $\text{ind}(E, A) = 1$, one has the following characterization of admissible perturbations, which is in agreement with (2.8).

Proposition 2.6 ([20]) *Consider the regular DAE (2.1) with $\text{ind}(E, A) = 1$, subject to a general unstructured perturbation,*

$$(E + F)\dot{x} = (A + H)x.$$

Then there exist an orthogonal matrix P and a permutation matrix Q such that

$$PEQ = \begin{bmatrix} E_{11} & E_{12} \\ 0 & 0 \end{bmatrix}, \quad PAQ = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $E_{11} \in \mathbb{K}^{r \times r}$, $E_{12} \in \mathbb{K}^{r \times (n-r)}$, A_{ij} ($i, j = 1, 2$) are of corresponding sizes, $\text{rank}[E_{11}, E_{12}] = \text{rank } E = r$, and $\text{rank } A_{22} = n - r$. Furthermore, if (F, H) is an admissible perturbation, then

$$PFQ = \begin{bmatrix} F_{11} & F_{12} \\ 0 & 0 \end{bmatrix}, \quad PHQ = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}.$$

Note that in Proposition 2.6, the transformation by the matrices P, Q does not change the structure, the stability, and consequently, the stability radii of (2.1). Note further that Proposition 2.5 can also be used to characterize admissible perturbations for the case $\text{ind}(E, A) > 1$.

After these observations we can introduce the *distance to the nearest pair with a different nilpotency structure*:

$$d_{\mathbb{K}}(E, A; B, C) = \inf \{ \|\Delta\|, \Delta \in \mathbb{K}^{m \times q} \text{ and (2.3) does not preserve the nilpotency structure} \},$$

and obtain the following result, see [11].

Theorem 2.7 *Consider a regular DAE with Weierstraß–Kronecker form (2.2), subject to transformed perturbations satisfying (2.8) for $\text{ind}(E, A) = 1$ and (2.9) for $\text{ind}(E, A) > 1$, respectively. Then the distance to the nearest system with a different nilpotency structure is given by*

$$d_{\mathbb{K}}(E, A; B, C) = \{ \mu_{\mathbb{K}} [C_1 B_{11} \quad C_2 B_{22}] \}^{-1}$$

if $\text{ind}(E, A) = 1$ and

$$d_{\mathbb{K}}(E, A; B, C) = \{ \mu_{\mathbb{K}}(C_1 B_{11}) \}^{-1},$$

if $\text{ind}(E, A) > 1$. Moreover, if the data set is real, then $d_{\mathbb{C}}(E, A; B, C) = d_{\mathbb{R}}(E, A; B, C)$.

Proof The proof is similar to that for the stability radii, see Remark 2.1. The nilpotency structure of the perturbed system (2.3) is preserved if and only if the perturbed

matrix

$$\begin{bmatrix} I + B_{11}\Delta_1C_1 & B_{11}\Delta_1C_2 \\ B_{22}\Delta_2C_1 & I + B_{22}\Delta_2C_2 \end{bmatrix}$$

is nonsingular in the case $\text{ind}(E, A) = 1$ and if $I + B_{11}\Delta_1C_1$ is nonsingular in the case $\text{ind}(E, A) > 1$. Thus, in both cases we have a problem of characterizing the distance of a matrix to singularity, which we obtain by taking $\mathbb{C}_g = \mathbb{C} \setminus \{0\}$. For the equality of the complex and the real stability radii, we note that if the data are real, then the smallest perturbation that makes a matrix singular can always be chosen to be real [42]. $\square$

Now we are in position to define the structured stability radii for the DAE (2.1) and state their formulas.

Definition 2.5 Consider a regular and asymptotically stable DAE (2.1). Then, the structured stability radii of system (2.1) with respect to structured perturbation (2.3) is defined by

$$r_{\mathbb{K}}(E, A; B, C) = \inf\{\|\Delta\|, \Delta \in \mathcal{V}(E, A; B, C) \text{ or (2.3) has different nilpotency structure}\}.$$

It is obvious that $r_{\mathbb{K}}(E, A; B, C) = \min\{r_{\mathbb{K}}^{\text{sp}}(E, A; B, C), d_{\mathbb{K}}(E, A; B, C)\}$, which follows directly from the definition of the stability radii.

Theorem 2.8 Consider a regular and asymptotically stable DAE (2.1) with Weierstraß–Kronecker form (2.2). Let the perturbation structure satisfy (2.8) for index $\text{ind}(E, A) = 1$ and (2.9) for $\text{ind}(E, A) > 1$, respectively. Then, the complex structured stability radius of system (2.1) and that of pair (E, A) coincide, i.e., $r_{\mathbb{C}}(E, A; B, C) = r_{\mathbb{C}}^{\text{sp}}(E, A; B, C)$. Furthermore, if the data set is real, then the real structured stability radius of system (2.1) and that of pair (E, A) are the same, too, that is, $r_{\mathbb{R}}(E, A; B, C) = r_{\mathbb{R}}^{\text{sp}}(E, A; B, C)$.

Proof Taking into consideration (2.8) for index $\text{ind}(E, A) = 1$ and (2.9) for $\text{ind}(E, A) > 1$, respectively, it is easy to check that

$$\lim_{s \rightarrow \infty} \mathcal{G}_1(s) = -C_1 B_{11}, \quad \text{and} \quad \lim_{s \rightarrow \infty} \mathcal{G}_2(s) = -C_2 B_{22}.$$

Since $\sup_{s \in i\mathbb{R}} \|\mathcal{G}(s)\| \geq \|\mathcal{G}(\infty)\|$ (due to the continuity of the norm), the statement for the case $\mathbb{K} = \mathbb{C}$ is obtained from Theorems 2.2 and 2.7.

If the data set is real then $\mathcal{G}(s)$ is real along the non-negative real semi-axis, and then due to the continuity of $\mu_{\mathbb{R}}(\mathcal{G}(s))$ on $\mathbb{R}^+$, we have $\sup_{\text{Re } s \geq 0} \mu_{\mathbb{R}}(\mathcal{G}(s)) \geq \sup_{s \in \mathbb{R}^+} \mu_{\mathbb{R}}(\mathcal{G}(s)) \geq \mu_{\mathbb{R}}(\mathcal{G}(\infty))$. Invoking again Theorems 2.2 and 2.7, the statement for the real stability radii immediately follows. $\square$

Theorem 2.8 shows that if $\|\Delta\| < r_{\mathbb{K}}^{\text{sp}}(E, A; B, C)$ then the perturbed DAE (2.3) preserves not only the stability, but also the nilpotency structure. Thus, the complex

structured stability radius of system (2.1) are given by (2.4) or (2.5), while if the data set is real, then the real structured stability radius can be computed by (2.6). For ODEs, the perturbed system becomes unstable if and only if at least one of its eigenvalues touches the imaginary axis, while it may happen with DAEs that under the effect of larger and larger perturbations, a finite eigenvalue moves to ∞ (which alters the nilpotency structure) and then appears again as a finite eigenvalue on the right-half complex plane (not necessarily on the imaginary axis).

