Chapter 1
Introduction

It is hard to overestimate the role of the Jacobian in the theory of smooth complex projective curves. The celebrated theorem of Torelli says that a curve of genus ≥ 2 is determined, up to isomorphism, by its Jacobian and its theta-divisor. Virtually all projective geometric features of a curve can be extracted from its Jacobian. But the Jacobian of a curve has its intrinsic importance and beauty. It is enough to recall that it is a principally polarized abelian variety with an incredibly rich and beautiful theory of theta-functions.

When one turns to higher dimensional projective varieties one quickly discovers that a comparable theory does not exist. However, in the end of 1960s Griffiths initiated a far reaching theory of Variation of Hodge structure (abbreviated in the sequel by VHS). Some of its goals include generalizations of the Theorem of Torelli and a study of algebraic cycles. From Griffiths' theory emerges a substitute for the Jacobian—Griffiths' period domain. This is an open subset of a certain flag variety (factored out by the action of a certain discrete group). In particular, the theory of VHS continues to have strong ties with the theory of Lie groups. Furthermore, a VHS comes with the period map and Griffiths suggested to view its derivative as a substitute for the principal polarization of the classical Jacobian (see [G] for an overview and references therein).

In [R1] we proposed a new version of Jacobian for a smooth complex projective surface X. We suggested to call it nonabelian Jacobian for the simple reason that it parametrizes a distinguished family of rank 2 bundles on X. More precisely, similar to its classical counterpart, our nonabelian Jacobian is, on the one hand, related to the moduli stack of torsion free sheaves[1] on X, and, on the other hand, to the Hilbert scheme of points on X. It also carries a distinguished divisor which can be viewed as a nonabelian analogue of the classical theta-divisor. But a new feature of our Jacobian is that it is also related to the Griffiths' ideas of the VHS and period maps.

[1]Sheaves are of rank 2, contrary to the classical situation of line bundles.

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One of the consequences of this is an appearance of a sheaf of reductive Lie algebras canonically attached to our Jacobian. This can be viewed as an analogue of the Lie algebraic structure of the classical Jacobian. What bearing does this Lie algebraic structure of the nonabelian Jacobian have on geometry of the underlying surface? This is the main question explored in this work.

Our study of the sheaf of Lie algebras and its ties to geometry of the underlying surface is naturally divided into two parts:

1. Establish a dictionary between the properties of the sheaf of reductive Lie algebras attached to our Jacobian and geometric properties of X.
2. Use the representation theory to define interesting objects (e.g. sheaves, complexes of sheaves) which can serve as new invariants of vector bundles on X as well as invariants of the surface itself.

For the first part we are able to uncover:

(a) A precise relationship between the center of the reductive Lie algebras in question and canonical decompositions of configurations of points on X into disjoint union of subconfigurations.
(b) How to use particular $\mathbf{sl_2}$-subalgebras of our reductive Lie algebras to gain an insight into the geometry of configurations of points on X.

For the second part we show how to use the sheaf of reductive Lie algebras associated to our nonabelian Jacobian to attach to X:

(a) A distinguished collection of objects in the category of representations of symmetric groups.
(b) A distinguished collection of objects in the category of perverse sheaves on the appropriate Hilbert schemes of points on X.
(c) A distinguished collection of irreducible representations of the Langlands dual group ${}^L\mathbf{SL}_n(\mathbf{C}) = \mathbf{PGL}_n(\mathbf{C})$, for appropriate values of n.

These results come from the fact that our Jacobian connects in a natural way to such fundamental objects in geometric representation theory as the Springer resolution of the nilpotent cone of simple Lie algebras (of type $\mathbf{A_n}$), Springer fibres, loop algebras and Infinite Grassmannians.

All of the above constitutes a substantial body of evidence that the sheaf of reductive Lie algebras naturally attached to our nonabelian Jacobian is indeed useful for revealing various aspects of geometry of surfaces as well as constructing invariants of the representation theoretic origin.

The ties of Hilbert schemes of points of algebraic surfaces with the representation theory of (affine) Lie algebras have emerged in the last 15 years through the influential works of Grojnowski and Nakajima [N]. The representation theoretic patterns in their works emerge by putting the cohomology rings of *all* Hilbert schemes together and by an explicit checking of the bracket relations of some natural incidence cycles. Until now one has no conceptual understanding why the relations hold and hence, why do we obtain representation theoretic patterns on Hilbert schemes of points of algebraic surfaces.

In our constructions the reductive Lie algebras and their representation theory emerge naturally as an integral part of the nonabelian Jacobian. As we mentioned above, the Jacobian is related to the Hilbert scheme of points (more details are given below) and it could be speculated that the representation theoretic patterns we observe on the Hilbert schemes are shadows of the representation theory on the Jacobian. This is something for future to tell. However, what should be clear, and this is the main message we try to pass across this work, is that the Lie algebraic structure of our Jacobian allows one to use the representation theory of reductive Lie algebras/reductive algebraic groups in a *systematic* way to gain insight into geometry of smooth projective surfaces. Heuristically speaking, our nonabelian Jacobian is a mechanism which reveals hidden symmetries of points of an algebraic surface and those hidden symmetries are useful for gaining insight into various algebro-geometric properties of smooth projective surfaces.

In the rest of this introduction, following a brief summary of [R1], we give a more detailed account of the results of this monograph.

