Chapter 3
The spin representation

The Clifford algebra for a vector space V with split bilinear form B has an (essentially unique) irreducible module called the *spinor module* S. The Clifford action restricts to a representation of the *Spin group* $\mathrm{Spin}(V)$, known as the *spin representation*. After developing the basic properties of spinor modules and the spin representation, we give a discussion of *pure spinors* and their relation with Lagrangian subspaces. Throughout we will assume that V is a finite-dimensional vector space over a field $\mathbb{K}$ of characteristic zero, and that the bilinear form B on V is nondegenerate. We will write $\mathrm{Cl}(V)$ in place of $\mathrm{Cl}(V;B)$.

3.1 The Clifford group and the spin group

3.1.1 The Clifford group

Recall that $\Pi:\ \mathrm{Cl}(V)\to\mathrm{Cl}(V),\, x\mapsto(-1)^{|x|}x$ denotes the parity automorphism of the Clifford algebra. Let $\mathrm{Cl}(V)^\times$ be the group of invertible elements in $\mathrm{Cl}(V)$.

Definition 3.1 The *Clifford group* $\Gamma(V)$ is the subgroup of $\mathrm{Cl}(V)^\times$, consisting of all $x\in\mathrm{Cl}(V)^\times$ such that

$$A_x(v):=\Pi(x)vx^{-1}\in V$$

for all $v\in V\subseteq\mathrm{Cl}(V)$.

Hence, by definition the Clifford group comes with a natural representation,

$$\Gamma(V)\to\mathrm{GL}(V),\quad x\mapsto A_x.$$

Let $S\Gamma(V)=\Gamma(V)\cap\mathrm{Cl}^{\bar 0}(V)^\times$ denote the *special Clifford group*. The Clifford group and the special Clifford group are extensions of the orthogonal and special orthogonal group, respectively. In fact we have:

E. Meinrenken, *Clifford Algebras and Lie Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics 58,
DOI 10.1007/978-3-642-36216-3_3, © Springer-Verlag Berlin Heidelberg 2013

Theorem 3.1 *The natural representation of the Clifford group takes values in* $\mathrm{O}(V)$, *and defines an exact sequence*

$$1 \longrightarrow \mathbb{K}^\times \longrightarrow \Gamma(V) \longrightarrow \mathrm{O}(V) \longrightarrow 1.$$

It restricts to a similar exact sequence for the special Clifford group

$$1 \longrightarrow \mathbb{K}^\times \longrightarrow S\Gamma(V) \longrightarrow \mathrm{SO}(V) \longrightarrow 1.$$

The elements of $\Gamma(V)$ *are all products*

$$x = v_1 \cdots v_k, \tag{3.1}$$

where $v_1, \ldots, v_k \in V$ *are non-isotropic. The corresponding element* A_x *is a product of reflections*:

$$A_x = R_{v_1} \cdots R_{v_k}. \tag{3.2}$$

In particular, every element $x \in \Gamma(V)$ *is either even or odd, depending on the parity of* k *in* (3.1). *In particular,* $S\Gamma(V)$ *is given by products* (3.1) *with* k *even.*

Proof Let $x \in \mathrm{Cl}(V)$. The transformation A_x is trivial if and only if $\Pi(x)v = vx$ for all $v \in V$, i.e., if and only if $[v, x] = 0$ for all $v \in V$. That is, it is the intersection of the center $\mathbb{K} \subseteq \mathrm{Cl}(V)$ with $\Gamma(V)$. (See Lemma 2.1.) This shows that the kernel of the homomorphism $\Gamma(V) \to \mathrm{GL}(V)$, $x \mapsto A_x$ is the group $\mathbb{K}^\times$ of invertible scalars.

Applying $-\Pi$ to the definition of A_x, we obtain $A_x(v) = xv\Pi(x)^{-1} = A_{\Pi(x)}(v)$. This shows that $A_{\Pi(x)} = A_x$ for $x \in \Gamma(V)$. Thus $\Pi(x)$ is a scalar multiple of x; in fact $\Pi(x) = \pm x$ since Π is the parity operator. This shows that elements of $\Gamma(V)$ have definite parity. For $x \in \Gamma(V)$ and $v, w \in V$ we have, using again $A_{\Pi(x)} = A_x$,

$$\begin{aligned} 2B(A_x(v), A_x(w)) &= A_x(v)A_x(w) + A_x(w)A_x(v) \\ &= A_x(v)A_{\Pi(x)}(w) + A_x(w)A_{\Pi(x)}(v) \\ &= \Pi(x)(vw + wv)\Pi(x^{-1}) \\ &= 2B(v, w)\Pi(x)\Pi(x^{-1}) \\ &= 2B(v, w). \end{aligned}$$

This proves that $A_x \in \mathrm{O}(V)$ for all $x \in \Gamma(V)$. Suppose now that $v \in V$ is non-isotropic. Then it is invertible in the Clifford algebra, with $v^{-1} = v/B(v, v)$ and $\Pi(v) = -v$. For all $w \in V$,

$$A_v(w) = -vwv^{-1} = (wv - 2B(v, w))v^{-1} = w - 2\frac{B(v, w)}{B(v, v)}v = R_v(w).$$

Hence $v \in \Gamma(V)$, with $A_v = R_v$ the reflection defined by v. More generally, this proves (3.2) whenever x is of the form (3.1). By the E. Cartan–Dieudonné Theorem 1.1, any $A \in \mathrm{O}(V)$ is a product of reflections R_{v_i}. This shows that the map $x \mapsto A_x$ is onto $\mathrm{O}(V)$, and that $\Gamma(V)$ is generated by the non-isotropic vectors in V. The remaining statements are clear. □

Since every $x \in \Gamma(V)$ can be written in the form (3.1), it follows that the element $x^\top x$ lies in $\mathbb{K}^\times$. This defines the *norm homomorphism*

$$\mathsf{N}\colon\ \Gamma(V) \to \mathbb{K}^\times,\ x \mapsto x^\top x. \tag{3.3}$$

It is a group homomorphism and has the property

$$\mathsf{N}(\lambda x) = \lambda^2 \mathsf{N}(x)$$

for $\lambda \in \mathbb{K}^\times$.

Example 3.1 The chirality element $\Gamma \in \mathrm{Cl}(V)$, defined by choice of a generator $\Gamma_\wedge \in \det(V)$, is an element of the Clifford group $\Gamma(V)$, and is contained in $S\Gamma(V)$ if and only if $\dim V = 2m$ is even. In the special case of a vector space with split bilinear form, and Γ normalized so that $\Gamma^2 = 1$ (see Eq. (2.12)), one has

$$\mathsf{N}(\Gamma) = \Gamma^\top \Gamma = (-1)^m.$$

Example 3.2 Consider $V = F^* \oplus F$ with $\dim F = 1$. Choose dual generators $e \in F$, $f \in F^*$ so that $B(e, f) = \frac{1}{2}$. One checks that an even element $x = s + tfe \in \mathrm{Cl}^{\bar{0}}(V)$ with $s, t \in \mathbb{K}$ lies in $S\Gamma(V)$ if and only if $s, s + t$ are both invertible, and in that case

$$A_x(e) = \frac{s}{s+t}\, e, \quad A_x(f) = \frac{s+t}{s}\, e.$$

We have $\mathsf{N}(x) = x^\top x = s(s + t)$.

3.1.2 *The groups* Pin(V) *and* Spin(V)

We give the following definitions.

Definition 3.2 The *Pin group* $\mathrm{Pin}(V)$ is the kernel of the norm homomorphism $\mathsf{N} : \Gamma(V) \to \mathbb{K}^\times$. Its intersection with $S\Gamma(V)$ is called the *Spin group*, and is denoted $\mathrm{Spin}(V)$.

The normalization $\mathsf{N}(x) = 1$ specifies $x \in \Gamma(V)$ up to sign. Hence one obtains exact sequences,

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Pin}(V) \longrightarrow \mathrm{O}(V),$$
$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(V) \longrightarrow \mathrm{SO}(V).$$

In general, the maps to $\mathrm{SO}(V)$, $\mathrm{O}(V)$ need not be surjective. A sufficient condition for surjectivity is that every element in $\mathbb{K}$ admits a square root, since one may then rescale any $x \in \Gamma(V)$ so that $\mathsf{N}(x) = 1$. Theorem 3.1 shows that in this case $\mathrm{Pin}(V)$ is the set of products $v_1 \cdots v_k$ of elements $v_i \in V$ with $B(v_i, v_i) = 1$, while $\mathrm{Spin}(V)$ consists of similar products with k even.

Remark 3.1 If V has non-zero isotropic vectors, then the condition that all elements in $\mathbb{K}$ have square roots is also necessary. Indeed, let $e \neq 0$ be isotropic, and let f be an isotropic vector with $B(e, f) = \frac{1}{2}$. Let $A \in \mathrm{SO}(V)$ be equal to the identity on $\mathrm{span}\{e, f\}^{\perp}$, and

$$A(e) = r\, e, \quad A(f) = r^{-1} f$$

with $r \in \mathbb{K}^{\times}$. As shown in Example 3.2, the lifts of A to $S\Gamma(V)$ are elements of the form $x = s + tfe$ with $r = s(s+t)^{-1}$. Since $\mathsf{N}(x) = s(s+t) = \frac{s^2}{r}$ we see that $x \in \mathrm{Spin}(V)$ if and only if $r = s^2$. The choice of square root of r specifies the lift x.