Example 2.1 Consider the linear constant coefficient DAE with structured perturbation to only the right-hand side $E\dot{x} = (A + B\Delta C)x$, where

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

This system is of index 1 and has only one finite eigenvalue $\lambda = -2$. Then, a simple calculation yields $\mathcal{G}(s) = 1 - 1/(2 + s)$. Thus, $\|\mathcal{G}(s)\| = \sqrt{1 - 3/(4 + |s|^2)}$ for $s \in i\mathbb{R}$, which implies $\sup_{s \in i\mathbb{R}} \|\mathcal{G}(s)\| = 1$ and $r_{\mathbb{C}}(E, A, B, C) = 1$. We also have

$$\mu_{\mathbb{R}}(\mathcal{G}(s)) = \begin{cases} 1 - \frac{1}{2+s}, & s \in \mathbb{R}, s \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

from which $\sup_{\operatorname{Re} s \geq 0} \mu_{\mathbb{R}}(\mathcal{G}(s)) = 1$ and $r_{\mathbb{R}}(E, A, B, C) = 1$ follow. However, we remark that $\sup_{\operatorname{Re} s \geq 0} \mu_{\mathbb{R}}(\mathcal{G}(s))$ is attainable only at $+\infty$, but not on the imaginary axis as in the case $\mathbb{K} = \mathbb{C}$. This is also reflected by the quite different effects of complex and real perturbations as the following simple calculations show. The perturbed DAE (2.3) reads

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 - \Delta & \Delta \\ -\Delta & -1 + \Delta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \quad (2.10)$$

For $\Delta = 1$, the system (2.10) is of index 2 and the associated finite spectrum is empty (the only finite eigenvalue moves to infinity), i.e., the pair $(E, A + B\Delta C)$ is stable, but the index is changed. With the perturbation $\Delta = 1 + 1/(1 + s)$, $\operatorname{Re} s \geq 0$, the pair $(E, A + B\Delta C)$ is again of index 1 and the only eigenvalue is s . This means that choosing $s \in i\mathbb{R}$, $|s| \gg 1$, the norm of the complex perturbation approximates the value of $r_{\mathbb{C}}(E, A, B, C)$ within arbitrary accuracy and the only finite eigenvalue appears on the imaginary axis. If we consider only real perturbations, which happens if and only if s is real, then by taking $s \gg 1$, the norm of the real perturbation approximates the value of $r_{\mathbb{R}}(E, A, B, C)$ within arbitrary accuracy and the finite eigenvalue is located on the positive real semi-axis.

Remark 2.2 The first results for stability radii for linear time-invariant DAEs of index 1 were given in [82] and [20]. In [82] only unstructured perturbations in A were considered and the formula for the unstructured complex stability radius measured in Euclidean norm is exactly a special case of (2.5) with restriction matrices $B_1 = 0$ and $B_2 = C = I$. A more general result was obtained in [20], where the

authors considered admissible perturbations as in Proposition 2.6 and formulated the complex stability radius using the Frobenius norm which is not a matrix norm induced by a vector norm. However, using the fact that the Frobenius norm gives an upper bound for the Euclidean norm, the stability radii in Frobenius norm are upper bounds for those in Euclidean norm. On the other hand, in the proof of Theorem 2.2, a rank one destabilizing perturbation can be constructed whose (Euclidean) norm approximates the true value of the stability radius with an arbitrarily small accuracy. Since the Euclidean and the Frobenius norm of a rank one matrix are equal, the formula of the complex stability radius given in [20] and (2.5) yield the same value, i.e., the Frobenius norm case can be considered as a special case of (2.4) and (2.5).

Remark 2.3 A somewhat more general and extensive analysis of stability radii for DAEs is given in [11], where the robustness of the structure and the spectrum are treated separately. The quantities $r_{\mathbb{K}}^{\text{sp}}(E, A; B, C)$ and $d_{\mathbb{K}}(E, A; B, C)$ are called the *spectral* and the *structure-preserving* stability radii, respectively. This approach makes the characterization of stability radii for higher index DAEs possible as well as that for the special uncertainty structure of affine perturbations. The latter one also extends the result in [69] to the case when both coefficient matrices are perturbed with a one-parameter family. In addition, the work in [11] is partially devoted to robust stability of second order DAEs with applications in electrical networks.

Remark 2.4 Since in general the real stability radius is more complicated than the complex one, the question when they are equal is of great practical interest. It has been shown in [87] that for the class of positive systems the complex and the real stability radii coincide and they are easily computable. Attempts to extend this result to DAEs and to other implicit dynamic equations are presented in [33] and [38], respectively.

Remark 2.5 In many applications, one does not only have static perturbations as in (2.3), but also linear time-varying, nonlinear or even nonlinear dynamic perturbations. In the context of regular DAEs (2.1) of index at most one, if perturbations are admitted in A only, then it is possible to extend the concept of structured stability radii with respect to linear time-varying, nonlinear, or nonlinear dynamic perturbations. It can be shown that all the complex structured stability radii with respect to different classes of perturbations are equal, as is well known in the ODE case [53], see also Corollary 3.4 below.

Remark 2.6 Numerical algorithms for computing the stability radii for ODEs are proposed in a number of works, e.g., see [10, 13, 18, 40, 44–46, 48, 57, 79–81, 89]. Some extensions to DAEs are discussed in [2–4, 11]. Since the robust stability of a linear time-invariant system is closely related to the sensitivity of the spectrum, the characterization and the computation of stability radii is also very closely related to the topics of spectral value sets [49, 54] and pseudospectra [17, 91].

Remark 2.7 Robustness questions can also be discussed for other fundamental concepts of control theory such as controllability, observability, stabilizability, or detectability. These concepts have been extended to DAEs in many different publications, see e.g., [14–16, 23, 27, 76, 78] and a recent survey [8]. It is natural to analyze the robustness of these properties when the control systems are subject to uncertain perturbations, which leads to similar distance problems as for robust stability. For ODEs, such distance problems are extensively investigated in a number of works, e. g., see [58, 79] and the references therein. Results on the controllability radius for linear time-invariant descriptor systems are given in [19, 66, 67, 88].

It is well known that solutions of DAEs are more sensitive to data than those of ODEs. This topic has been discussed in [75] for a perturbed index two DAE in semi-explicit form and in [30, 90] for general singularly perturbation problems of DAEs. But a general perturbation theory for linear time-invariant DAEs is *still open*. This is partly due to the fact that no complete characterization of the distance to the nearest singular pencil is available [21].