1.1 Nonabelian Jacobian $\mathbf{J}(X; L, d)$ (a Summary of [R1])

A new version of the Jacobian for smooth projective surfaces was proposed in [R1]. Our construction is based on viewing the Jacobian of a smooth projective curve as the parameter space for line bundles with a fixed Chern class. We suggested that for a smooth projective variety X of dimension $n \geq 2$, the Jacobian could be the parameter space of a distinguished family of vector bundles of rank $n = dim_{\mathbf{C}} X$ with fixed Chern invariants. Using this analogy for a smooth projective surface X, we have constructed the scheme $\mathbf{J}(X; L, d)$, whose closed points are pairs $(\mathcal{E}, [e])$, where $\mathcal{E}$ is a torsion free sheaf of rank 2 on X with Chern invariants $c_1(\mathcal{E}) = L$ and $c_2(\mathcal{E}) = d$, where L is a suitably fixed divisor on X and d is a fixed positive integer, and where $[e]$ is the homothety class of a global section e of $\mathcal{E}$, whose zero-locus $Z_e = (e = 0)$ is a subscheme of codimension 2 (equivalently, dimension 0) of X. We suggested to call $\mathbf{J}(X; L, d)$ a nonabelian Jacobian of X (of type (L, d)).

By definition $\mathbf{J}(X; L, d)$ is a scheme over the Hilbert scheme $X^{[d]}$, the scheme parametrizing the subschemes Z of X having dimension zero and length d. The natural morphism

$$\pi : \mathbf{J}(X; L, d) \longrightarrow X^{[d]} \tag{1.1}$$

sends a pair $(\mathcal{E}, [e])$ to the point $[Z_e] \in X^{[d]}$ corresponding to the subscheme $Z_e = (e = 0)$ of X.

As in the classical case, $\mathbf{J}(X;L,d)$, over a suitable subscheme of $X^{[d]}$, comes with a distinguished Cartier divisor $\boldsymbol{\Theta}(X;L,d)$, whose closed points parametrize pairs $(\mathcal{E},[e])$, where the sheaf $\mathcal{E}$ is not locally free. But there is also a new phenomenon: $\mathbf{J}(X;L,d)$ carries a natural structure resembling a VHS à la Griffiths. More precisely, for every point $(\mathcal{E},[e]) \in \mathbf{J}(X;L,d)$, one has a distinguished filtration on $H^0(\mathcal{O}_{Z_e})$

$$0 = \tilde{\mathcal{H}}_0(\mathcal{E},[e]) \subset \tilde{\mathcal{H}}_{-1}(\mathcal{E},[e]) \subset \ldots \subset \tilde{\mathcal{H}}_{-l_{Z_e}-1}(\mathcal{E},[e]) = H^0(\mathcal{O}_{Z_e}), \tag{1.2}$$

where the integer l_{Z_e} is intrinsically associated to Z_e.

Furthermore, if $(\mathcal{E},[e])$ is in a certain constructible subset $\breve{\mathbf{J}}$ of $\mathbf{J}(X;L,d)$, the filtration (1.2) splits. By this we mean that $H^0(\mathcal{O}_{Z_e})$ admits a distinguished direct sum decomposition

$$H^0(\mathcal{O}_{Z_e}) = \bigoplus_{p=0}^{l_{Z_e}} \mathbf{H}^p(\mathcal{E},[e]) \tag{1.3}$$

with a natural identification

$$\mathbf{H}^p(\mathcal{E},[e]) \cong \tilde{\mathcal{H}}_{-(p+1)}(\mathcal{E},[e])/\tilde{\mathcal{H}}_{-p}(\mathcal{E},[e]),$$

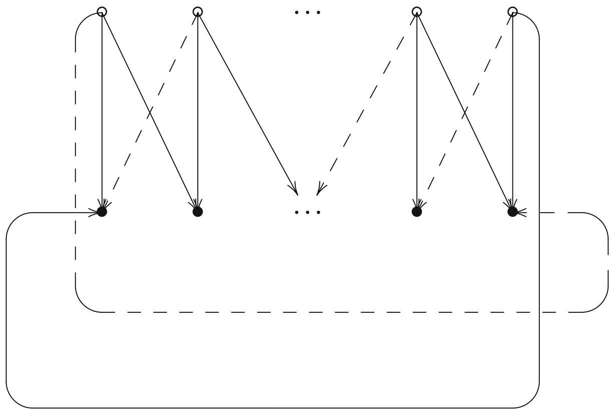
for $p = 0,\ldots,l_{Z_e}$. This direct sum decomposition could be thought of as some kind of periods for the points in $\breve{\mathbf{J}}$. Thus our nonabelian Jacobian possesses features of the classical Jacobian as well as a period map in the spirit of Griffiths theory of VHS.

The decomposition (1.3) together with the obvious ring structure on $H^0(\mathcal{O}_{Z_e})$ gives rise to a reductive Lie subalgebra $\tilde{\boldsymbol{\mathcal{G}}}(\mathcal{E},[e])$ of $\mathbf{gl}(H^0(\mathcal{O}_{Z_e}))$. By varying $(\mathcal{E},[e])$ in $\breve{\mathbf{J}}$ we obtain the sheaf $\tilde{\boldsymbol{\mathcal{G}}}(X;L,d)$ of reductive Lie algebras naturally associated to $\mathbf{J}(X;L,d)$. This could be viewed as a generalization of the Lie algebraic nature of the classical Jacobian.

One of the features of the sheaf $\tilde{\boldsymbol{\mathcal{G}}}(X;L,d)$ is that it gives rise to a natural family of Higgs structures in the sense of Simpson [S]. The parameter space H of this family turns out to be a toric (singular) Fano variety whose hyperplane sections are, in general, singular Calabi-Yau varieties. This H could be viewed as a nonabelian $(1,0)$-Dolbeault variety[2] of $\mathbf{J}(X;L,d)$.

It should be pointed out that H depends only on the properties of the sheaf $\tilde{\boldsymbol{\mathcal{G}}}(X;L,d)$ of reductive Lie algebras and the decomposition (1.3). All this can be encapsulated in the following trivalent graph

[2] In [R1], §4, this variety was called "nonabelian Albanese". This terminology is not quite appropriate, since classically, the Albanese variety involves taking the dual of the space $H^{1,0}$ of holomorphic 1-forms. The variety H parametrizing Higgs structures is certainly more like a direct analogue of the space of holomorphic 1-forms itself. Hence the change of terminology.