Remark 3.2 If $\mathbb{K} = \mathbb{R}$, the Pin and Spin groups are sometimes defined using a weaker condition $\mathsf{N}(x) = \pm 1$. This then guarantees that the maps to $\mathrm{O}(V)$, $\mathrm{SO}(V)$ are surjective.

For $\mathbb{K} = \mathbb{R}$, $V = \mathbb{R}^{n,m}$ we use the notation

$$\mathrm{Pin}(n, m) = \mathrm{Pin}(\mathbb{R}^{n,m}), \quad \mathrm{Spin}(n, m) = \mathrm{Spin}(\mathbb{R}^{n,m}).$$

If $m = 0$, we simply write $\mathrm{Pin}(n) = \mathrm{Pin}(n, 0)$ and $\mathrm{Spin}(n) = \mathrm{Spin}(n, 0)$.

Theorem 3.2 *Let $\mathbb{K} = \mathbb{R}$. Then* $\mathrm{Spin}(n, m)$ *is a double cover of the identity component* $\mathrm{SO}_0(n, m)$. *If $n \geq 2$ or $m \geq 2$, the group* $\mathrm{Spin}(n, m)$ *is connected.*

Proof The cases of $(n, m) = (0, 1)$, $(1, 0)$ are trivial. If $(n, m) = (1, 1)$ one has $\mathrm{SO}_0(1, 1) = \mathbb{R}_{>0}$, and $\mathrm{Spin}(1, 1) = \mathbb{Z}_2 \times \mathbb{R}_{>0}$ (see Example 3.2). Suppose $n \geq 2$ or $m \geq 2$. To show that $\mathrm{Spin}(n, m)$ is connected, it suffices to show that the elements ± 1 (the pre-image of the group unit in $\mathrm{SO}(n, m)$) are in the same connected component. Let

$$v(\theta) \in \mathbb{R}^{n,m}, \quad 0 \leq \theta \leq \pi$$

be a continuous family of non-isotropic vectors with the property $v(\pi) = -v(0)$. Such a family exists, since V contains a 2-dimensional subspace isomorphic to $\mathbb{R}^{2,0}$ or $\mathbb{R}^{0,2}$. Rescale the vectors $v(\theta)$ to satisfy $B(v(\theta), v(\theta)) = \pm 1$. Then

$$[0, \pi] \to \mathrm{Spin}(n, m), \ \theta \mapsto x(\theta) = v(\theta) v(0)$$

is a path connecting $+1$ and -1. $\square$

The groups $\mathrm{Spin}(n, m)$ are usually not simply connected. Indeed since $\mathrm{SO}_0(n, m)$ has maximal compact subgroup $\mathrm{SO}(n) \times \mathrm{SO}(m)$, the fundamental group is

$$\pi_1(\mathrm{SO}_0(n, m)) = \pi_1(\mathrm{SO}(n)) \times \pi_1(\mathrm{SO}(m)).$$

In particular, if $n, m > 2$ the fundamental group of $\mathrm{SO}(n, m)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$, and hence that of its double cover $\mathrm{Spin}(n, m)$ is $\mathbb{Z}_2$. The spin group is simply connected only in the cases $n > 2$ and $m = 0, 1$, or $n = 0, 1$ and $m > 2$, and only in those cases is $\mathrm{Spin}(n, m)$ the universal cover of $\mathrm{SO}_0(n, m)$. Of particular interest is the case $m = 0$, where $\mathrm{Spin}(n)$ defines the universal cover of $\mathrm{SO}(n)$ for $n > 2$. In low dimensions, one has the exceptional isomorphisms

$$\begin{aligned}
\mathrm{Spin}(2) &= \mathrm{SO}(2),\\
\mathrm{Spin}(3) &= \mathrm{SU}(2),\\
\mathrm{Spin}(4) &= \mathrm{SU}(2) \times \mathrm{SU}(2),\\
\mathrm{Spin}(5) &= \mathrm{Sp}(2),\\
\mathrm{Spin}(6) &= \mathrm{SU}(4).
\end{aligned}$$

Here $\mathrm{Sp}(n)$ is the compact symplectic group, i.e., the group of norm-preserving automorphisms of the n-dimensional quaternionic vector space $\mathbb{H}^n$. The isomorphisms for $\mathrm{Spin}(3)$, $\mathrm{Spin}(4)$ follow from Proposition 1.11, while the isomorphisms for $\mathrm{Spin}(5)$, $\mathrm{Spin}(6)$ are obtained from a discussion of the spin representation of these groups; see Section 3.7.6 below. For $n \geq 7$, there are no further accidental isomorphisms of this type.

Let us now turn to the case $\mathbb{K} = \mathbb{C}$, so that $V \cong \mathbb{C}^n$ with the standard bilinear form. We write $\mathrm{Pin}(n, \mathbb{C}) = \mathrm{Pin}(\mathbb{C}^n)$ and $\mathrm{Spin}(n, \mathbb{C}) = \mathrm{Spin}(\mathbb{C}^n)$.

Proposition 3.1 *The Lie groups* $\mathrm{Pin}(n, \mathbb{C})$ *and* $\mathrm{Spin}(n, \mathbb{C})$ *are double covers of* $\mathrm{O}(n, \mathbb{C})$ *and* $\mathrm{SO}(n, \mathbb{C})$. *Furthermore,* $\mathrm{Spin}(n, \mathbb{C})$ *is connected and simply connected, i.e., it is the universal cover of* $\mathrm{SO}(n, \mathbb{C})$. *The group* $\mathrm{Spin}(n)$ *is the maximal compact subgroup of* $\mathrm{Spin}(n, \mathbb{C})$.

Proof The first part is clear, since for $x \in \Gamma(\mathbb{C}^n)$ the condition $N(\lambda x) = 1$ determines λ up to a sign. The second part follows by the same argument as in the real case, or alternatively by observing that ± 1 are in the same component of $\mathrm{Spin}(n, \mathbb{R}) \subseteq \mathrm{Spin}(n, \mathbb{C})$. Finally, since $\mathrm{SO}(n)$ is the maximal compact subgroup of $\mathrm{SO}(n, \mathbb{C})$, its pre-image $\mathrm{Spin}(n)$ is the maximal compact subgroup of $\mathrm{Spin}(n, \mathbb{C})$. $\square$

Suppose V is a vector over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, with non-degenerate symmetric bilinear form B. Since $\mathrm{Spin}(V)$ is a double cover of the identity component of $\mathrm{SO}(V)$, its Lie algebra is $\mathfrak{o}(V)$. The following result realizes the exponential map for $\mathrm{Spin}(V)$ directly in terms of the Clifford algebra. View $\mathrm{Cl}^{\bar{0}}(V)$ as a Lie algebra under Clifford commutation, where the corresponding Lie group is $\mathrm{Cl}^{\bar{0}}(V)^\times$. The exponential map $\exp\colon \mathrm{Cl}^{\bar{0}}(V) \to \mathrm{Cl}^{\bar{0}}(V)^\times$ for this Lie group is given by the usual power series. Recall the Lie algebra homomorphism $\gamma\colon \mathfrak{o}(V) \to \mathrm{Cl}^{\bar{0}}(V)$ from Section 2.2.10.

Proposition 3.2 *The following diagram commutes*:

$$\begin{tikzcd}
\mathrm{Spin}(V) \arrow[r] & \mathrm{Cl}^{\bar{0}}(V)^\times \\
\mathfrak{o}(V) \arrow[u, "\exp"] \arrow[r, "\gamma"'] & \mathrm{Cl}^{\bar{0}}(V). \arrow[u, "\exp"']
\end{tikzcd}$$

Proof For $A \in \mathfrak{o}(V)$ we have $A(v) = [\gamma(A), v]$ for $v \in V$, and accordingly

$$\exp(A)(v) = \exp(\mathrm{ad}(\gamma(A))v.$$

Using the identity $\exp(a)b\exp(-a) = \exp(\mathrm{ad}(a))b$ for elements a, b in a finite-dimensional (ordinary) algebra, we obtain

$$\exp(A)(v) = \mathrm{e}^{\gamma(A)} v \mathrm{e}^{-\gamma(A)}.$$

Since the left-hand side lies in V, this shows that $\mathrm{e}^{\gamma(A)} \in S\Gamma(V)$, by definition of the Clifford group. Furthermore, since $\gamma(A)^\top = -\gamma(A)$ we have

$$(\mathrm{e}^{\gamma(A)})^\top = \mathrm{e}^{\gamma(A)^\top} = \mathrm{e}^{-\gamma(A)},$$

and therefore $N(\mathrm{e}^{\gamma(A)}) = 1$. That is,

$$\mathrm{e}^{\gamma(A)} \in \mathrm{Spin}(V).$$

This shows that the group $\mathrm{Spin}(V) \subseteq \mathrm{Cl}(V)^\times$ has Lie algebra $\gamma(\mathfrak{o}(V)) \subseteq \mathrm{Cl}^{\bar{0}}(V)$. □

Example 3.3 Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $V = \mathbb{K}^2$ with the standard bilinear form. Consider the element $A \in \mathfrak{o}(V)$ defined by $\lambda(A) = e_1 \wedge e_2$. Then $\gamma(A) = e_1 e_2$. Since $(e_1 e_2)^2 = -1$, the 1-parameter group of elements

$$x(\theta) = \exp(\theta/2 e_1 e_2) \in \mathrm{Spin}(V)$$

is given by the formula,

$$x(\theta) = \cos(\theta/2) + \sin(\theta/2) e_1 e_2.$$

To find its action $A_{x(\theta)}$ on V, we compute

$$\begin{aligned} x(\theta) e_1 x(-\theta) &= (\cos(\theta/2) + \sin(\theta/2) e_1 e_2) e_1 (\cos(\theta/2) - \sin(\theta/2) e_1 e_2) \\ &= (\cos(\theta/2) e_1 - \sin(\theta/2) e_2)(\cos(\theta/2) - \sin(\theta/2) e_1 e_2) \\ &= (\cos^2(\theta/2) - \sin^2(\theta/2)) e_1 - 2 \sin(\theta/2)\cos(\theta/2) e_2 \\ &= \cos(\theta) e_1 - \sin(\theta) e_2. \end{aligned}$$

This verifies that $A_{x(\theta)}$ is given as rotations by θ. We see explicitly that $A_{x(\theta+2\pi)} = A_{x(\theta)}$ while $x(\theta + 2\pi) = -x(\theta)$.

