

Preface

The subject of mathematical finance has undergone rapid development in recent years, with mathematical descriptions of financial markets evolving both in volume and technical sophistication. Pivotal in this development have been *quantitative models* and *computational methods* for calibrating mathematical models to market data, and for obtaining option prices of concrete products from the calibrated models.

In this development, two broad classes of computational methods have emerged: statistical sampling approaches and grid-based methods. They correspond, roughly speaking, to the characterization of arbitrage-free prices as conditional expectations over all sample paths of a stochastic process model of the market behavior, or to the characterization of prices as solutions (in a suitable sense) of the corresponding Kolmogorov forward and/or backward partial differential equations, or PDEs for short, the canonical example being the Black–Scholes equation and its extensions.

Sampling methods contain, for example, Monte-Carlo and Quasi-Monte-Carlo Methods, whereas grid-based methods contain, for example, Finite Difference, Finite Element, Spectral and Fourier transformation methods (which, by the use of the Fast Fourier Transform, require approximate evaluation of Fourier integrals on grids). The present text discusses the analysis and implementation of grid-based methods.

The importance of numerical methods for the efficient valuation of derivative contracts cannot be overstated: often, the selection of mathematical models for the valuation of derivative contracts is determined by the ease and efficiency of their *numerical evaluation* to the extent that computational efficiency takes priority over mathematical sophistication and general applicability.

Having said this, we hasten to add that the computational methods presented in these notes approximate the (forward and backward) pricing partial (integro) differential equations and inequalities by finite dimensional *discretizations* of these equations which are amenable to numerical solution on a computer. The methods incur, therefore, naturally an error due to this replacement of the forward pricing equation by a discretization, the so-called *discretization error*. One main message to be conveyed by these notes is that, using numerical analysis and advanced solution

methods, efficient discretizations of the pricing equations for a wide range of market models and term sheets are available, and there is no obvious necessity to confine financial modeling to processes which entail “exactly solvable” PIDEs.

We caution the reader, however, that this reasoning implies that the error estimates presented in these notes are *bounds on the discretization error*, i.e. the error in the computed solution with respect to the *exact solution of one particular market model under consideration*. An equally important theme is the quantitative analysis of the error inherent in the financial models themselves, i.e. the so-called *modeling errors*. Such errors are due to assumptions on the markets which were (explicitly or implicitly) used in their derivations, and which may or may not be valid in the situations where the models are used. It is our view that a unified, numerical pricing methodology that accommodates a wide range of market models can facilitate quantitative verification of dependence of prices on various assumptions implicit in particular classes of market models.

Thus, to give “non-experts” in computational methods and in numerical analysis an introduction to grid-based numerical solution methods for option pricing problems is one purpose of the present volume. Another purpose is to acquaint numerical analysts and computational mathematicians with formulation and numerical analysis of typical initial-boundary value problems for partial integro-differential equations (PIDEs) that arise in models of financial markets with jumps. Financial contracts with early exercise features lead to optimal stopping problems which, in turn, lead to unilateral boundary value problems for the corresponding PIDEs. Efficient numerical solution methods for such problems have been developed over many years in solvers for contact problems in mechanics. Contrary to the differential operators which arise with obstacle problems in mechanics, however, the PIDEs in financial models with jumps are, as a rule, *nonsymmetric* (due to the presence of a drift term which, in turn, is mandated by no-arbitrage conditions in the pricing of derivative contracts). The numerical analysis of the corresponding algorithms in financial applications cannot rely, therefore, on energy minimization arguments so that many well-established algorithms are ruled out.

Rather than trying to cover all possible numerical approaches for the computational solution of pricing equations, we decided to focus on Finite Difference and on Finite Element Methods. Finite Element Methods (FEM for short) are based on particularly general, so-called *weak, or variational formulations* of the pricing equation. This is, on the one hand, the natural setting for FEM; on the other hand, as we will try to show in these notes, the variational formulation of the forward and backward equations (in price or in log-price space) on which the FEM is based has a very natural correspondence on the “stochastic side”, namely the so-called *Dirichlet form* of the stochastic process model for the dynamics of the risky asset(s) underlying the derivative contracts of interest. As we show here, FEM based numerical solution methods allow for a unified numerical treatment of rather *general classes of market models*, including local and stochastic volatility models, square root driving processes, jump processes which are either stationary (such as Lévy processes) or nonstationary (such as affine and polynomial processes or processes which are additive in the sense of Sato), for which transform based numerical schemes are not immediately applicable due to lack of stationarity.

In return for this restriction in the types of methods which are presented here, we tried to accommodate within a single mathematical solution framework a wide range of mathematical models, as well as a reasonably large number of term sheet features in the contracts to be valued.

The presentation of the material is structured in two parts: Part I “Basic Methods”, and Part II “Advanced Methods”. The material in the first part of these notes has evolved over several years, in graduate courses which were taught to students in the joint ETH and Uni Zürich MSc programme in quantitative finance, whereas Part II is based on PhD research projects in computational finance.

This distinction between Parts I and II is certainly subjective, and we have seen it evolve over time, in line with the development of the field. In the formulation of the methods and in their analysis, we have tried to maintain mathematical rigor whenever possible, without compromising ease of understanding of the computational methods per se. This has, in particular in Part I, lead to an engineering style of method presentation and analysis in many places. In Part II, fewer such compromises have been made. The formulation of forward and backward equations for rather large classes of jump processes has entailed a somewhat heavy machinery of Sobolev spaces of fractional and variable, state dependent order, of Dirichlet forms, etc. There is a close correspondence of many notions to objects on the stochastic side where the stochastic processes in market models are studied through their Dirichlet forms.

We are convinced that many of the numerical methods presented in these notes have applications beyond the immediate area of computational finance, as Kolmogorov forward and backward equations for stochastic models with jumps arise naturally in many contexts in engineering and in the sciences. We hope that this broader scope will justify to the readers the analytical apparatus for numerical solution methods in particular in Part II.

The present material owes much in style of presentation to discussions of the authors with students in the UZH and ETH MSc quantitative finance and in the ETH MSc Computational Science and Engineering programmes who, during the courses given by us during the past years, have shaped the notes through their questions, comments and feedback. We express our appreciation to them. Also, our thanks go to Springer Verlag for their swift and easy handling of all nonmathematical aspects at the various stages during the preparation of this manuscript.

Winterthur, Switzerland
Zurich, Switzerland
Zurich, Switzerland
Munich, Germany

Norbert Hilber
Oleg Reichmann
Christoph Schwab
Christoph Winter

Computational Methods for Quantitative Finance

Finite Element Methods for Derivative Pricing

Hilber, N.; Reichmann, O.; Schwab, C.; Winter, C.

2013, XIII, 299 p. 56 illus., 47 illus. in color., Hardcover

ISBN: 978-3-642-35400-7