(1.4)

where the vertical levels represent the first l_{Z_e} summands of the decomposition (1.3) and the slanted arrows represent certain degree ± 1 operators which are among the generators of $\tilde{\boldsymbol{\mathcal{G}}}(X; L, d)$.

The features of $\mathbf{J}(X; L, d)$ enumerated above show that our Jacobian relates in a natural way to

- Lie algebras and their representations (the sheaf of reductive Lie algebras $\tilde{\boldsymbol{\mathcal{G}}}(X; L, d)$).
- Toric geometry and Calabi-Yau varieties (the nonabelian $(1, 0)$-Dolbeault variety H).
- Low dimensional topology (trivalent graph (1.4)).

Being such a multifaceted object it seems to us that $\mathbf{J}(X; L, d)$ is worthy of a serious study.

In this monograph we undertake a study of the Lie algebraic aspect of our Jacobian with a view toward gaining insights into various algebro-geometric aspects of underlying surface. In the following subsections of the introduction we summarize the key results of this work.

1.2 The Center of the Lie Algebra $\tilde{\boldsymbol{\mathcal{G}}}(\mathcal{E}, [e])$ and Geometry of Z_e

We determine the reductive algebras $\tilde{\boldsymbol{\mathcal{G}}}(\mathcal{E}, [e])$ attached to points of the Jacobian $\breve{\mathbf{J}}$. It turns out that the center of these algebras completely determines the Lie algebra $\tilde{\boldsymbol{\mathcal{G}}}(\mathcal{E}, [e])$ and is related to the geometry of the zero-locus $Z_e = (e = 0)$ associated to $(\mathcal{E}, [e]) \in \breve{\mathbf{J}}$. More precisely, we show

Theorem 1.1. *The zero locus $Z_e = (e = 0)$ decomposes into the disjoint union*

$$Z_e = \bigcup_{i=1}^{\nu} Z_e^{(i)}, \tag{1.5}$$

where ν is the dimension of the center of the Lie algebra $\tilde{\mathcal{G}}(\mathcal{E},[e])$ attached to $(\mathcal{E},[e]) \in \breve{\mathbf{J}}$. Furthermore, the Lie algebra $\tilde{\mathcal{G}}(\mathcal{E},[e])$ and hence, its center act on the subspace $\tilde{\mathbf{H}}_{-l_\Gamma}(\mathcal{E},[e])$ of the filtration of $H^0(\mathcal{O}_{Z_e})$ in (1.2). This action of the center determines the weight decomposition

$$\tilde{\mathbf{H}}_{-l_\Gamma}(\mathcal{E},[e]) = \bigoplus_{i=1}^{\nu} V_i(\mathcal{E},[e])$$

which possesses the following properties:

1) $H^0(\mathcal{O}_{Z_e^{(i)}}) \cong V_i(\mathcal{E},[e]) \cdot H^0(\mathcal{O}_{Z_e})$,
2) one has a natural isomorphism

$$\tilde{\mathcal{G}}(\mathcal{E},[e]) \cong \bigoplus_{i=1}^{\nu} \mathbf{gl}(V_i(\mathcal{E},[e])) . \tag{1.6}$$

This result establishes a precise dictionary between the decomposition of the Lie algebra $\tilde{\mathcal{G}}(\mathcal{E},[e])$ into the direct sum of matrix algebras and the geometric decomposition of Z into the disjoint union of subschemes in (1.5).

It turns out that the Lie algebra $\tilde{\mathcal{G}}(\mathcal{E},[e])$ also controls the properties of the derivative of the period map associated to $\breve{\mathbf{J}}$.

Theorem 1.2. *The derivative of the period map attached to $\breve{\mathbf{J}}$ is injective precisely at the points $(\mathcal{E},[e])$ for which $\tilde{\mathcal{G}}(\mathcal{E},[e]) \cong \mathbf{gl_{d'}}(\mathbf{C})$, where $d' = dim(\tilde{\mathbf{H}}_{-l_\Gamma}(\mathcal{E},[e]))$ and where $\tilde{\mathbf{H}}_{-l_\Gamma}(\mathcal{E},[e])$ is as in the filtration in (1.2).*

This is a version of the Infinitesimal Torelli Theorem for $\mathbf{J}(X;L,d)$. Thus in our story the Infinitesimal Torelli property, i.e. the injectivity of the differential of the period map, has a precise geometric meaning: it fails exactly when the decomposition (1.5) is non-trivial.

These results constitute a semisimple aspect of the representation theory of $\tilde{\mathcal{G}}(\mathcal{E},[e])$ in a sense that it takes into account the action on the space $H^0(\mathcal{O}_{Z_e})$ of the center of $\tilde{\mathcal{G}}(\mathcal{E},[e])$, which is composed of semisimple elements. There is also a nilpotent aspect which is much more involved.

1.3 Nilpotent Aspect of $\mathcal{G}(\mathcal{E}, [e])$

Let $\mathcal{G}(\mathcal{E}, [e])$ be the semisimple part of $\tilde{\mathcal{G}}(\mathcal{E}, [e])$. From the construction of the Lie algebra $\mathcal{G}(\mathcal{E}, [e])$ it follows that we can attach a nilpotent element $D^+(v)$ of $\mathcal{G}(\mathcal{E}, [e])$ to every vertical[3] tangent vector v of $\breve{\mathbf{J}}$ at a point $(\mathcal{E}, [e]) \in \breve{\mathbf{J}}$. On the diagrammatic representation (1.4) the elements $D^+(v)$ are depicted by the right-handed arrows. As v runs through the space $T_\pi(\mathcal{E}, [e])$ of the vertical tangent vectors of $\breve{\mathbf{J}}$ at $(\mathcal{E}, [e])$ we obtain the linear map

$$D^+_{(\mathcal{E},[e])} : T_\pi(\mathcal{E}, [e]) \longrightarrow \mathcal{N}(\mathcal{G}(\mathcal{E}, [e])) \tag{1.7}$$

into the nilpotent cone $\mathcal{N}(\mathcal{G}(\mathcal{E}, [e]))$ of $\mathcal{G}(\mathcal{E}, [e])$.