3.2 Clifford modules

3.2.1 Basic constructions

Let V be a vector space with symmetric bilinear form B, and $\mathrm{Cl}(V)$ the corresponding Clifford algebra. A module over the super algebra $\mathrm{Cl}(V)$ is called a *Clifford module*, or simply a $\mathrm{Cl}(V)$-module. That is, a Clifford module is a finite-dimensional super vector space E together with a morphism of super algebras,

$$\rho_E : \mathrm{Cl}(V) \to \mathrm{End}(E).$$

Equivalently, a Clifford module is given by a linear map $\rho_E : V \to \mathrm{End}^{\bar{1}}(E)$ such that

$$\rho_E(v)\rho_E(w) + \rho_E(w)\rho_E(v) = 2\,B(v, w)1$$

for all $v, w \in V$. A morphism of Clifford modules E, E' is a morphism of super vector spaces $f : E \to E'$ intertwining the Clifford actions.

Remark 3.3 If E has a filtration, compatible with the $\mathbb{Z}_2$-grading in the sense of Appendix A and such that the Clifford action is filtration-preserving, then we call E a *filtered* Clifford module. To construct a compatible filtration on a given Clifford module, choose any subspace $E' \subseteq E$ of definite parity, and pick $l \in \mathbb{Z}$, even or odd depending on the parity of E'. Then put $E^{(l+m)} = \mathrm{Cl}(V)^{(m)} E'$ for $m \in \mathbb{Z}$.

Remark 3.4 One can also consider modules over $\mathrm{Cl}(V)$, viewed as an *ordinary* (rather than super) algebra. These will be referred to as *ungraded Clifford modules*.

There are several standard constructions with Clifford modules:

1. *Submodules, quotient modules*. A submodule of a $\mathrm{Cl}(V)$-module E is a super subspace E_1 which is stable under the module action. In this case the quotient E/E_1 becomes a $\mathrm{Cl}(V)$-module in an obvious way. A $\mathrm{Cl}(V)$-module E is called *irreducible* if there are no submodules other than E and $\{0\}$.
2. *Direct sum*. The direct sum of two $\mathrm{Cl}(V)$-modules E_1, E_2 is again a $\mathrm{Cl}(V)$-module, with $\rho_{E_1 \oplus E_2} = \rho_{E_1} \oplus \rho_{E_2}$.
3. *Dual modules*. If E is any Clifford module, then the dual space $E^* = \mathrm{Hom}(E, \mathbb{K})$ becomes a Clifford module, with module structure defined in terms of the canonical anti-automorphism of $\mathrm{Cl}(V)$ by
$$\rho_{E^*}(x) = \rho_E(x^\top)^*,\ x \in \mathrm{Cl}(V).$$
That is, $\langle \rho_{E^*}(x)\psi, \beta\rangle = \langle \psi, \rho_E(x^\top)\beta\rangle$ for $\psi \in E^*$ and $\beta \in E$. If E is a filtered $\mathrm{Cl}(V)$-module, then E^* with the dual filtration (see Appendix A) is again a filtered $\mathrm{Cl}(V)$-module.
4. *Tensor products*. Suppose V_1, V_2 are vector spaces with symmetric bilinear forms B_1, B_2. If E_1 is a $\mathrm{Cl}(V_1)$-module and E_2 is a $\mathrm{Cl}(V_2)$-module, the tensor product $E_1 \otimes E_2$ is a module over $\mathrm{Cl}(V_1) \otimes \mathrm{Cl}(V_2) = \mathrm{Cl}(V_1 \oplus V_2)$, with
$$\rho_{E_1 \otimes E_2}(x_1 \otimes x_2) = \rho_{E_1}(x_1) \otimes \rho_{E_2}(x_2).$$
In particular, $\mathrm{Cl}(V)$-modules E can be tensored with super vector spaces, viewed as modules over the Clifford algebra for the trivial vector space $\{0\}$.
5. *Opposite grading*. If E is any $\mathrm{Cl}(V)$-module, then the same space E with the opposite $\mathbb{Z}_2$-grading is again a $\mathrm{Cl}(V)$-module, denoted by E^{op}.

Given a $\mathrm{Cl}(V)$-module E, one obtains a group representation of the Clifford group $\Gamma(V)$ by restriction, and $S\Gamma(V)$ acquires two representations $E^{\bar{0}}, E^{\bar{1}}$.

The first example of a Clifford module is the Clifford algebra $\mathrm{Cl}(V)$ itself where the module structure is given by multiplication from the left. The exterior algebra

$\wedge(V)$ is a Clifford module, with the action given on generators by (2.10). The symbol map $\sigma:\ \mathrm{Cl}(V)\cong\wedge(V)$ from Section 2.2.5 is characterized as the unique isomorphism of Clifford modules taking $1\in\mathrm{Cl}(V)$ to $1\in\wedge(V)$. The Clifford module $\mathrm{Cl}(V)\cong\wedge(V)$ is self-dual:

Proposition 3.3 *The* $\mathrm{Cl}(V)$*-module* $E=\mathrm{Cl}(V)$ (*with action by left-multiplication*) *is canonically isomorphic to its dual.*

Proof The map

$$\mathrm{Cl}(V)\to\mathrm{Cl}(V)^*,\ \ y\mapsto\phi_y,$$

where $\langle\phi_y,z\rangle=\mathrm{tr}(y^\top z)$, is a linear isomorphism of super spaces. For $x\in\mathrm{Cl}(V)$ we have

$$\langle\phi_{xy},\ z\rangle=\mathrm{tr}(y^\top x^\top z)=\langle\phi_y,\ x^\top z\rangle=\langle x.\phi_y,z\rangle;$$

hence ϕ is $\mathrm{Cl}(V)$-equivariant. □

Note however that the isomorphism described above does *not* preserve filtrations.

3.2.2 The spinor module S_F

Let V be a vector space of dimension $n=2m$, equipped with a split bilinear form B. View $\mathrm{Cl}(V)$ as a Clifford module under multiplication from the left, and let $F\subseteq V$ be a Lagrangian subspace. Then the left-ideal $\mathrm{Cl}(V)F$ is a submodule of $\mathrm{Cl}(V)$. The *spinor module associated to* F is the quotient $\mathrm{Cl}(V)$-module,

$$\mathsf{S}_F=\mathrm{Cl}(V)/\mathrm{Cl}(V)F.$$

The spinor module S_F may also be viewed as an *induced module*. View $\wedge(F)=\mathrm{Cl}(F)$ as a subalgebra of $\mathrm{Cl}(V)$, by the natural homomorphism extending the inclusion $F\subseteq V$.

Proposition 3.4 *Let* $\mathbb{K}$ *be the trivial* $\wedge(F)$*-module, that is,*

$$\phi.t=\phi_{[0]}t,\quad\ \phi\in\wedge(F),\ t\in\mathbb{K}.$$

Then S_F *is the corresponding induced module*

$$\mathsf{S}_F=\mathrm{Cl}(V)\otimes_{\wedge(F)}\mathbb{K}.$$

Proof By definition of the tensor product over $\wedge(F)$, the right-hand side is the quotient of $\mathrm{Cl}(V)\otimes\mathbb{K}$ by the subspace generated by all $x\otimes\phi.t-x\phi\otimes t$. But this is the same as the subspace of $\mathrm{Cl}(V)$ by the subspace generated by all $x(\phi-\phi_{[0]})$ for $\phi\in\wedge(F)$, which is exactly $\mathrm{Cl}(V)F$. □

Proposition 3.5 *The choice of a Lagrangian complement $F' \cong F^*$ to F identifies*

$$\mathsf{S}_F = \wedge(F^*),$$

where the Clifford action is given on generators by $\rho(\mu, v) = \varepsilon(\mu) + \iota(v)$ for $v \in F$ and $\mu \in F^*$.