2.2 Dependence of Stability Radii on the Data

In view of the numerical computation of the stability radii, a natural question is whether the structured stability radii $r_{\mathbb{K}}(E, A; B, C)$ depend continuously on the data E, A, B, C . In the ODE case, i.e., $E = I$ and if only A is perturbed, it was shown in [52, 83] that the complex structured stability radius depends continuously on data, but the real one does not. This is due to the continuity (discontinuity) of the complex (real) structured singular value, see [83]. Extending these results to DAEs, it follows that the complex structured stability radius $r_{\mathbb{C}}(E, A; B, C)$ depends continuously on the data, provided that the nilpotency structure is preserved, i.e., we are restricted to the set of DAEs (2.1) that have the same nilpotency structure and to the set of admissible perturbations. In [34, 35] the robust stability of the parameterized DAE system

$$(E + \varepsilon F)\dot{x} = Ax, \quad (2.11)$$

is considered, where $\varepsilon > 0$ is a small parameter and the unperturbed DAE ($\varepsilon = 0$) is assumed to be regular, of index at most one and asymptotically stable. The classical singularly perturbed system (1.8) is a special case of this more general system. If εF belongs to the class of admissible perturbations characterized by Proposition 2.6, then it is easily shown that the complex structured stability radius depends continuously on the parameter ε . This, however, is not the case when the appearance of εF changes the index and/or the number of finite eigenvalues, i.e., the nilpotency structure of (E, A) .

Sufficient conditions can be given to ensure that (2.11) is asymptotically stable for all sufficiently small and positive ε . Namely, if the unperturbed DAE ($\varepsilon = 0$) and the fast subsystem (which is associated with the algebraic part of the DAE) are simultaneously asymptotically stable, then for all sufficiently small ε , so is the

parameterized system (2.11). Furthermore, the complex structured stability radius of (2.11) converges to the minimum of the stability radius of the reduced system and that of the fast subsystem. This implies that the stability radius of (2.11) may be discontinuous at $\varepsilon = 0$, when the nilpotency structure is no longer invariant. As a special case, the result for the robust stability of (1.8), investigated in [31] by a different approach, then follows immediately. The asymptotic behavior of the real structured stability radius for (2.11) is of interest as well, but it is still an open problem.

Remark 2.8 In [69], the robust stability of a DAE subject to perturbations of the form $E\dot{x} = (A + \varepsilon H)x$ is considered, where E, A are given as in (2.1), H is a given matrix, and ε is an uncertain parameter. Assuming that the unperturbed system is regular, of index at most one and asymptotically stable, a computable formula for the maximal stability interval $(\varepsilon_1, \varepsilon_2)$ is derived, i.e., the perturbed system retains the index and is asymptotically stable for all $\varepsilon \in (\varepsilon_1, \varepsilon_2)$.

In [37], complex structured stability radii for the discrete-time analog of DAEs, i.e., singular difference equations are analyzed. In particular, again due to the continuity, the complex structured stability radius of the discretized system (using the implicit Euler method) of (2.1) is shown to converge to the corresponding one of the continuous-time system as the stepsize tends to zero. The analogous question concerning the real structured stability radii is **still open**.

3 Robust Stability of Linear Time-Varying DAEs

In this section, we investigate the exponential and asymptotic stability and its robustness for linear time-varying DAEs of the form

$$E(t)\dot{x}(t) = A(t)x(t), \quad t \in \mathbb{I}, \quad (3.1)$$

with matrix functions $E, A \in C(\mathbb{I}, \mathbb{K}^{n \times n})$, $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$.

Analyzing the different stability concepts for (3.1) is, however, much more complicated than for linear time-invariant systems. Even if for all $t \in \mathbb{I}$, the finite eigenvalues of $(E(t), A(t))$ have negative real part, system (3.1) may be unstable, as many well-known examples demonstrate for the ODE case, see e.g., [55].

Example 3.1 For all $t \in \mathbb{R}$

$$A(t) = \begin{bmatrix} \cos^2(3t) - 5 \sin^2(3t) & -6 \cos(3t) \sin(3t) + 3 \\ -6 \cos(3t) \sin(3t) + 3 & \sin^2(3t) - 5 \cos^2(3t) \end{bmatrix}$$

has a double eigenvalue at -2 but the solution of $\dot{x} = Ax$, with $x(0) = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$ is given by $x(t) = \begin{bmatrix} c_1 e^{t \cos(3t)} \\ -c_1 e^{t \sin(3t)} \end{bmatrix}$, which is obviously unstable.

We use the following standard notation as in [36, 60]. Let X, Y be finite dimensional vector spaces. For every $p, 1 \leq p < \infty$ and $s, t, t_0 \leq s < t < \infty$, we denote by $L_p(s, t; X)$ the space of measurable functions f with values in X and norm $\|f\|_p := (\int_s^t \|f(\rho)\|^p d\rho)^{1/p} < \infty$ and by $L_\infty(s, t; X)$ the space of measurable and essentially bounded functions f with $\|f\|_\infty := \text{ess sup}_{\rho \in [s, t]} \|f(\rho)\|$. We also consider the spaces $L_p^{\text{loc}}(t_0, \infty; X)$ and $L_\infty^{\text{loc}}(t_0, \infty; X)$, which contain all functions $f \in L_p(s, t; X)$ and $f \in L_\infty(s, t; X)$ for some $s, t, t_0 \leq s < t < \infty$, respectively. We, furthermore, use the notation $\mathcal{L}(L_p(t_0, \infty; X), L_p(t_0, \infty; Y))$ to denote the Banach space of linear bounded operators $\mathbb{P}$ from $L_p(t_0, \infty; X)$ to $L_p(t_0, \infty; Y)$ supplied with the norm

$$\|\mathbb{P}\| := \sup_{x \in L_p(t_0, \infty; X), \|x\|=1} \|\mathbb{P}x\|_{L_p(t_0, \infty; Y)}.$$

In the following we assume that (3.1) is of index 1, in the sense of the tractability index [43] or that it is strangeness-free in the sense of the strangeness-index [63]. Let us briefly introduce these index concepts. First, to introduce the tractability index and the projector chain approach, see e.g., [43, 68], we assume that $E(t)$ is singular for all t and $S := \ker E$ is absolutely continuous. Then, there exists an absolutely continuous projector Q onto S , i.e., $Q \in C(0, \infty; \mathbb{K}^{n \times n})$, Q is differentiable almost everywhere, $Q^2 = Q$, and $\text{Im } Q = S$ for all $t \in \mathbb{I}$. If we assume in addition that $\dot{Q} \in L_\infty^{\text{loc}}(0, \infty; \mathbb{K}^{n \times n})$, then $P = I - Q$ is a projector along S and system (3.1) can be rewritten in the form

$$E \frac{d}{dt}(Px) = \hat{A}x, \quad (3.2)$$

where $\hat{A} := A + E\dot{P} \in L_\infty^{\text{loc}}(0, \infty; \mathbb{K}^{n \times n})$. Setting

$$G := E - \hat{A}Q. \quad (3.3)$$

Definition 3.1 The linear DAE (3.1) is said to be *tractable of index 1*, if $G(t)$ defined by (3.3) is invertible for almost every $t \in [0, \infty)$ and $G^{-1} \in L_\infty^{\text{loc}}(0, \infty; \mathbb{K}^{n \times n})$.