From the well-known fact that $\mathcal{N}(\mathcal{G}(\mathcal{E}, [e]))$ is partitioned into a finite set of nilpotent orbits we deduce that the map $D^+_{(\mathcal{E},[e])}$ assigns to $(\mathcal{E}, [e])$ a finite collection of nilpotent orbits of $\mathcal{N}(\mathcal{G}(\mathcal{E}, [e]))$. These are the orbits intersecting the image of $D^+_{(\mathcal{E},[e])}$. Varying $(\mathcal{E}, [e])$ in the suitable subvarieties of $\breve{\mathbf{J}}$ we deduce the following.

Theorem 1.3. *The Jacobian $\mathbf{J}(X; L, d)$ gives rise to a finite collection $\mathcal{V}$ of quasi-projective subvarieties of $X^{[d]}$ such that every $\Gamma \in \mathcal{V}$ determines a finite collection $O(\Gamma)$ of nilpotent orbits in $\mathbf{sl}_{\mathbf{d}'_\Gamma}(\mathbf{C})$, where $d'_\Gamma \leq d$ is an integer intrinsically associated to Γ.*

Recalling that nilpotent orbits in $\mathbf{sl_n}(\mathbf{C})$ are parametrized by the set of partitions P_n of n, the above result can be rephrased by saying that every Γ in $\mathcal{V}$ distinguishes a finite collection $P(\Gamma)$ of partitions of d'_Γ. Since partitions of n also parametrize isomorphism classes of irreducible representations of the symmetric group S_n we obtain the following equivalent version of Theorem 1.3.

Theorem 1.4. *The Jacobian $\mathbf{J}(X; L, d)$ gives rise to a finite collection $\mathcal{V}$ of quasi-projective subvarieties of $X^{[d]}$ such that every $\Gamma \in \mathcal{V}$ determines a finite collection $R_{d'_\Gamma}(\Gamma)$ of irreducible representations of the symmetric group $S_{d'_\Gamma}$, where $d'_\Gamma \leq d$ is an integer intrinsically associated to Γ.*

One way to express this result is by saying that the Jacobian $\mathbf{J}(X; L, d)$ elevates a single topological invariant d, the degree of the second Chern class of sheaves parametrized by certain subvarieties of $\mathbf{J}(X; L, d)$, to the level of modules of symmetric groups. Thus our Jacobian gives rise to new invariants with values in the categories of modules of symmetric groups.

But there is more to it. The partitions distinguished by $\mathbf{J}(X; L, d)$ contain a great deal of geometry of subschemes parametrized by Γ's in Theorem 1.3. In down to earth terms one can say that the partitions picked out by points $(\mathcal{E}, [e])$ of $\breve{\mathbf{J}}$ yield equations defining the image of Z_e under certain morphisms into appropriate projective spaces.

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[3] Throughout the monograph "vertical" means in the direction of the fibres of the projection π in (1.1).

The process of obtaining these equations is somewhat evocative of the classical method of Petri (see [Mu] for an overview). However, the essential ingredient in our approach is representation theoretic. It turns on the use of $\mathbf{sl_2}$-subalgebras of $\mathcal{G}(\mathcal{E},[e])$ associated to the nilpotent elements $D^{+}_{(\mathcal{E},[e])}(v)$, the values of the map $D^{+}_{(\mathcal{E},[e])}$ in (1.7). The operator $D^{+}_{(\mathcal{E},[e])}(v)$ in our considerations plays the role of the operator L in the Lefschetz decomposition in the Hodge theory. Completing it to an $\mathbf{sl_2}$-subalgebra of $\mathcal{G}(\mathcal{E},[e])$ in an appropriate way and considering its representation on $H^0(\mathcal{O}_{Z_e})$, gives a sort of Lefschetz decomposition of $H^0(\mathcal{O}_{Z_e})$. This combined with the orthogonal decomposition in (1.3) yields a bigrading of $H^0(\mathcal{O}_{Z_e})$ thus revealing a much finer structure than the initial grading (1.3).

Once this bigrading is in place, writing down the equations defining Z_e in a certain projective space is rather straightforward. This is discussed in details in §10. The equations themselves can be complicated and, in general, not very illuminating. What is essential in our approach is that this complicated set of equations is encoded in an appropriate $\mathbf{sl_2}$-decomposition of $H^0(\mathcal{O}_{Z_e})$. This in turn can be neatly "packaged" in the properties of the partitions singled out by the points $(\mathcal{E},[e])$ of $\breve{\mathbf{J}}$ "polarized" by operators $D^{+}_{(\mathcal{E},[e])}(v)$, with v varying in $T_{\pi}(\mathcal{E},[e])$ as in (1.7).

To summarize, one can say that the nilpotent aspect of the representation theory of $\mathcal{G}(X;L,d)$ provides new geometric insights as well as new invariants of the representation theoretic nature.

This turns out to be only a part of the story. In fact, we can go further by relating $\mathbf{J}(X;L,d)$ to the category of perverse sheaves on $X^{[d]}$.