Proof The choice of F' identifies $V = F^* \oplus F$, with the bilinear form

$$B((\mu_1, v_1), (\mu_2, v_2)) = \frac{1}{2}(\langle \mu_1, v_2 \rangle + \langle \mu_2, v_1 \rangle).$$

Both $\wedge(F)$ and $\wedge(F^*)$ are embedded as subalgebras of $\mathrm{Cl}(V)$, and the multiplication map defines a homomorphism of filtered super vector spaces

$$\wedge(F^*) \otimes \wedge(F) \to \mathrm{Cl}(F^* \oplus F). \tag{3.4}$$

The associated graded map is the isomorphism

$$\wedge(F^*) \otimes \wedge(F) \to \wedge(F^* \oplus F)$$

given by wedge product. Hence (3.4) is a linear isomorphism. Under this identification, $\mathrm{Cl}(V)F = \wedge(F^*) \otimes \bigoplus_{k \geq 1} \wedge^k(F)$, which has a natural complement $\wedge(F^*)$. Consequently

$$\mathsf{S}_F = \mathrm{Cl}(V)/\mathrm{Cl}(V)F \cong \wedge(F^*).$$

The Clifford action of $(\mu, v) \in F^* \oplus F$ on any $\psi \in \wedge(F^*)$ is given by Clifford multiplication by $\mu + v$ from the left, followed by projection to $\wedge(F^*)$ along $\mathrm{Cl}(V)F$. Since

$$(\mu + v)\psi = \mu\psi + [v, \psi] + (-1)^{|\psi|}\psi v = (\mu \wedge \psi + \iota(v)\psi) + (-1)^{|\psi|}\psi v,$$

and $\psi v \in \mathrm{Cl}(V)F$, this confirms our description of the action on $\wedge(F^*)$. □

The spinor module S_F has a canonical filtration, compatible with the $\mathbb{Z}_2$-grading, given as the quotient of the filtration of the Clifford algebra:

$$\mathsf{S}_F^{(k)} = \mathrm{Cl}(V)^{(k)}/\mathrm{Cl}(V)^{(k-1)}F.$$

Proposition 3.6

1. *The associated graded space for the filtration on the spinor module is*

$$\mathrm{gr}(\mathsf{S}_F) = \wedge(F^*).$$

2. *For $v \in V$, the operator $\rho(v)$ on S_F has filtration degree 1, and the associated graded operator $\mathrm{gr}^1(\rho(v))$ on $\mathrm{gr}(\mathsf{S}_F) = \wedge(F^*)$ is the wedge product with the image of v in $F^* = V/F$.*
3. *For $v \in F \subseteq V$, the operator $\rho(v)$ has filtration degree -1, and the associated graded operator is contraction: $\mathrm{gr}^{-1}(\rho(v)) = \iota(v)$. That is, for all $\phi \in \mathsf{S}_F^{(k)}$ the leading term of $\rho(v)\phi \in \mathsf{S}_F^{(k-1)}$ is*

$$\mathrm{gr}^{k-1}(\rho(v)\phi) = \iota(v)\mathrm{gr}^k(\phi).$$

Proof Since $\mathrm{gr}(\mathrm{Cl}(V)) = \wedge(V)$, the associated graded space is

$$\mathrm{gr}(\mathsf{S}_F) = \wedge(V)/\wedge(V)F = \wedge(V/F) \cong \wedge(F^*).$$

Choose a Lagrangian complement $F' \cong F^*$ to F to identify $\mathsf{S}_F = \wedge(F^*)$. Then $\mathsf{S}_F^{(k)} = \bigoplus_{i \le k} \wedge^i F^*$, and the remaining claims are immediate from Proposition 3.5. □

3.2.3 The dual spinor module S^F

We define a *dual spinor module associated to F* as

$$\mathsf{S}^F = \mathrm{Cl}(V)\det(F),$$

where $\det(F) = \wedge^m(F)$ is the determinant line.

Proposition 3.7 *The choice of a Lagrangian complement $F' \cong F^*$ to F identifies*

$$\mathsf{S}^F = \wedge(F),$$

where the Clifford action is given on generators by $\rho(\mu, v) = \iota(\mu) + \varepsilon(v)$.

Proof Following the notation from the proof of Proposition 3.5, we have

$$\mathsf{S}^F = \mathrm{Cl}(V)\det(F) = \wedge(F^*) \otimes \det(F) \cong \wedge(F),$$

where the isomorphism is given by the contraction $\wedge(F^*) \to \mathrm{End}(\wedge(F))$. One readily checks that this identification takes the Clifford action to $\iota(\mu) + \varepsilon(v)$. □

Propositions 3.5 and 3.7 suggest that the spinor modules $\mathsf{S}_F, \mathsf{S}^F$ are in duality. In fact, this duality does not depend on the choice of complement.

Proposition 3.8 *There is a non-degenerate pairing*

$$\mathsf{S}^F \times \mathsf{S}_F \to \mathbb{K}, \quad y \times [x] \mapsto (y, [x]) = \mathrm{tr}(y^\top x)$$

for $y \in \mathsf{S}^F \subseteq \mathrm{Cl}(V)$ and $[x] \in \mathrm{Cl}(V)/\mathrm{Cl}(V)F$ (represented by an element $x \in \mathrm{Cl}(V)$). The pairing satisfies

$$(y,\ \rho(z)[x]) = (\rho(z^T)y, [x]),$$

hence it defines an isomorphism of Clifford modules $\mathsf{S}^F \cong \mathsf{S}_F^*$. Choosing a Lagrangian complement F' to identify $\mathsf{S}_F = \wedge(F^*)$ and $\mathsf{S}^F = \wedge(F)$, the pairing is just the usual pairing between $\wedge(F^*)$ and $\wedge(F)$.

Proof The pairing is well defined, since

$$y \in \mathrm{Cl}(V)\det(F),\ x \in \mathrm{Cl}(V)F \ \Rightarrow\ xy^\top = 0 \Rightarrow\ \mathrm{tr}(y^\top x) = 0.$$

The pairing satisfies

$$(y,\ \rho(z)[x]) = \mathrm{tr}(y^\top zx) = \mathrm{tr}((z^\top y)^\top x) = (\rho(z^T)y, [x]).$$

Choose a Lagrangian complement F' to F, and view S_F as a subspace $\wedge(F^*) \subseteq \mathrm{Cl}(V)$ as in (3.4). The pairing between

$$[x] = \phi \in \mathsf{S}_F, \qquad y = \phi'\chi \in \mathsf{S}^F = \wedge(F^*)\det(F),$$

corresponding to $\psi = \iota(\phi')\chi \in \wedge(F)$, reads

$$(\psi, \phi) = \mathrm{tr}(y^\top x) = \mathrm{tr}(x^\top y) = \sigma(\phi^\top \phi'\chi)_{[0]} = (\iota(\phi^\top)\iota(\phi')\chi)_{[0]} = (\iota(\phi^\top)\psi)_{[0]}$$

which is just the standard pairing between $\wedge(F^*)$ and $\wedge(F)$. □

3.2.4 Irreducibility of the spinor module

We encountered special cases of the following result in our discussion of the Clifford algebras for $\mathbb{K} = \mathbb{C}$. See Proposition 2.4.

Theorem 3.3 *Let V be a vector space with split bilinear form B, and $F \subseteq V$ a Lagrangian subspace. The spinor module S_F is irreducible, and the module map*

$$\rho:\ \mathrm{Cl}(V) \to \mathrm{End}(\mathsf{S}_F)$$

is an isomorphism of super algebras. It restricts to an isomorphism

$$\mathrm{Cl}(V)^{\bar{0}} \to \mathrm{End}(\mathsf{S}_F^{\bar{0}}) \oplus \mathrm{End}(\mathsf{S}_F^{\bar{1}}).$$

Hence both $\mathsf{S}_F^{\bar{0}}, \mathsf{S}_F^{\bar{1}}$ are irreducible modules over $\mathrm{Cl}(V)^{\bar{0}}$. These two modules are non-isomorphic.

Proof We may use the model $V = F^* \oplus F$, $\mathsf{S}_F = \wedge(F^*)$. To prove $\mathrm{Cl}(V) \cong \mathrm{End}(\mathsf{S}_F)$, note that both spaces have the same dimension. Hence it suffices to show that ρ is surjective. That is, we have to show that $\mathrm{End}(\wedge F^*)$ is generated by exterior multiplications and contractions. Suppose first that $\dim F = 1$, and let $e \in F$, $f \in F^*$ be dual generators, so that $B(e, f) = \frac{1}{2}$. Then $\wedge F^*$ has basis $1, f$. In terms of this basis,

$$\varepsilon(f) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \iota(e) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \varepsilon(f)\iota(e) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Together with the identity map of $\wedge F^*$, these form a basis of $\mathrm{End}(\wedge F^*) \cong \mathrm{Mat}_2(\mathbb{K})$, as claimed. The general case follows from the 1-dimensional case, using

$$\mathrm{End}(\wedge(F_1^* \oplus F_2^*)) = \mathrm{End}(\wedge F_1^*) \otimes \mathrm{End}(\wedge F_2^*).$$

This proves $\mathrm{Cl}(V) \cong \mathrm{End}(\mathsf{S}_F)$, which also implies that the spinor module is irreducible. It also yields

$$\mathrm{Cl}(V)^{\bar{0}} \cong \mathrm{End}^{\bar{0}}(\mathsf{S}_F) = \mathrm{End}(\mathsf{S}_F^{\bar{0}}) \oplus \mathrm{End}(\mathsf{S}_F^{\bar{1}}),$$

a direct sum of two irreducible modules. To see that the even and odd part of the spinor module are non-isomorphic modules over $\mathrm{Cl}^{\bar{0}}(V)$, choose bases e_i of F and f^i of F^* such that $B(e_i, f^j) = \frac{1}{2}\delta_i^j$, thus $e_i f^j = \delta_i^j - f^j e_i$. Consider the chirality element (2.12), written in the "normal-ordered" form

$$\Gamma = (1 - 2f^1 e_1) \cdots (1 - 2f^m e_m). \tag{3.5}$$

Since $\rho(1 - 2f^i e_i) f^I = \pm f^I$, with a $-$ sign if $i \in I$ and a minus sign if $i \notin I$, we find that $\rho(\Gamma)$ is the parity operator on S_F: it acts as $+1$ on $\mathsf{S}_F^{\bar{0}}$ and as -1 on $\mathsf{S}_F^{\bar{1}}$. In particular, these two representations are non-isomorphic. $\square$