Multiplying both sides of (3.2) by PG^{-1} , QG^{-1} , and taking into account the identities $G^{-1}E = P$, $G^{-1}\hat{A} = -Q + G^{-1}\hat{A}P$, we obtain

$$\begin{aligned} \frac{d}{dt}(Px) &= \left(\frac{d}{dt}P + PG^{-1}\hat{A} \right) Px, \\ Qx &= QG^{-1}\hat{A}Px, \end{aligned}$$

which decomposes the DAE into a differential part and an algebraic part. With $z = Px$, the dynamics of the system is given by the *inherent ODE*

$$\dot{z} = (\dot{P} + PG^{-1}\hat{A})z \quad (3.4)$$

of (3.1). Let $\Phi_0(t, s)$ denote the *fundamental solution matrix* associated with the inherent ODE (3.4), i.e., the matrix function satisfying

$$\frac{d}{dt}\Phi_0(t, s) = (\dot{P} + PG^{-1}\hat{A})\Phi_0(t, s), \quad \Phi_0(s, s) = I,$$

for $t > s \geq 0$, then the fundamental solution operator generated by (3.1) is defined by

$$E \frac{d}{dt}\Phi(t, s) = A\Phi(t, s), \quad P(s)(\Phi(s, s) - I) = 0$$

and Φ can be expressed as $\Phi(t, s) = (I + QG^{-1}\hat{A}(t))\Phi_0(t, s)P(s)$. It can be shown by direct calculations that, despite its construction, the fundamental solution operator is in fact independent of the choice of projector Q .

The concept of the strangeness index uses the DAE and its derivatives to construct a so-called strangeness-free DAE with the same solution [63]. In the homogeneous case this *strangeness-free* system has the form (3.1), where

$$E = \begin{bmatrix} E_1 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \quad (3.5)$$

with $E_1 \in C(\mathbb{I}, \mathbb{K}^{d \times n})$ and $A_2 \in C(\mathbb{I}, \mathbb{K}^{(n-d) \times n})$ such that the matrix $\hat{E}(t) := \begin{bmatrix} E_1(t) \\ A_2(t) \end{bmatrix}$ is invertible for all $t \in \mathbb{I}$.

By using a global kinematic equivalence transformation, see [70, Remark 13], (3.1) can be transformed to the special form,

$$\begin{bmatrix} \tilde{E}_{11} & \tilde{E}_{12} \\ 0 & 0 \end{bmatrix} \dot{\tilde{x}} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} \tilde{x}, \quad (3.6)$$

so that the *essential underlying ODE* is readily given by $\tilde{E}_{11}\dot{\tilde{x}}_1 = \tilde{A}_{11}\tilde{x}_1$ with non-singular $\tilde{E}_{11}$.

A matrix function $\Phi \in C^1(\mathbb{I}, \mathbb{R}^{n \times d})$ is called *minimal fundamental solution matrix of the strangeness-free DAE* (3.1) if each of its columns is a solution to (3.1) and $\text{rank } \Phi(t) = d$ for all $t \in \mathbb{I}$.

In the following we assume that the DAE is of index at most one or alternatively strangeness-free. These conditions are equivalent if the coefficients are sufficiently smooth [77].

The characterization of the different stability concepts for linear variable coefficient ODEs is well established via the concepts of *Bohl and Lyapunov exponents* [1, 9, 28] and Sacker–Sell spectra [29, 86]. These concepts were extended from ODEs to DAEs in [24, 70, 71].

To analyze exponential stability, we introduce first the Bohl exponent.

Definition 3.2 The *Bohl exponent* for an index 1 system of the form (3.1) with fundamental solution Φ is given by

$$k_B(E, A) = \inf \{ -\alpha \in \mathbb{R}; \text{ there exists } L_\alpha > 0 \\ \text{such that for all } t \geq t_0 \geq 0 : \|\Phi(t, t_0)\| \leq L_\alpha e^{-\alpha(t-t_0)} \}.$$

It follows that (3.1) is exponentially stable if and only if its Bohl exponent is negative (including the case $k_B(E, A) = -\infty$).

Analogous to the ODE case (see [28]), using the fundamental solution operator Φ , it follows that the Bohl exponent of (3.1) is bounded from above, i.e., $k_B(E, A) < \infty$ if and only if $\sup_{0 \leq |t-s| \leq 1} \|\Phi(t, s)\| < \infty$. Furthermore, the Bohl exponent of (3.1) can be determined by

$$k_B(E, A) = \limsup_{s, t-s \rightarrow \infty} \frac{\ln \|\Phi(t, s)\|}{t-s}.$$

In [24], various properties of the Bohl exponent, as well as the connection between the exponential stability of (3.1) and the boundedness of solutions to nonhomogeneous DAE with bounded inhomogeneity are investigated.

Remark 3.1 In the ODE case, the boundedness of the coefficient function A ensures the finiteness of Bohl exponent. This is not true for DAEs (3.1) even with both bounded coefficient functions E and A , where the Bohl exponent may be $+\infty$ or $-\infty$. We note also that, by assuming that (3.1) is of index 1, we exclude degenerate cases such as non-uniqueness or finite escape time of solutions, which may happen with nonregular DAEs and are discussed in [6, 7].

For ODEs, the asymptotic stability of solutions can be characterized by the Lyapunov exponents, see [74]. The extension of the theory of Lyapunov exponents to linear time-varying DAEs has been given in [25, 26, 70–72], using either the projector-based tractability index or the derivative array-based strangeness index approach.

Definition 3.3 For a given minimal fundamental solution matrix Φ of a strangeness-free DAE system of the form (3.1), and for $1 \leq i \leq d$, we introduce

$$\lambda_i^u = \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \|\Phi(t) e_i\| \quad \text{and} \quad \lambda_i^\ell = \liminf_{t \rightarrow \infty} \frac{1}{t} \ln \|\Phi(t) e_i\|,$$

where e_i denotes the i th unit vector and $\|\cdot\|$ denotes the Euclidean norm. Let all the Lyapunov exponents be finite, then the columns of a minimal fundamental solution matrix form a *normal basis* if $\sum_{i=1}^d \lambda_i^u$ is minimal. The $\lambda_i^u, i = 1, 2, \dots, d$ belonging to a normal basis are called (*upper*) *Lyapunov exponents* and the intervals $[\lambda_i^\ell, \lambda_i^u], i = 1, 2, \dots, d$, are called *Lyapunov spectral intervals*.