Theorem 1.5. *The Jacobian $\mathbf{J}(X;L,d)$ determines a finite collection $\mathcal{P}(X;L,d)$ of perverse sheaves on $X^{[d]}$. These perverse sheaves are parametrized by pairs (Γ,λ), where Γ is a subvariety in $\mathcal{V}$ as in Theorem 1.3 and λ is a partition in $P(\Gamma)$.*

This result subsumes two previous theorems since the perverse sheaves $\mathcal{C}(\Gamma,\lambda)$ in $\mathcal{P}(X;L,d)$ have the following properties:

(a) $\mathcal{C}(\Gamma,\lambda)$ is the Intersection Cohomology complex $IC(\Gamma,\mathcal{L}_{\lambda})$ associated to the local system $\mathcal{L}_{\lambda}$ on Γ.
(b) The local system $\mathcal{L}_{\lambda}$ corresponds to a representation

$$\rho_{\Gamma,\lambda} : \pi_1(\Gamma,[Z]) \longrightarrow Aut(H^{\bullet}(B_{\lambda},\mathbf{C})) \tag{1.8}$$

of the fundamental group $\pi_1(\Gamma,[Z])$ of Γ based at a point $[Z]\in\Gamma$ and where $H^{\bullet}(B_{\lambda},\mathbf{C})$ is the cohomology ring (with coefficients in $\mathbf{C}$) of a Springer fibre[4] B_{λ} over the nilpotent orbit $\boldsymbol{O}_{\lambda}$ of $\mathbf{sl}_{d'_{\Gamma}}(\mathbf{C})$ corresponding to the partition λ.

[4] A Springer fibre B_{λ} is a fibre of the Springer resolution

$$\sigma : \tilde{\mathcal{N}} \longrightarrow \mathcal{N}(\mathbf{sl}_{d'_{\Gamma}}(\mathbf{C}))$$

of the nilpotent cone $\mathcal{N}(\mathbf{sl}_{d'_{\Gamma}}(\mathbf{C}))$ of $\mathbf{sl}_{d'_{\Gamma}}(\mathbf{C})$ and where a fibre B_{λ} is taken over the nilpotent orbit $\boldsymbol{O}_{\lambda}$ in $\mathcal{N}(\mathbf{sl}_{d'_{\Gamma}}(\mathbf{C}))$ corresponding to a partition λ of d'_{Γ}.

(c) The representation $\rho_{\Gamma,\lambda}$ admits the following factorization

$$\rho_{\Gamma,\lambda} : \pi_1(\Gamma, [Z]) \xrightarrow{\rho'} S_{d'_\Gamma} \xrightarrow{sp_\lambda} Aut(H^\bullet(B_\lambda, \mathbf{C})), \tag{1.9}$$

where $S_{d'_\Gamma} \xrightarrow{sp_\lambda} Aut(H^\bullet(B_\lambda, \mathbf{C}))$ is the Springer representation of the Weyl group $W = S_{d'_\Gamma}$ of $\mathbf{sl}_{d'_\Gamma}(\mathbf{C})$ on the cohomology of a Springer fibre B_λ.

Taking the irreducible constituents of the perverse sheaves in $\boldsymbol{\mathcal{P}}(X; L, d)$, gives rise to a distinguished collection, denoted $\boldsymbol{C}(X; L, d)$, of *irreducible* perverse sheaves on $X^{[d]}$. This in turn defines the abelian category $\mathcal{A}(X; L, d)$ whose objects are isomorphic to finite direct sums of complexes of the form $\mathcal{C}[n]$, where $\mathcal{C} \in \boldsymbol{C}(X; L, d)$ and $n \in \mathbb{Z}$.

This construction parallels the construction of local systems on the classical Jacobian. Recall that if $J(C)$ is the Jacobian of a smooth projective curve C, then isomorphism classes of irreducible local systems on $J(C)$ are parametrized by the group of characters $Hom(\pi_1(J(C)), \mathbf{C}^\times)$. So we suggest to view the collection of irreducible perverse sheaves $\boldsymbol{C}(X; L, d)$ as a nonabelian analogue of the group of characters of the classical Jacobian, while the abelian category $\mathcal{A}(X; L, d)$ could be envisaged as an analogue of the group-ring of $Hom(\pi_1(J(C)), \mathbf{C}^\times)$.

Though objects of $\mathcal{A}(X; L, d)$ are complexes of sheaves on the Hilbert scheme $X^{[d]}$, they really descend from $\mathbf{J}(X; L, d)$ and one of the ways to remember this is the following

Theorem 1.6. *Let* $\overset{\circ}{\mathbf{J}}(X; L, d) = \mathbf{J}(X; L, d) \setminus \boldsymbol{\Theta}(X; L, d)$ *be the complement of the theta-divisor* $\boldsymbol{\Theta}(X; L, d)$ *in* $\mathbf{J}(X; L, d)$ *and let* $\mathcal{T}^*_{\overset{\circ}{\mathbf{J}}(X;L,d)/X^{[d]}}$ *be the sheaf of relative differentials of* $\overset{\circ}{\mathbf{J}}(X; L, d)$ *over* $X^{[d]}$*. Then there is a natural map*

$$exp\left(\int\right) : H^0(\mathcal{T}^*_{\overset{\circ}{\mathbf{J}}(X;L,d)/X^{[d]}}) \longrightarrow \mathcal{A}(X; L, d).$$

The map in the above theorem could be viewed as a reincarnation of the classical map

$$H^0(\mathcal{T}^*_{J(C)}) \longrightarrow Hom(\pi_1(J(C)), \mathbf{C}^\times),$$

where $\mathcal{T}^*_{J(C)}$ is the cotangent bundle of $J(C)$. This map sends a holomorphic 1-form ω on $J(C)$ to the exponential of the linear functional

$$\int(\omega) : \pi_1(J(C)) = H_1(J(C), \mathbb{Z}) \longrightarrow \mathbf{C}$$

given by integrating ω over 1-cycles on $J(C)$ (the notation "$exp(\int)$" in Theorem 1.6 is an allusion to this classical map).