Remark 3.5 Let $V = F^* \oplus F$ and $\mathsf{S}_F = \wedge(F^*)$, as in the proof above. On the subalgebra $\wedge(F) \subseteq \mathrm{Cl}(V)$, the homomorphism ρ coincides with the extension of the contraction operation $\iota: \ F \to \mathrm{End}(\wedge F^*)$ to an algebra morphism (still denoted by ι). Similarly, on $\wedge(F^*)$ it coincides with the morphism (still denoted by ι). Similarly, on $\wedge(F^*)$ it coincides with the extension of exterior multiplication $\varepsilon: \ F^* \to \mathrm{End}(\wedge F^*)$ to an algebra morphism (still denoted by ε). Using the proposition, we obtain that the linear map

$$\wedge(F^*) \otimes \wedge(F) \to \mathrm{End}(\wedge(F^*)), \ \ \sum_i \phi_i \otimes \psi_i \mapsto \sum_i \varepsilon(\phi_i)\iota(\psi_i^\top) \tag{3.6}$$

is an isomorphism of super vector spaces. The operators on the right-hand side may be thought of as differential operators on the super algebra $\wedge(F^*)$.

3.2.5 Abstract spinor modules

Theorem 3.3 motivates the following definition.

Definition 3.3 Let V be a vector space with split bilinear form. A *spinor* module over $\mathrm{Cl}(V)$ is a Clifford module S for which the Clifford action

$$\rho: \ \mathrm{Cl}(V) \to \mathrm{End}(\mathsf{S})$$

is an isomorphism of super algebras. An *ungraded spinor module* is defined as an ungraded Clifford module such that ρ is an isomorphism of (ordinary) algebras.

We stress that we take Clifford modules, spinor modules, etc., to be $\mathbb{Z}_2$-graded unless specified otherwise. By Theorem 3.3 the standard spinor module S_F is an example of a spinor module, as is its dual S^F.

Theorem 3.4 *Let V be a vector space with split bilinear form.*

1. *There is a unique isomorphism class of ungraded spinor modules over* $\mathrm{Cl}(V)$.
2. *There are exactly two isomorphism classes of spinor modules over* $\mathrm{Cl}(V)$, *represented by* $\mathsf{S}_F, \mathsf{S}_F^{\mathrm{op}}$.

3. *A given ungraded spinor module admits exactly two compatible $\mathbb{Z}_2$-gradings. The corresponding parity operator is given by the Clifford action of the chirality element Γ, normalized (up to sign) by the condition $\Gamma^2 = 1$.*

Proof Theorem 3.3 shows that as an ungraded algebra, $\mathrm{Cl}(V)$ is isomorphic to a matrix algebra. Hence it admits, up to isomorphism, a *unique* ungraded spinor module, proving (1). The chirality element $\Gamma \in \mathrm{Cl}(V)$ (cf. (3.5)) satisfies $v\Gamma = -\Gamma v$ for all $v \in V$. Hence $\rho(v)$ exchanges the ± 1 eigenspaces of $\rho(\Gamma)$, showing that $\rho(\Gamma)$ defines a compatible $\mathbb{Z}_2$-grading. Now suppose $\mathsf{S} = \mathsf{S}^{\bar{0}} \oplus \mathsf{S}^{\bar{1}}$ is any compatible $\mathbb{Z}_2$-grading. Since $\rho(v)$ for $v \neq 0$ exchanges the odd and even summands, they both have dimension $\frac{1}{2}\dim \mathsf{S}$. Hence they are both irreducible under the action of $\mathrm{Cl}(V)^{\bar{0}}$. Since Γ is in the center of $\mathrm{Cl}(V)^{\bar{0}}$, it acts as a scalar on each summand. It follows that $\mathsf{S}^{\bar{0}}$ must be one of the two eigenspaces of $\rho(\Gamma)$, and $\mathsf{S}^{\bar{1}}$ is the other. This proves Part 3. Part 2 is immediate from Part 1 and Part 3. □

The theorem shows that if S, S' are two spinor modules, then the space

$$\mathrm{Hom}_{\mathrm{Cl}(V)}(\mathsf{S}, \mathsf{S}')$$

of intertwining operators is a 1-dimensional super vector space.

Remark 3.6 For $\mathbb{K} = \mathbb{R}$, the choice of $\mathbb{Z}_2$-grading on an ungraded spinor module over S is equivalent to a choice of orientation of V. Indeed, the definition (2.12) of the chirality element $\Gamma \in \mathrm{Spin}(V)$ shows that the choice of sign of Γ is equivalent to a choice of orientation on V.

As a special case, it follows that the spinor modules defined by two Lagrangian subspaces F, F' are isomorphic, possibly up to parity reversal, where the isomorphism is unique up to a scalar. Recall that $\mathrm{O}(V)$ acts transitively on the set $\mathrm{Lag}(V)$ of Lagrangian subspaces of V. Furthermore, the stabilizer group $\mathrm{O}(V)_F$ of any $F \in \mathrm{Lag}(V)$ is contained in $\mathrm{SO}(V)$.

Definition 3.4 We say that $F, F' \in \mathrm{Lag}(V)$ have *equal parity* if they are related by a transformation $g \in \mathrm{SO}(V)$ and *opposite parity* otherwise.

(For $\mathbb{K} = \mathbb{R}, \mathbb{C}$, the relative parity indicates if F, F' are in the same component of $\mathrm{Lag}(V)$.)

Proposition 3.9 *Let $g \in \mathrm{O}(V)$ with $g.F = F'$. Then any lift $x \in \Gamma(V)$ of g determines an isomorphism of Clifford modules $\mathsf{S}_F \to \mathsf{S}_{F'}$. This isomorphism preserves parity if and only if F, F' have the same parity.*

Proof Suppose $x \in \Gamma(V)$ lifts g, so that $A_x = g$. Then $F' = A_x(F) = \Pi(x)Fx^{-1}$ (as subsets of $\mathrm{Cl}(V)$). Hence

$$\mathrm{Cl}(V)Fx^{-1} = \mathrm{Cl}(V)F'.$$

Thus, right-multiplication by x^{-1} on $\mathrm{Cl}(V)$ descends to an isomorphism of Clifford modules $\mathsf{S}_F \to \mathsf{S}_{F'}$. Note that this isomorphism preserves parity if and only if x is even, i.e., $g \in \mathrm{SO}(V)$. □

Given a spinor module S over $\mathrm{Cl}(V)$, one obtains by restriction a group representation of the Clifford group $\Gamma(V)$ and its subgroup $\mathrm{Pin}(V)$. This is called the *spin representation* of $\Gamma(V)$. The action of $S\Gamma(V)$ preserves the splitting $\mathsf{S} = \mathsf{S}^{\bar{0}} \oplus \mathsf{S}^{\bar{1}}$; the two summands are called the *half-spin representations* of the special Clifford group $S\Gamma(V)$ and of its subgroup $\mathrm{Spin}(V)$.

Theorem 3.5 *The spin representation of $\Gamma(V)$ on S is irreducible. Similarly, each of the half-spin representations $\mathsf{S}^{\bar{0}}$ and $\mathsf{S}^{\bar{1}}$ is an irreducible representation of $S\Gamma(V)$. The two half-spin representations of $S\Gamma(V)$ are non-isomorphic. If $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ we can replace $\Gamma(V)$ with* $\mathrm{Pin}(V)$ *and $S\Gamma(V)$ with* $\mathrm{Spin}(V)$.

Proof If a subspace of S is invariant under the action of $\Gamma(V)$, then it is also invariant under the subalgebra generated by $\Gamma(V)$. But $\Gamma(V)$ contains in particular all non-isotropic vectors, and linear combinations of non-isotropic vectors span all of V and hence generate all of $\mathrm{Cl}(V)$. Hence the subalgebra generated by $\Gamma(V)$ is all of $\mathrm{Cl}(V)$, and the irreducibility under $\Gamma(V)$ follows from that under $\mathrm{Cl}(V)$. Similarly, the subalgebra generated by $S\Gamma(V)$ equals $\mathrm{Cl}^{\bar{0}}(V)$, and the irreducibility of the half-spin representations under $S\Gamma(V)$ follows from that under $\mathrm{Cl}^{\bar{0}}(V)$. □

3.3 Pure spinors

Let $\rho : \mathrm{Cl}(V) \to \mathrm{End}(\mathsf{S})$ be a spinor module. If $\phi \in \mathsf{S}$ is a non-zero spinor, we can consider the space of vectors in V which annihilate ϕ under the Clifford action:

$$F(\phi) = \{v \in V \mid \rho(v)\phi = 0\}.$$

Lemma 3.1 *For all non-zero spinors $\phi \in \mathsf{S}$, the space $F(\phi)$ is an isotropic subspace of V.*

Proof If $v_1, v_2 \in F(\phi)$ we have

$$0 = (\rho(v_1)\rho(v_2) + \rho(v_2)\rho(v_1))\phi = 2B(v_1, v_2)\phi,$$

hence $B(v_1, v_2) = 0$. □

Definition 3.5 A non-zero spinor $\phi \in \mathsf{S}$ is called *pure* if the subspace $F(\phi)$ is Lagrangian.