The strangeness-free DAE system (3.1) then is asymptotically stable if and only if the largest upper Lyapunov exponent is negative. Note that for linear time-invariant DAEs (2.1), the Lyapunov exponents are exactly the real parts of the finite eigenvalues of pencil (E, A) .

This brings us to another major difference between linear time-invariant and linear time-varying DAEs. The following example shows that an infinitesimally small time-varying perturbation applied to both coefficient matrices may change the asymptotic stability, even if the perturbation does not change the index.

Example 3.2 ([71]) Consider the system $\dot{x}_1 = -x_1$, $0 = x_2$, which is strangeness-free and asymptotically stable. For the perturbed DAE

$$(1 + \varepsilon^2 \sin(2nt))\dot{x}_1 - \varepsilon \cos(nt)\dot{x}_2 = -x_1, \quad 0 = -2\varepsilon \sin(nt)x_1 + x_2, \quad (3.7)$$

where ε is a small parameter and n is a given number, from the second equation of (3.7), we obtain $x_2 = 2\varepsilon \sin nt x_1$. Differentiating this expression for x_2 and inserting the result into the first equation, after some elementary calculations, we obtain $\dot{x}_1 = (-1 + n\varepsilon^2 + n\varepsilon^2 \cos(2nt))x_1$. Explicit integration yields $x_1(t) = e^{(-1+n\varepsilon^2)t + \varepsilon^2 \sin(2nt)/2} x_1(0)$. Clearly, even if ε is arbitrarily small (hence the perturbation in the coefficient matrices is arbitrarily small in the sup-norm), (3.7) may become unstable if n is sufficiently large.

3.1 Stability Radii for Linear Time-Varying DAEs

In this section we discuss the stability radii for linear DAEs with variable coefficients. Formulas for the stability radii of exponential stability were derived in [24, 36] extending the results for ODEs in [56, 60].

We assume that the DAE is of index at most one and discuss perturbed systems

$$E(t)\dot{x}(t) = (A(t) + H(t))x(t), \quad t \in \mathbb{I}, \quad (3.8)$$

with a perturbation function $H \in L_\infty(0, \infty; \mathbb{K}^{n \times n})$ as well as structured perturbations of the form

$$E(t)\dot{x}(t) = A(t)x(t) + B(t)\Delta(C(\cdot)x(\cdot))(t), \quad t \in \mathbb{I}, \quad (3.9)$$

where $B \in L_\infty(0, \infty; \mathbb{K}^{n \times m})$ and $C \in L_\infty(0, \infty; \mathbb{K}^{q \times n})$ are given matrix functions, restricting the structure of the perturbation and $\Delta : L_p(0, \infty; \mathbb{K}^q) \rightarrow L_p(0, \infty; \mathbb{K}^m)$ is an unknown perturbation operator.

To obtain formulas for the exponential stability radius, we assume that (3.1) is exponentially stable, i.e., there exist constants $L > 0$, $\gamma > 0$ such that $\|\Phi_0(t, s)P(s)\| \leq Le^{-\gamma(t-s)}$, for $t \geq s \geq 0$, and that it is robustly index 1 in the following sense.

Definition 3.4 Consider the DAE (3.1) in the form (3.2) and let G be as in (3.3). Then the DAE is said to be *robustly index 1* if, supplied with a bounded projection Q , the matrix functions G^{-1} and $Q_s := -QG^{-1}\hat{A}$ are essentially bounded on $\mathbb{I}$.

To extend the tractability index concept to the perturbed system (3.9), we assume that the perturbation operator $\Delta \in \mathcal{L}(L_p(0, \infty; \mathbb{K}^q), L_p(0, \infty; \mathbb{K}^m))$ is causal which is defined as follows. For $T \in \mathbb{I}$, the *truncation operator* π_T at T on $L_p(0, \infty; X)$ is defined via

$$\pi_T u(t) := \begin{cases} u(t), & t \in [0, T], \\ 0, & t > T. \end{cases}$$

An operator $\mathbb{P} \in \mathcal{L}(L_p(0, \infty; X), L_p(0, \infty; Y))$ then is said to be *causal*, if $\pi_T \mathbb{P} \pi_T = \pi_T \mathbb{P}$ for every $T \in \mathbb{I}$. Let the linear operator $\tilde{G} \in \mathcal{L}(L_p^{\text{loc}}(0, \infty; \mathbb{K}^n), L_p^{\text{loc}}(0, \infty; \mathbb{K}^n))$ be defined via

$$(\tilde{G}z) = (E - \hat{A}Q)z - B\Delta(CQz).$$

If for every $T > 0$, the operator $\tilde{G}$ restricted to $L_p(0, T; \mathbb{K}^n)$ is invertible and the inverse operator $\tilde{G}^{-1}$ is bounded, then we say the perturbed DAE (3.9) is of *index 1 (in a generalized sense)*. The structured perturbation in (3.9) is called *dynamic perturbation*, which is different from static perturbations considered in Sect. 2. Then we can employ the concept of *mild solution*.

Definition 3.5 We say that the initial value problem for the perturbed system (3.9) with initial condition

$$P(t_0)(x(t_0) - x_0) = 0, \quad (3.10)$$

admits a *mild solution* if there exists $x \in L_p^{\text{loc}}(t_0, \infty; \mathbb{K}^n)$ satisfying

$$\begin{aligned} x(t) &= \Phi(t, t_0)P(t_0)x_0 \\ &+ \int_{t_0}^t \Phi(t, \rho)PG^{-1}B\Delta([Cx(\cdot)]_{t_0})(\rho) d\rho + QG^{-1}B\Delta([Cx(\cdot)]_{t_0})(t) \end{aligned}$$

for $t \geq t_0$, where

$$[Cx(\cdot)]_{t_0} = \begin{cases} 0, & t \in [0, t_0), \\ C(t)x(t), & t \in [t_0, \infty). \end{cases}$$

The mild solution plays an important role in the characterization of robust stability, since solutions of the perturbed DAE (3.9) in the classical sense do not exist, in general. The existence and uniqueness of mild solutions is given by the following result.