Relations of the Hilbert schemes of points of surfaces to partitions is not new. Notably, Haiman's work on the Macdonald positivity conjecture [Hai], makes an essential use of such a relation. The same goes for an appearance of perverse sheaves on $X^{[d]}$: the work of Göttsche and Soergel [Go-So], uses the decomposition theorem of [BBD] for the direct image of the Intersection cohomology complex $IC(X^{[d]})$ under the Hilbert-Chow morphism to compute the cohomology of Hilbert schemes. In both of these works the partitions appear from the outset because the authors exploit the points of the Hilbert scheme corresponding to the zero-dimensional subschemes Z of X, where the points in Z are allowed to collide according to the pattern determined by partitions. In our constructions it is essential to work over the open part $Conf_d(X)$ of $X^{[d]}$, parametrizing configurations of d distinct points of X. So there are no partitions seen on the level of the Hilbert scheme. The partitions become visible only on the Jacobian $\mathbf{J}(X; L, d)$ via the Lie algebraic invariants attached to it. One can say that our constructions turn a configuration of distinct points with no interesting structure on it into a dynamical object. The dynamics is given by certain linear operators acting on the space of complex valued functions on a configuration. In particular, the operators $D^+(v)$ obtained as values of the morphism D^+ in (1.7) give rise to the "propagations" and "collisions" in the direct sum decomposition (1.3). This is not an actual, physical, collision of points in a configuration but rather algebro-geometric constraints for a configuration to lie on hypersurfaces in the appropriate projective spaces. The partitions attached to the nilpotent operators $D^+(v)$ can be viewed as a combinatorial (or representation theoretic) measure of this phenomenon, while the perverse sheaves in Theorem 1.5 could be envisaged as its categorical manifestation.

1.4 From $\mathbf{J}(X; L, d)$ to Affine Lie Algebras

One of the major developments of the last 15 years about the Hilbert schemes of points of complex projective surfaces is the discovery of Grojnowski and Nakajima of the action of affine Lie algebras on the direct sum of the cohomology rings (with rational coefficients) of the Hilbert schemes $X^{[n]} (n \in \mathbf{Z}_+)$ (see [N] and the references therein for more details). However, as Nakajima points out in the Introduction of [N], until now one has no good explanation of this phenomenon. In this subsection we explain how our Jacobian can be used to address this problem.

It is clear that formally we can replace the Lie algebra $\mathcal{G}(\mathcal{E}, [e])$ attached to a point $(\mathcal{E}, [e]) \in \mathbf{J}(X; L, d)$ by its loop Lie algebra $\mathcal{G}(\mathcal{E}, [e])[z^{-1}, z]$, where z is a formal variable. However, there is a more natural and explicit reason for appearance of loop Lie algebras in our story. To explain this we recall that the Lie algebra $\tilde{\mathcal{G}}(\mathcal{E}, [e])$ is obtained as follows.

For every h in the summand $\mathbf{H}^0(\mathcal{E}, [e])$ of the decomposition (1.3), we consider the operator $D(h)$ of multiplication by h in the ring $H^0(\mathcal{O}_{Z_e})$. Decomposing this operator according to the direct sum in (1.3) yields a triangular decomposition

$$D(h) = D^-(h) + D^0(h) + D^+(h)\,, \tag{1.10}$$

where $D^{\pm}(h)$ are linear operators of degree ± 1 with respect to the grading in (1.3) and $D^0(h)$ is a grading preserving operator. In particular, the operators $D^+(h)$, for $h \in \mathbf{H}^0(\mathcal{E}, [e])$, are essentially the same as the values of the morphism in (1.7), due to the canonical identification of the relative tangent space $T_\pi(\mathcal{E}, [e])$ with a codimension one subspace of $\mathbf{H}^0(\mathcal{E}, [e])$.

It is quite natural and immediate to turn (1.10) into a loop

$$D(h, z) = z^{-1} D^-(h) + D^0(h) + z D^+(h)\,, \tag{1.11}$$

where z is a formal parameter. Morally, this natural one-parameter deformation of the multiplication in $H^0(\mathcal{O}_{Z_e})$ is behind the following loop version of the map (1.7):

$$LD^+_{(\mathcal{E},[e])} : \overset{\circ}{T}_\pi\,(\mathcal{E}, [e]) \longrightarrow Gr(\mathcal{G}(\mathcal{E}, [e]))\,, \tag{1.12}$$

where $Gr(\mathcal{G}(\mathcal{E}, [e]))$ is the loop or Infinite Grassmannian of the semisimple Lie algebra $\mathcal{G}(\mathcal{E}, [e])$ and $\overset{\circ}{T}_\pi\,(\mathcal{E}, [e])$ is an appropriate Zariski open subset of the vertical tangent space $T_\pi(\mathcal{E}, [e])$ of $\mathbf{J}(X; L, d)$ at $(\mathcal{E}, [e])$. This gives the following "loop" version of Theorem 1.3

Theorem 1.7. *The Jacobian $\mathbf{J}(X; L, d)$ gives rise to a finite collection $\mathcal{V}$ (the same as in Theorem 1.3) of subvarieties Γ of $X^{[d]}$. Every such Γ determines a finite collection $LO(\Gamma)$ of orbits of the Infinite Grassmannian $Gr(\mathbf{SL}_{d'_\Gamma}(\mathbf{C}))$ of $\mathbf{SL}_{d'_\Gamma}(\mathbf{C})$, where d'_Γ is the same as in Theorem 1.3.*

Taking the Intersection Cohomology complexes $IC(O)$ of the orbits O in $LO(\Gamma)$, for every Γ in $\mathcal{V}$, we pass to the category of perverse sheaves on $Gr(\mathbf{SL}_{d'_\Gamma}(\mathbf{C}))$. A beautiful and profound result of Ginzburg [Gi], and Mirkovič and Vilonen [M-V], which establishes an equivalence between the category of perverse sheaves (subject to a certain equivariance condition) on the Infinite Grassmannian $Gr(\mathbf{G})$ of a semisimple Lie group $\mathbf{G}$ and the category of finite dimensional representations of the Langlands dual group $^{\mathbf{L}}\mathbf{G}$ of $\mathbf{G}$, gives a Langlands dual version of Theorem 1.3.