Consider, for instance, the standard spinor module $\mathsf{S}_F = \mathrm{Cl}(V)/\mathrm{Cl}(V)F$ defined by a Lagrangian subspace F. Let $\phi_0 \in \mathsf{S}_F$ be the image of $1 \in \mathrm{Cl}(V)$. Then ϕ_0 is a pure spinor, with $F(\phi_0) = F$.

Theorem 3.6 *The representation of $\Gamma(V)$ on a spinor module* S *restricts to a transitive action on the set of pure spinors. The map*

$$\left\{ \begin{array}{c} \textit{pure} \\ \textit{spinors} \end{array} \right\} \to \mathrm{Lag}(V), \ \phi \mapsto F(\phi) \tag{3.7}$$

is a $\Gamma(V)$-equivariant surjection, with fibers $\mathbb{K}^\times$. That is, if $F(\phi) = F(\phi')$, then ϕ, ϕ' coincide up to a non-zero scalar. All pure spinors ϕ are either even or odd. The relative parity of pure spinors ϕ, ϕ' is equal to the relative parity of the Lagrangian subspaces $F(\phi)$, $F(\phi')$.

Proof For any $x \in \Gamma(V)$, mapping to $g \in \mathrm{O}(V)$,

$$F(\rho(x)\phi) = xF(\phi)x^{-1} = \Pi(x)F(\phi)x^{-1} = A_x.F(\phi) = g.F(\phi).$$

It follows that for any pure spinor ϕ, the element $\rho(x)\phi$ is again a pure spinor.

To prove the remaining claims, we work with the standard spinor module S_F defined by a fixed Lagrangian subspace F. Let $\phi_0 \in \mathsf{S}_F$ be the image of $1 \in \mathrm{Cl}(V)$, so that $F(\phi_0) = F$. Suppose ϕ is a pure spinor with $F(\phi) = F$, and consider the standard filtration on S_F. By Proposition 3.6, $\rho(v)$ for $v \in F$ has filtration degree -1, and $\mathrm{gr}^{-1}(\rho(v))$ is the operator of contraction by $\mathrm{gr}(\mathsf{S}_F) = \wedge(F^*)$ given by contraction with v. Since $\bigcap_{v\in V} \ker(\iota(v)) = \wedge^0(F^*) = \mathbb{K}$, we conclude that $\bigcap_{v\in V} \ker(\rho(v)) = \mathsf{S}_F^{(0)} = \mathbb{K}\phi_0$. That is, $F \subseteq F(\phi)$ if and only if ϕ is a scalar multiple of ϕ_0.

Consider now a general pure spinor ϕ. Pick $g \in \mathrm{O}(V)$ with $g.F(\phi) = F$, and choose a lift $x \in \Gamma(V)$ of g. Then $F(\rho(x)\phi) = g.F(\phi) = F$, so that $\rho(x)\phi$ is a scalar multiple of ϕ_0. Since $\mathbb{K}^\times \subseteq \Gamma(V)$, this shows that $\Gamma(V)$ acts transitively on the set of pure spinors. The last statement follows since any element of the Clifford group is either even or odd; thus $\rho(x)^{-1}\phi_0$ is even or odd depending on the parity of x. $\square$

The following proposition shows how the choice of a pure spinor identifies S with a spinor module of the form S_F.

Proposition 3.10 *Let* S *be a spinor module over* $\mathrm{Cl}(V)$.

(i) *For any pure spinor $\phi \in \mathsf{S}$, one has*

$$\{x \in \mathrm{Cl}(V) |\ \rho(x)\phi = 0\} = \mathrm{Cl}(V)F(\phi).$$

Hence, there is a unique isomorphism of spinor modules $\mathsf{S} \to \mathsf{S}_{F(\phi)}$ taking ϕ to the image of 1 *in* $\mathrm{Cl}(V)/\mathrm{Cl}(V)F(\phi)$. *This identification preserves or reverses the $\mathbb{Z}_2$-grading depending on the parity of ϕ.*

(ii) *Suppose $F \subseteq V$ is a Lagrangian subspace. Then the pure spinors defining F are exactly the non-zero elements of the* pure spinor line

$$l_F = \{\phi \in \mathsf{S} |\ \rho(v)\phi = 0 \ \ \forall v \in F\}.$$

There is a canonical isomorphism,

$$l_F \cong \mathrm{Hom}_{\mathrm{Cl}(V)}(\mathsf{S}_F, \mathsf{S}).$$

(iii) *If* S' *is another spinor module, and* l'_F *the pure spinor line for* F, *then*

$$\mathrm{Hom}_{\mathrm{Cl}(V)}(\mathsf{S}, \mathsf{S}') = \mathrm{Hom}_{\mathbb{K}}(\mathsf{l}_F, \mathsf{l}'_F)$$

canonically.

Proof (i) The left ideal $\mathrm{Cl}(V)F(\phi)$ annihilates ϕ, defining a non-zero $\mathrm{Cl}(V)$-equivariant homomorphism $\mathsf{S}_{F(\phi)} = \mathrm{Cl}(V)/\mathrm{Cl}(V)F(\phi) \to \mathsf{S}$, $[x] \mapsto \rho(x)\phi$. Since $\mathrm{Hom}_{\mathrm{Cl}(V)}(\mathsf{S}_{F(\phi)}, \mathsf{S})$ is 1-dimensional, this map is an isomorphism. In particular, $\rho(x)\phi = 0$ if and only if $[x] = 0$, i.e., $x \in \mathrm{Cl}(V)F(\phi)$. (ii) By definition, l_F consists of spinors ϕ with $F \subseteq F(\phi)$. If ϕ is non-zero, this must be an equality since $F(\phi)$ is isotropic. This shows that the non-zero elements of l_F are precisely the pure spinors defining F, and (using Theorem 3.6) that $\dim l_F = 1$. The isomorphism $l_F \cong \mathrm{Hom}_{\mathrm{Cl}(V)}(\mathsf{S}_F, \mathsf{S})$ is defined by the map taking $\phi \in l_F$ to the homomorphism $\mathsf{S}_F = \mathrm{Cl}(V)/\mathrm{Cl}(V)F \to \mathsf{S}$, $[x] \mapsto \rho(x)\phi$. (iii) A $\mathrm{Cl}(V)$-equivariant isomorphism $\mathsf{S} \to \mathsf{S}'$ must restrict to an isomorphism of the pure spinor lines for any Lagrangian subspace F. This defines a non-zero map $\mathrm{Hom}_{\mathrm{Cl}(V)}(\mathsf{S}, \mathsf{S}') \to \mathrm{Hom}_{\mathbb{K}}(\mathsf{l}_F, \mathsf{l}'_F)$. Since both sides are 1-dimensional, it is an isomorphism. □

Having established these general results, we proceed to give an explicit description of all pure spinors for the standard spinor module S_F, using a Lagrangian complement to F to identify $V = F^* \oplus F$ and $\mathsf{S}_F = \wedge(F^*)$.

Proposition 3.11 *Let* $V = F^* \oplus F$. *Then any triple* (N, χ, ω_N) *consisting of a subspace* $N \subseteq F$, *a volume form* $\kappa \in \det(\mathrm{ann}(N))^\times$ *on* V/N, *and a 2-form* $\omega_N \in \wedge^2 N^*$ *on* N, *defines a pure spinor*

$$\phi = \exp(-\tilde{\omega}_N)\kappa \in \wedge(F^*).$$

Here $\tilde{\omega}_N \in \wedge^2 F^*$ *is an arbitrary extension of* ω_N *to a 2-form on* F. (*Note that* ϕ *does not depend on the choice of this extension.*) *The corresponding Lagrangian subspace is*

$$F(\phi) = \{(\mu, v) \in F^* \oplus F \,|\, v \in N,\ \forall w \in N :\ \langle \mu, w\rangle = \omega_N(v, w)\}.$$

Every pure spinor in S_F *arises in this way from a unique triple* (N, κ, ω_N).

Proof We first observe that Lagrangian subspaces $L \subseteq F^* \oplus F$ are in bijective correspondence with pairs (N, ω_N). Indeed, any such pair defines a subspace of dimension $\dim F$,

$$L = \{(\mu, v) \in F^* \oplus F \,|\, v \in N,\ \mu|_N = \omega_N(v, \cdot)\}.$$

If (μ, v), $(\mu', v') \in L$, then

$$\langle \mu, v'\rangle + \langle \mu', v\rangle = \omega_N(v, v') + \omega_N(v', v) = 0,$$

hence L is Lagrangian. Conversely, given $L \subseteq F^* \oplus F$ let $N \subseteq F$ be its projection, and define ω_N by

$$\omega_N(v, v') = \langle \mu, v'\rangle = -\langle \mu', v\rangle,$$

where (μ, v), $(\mu', v') \in L$ are pre-images of v, v'.