Theorem 3.1 ([36]) *Consider the initial value problem (3.9)–(3.10). If (3.9) is of index at most one, then it admits a unique mild solution $x \in L_p^{\text{loc}}(t_0, \infty; \mathbb{K}^n)$ with absolutely continuous $z = Px$ for all $t_0 \in \mathbb{I}$, $x_0 \in \mathbb{K}^n$. Furthermore, for an arbitrary $T > 0$, there exists a constant L_1 such that pointwise*

$$\|P(t)x(t)\| \leq L_1 \|P(t_0)x_0\| \quad \text{for all } t \in [t_0, T].$$

With (3.9), we associate the *perturbation operator* (in [35, 60] it is called the input–output operator)

$$\mathbb{L}_{t_0} z = C \int_{t_0}^t \Phi(t, \rho) P G^{-1} B(\rho) z(\rho) d\rho + C Q G^{-1} B z, \quad (3.11)$$

which is defined for all $t \geq t_0 \geq 0, z \in L_p(0, \infty; \mathbb{K}^m)$. It is important to note that, by direct algebraic verifications, the mild solution as well as the perturbation operator are independent of the choice of projector Q . It is shown that, analogously to the ODE case [56, 60], there exists a close relationship between the perturbation operator associated with (3.9) and the robust stability of (3.1) with respect to structured and dynamic perturbations given by (3.9). For exponentially stable robustly index 1 systems this operator is linear, bounded, and monotonically non-increasing with respect to t , i.e.,

$$\|\mathbb{L}_{t_0}\| \geq \|\mathbb{L}_{t_1}\| \quad \text{for all } t_1 \geq t_0 \geq 0,$$

and for all $t \in \mathbb{I}$ the bound

$$\|\mathbb{L}_t\| \leq \frac{L}{\gamma} \|P G^{-1}\|_{\infty} \|B\|_{\infty} \|C\|_{\infty} + \|C Q G^{-1} B\|_{\infty}$$

holds.

Definition 3.6 ([36]) Consider a DAE of the form (3.9) of index 1 and denote by $x(t; t_0, x_0)$ the solution satisfying initial condition (3.10). The trivial solution of (3.9) is said to be *globally L_p -stable*, if there exist positive constant L_2 and L_3 such that

$$\begin{aligned} \|P(t)x(t; t_0, x_0)\|_{\mathbb{K}^n} &\leq L_2 \|P(t_0)x_0\|_{\mathbb{K}^n}, \\ \|x(\cdot; t_0, x_0)\|_{L_p(t_0, \infty; \mathbb{K}^n)} &\leq L_3 \|P(t_0)x_0\|_{\mathbb{K}^n}, \end{aligned} \quad (3.12)$$

for all $t \geq t_0, x_0 \in \mathbb{K}^n$.

Note that this kind of stability notion is equivalent to the concept of output stability, see [59] for various stability concepts in the ODE case.

We then have the following definition of stability radii for time-varying DAEs.

Definition 3.7 If system (3.1) is exponentially stable and robustly index 1, then the *complex/real structured stability radii* of (3.1) subject to dynamic structured perturbation as in (3.9), are defined by

$$\begin{aligned} r_{\mathbb{K}}(E, A; B, C) \\ = \inf \left\{ \|\Delta\|, \begin{array}{l} \text{the trivial solution of (3.9) is not globally } L_p\text{-stable} \\ \text{or (3.9) is not of index 1} \end{array} \right\}, \end{aligned}$$

where $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, respectively.

In [36] the following formulas for the stability radii were derived.

Theorem 3.2 ([36]) *If system (3.1) is exponentially stable and robustly index 1, then*

$$r_{\mathbb{K}}(E, A; B, C) = \min \left\{ \lim_{t_0 \rightarrow \infty} \|\mathbb{L}_{t_0}\|^{-1}, \left(\operatorname{ess\,sup}_{t \in \mathbb{I}} \|C Q G^{-1} B(t)\| \right)^{-1} \right\}.$$

This implies the following corollary.

Corollary 3.3 ([36]) *If system (3.1) is exponentially stable and robustly index 1 and the data $(E, A; B, C)$ are real, then*

$$r_{\mathbb{C}}(E, A; B, C) = r_{\mathbb{R}}(E, A; B, C).$$

As special case we obtain the formula (2.4) for the complex stability radius of time-invariant systems (with respect to dynamic perturbations).

Corollary 3.4 ([36]) *Let E, A, B, C be time-invariant, let the system (3.1) be index 1 and exponentially stable. If $p = 2$, i.e., for the space L_2 of square integrable functions, then*

$$r_{\mathbb{C}}(E, A; B, C) = \|\mathbb{L}_0\|^{-1} = \left(\sup_{\omega \in i\mathbb{R}} \|C(\omega E - A)^{-1} B\| \right)^{-1}.$$

Comparing with a special case of Theorem 2.2 when only A is subject to perturbations, Corollary 3.4 states that the complex stability radius with respect to dynamic perturbations and that to static perturbations are equal. This statement in fact generalizes a previous result for the ODE case in [53].

Example 3.3 ([36, Sect. 5.1]) Consider a DAE in semi-explicit form

$$E = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad (3.13)$$

with appropriate partitioning. The index 1 assumption means that $A_{22}(t)$ is invertible almost everywhere in $\mathbb{I}$. One gets $Q = \operatorname{diag}(0, I_{n-r})$,

$$G = \begin{bmatrix} I_r & -A_{12} \\ 0 & -A_{22} \end{bmatrix}, \quad \Phi(t, s) = \begin{bmatrix} \widehat{\Phi}(t, s) \\ -A_{22}^{-1} A_{21} \widehat{\Phi}(t, s) \end{bmatrix},$$

where $\widehat{\Phi}(t, s)$ is the fundamental solution operator of the underlying ordinary differential equation $\dot{y} = (A_{11} - A_{12} A_{22}^{-1} A_{21})y$, which is assumed to be exponentially stable. The assumption that the system is robustly index 1 means the essential boundedness of A_{22}^{-1} , $A_{22}^{-1} A_{21}$, and $A_{12} A_{22}^{-1}$. Partitioning the restriction matrices B, C as

$B = [B_1^T \ B_2^T]^T$, $C = [C_1 \ C_2]$, analogously, we obtain

$$\begin{aligned} (\mathbb{L}_{t_0} u)(t) &= (C_1 - C_2 A_{22}^{-1} A_{21}) \int_{t_0}^t \widehat{\Phi}(t, \rho) (B_1(\rho) - A_{12} A_{22}^{-1} B_2(\rho)) u(\rho) d\rho \\ &\quad - C_2 A_{22}^{-1} B_2 u(t), \end{aligned}$$

and by Theorem 3.2 we have

$$r_{\mathbb{K}}(E, A; B, C) = \min \left\{ \lim_{t_0 \rightarrow \infty} \|\mathbb{L}_{t_0}\|^{-1}, \left(\operatorname{ess\,sup}_{t \in \mathbb{I}} \|C_2 A_{22}^{-1} B_2(t)\| \right)^{-1} \right\}.$$

In summary, we have seen in this section that in the index 1 case the exponential stability radii in the linear time-varying case are similar to the ones in the linear time-invariant case.

A similar analysis can be performed in principle for the asymptotic stability, by studying the Lyapunov exponents under perturbations, see the next section. Unfortunately, however, in contrast to the Bohl exponents, the Lyapunov exponents themselves may be very sensitive to small changes in the data.