Theorem 1.8. *For every subvariety Γ in $\mathcal{V}$ in Theorem 1.7 the Jacobian $\mathbf{J}(X; L, d)$ determines a finite collection $^L R(\Gamma)$ of irreducible representations of the Langlands dual group $^L\mathbf{SL}_{d'_\Gamma}(\mathbf{C}) = \mathbf{PGL}_{d'_\Gamma}(\mathbf{C})$.*

In retrospect a connection of our Jacobian with the Langlands duality could have been foreseen. After all, the nature of $\mathbf{J}(X; L, d)$ as the moduli space of pairs $(\mathcal{E}, [e])$ resembles the moduli space of pairs of Drinfeld in [Dr]. The fundamental difference is that the groups $\mathbf{SL}_{d'_\Gamma}(\mathbf{C})$ and their Langlands duals in our story have nothing to do with the structure group $(\mathbf{GL_2}(\mathbf{C}))$ of bundles parametrized by $\mathbf{J}(X; L, d)$. These groups rather reflect the geometric underpinnings of our construction related to the

Hilbert scheme $X^{[d]}$. Noting this difference, we also point out one of the key features of $\mathbf{J}(X; L, d)$:

> *it transforms the vertical vector fields of* $\mathbf{J}(X; L, d)$ (*i.e. sections of the relative tangent sheaf* $\mathcal{T}_\pi = \mathcal{T}_{\mathbf{J}(X;L,d)/X^{[d]}}$) *to perverse sheaves on* $X^{[d]}$.

This feature is essentially the map in Theorem 1.6 and it can be viewed as a "tangent" version of Grothendieck's "functions-faisceaux dictionnaire", which plays an important role in a reformulation of the classical, number theoretic, Langlands correspondence into the geometric one (see [Fr], for an excellent introduction to the subject of the geometric Langlands program).

1.5 Concluding Remarks and Speculations

The results of this work show that the Lie algebraic aspects of our Jacobian are useful in addressing various issues related to algebro-geometric properties of configurations of points on surfaces. It also enables us to attach to the degree of the second Chern class of vector bundles such objects as irreducible representations of symmetric groups and perverse sheaves of the representation theoretic origin. In fact, we believe that the tools developed in the monograph allow one to transfer virtually any object/invariant of the geometric representation theory to the realm of smooth projective surfaces. For example, one should be able to have a version of Theorem 1.4, where the representations of the symmetric groups are replaced by the representations of the corresponding Hecke algebras as well as Affine Hecke algebras.

To our mind all these invariants fit into a sort of "secondary" type invariants for vector bundles in the sense of Bott and Chern in [B-C]. Indeed, our construction begins by replacing the second Chern class of a bundle $\mathcal{E}$ (of rank 2) by its geometric realization, i.e. the zero-locus Z of a suitable global section e of $\mathcal{E}$. This is followed by a distinguished orthogonal decomposition (1.3) of the space of functions $H^0(\mathcal{O}_Z)$ on Z. The decomposition gives rise to the Lie subalgebra $\tilde{\mathcal{G}}(\mathcal{E}, [e])$ of $\mathbf{gl}(H^0(\mathcal{O}_Z))$ which is intrinsically associated to the pair $(\mathcal{E}, [e])$. This Lie subalgebra could be viewed as the "secondary" structure Lie algebra associated to $\mathcal{E}$. While the structure group ($\mathbf{GL_2(C)}$) with its Lie algebra provide the topological invariants of $\mathcal{E}$, i.e. its Chern classes, the secondary structure Lie algebra detects various algebro-geometric properties of the subscheme Z. For example, Theorem 1.1 can be interpreted as a statement of reduction of the secondary structure Lie algebra to a proper Lie subalgebra of $\mathbf{gl}(H^0(\mathcal{O}_Z))$ [see (1.6)]. A geometric significance of such a reduction is the decomposition of Z in (1.5). Furthermore, if the structure group and its Lie algebra yield the Chern invariants of $\mathcal{E}$ by evaluating the basic structure group-invariant polynomials on a curvature form of $\mathcal{E}$, it is plausible to expect that our secondary Lie algebra should provide many more representation theoretic invariants of $(\mathcal{E}, [e])$, which would reflect properties of geometric representatives of the Chern invariants of $\mathcal{E}$. Other theorems stated in the introduction could be viewed as a confirmation of this heuristic reasoning.

Theorems 1.4 and 1.5 could also be viewed as two kinds of categorifications of the second Chern class of rank 2 vector bundles on projective surfaces. The latter result and the tools developed to obtain it suggest that there might be a categorification of the representation of affine Lie algebras on the direct sum of the cohomology rings of the Hilbert schemes discovered by I. Grojnowski and H. Nakajima (see the discussion in §1.4)

The results of §1.4 indicate a relation of our Jacobian to the Langlands duality. On the other hand it is conceptually sound to suggest that a formulation of the geometric Langlands program for higher dimensional varieties could involve correspondences in the middle dimension.[5] Now the very idea of the Jacobian as a tool to study correspondences goes back to A.Weil (see [W]). In fact, one of our main motivations for introducing and studying $\mathbf{J}(X; L, d)$ was to study correspondences in the case of projective surfaces. Thus what emerges from our considerations is the following triangular relation

$$\begin{tikzcd} & \mathbf{J}(X;L,d) \arrow[dl] \arrow[dr] & \\ \text{Correspondences of X} \arrow[rr] & & \text{Langlands Duality} \end{tikzcd} \tag{1.13}$$

A precise discussion of these interrelations will appear elsewhere but we hope that the results and tools developed in this work will convince the reader that the nonabelian Jacobian $\mathbf{J}(X; L, d)$ exhibits strong ties with the base of the above triangle.

1.6 Organization of the Monograph

There is a number of different topics discussed in this work and we would like to summarize here how they fit together in our exposition.

To begin with the work draws heavily on the results of [R1]. For this reason §2 is entirely devoted to a concise summary of the main properties of our nonabelian Jacobian obtained in that paper. This is also a place to introduce the main notation and conventions used throughout the monograph.

With these preliminaries out of the way, the development of our theory truly begins with §3. The essential results here are Lemma 3.1 and its geometric realization in Corollary 3.3. These results are of technical nature and are in preparation for the determination of the Lie algebras attached to points of $\mathbf{J}(X; L, d)$.