Suppose now that (N, ω_N, κ) are given and define ϕ as above. It is straightforward to check that elements (μ, v) with $v \in N$ and $\omega|_N = \omega_N(v, \cdot)$ annihilate ϕ. Hence $F(\phi)$ contains all such elements, and equality follows for dimension reasons. Conversely, suppose ϕ is a pure spinor. Let $N \subseteq F$ be the projection of $F(\phi) \subseteq F^* \oplus F$ to F. Then $\mathrm{ann}(N) \subseteq F(\phi)$. Pick $\kappa \in \det(\mathrm{ann}(N))^\times$. For $v, w \in N$, let $\mu, \nu \in F^*$ such that $(\mu, v), (\nu, w) \in F(\phi)$. Since $F(\phi)$ is isotropic, we have $\langle \mu, w \rangle + \langle \nu, v \rangle = 0$. Hence

$$\omega_N(v, w) = \langle \mu, w \rangle$$

is a well-defined skew-symmetric 2-form on N. Let $\tilde{\omega}_N$ be an arbitrary extension to a 2-form on V. Then $F(\phi)$ has the description given in the proposition, and hence coincides with $F(\mathrm{e}^{-\tilde{\omega}_N} \kappa)$. It follows that ϕ is a non-zero scalar multiple of $\mathrm{e}^{-\tilde{\omega}_N} \kappa$, where the scalar can be absorbed in the choice of κ. □

In low dimensions it is easy to be pure:

Proposition 3.12 *Suppose V is a vector space with split bilinear form, with* $\dim V \leq 6$, *and* S *a spinor module. Then all non-zero even or odd elements in* S *are pure spinors.*

Proof Consider the case $\dim V = 6$ (the case $\dim V < 6$ is even easier). We may use the model $V = F^* \oplus F$, $\mathsf{S} = \wedge(F^*)$. Suppose $\phi = \phi_{[0]} + \phi_{[2]} \in \mathsf{S}^{\bar{0}}$ is non-zero. Put $t = \phi_{[0]}$. If $t \neq 0$, we have $\phi = t \exp(\phi_{[2]}/t)$, which is a pure spinor by Proposition 3.11. If $\phi_{[0]} = 0$, then $\chi := \phi_{[2]}$ is a non-zero element of $\wedge^2 F^*$. Since $\dim F = 3$, it has a 1-dimensional kernel $N \subseteq F$, with χ a generator of $\det(\mathrm{ann}(N))$. But this again is a pure spinor by Proposition 3.11.

For non-zero odd spinors $\phi \in \mathsf{S}^{\bar{1}}$, choose a non-isotropic $v \in V$ with $\rho(v)\phi \neq 0$. Since $\phi' = \rho(v)\phi \in \mathsf{S}^{\bar{0}}$ is pure, the same is true of $\phi = B(v, v)^{-1} \rho(v) \phi'$. □

3.4 The canonical bilinear pairing on spinors

Given a spinor module S, the dual S^* is again a spinor module. The 1-dimensional super vector space

$$K_{\mathsf{S}} := \mathrm{Hom}_{\mathrm{Cl}(V)}(\mathsf{S}^*, \mathsf{S})$$

is called the *canonical line* for the spinor module. Its parity is even or odd depending on the parity of $\frac{1}{2} \dim V$. The evaluation map defines an isomorphism of Clifford modules,

$$\mathsf{S}^* \otimes K_{\mathsf{S}} \to \mathsf{S}.$$

Note also that $K_{\mathsf{S}^*} = K_{\mathsf{S}}^*$.

By Proposition 3.10, if $F \subseteq V$ is a Lagrangian subspace and $\mathsf{l}_{\mathsf{S}^*, F}$, $\mathsf{l}_{\mathsf{S}, F}$ are the corresponding pure spinor lines, we have

$$K_{\mathsf{S}} = \mathrm{Hom}_{\mathbb{K}}(\mathsf{l}_{\mathsf{S}^*, F}, \mathsf{l}_{\mathsf{S}, F}) = \mathsf{l}_{\mathsf{S}, F} \otimes (\mathsf{l}_{\mathsf{S}^*, F})^*.$$

Example 3.4 Let $F \subseteq V$ be a Lagrangian subspace. We saw above that the dual of $\mathsf{S}_F = \mathrm{Cl}(V)/\mathrm{Cl}(V)F$ is canonically isomorphic to $\mathsf{S}^F = \mathrm{Cl}(V)\det(F)$. The pure spinor lines l_F for these two spinor modules are

$$\mathsf{l}_{\mathsf{S}^F,F} = \det(F), \qquad \mathsf{l}_{\mathsf{S}_F,F} = \mathbb{K}.$$

Hence

$$K_{\mathsf{S}^F} = \det(F), \quad K_{\mathsf{S}_F} = \det(F^*).$$

In terms of the identifications $\mathsf{S}_F = \wedge(F^*)$, $\mathsf{S}^F = \wedge(F)$ given by the choice of a complementary Lagrangian subspace, the isomorphism

$$K_{\mathsf{S}^F} = \mathrm{Hom}_{\mathrm{Cl}(V)}(\wedge(F), \wedge(F^*)) = \det(F^*)$$

is given by the contraction $\wedge(F) \otimes \det(F^*) \to \wedge(F^*)$. Indeed, given a generator of $\det(F^*)$, the resulting map $\wedge(F) \to \wedge(F^*)$ intertwines $\iota(\mu), \varepsilon(v)$ with $\varepsilon(\mu), \iota(v)$.

Definition 3.6 The canonical bilinear pairing

$$(\cdot,\cdot)_{\mathsf{S}}: \quad \mathsf{S} \otimes \mathsf{S} \to K_{\mathsf{S}}, \quad \phi \otimes \psi \mapsto (\phi, \psi)_{\mathsf{S}}$$

is the isomorphism $\mathsf{S} \otimes \mathsf{S} \to \mathsf{S}^* \otimes \mathsf{S} \otimes K_{\mathsf{S}}$, followed by the duality pairing $\mathsf{S}^* \otimes \mathsf{S} \to \mathbb{K}$.

The pairing $(\cdot,\cdot)_{\mathsf{S}}$ satisfies

$$(\rho(x)\phi, \psi)_{\mathsf{S}} = (\phi, \rho(x^\top)\psi)_{\mathsf{S}}, \quad x \in \mathrm{Cl}(V), \tag{3.8}$$

by a similar property of the pairing between S^* and S (defining the Clifford action on S^*). Restricting to the Clifford group and replacing ψ with $\rho(x)\psi$ it follows that

$$(\rho(x)\phi,\ \rho(x)\psi)_{\mathsf{S}} = \mathsf{N}(x)\ (\phi, \psi)_{\mathsf{S}}, \quad x \in \Gamma(V), \tag{3.9}$$

where $\mathsf{N}: \ \Gamma(V) \to \mathbb{K}^\times$ is the norm homomorphism (3.3). The bilinear form is uniquely determined, up to a non-zero scalar, by its invariance property:

Proposition 3.13 *Let* S *be a spinor module over* $\mathrm{Cl}(V)$, *and* $(\cdot,\cdot): \ \mathsf{S} \times \mathsf{S} \to \mathbb{K}$ *is a bilinear pairing with the property*

$$(\rho(x)\phi,\ \rho(x)\psi)_{\mathsf{S}} = \mathsf{N}(x)\ (\phi, \psi)_{\mathsf{S}}, \quad x \in \Gamma(V),$$

where $\rho: \ \mathrm{Cl}(V) \to \mathrm{End}(\mathsf{S})$ *is the module action. Then* $(\cdot,\cdot)$ *coincides with the canonical pairing* $(\cdot,\cdot)_{\mathsf{S}}$, *for some trivialization* $K_{\mathsf{S}} \cong \mathbb{K}$.

Proof The invariance property implies that $(\rho(x)\phi, \psi) = (\phi, \rho(x^\top)\psi)$ for all $x \in \Gamma(V)$, hence (by linearity) for all $x \in \mathrm{Cl}(V)$. This shows that the bilinear pairing gives an isomorphism of Clifford modules $\mathsf{S} \to \mathsf{S}^*$. It hence provides a trivialization of $K_{\mathsf{S}} = \mathrm{Hom}_{\mathrm{Cl}(V)}(\mathsf{S}^*, \mathsf{S})$, identifying $(\cdot,\cdot)$ with the pairing $(\cdot,\cdot)_{\mathsf{S}}$. □

Example 3.5 For the Clifford module $\mathsf{S}_F = \mathrm{Cl}(V)/\mathrm{Cl}(V)F$, defined by a Lagrangian subspace F, one has $K_{\mathsf{S}_F} = \det(F^*)$. To explicitly write the pairing, choose a Lagrangian complement in order to identify $\mathsf{S}_F = \wedge(F^*)$. Then the pairing is given by

$$(\phi, \psi)_{\mathsf{S}} = (\phi^\top \wedge \psi)_{[\mathrm{top}]}. \tag{3.10}$$

We next turn to the symmetry properties of the bilinear form.

Proposition 3.14 *Let* $\dim V = 2m$. *The canonical pairing* $(\cdot,\cdot)_{\mathsf{S}}$ *is*

- *symmetric if* $m = 0, 1 \bmod 4$,
- *skew-symmetric if* $m = 2, 3 \bmod 4$.