3.2 Dependence of Stability Radii on the Data

The robustness analysis of the Bohl exponent was extended from ODEs in [56] to DAEs in [24]. In the context of deriving numerical methods for the numerical computation of Bohl and Lyapunov exponents for linear time-varying DAEs, see [70, 71, 73] the robustness of these exponents under perturbations was studied and the concept of admissible perturbations was extended to the variable coefficient case. In this section we summarize these results and study how the stability radii depend on the data.

To see that the Bohl exponent is robust under sufficiently small index-1-preserving perturbations, we consider the perturbed equation

$$E \frac{d}{dt} (Px) = (\widehat{A} + H)x, \quad (3.14)$$

where $\widehat{A}$ is the same as in (3.2). Here H is assumed to be piecewise continuous (just for sake of simplicity) and to satisfy

$$\sup_{t \in \mathbb{I}} \|H(t)\| < \left(\sup_{t \in \mathbb{I}} \|QG^{-1}(t)\| \right)^{-1}. \quad (3.15)$$

Note that this condition implies the inequality $\sup_{t \in \mathbb{I}} \|QG^{-1}H(t)\| < 1$, which is essential in the analysis.

Theorem 3.5 ([24]) *Suppose that system (3.1) is robustly index 1, the perturbation H satisfies (3.15), and suppose further that for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for all H satisfying*

$$\limsup_{s, t-s \rightarrow \infty} \frac{1}{t-s} \int_s^t \|PG^{-1}H(\tau)\| d\tau < \delta,$$

then

$$k_B(E, A + H) < k_B(E, A) + \varepsilon.$$

As a consequence of Theorem 3.5, the exponential stability is preserved under all sufficiently small perturbations H .

Remark 3.2 The robustness analysis of the Bohl exponent is extendable to the case of general perturbations arising in both coefficients of (3.1). It is clear that additional assumptions on the perturbation structure and/or the smoothness of the admissible perturbations are necessary in this case. The same can be said for the analysis of Bohl exponents for general higher-index DAEs and for nonlinear perturbations, see [6].

We now discuss how the structured stability radius of (3.8) with respect to the same structured perturbation as in (3.9) depends on the perturbation H and the restriction matrices B, C . To this end, we first establish the asymptotic behavior of the norm of the input-output operator defined in (3.11).

Theorem 3.6 [24] *Suppose that system (3.1) is exponentially stable, robustly index 1 and satisfies (3.15) and suppose, in addition, that the perturbation function H satisfies*

$$\lim_{t \rightarrow \infty} \|H(t)\| = 0.$$

Then the operator $\mathbb{L}_t$ defined in (3.11) and the corresponding operator associated with (3.14), denoted by $\tilde{\mathbb{L}}_t$, satisfy

$$\lim_{t \rightarrow \infty} \|\mathbb{L}_t\| = \lim_{t \rightarrow \infty} \|\tilde{\mathbb{L}}_t\|.$$

By invoking Theorem 3.2, we obtain sufficient conditions for a sequence of perturbations H_k under which the structured stability radius of the perturbed systems converges to that of the unperturbed system.

Theorem 3.7 ([24]) *Suppose that system (3.1) is exponentially stable, robustly index 1 and satisfies (3.15). Let $\{H_k(\cdot)\}_{k=1}^\infty$ be a sequence of measurable matrix functions and suppose that $\sup_{t \in \mathbb{I}} \|H_k(t)\| < (\sup_{t \in \mathbb{I}} \|QG^{-1}(t)\|)^{-1}$ for all $k = 1, 2, \dots$ and $\lim_{k \rightarrow \infty} \sup_{t \in \mathbb{I}} \|H_k(t)\| = 0$. Then*

$$\lim_{k \rightarrow \infty} r_{\mathbb{C}}(E, A + H_k; B, C) = r_{\mathbb{C}}(E, A; B, C).$$

Theorem 3.7 implies that the stability radius for the system (3.1) depends continuously on the coefficient matrix function A . As a consequence of Theorem 3.7 and Corollary 3.4, we get the following result.

Corollary 3.8 *Let E, A, B, C be constant matrices, let system (3.1) be of index 1 and exponentially stable. Furthermore, assume that the sequence of time-varying perturbation $\{H_k\}_{k=1}^{\infty}$ fulfills the conditions of Theorem 3.7. Then, for the Euclidean norm, one has*

$$\lim_{k \rightarrow \infty} r_{\mathbb{C}}(E, A + H_k; B, C) = \left(\sup_{w \in i\mathbb{R}} \|C(wE - A)^{-1}B\| \right)^{-1}.$$

For illustration, the following numerical example shows that the complex stability radius of a linear time-invariant system under time-varying perturbations can be well approximated by that of the corresponding time-invariant DAE.

Example 3.4 ([24, Example 5.13]) Consider the simple example of a linear constant coefficient DAE with data

$$E = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -2 & -1 \\ 2 & 2 \end{bmatrix}, \quad B = I, \quad C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}. \quad (3.16)$$

Let a sequence of time-varying perturbations be defined by

$$F_k(t) = \begin{bmatrix} -\frac{1}{3+t^2} & \frac{1}{4\sqrt{1+t}} \\ \frac{e^{-2t}}{k+1} & -\frac{e^{-t}}{2k} \end{bmatrix}, \quad k = 1, 2, \dots \quad (3.17)$$

Here, we choose

$$Q = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}, \quad G = E - AQ = \begin{bmatrix} 2 & 1 \\ -2 & -2 \end{bmatrix}.$$

Then it is easy to check that $\lim_{t \rightarrow \infty} \|F_k(t)\| = 0$ and

$$\sup_{t \geq 0} \|F_k(t)\| < \left\| \begin{bmatrix} 1/3 & 1/4 \\ 1/2 & 1/2 \end{bmatrix} \right\| \approx 0.8192 < \|QG^{-1}\|^{-1} \approx 0.8945.$$

Furthermore, we have

$$QG^{-1}F_k(t) = \begin{bmatrix} \frac{e^{-2t}}{2(k+1)} & -\frac{e^{-t}}{4k} \\ -\frac{e^{-2t}}{k+1} & \frac{e^{-t}}{2k} \end{bmatrix},$$

thus $\lim_{k \rightarrow \infty} \sup_{t \geq 0} \|QG^{-1}F_k(t)\| = 0$. On the other hand, by elementary calculations, we obtain

$$\left(\sup_{w \in i\mathbb{R}} \|C(wE - A)^{-1}B\| \right)^{-1} = 1.$$

Invoking Corollary 3.8, we have

$$\lim_{k \rightarrow \infty} r_{\mathbb{C}}(E, A + F_k; B, C) = 1.$$

Finally, one also obtains that the complex structured stability radius of (3.1) depends continuously on the restriction matrices B and C as in the time-invariant case.