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[5] What we have in mind here is that correspondences in the middle dimension could be taken as a geometric substitute for the Galois side of the Langlands correspondence.

In §4 these Lie algebras are explicitly determined. This is done in two stages:

- In §4.1 we consider the center of the Lie algebras in question; the geometric consequences of this study are given in Corollary 4.13.
- In §4.2 we determine the semisimple part of the Lie algebras attached to points of $\mathbf{J}(X; L, d)$: the main technical result here is Proposition 4.14.

A combination of these two stages constitutes the results of Theorem 1.1 of the Introduction.

In §5 we switch to a more geometric point of view on our constructions by defining the period maps for our Jacobian. We show that the period maps satisfy Griffiths transversality condition (Proposition 5.4) and compute their differentials in terms of the operators $D^{\pm}(h)$ of the triangular decomposition in (1.10). This gives a purely algebraic formulas to compute the derivatives of our period maps (Lemma 5.7, Proposition 5.9) and links the geometry of the periods maps with the Lie algebraic considerations of the previous sections.

In §5.3 we define Torelli property for our period maps and show that it is entirely controlled by the center of the Lie algebras attached to points of $\mathbf{J}(X; L, d)$ (Corollary 5.15, Theorem 5.16).

Next three sections are devoted to $\mathbf{sl_2}$-subalgebras associated to the operators $D^{\pm}(h)$ of the triangular decomposition in (1.10).

In §6.1 we consider $\mathbf{sl_2}$-subalgebras associated to the operators $D^{+}(h)$. This gives rise to bigraded structures on $H^0(\mathcal{O}_{Z_e})$ in (1.3). The main properties of these bigradings and the action of $D^{+}(h)$ are given in Proposition 6.2. In §6.2 we give a sheaf version of the above structures.

In §7 we consider the adjoint action of the $\mathbf{sl_2}$-subalgebras in §6 on the sheaf of Lie algebras attached to $\mathbf{J}(X; L, d)$. This results in a bigraded structure of the Lie algebras attached to points of $\mathbf{J}(X; L, d)$. The properties of this bigrading can be found in Lemma 7.2 and in Proposition-Definition 7.7.

In §8 we change from operator $D^{+}(h)$ to $D^{-}(h)$ and consider $\mathbf{sl_2}$-subalgebras associated to $D^{-}(h)$. The formalism is of course the same and the main issue here is the interaction of the two structures. In Proposition 8.5 and Corollary 8.6 it is shown how the two $\mathbf{sl_2}$-structures are related. The result is reminiscent of the Hodge-Riemann bilinear relations in Hodge theory.

In §9 we return to geometric considerations. In particular, we show how to use $\mathbf{sl_2}$-subalgebras studied in previous sections to define a stratification of the relative tangent sheaf of $\mathbf{J}(X; L, d)$. The resulting strata are indexed by certain upper triangular, integer-valued matrices which we call multiplicity matrices (Definition 9.5, Proposition 9.6) or, equivalently, by partitions associated to the nilpotent operators $D^{+}(h)$ (Proposition 9.14).

§10 is devoted to applications of the theory built so far to various algebro-geometric questions concerning configurations of points on X.

In §§10.2–10.3 we present a general method of using $\mathbf{sl_2}$-subalgebras considered in §6 to obtain equations of hypersurfaces cutting out configurations in an appropriate projective space. In §10.5 the general method is applied to a particular case: complete intersections on a $K3$-surface. In this case everything can be computed quite explicitly. In particular, one obtains a complete list of very simple quadratic hypersurfaces (of rank ≤ 4) cutting out complete intersections (see Proposition 10.16). This gives a *hyperplane section* version of Mark Green's theorem on quadrics of rank 4 in the ideal of a canonical curve in [Gr].

In §10.6 the $\mathbf{sl_2}$-subalgebras considered in §8 are put to use to study geometry of configurations of points on X with respect to the adjoint linear system $|L + K_X|$. Our considerations show how the partition associated to the nilpotent operator $D^-(h)$ in (1.10) determines a special subvariety in $\mathbb{P}(H^0(\mathcal{O}_X(L + K_X))^*)$, passing through the image of a configuration under the morphism defined by $|L + K_X|$. This is Theorem 10.23 which generalizes a well-known classical result saying that d points ($d \geq 4$) in general position in the projective space $\mathbb{P}^{d-3}$ lie on a rational normal curve.

In §11 we return to general considerations with the intention to use nilpotent elements $D^+(h)$ in a more conceptual way. This leads to a relation of $\mathbf{J}(X; L, d)$ to the nilpotent cone and the Springer resolution of simple Lie algebras of type $\mathbf{sl_n}$. The main results in §11.2 are Proposition 11.4 and Theorem 11.5 (which is equivalent to Theorem 1.3 of the Introduction).

In §11.3 the Springer resolution and Springer fibres are used to construct perverse sheaves on the Hilbert scheme $X^{[d]}$ (Theorem 11.9). This yields the collection $\mathcal{P}(X; L, d)$ of perverse sheaves on $X^{[d]}$ as in Theorem 1.5 of the Introduction.

In §11.4 the collection $\mathcal{P}(X; L, d)$ is put to use to construct the abelian category $\mathcal{A}(X; L, d)$ appearing in Theorem 1.6. The relation of relative differentials of $\mathbf{J}(X; L, d)$ with objects of $\mathcal{A}(X; L, d)$ (the map *exp*($\int$) in Theorem 1.6) is given in Theorem 11.16 (see also Proposition 11.14 and Remark 11.13).

In §12 a relation of $\mathbf{J}(X; L, d)$ and the Infinite Grassmannian of type $\mathbf{SL_n(C)}$ is established (Proposition 12.8). This leads to Theorems 1.7 and 1.8 of the Introduction (stated respectively as Propositions 12.9 and 12.10).

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