Theorem 3.9 ([24]) *Suppose that system (3.1) is exponentially stable, robustly index 1 and satisfies (3.15). Let B_k and C_k be two sequences of measurable and essentially bounded matrix functions satisfying*

$$\lim_{k \rightarrow \infty} \operatorname{ess\,sup}_{t \in \mathbb{I}} \|B_k(t) - B(t)\| = 0, \quad \lim_{k \rightarrow \infty} \operatorname{ess\,sup}_{t \in \mathbb{I}} \|C_k(t) - C(t)\| = 0$$

then,

$$\lim_{k \rightarrow \infty} r_{\mathbb{C}}(E, A; B_k, C_k) = r_{\mathbb{C}}(E, A; B, C).$$

We stress once more that, since the dynamics of DAEs is constrained and the index-1 property should be preserved, only weaker results hold for the continuity of the stability radius and more restrictive assumptions are required than in the ODE case. Furthermore, for time-varying DAEs, we have to restrict the analysis to perturbations in A only, see (3.8) and (3.9), simply because the study of perturbations associated with the leading term E is a long-standing *open problem*. Just very recently, in [5], a result on robust stability with respect to perturbations associated with the leading term E has been derived within the framework of DAEs of tractability-index 1.

In order to study the robustness of Lyapunov exponents, we consider the specially perturbed system for (3.1) given in the form (3.5)

$$[E + F]\dot{x} = [A + H]x, \quad t \in \mathbb{I}, \quad (3.18)$$

where $F = \begin{bmatrix} F_1 \\ 0 \end{bmatrix}$ and $H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$, and where F_1 and H_1, H_2 are assumed to have the same order of smoothness as E_1 and A_1, A_2 , respectively. Perturbations of this structure are called *admissible*, generalizing the concept for the constant coefficient DAEs studied in [20].

The DAE (3.1) is said to be *robustly strangeness-free* if it stays strangeness-free under all sufficiently small admissible perturbations and it is easy to see that this property holds if and only if the matrix function $\hat{E}$ is boundedly invertible.

If we assume that (3.1) is already given in the form (3.6), then the perturbed DAE has the form

$$(E_{11} + F_{11}) \frac{d}{dt} \tilde{x}_1 + (E_{12} + F_{12}) \frac{d}{dt} \tilde{x}_2 = (A_{11} + H_{11}) \tilde{x}_1 + (A_{12} + H_{12}) \tilde{x}_2, \\ 0 = H_{21} \tilde{x}_1 + (A_{22} + H_{22}) \tilde{x}_2.$$

In the following we restrict ourselves to robustly strangeness-free DAE systems under admissible perturbations.

Definition 3.8 The upper Lyapunov exponents $\lambda_1^u \geq \dots \geq \lambda_d^u$ of (3.1) are said to be *stable* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that the conditions $\sup_t \|F(t)\| < \delta$, $\sup_t \|H(t)\| < \delta$, and $\sup_t \|\dot{H}_2(t)\| < \delta$ on the perturbations imply that the perturbed DAE system (3.18) is strangeness-free and

$$|\lambda_i^u - \gamma_i^u| < \varepsilon, \quad \text{for all } i = 1, 2, \dots, d,$$

where the γ_i^u are the ordered upper Lyapunov exponents of (3.18).

The boundedness condition on $\dot{H}_2$, which is obviously satisfied in the time-invariant setting [20], is an extra condition and it seems to be somehow unusual. However, the DAE (3.7) shows the necessity.

As in the ODE case, see [1, 29], to have stability of the Lyapunov spectrum, one needs the property of *integral separation*, i.e., for the columns of the minimal fundamental solution matrix Φ of (3.1) there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$\frac{\|X(t)e_i\|}{\|X(s)e_i\|} \cdot \frac{\|X(s)e_{i+1}\|}{\|X(t)e_{i+1}\|} \geq c_2 e^{c_1(t-s)},$$

for all t, s with $t \geq s \geq 0$ and $i = 1, \dots, d-1$. Then we have the following sufficient conditions for the stability of the upper Lyapunov exponents of (3.1).

Theorem 3.10 ([70]) *Consider the DAE (3.1) in the form (3.6). Suppose that the matrix $\hat{E}$ is boundedly invertible and that $E_{11}^{-1}A_{11}$, $A_{12}A_{22}^{-1}$ and the derivative of A_{22} are bounded on $\mathbb{I}$. Then, the upper Lyapunov exponents of (3.1) are distinct and stable if and only if the system has the integral separation property.*

This shows that if perturbations are performed in E as well, then the perturbation analysis of time-varying DAEs requires more restrictive conditions than in the time-invariant case. However, for some classes of structured problems and/or structured perturbation, parts of these conditions can be relaxed.

If the perturbation block H_{21} disappears, i.e., if H and A have the same block triangular structure, then for example the restrictive conditions on the derivatives in Definition 3.8 and Theorem 3.10 can be omitted. Similar situations happen in the case that $E_{12} = F_{12} = 0$ as discussed in [70, Sect. 3.2] and the case of perturbations in A only as in (3.8).

In [70] another stability concept, the *Sacker–Sell spectrum* has been extended to linear DAEs with variable coefficients. It is also shown that unlike the Lyapunov spectrum, the Sacker–Sell spectrum is robust in the sense that it is stable without requiring integral separation.

This means that for general strangeness-free time-varying systems, the exponential stability is robust, but the asymptotic stability is not. Note that this remark does not apply to time-invariant systems, for which the two stability notions are equivalent.

Remark 3.3 In [56], the robust stability of time-varying ODEs with respect to time-varying perturbations was characterized in terms of differential Riccati equations. In particular, the relation between the perturbation operator and the existence of solutions to the Riccati equations was pointed out. A similar analysis for time-varying DAEs would be of interest as well. However, this gives rise to differential-algebraic Riccati equations, which may have a very complicated solution structure, see e.g. [62, 65].

Remark 3.4 The robustness analysis for linear DAEs of index higher than one and under general perturbations is essentially an *open problem*. The same is true for the distances to the other important control properties such as controllability and observability. The robustness of these concepts for linear DAEs with variable coefficients presents a major challenge.

4 Discussion

In this paper we have surveyed recent results on the robustness of asymptotic and exponential stability for linear time-invariant and time-varying DAEs. We have analyzed robust stability and its distance measures, the real or complex structured stability radius and presented formulas and various properties of the stability radii. We have seen that the robustness analysis for DAEs is much more complicated than that for ODEs. In general, results already known for ODEs now hold for DAEs only under extra assumptions, mainly restricting the set of admissible perturbations. DAE aspects also give rise to new robustness and distance problems. While for time-invariant DAEs, most of the robustness and distance problems are well understood, many problems for time-varying DAEs are still open. These and robustness analysis for general nonlinear and/or high-index DAEs and time-delay DAEs are interesting and challenging topics for future work.

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