Furthermore, if $m = 0 \bmod 4$ (*resp.* $m = 2 \bmod 4$) *it restricts to a non-degenerate symmetric* (*resp. skew-symmetric*) *form on both* $\mathsf{S}^{\bar{0}}$ *and* $\mathsf{S}^{\bar{1}}$. *If* m *is odd, then the bilinear form vanishes on both* $\mathsf{S}^{\bar{0}}$, $\mathsf{S}^{\bar{1}}$, *and hence gives a non-degenerate pairing between them.*

Proof We may use the model $V = F^* \oplus F$, $\mathsf{S} = \wedge(F^*)$. Let $\phi \in \wedge^k(F^*)$ and $\psi \in \wedge^{m-k}(F^*)$. Then

$$\begin{aligned}
(\psi, \phi)_{\mathsf{S}} &= \psi^\top \wedge \phi \\
&= (-1)^{(m-k)(m-k-1)/2} \psi \wedge \phi \\
&= (-1)^{(m-k)(m-k-1)/2 + k(m-k)} \phi \wedge \psi \\
&= (-1)^{(m-k)(m-k-1)/2 + k(m-k) + k(k-1)/2} \phi^\top \wedge \psi \\
&= (-1)^{m(m-1)/2} (\phi, \psi)_{\mathsf{S}}.
\end{aligned}$$

This gives the symmetry property of the bilinear form. If m is even, then $\mathsf{S}^{\bar{0}}$ and $\mathsf{S}^{\bar{1}}$ are orthogonal under this bilinear form, and hence the bilinear form is non-degenerate on both. □

Remark 3.7 Suppose $m = 0, 1 \bmod 4$, so that $(\cdot,\cdot)_{\mathsf{S}}$ is symmetric. Then

$$(\rho(v)\phi, \rho(w)\phi)_{\mathsf{S}} = B(v, w)(\phi, \phi)_{\mathsf{S}}. \tag{3.11}$$

If $v = w$, this follows from the invariance property since $\mathsf{N}(v) = v^\top v = B(v, v)$, and in general by polarization. The identity shows that if $(\phi, \phi)_{\mathsf{S}} \neq 0$, then the null space $F(\phi)$ is trivial. Indeed, for all $v \in F(\phi)$ the identity implies $B(v, w) = 0$ for all w, hence $v = 0$.

Theorem 3.7 (E. Cartan, Chevalley) *Let* S *be a spinor module. Let* $\phi, \psi \in \mathsf{S}$ *be pure spinors. Then the pairing* $(\phi, \psi)_{\mathsf{S}}$ *is non-zero if and only if the Lagrangian subspaces* $F(\phi)$ *and* $F(\psi)$ *are transverse.*

Proof We use the model $V = F^* \oplus F$ and $\mathsf{S} = \wedge F^*$, using the formula (3.10) for the pairing.

'$\Leftarrow$'. Suppose $F(\phi) \cap F(\psi) = 0$. Choose $A \in \mathrm{O}(V)$ such that A^{-1} takes $F(\phi), F(\psi)$ to F^*, F, respectively. Let $x \in \Gamma(V)$ be a lift, i.e., $A_x = A$. Then $\rho(x)^{-1}\phi$ and $\rho(x)^{-1}\psi$ are pure spinors representing F^* and F, hence they are elements of $\det(F^*)^\times$ and $\mathbb{K}^\times$ respectively. By (3.10) their pairing is non-zero, hence also

$$(\phi, \psi)_{\mathsf{S}} = \mathsf{N}(x)\ (\rho(x)^{-1}\phi, \rho(x)^{-1}\psi)_{\mathsf{S}} \neq 0.$$

'$\Rightarrow$'. Suppose $(\phi, \psi)_{\mathsf{S}} \neq 0$. Choose $x \in \Gamma(V)$ with $\psi = \rho(x).1$. Then

$$0 \neq \mathsf{N}(x)(\phi, \psi)_{\mathsf{S}} = (\rho(x)^{-1}\phi, 1)_{\mathsf{S}} = (\rho(x)^{-1}\phi)_{[\mathrm{top}]}.$$

In particular, $\rho(x)^{-1}\phi$ is not annihilated by any non-zero $\rho(v)$, $v \in F$. Hence $F(\rho(x)^{-1}\phi) \cap F = 0$, and consequently $F(\phi) \cap F(\psi) = F(\phi) \cap A_x(F) = 0$. □

Remark 3.8 More generally, we could also consider two different spinor modules $\mathscr{S}, \mathscr{S}'$. One obtains a pairing

$$(\cdot,\cdot):\ \mathscr{S}' \otimes \mathscr{S} \cong \mathscr{S}^* \otimes \mathrm{Hom}(\mathscr{S}^*, \mathscr{S}') \otimes \mathscr{S} \to \mathrm{Hom}(\mathscr{S}^*, \mathscr{S}').$$

As before, the Lagrangian subspaces defined by $\phi \in \mathscr{S}, \phi' \in \mathscr{S}'$ are transverse if and only if $(\phi, \phi') \neq 0$.

In particular, Theorem 3.7 shows that pure spinors satisfy $(\phi, \phi)_{\mathsf{S}} = 0$. In dimension 8, the converse is true.

Proposition 3.15 (Chevalley [41, IV.1.1]) *Suppose V is a vector space with split bilinear form, with* $\dim V = 8$, *and* S *a spinor module. Then a non-zero even or odd spinor $\phi \in \mathsf{S}$ is pure if and only if $(\phi, \phi)_{\mathsf{S}} = 0$. Furthermore, the spinors ϕ with $(\phi, \phi)_{\mathsf{S}} \neq 0$ satisfy $F(\phi) = 0$.*

Proof We work in the model $V = F^* \oplus F$, with the spinor module $\mathsf{S}_F = \wedge F^*$. Suppose $\phi \in \mathsf{S}_F$ is an even or odd non-zero spinor with $(\phi, \phi)_{\mathsf{S}} = 0$. We will show that ϕ is pure. Suppose first that ϕ is even. Then

$$0 = (\phi, \phi)_{\mathsf{S}} = 2\phi_{[0]}\phi_{[4]} - \phi_{[2]} \wedge \phi_{[2]}.$$

If $\phi_{[0]} \neq 0$, we may rescale ϕ to arrange $\phi_{[0]} = 1$. The property $\phi_{[4]} = \frac{1}{2}\phi_{[2]} \wedge \phi_{[2]}$ then means $\phi = \exp(\phi_{[2]})$, which is a pure spinor. If $\phi_{[0]} = 0$, the property $\phi_{[2]} \wedge \phi_{[2]} = 0$ tells us that $\phi_{[2]} = \mu^1 \wedge \mu^2$ for suitable $\mu^1, \mu^2 \in F^*$. Let ω be a 2-form such that $\phi_{[2]} \wedge \omega = \phi_{[4]}$; then $\phi = \mu_1 \wedge \mu_2 \wedge \exp(\omega)$ which is again a pure spinor. If ϕ is odd, choose any non-isotropic v. Then $\rho(v)\phi$ is an even spinor, with $(\rho(v)\phi, \rho(v)\phi)_{\mathsf{S}} = B(v, v)(\phi, \phi)_{\mathsf{S}} = 0$. Hence $\rho(v)\phi$ is pure, and consequently ϕ is pure.

On the other hand, if $(\phi, \phi)_{\mathsf{S}} \neq 0$, and $v \in F(\phi)$, then (3.11) shows that

$$0 = (\rho(v)\phi, \rho(w)\phi)_{\mathsf{S}} = B(v, w)(\phi, \phi)_{\mathsf{S}}$$

for all $w \in V$. Hence $B(v, w) = 0$ for all w and therefore $v = 0$. □

3.5 The character $\chi : \Gamma(V)_F \to \mathbb{K}^\times$

Let $F \subseteq V$ be a Lagrangian subspace. By Proposition 1.5 the group $\mathrm{O}(V)_F$ of orthogonal transformations preserving F is contained in $\mathrm{SO}(V)$, and fits into an exact sequence

$$1 \to \wedge^2(F) \to \mathrm{O}(V)_F \to \mathrm{GL}(F) \to 1.$$

Let $\Gamma(V)_F \subseteq S\Gamma(V)$ be the pre-image of $\mathrm{O}(V)_F$ in the Clifford group, and

$$\mathrm{Spin}(V)_F = \Gamma(V)_F \cap \mathrm{Pin}(V).$$

Thus $x \in \Gamma(V)_F$ if and only if A_x preserves F. The action $\rho(x)$ of such an element on the spinor module S must preserve the pure spinor line l_F. This defines a group homomorphism

$$\chi : \Gamma(V)_F \to \mathbb{K}^\times,$$

with $\rho(x)\phi = \chi(x)\phi$ for all $x \in \Gamma(V)_F$, $\phi \in l_F$. Clearly, this character is independent of the choice of S.

Proposition 3.16 *The character χ satisfies*

$$\chi(x)^2 = \mathsf{N}(x)\ \det(A_x|_F)$$

for all $x \in \Gamma(V)_F$. Hence, the restriction of χ to the group $\mathrm{Spin}(V)_F$ *defines a square root of the function $x \mapsto \det(A_x|_F)$.*

Proof We use the model $V = F^* \oplus F$, $\mathsf{S} = \wedge(F^*)$, so that $\chi(x) = \rho(x).1$. It suffices to check the following two cases: (i) A_x fixes F pointwise, and (ii) A_x preserves both F and F^*.

In case (i), A_x is given by an element $\lambda \in \wedge^2(F)$, and hence $x = t\exp(-\lambda)$ for some non-zero $t \in \mathbb{K}$. We have $\det(A_x|_F) = 1$. The action of x on $1 \in \wedge(F^*)$ is multiplication by t, hence $\chi(x) = t$, while $\mathsf{N}(x) = t^2$. This verifies the formula in case (i).

In case (ii), let $Q = A_x|_F \in \mathrm{GL}(F)$. Then $A_x(\mu, v) = ((Q^{-1})^*\mu, Qv)$. For all $\nu \in F^*$ we have $x\nu x^{-1} = A_x(\nu) = (Q^{-1})^*\nu$, where we used $\Pi(x) = x$. It follows that

$$x\psi x^{-1} = (Q^{-1})^*\psi$$

for all $\psi \in \wedge(F^*) \subseteq \mathrm{Cl}(V)$. Take $\psi \in \det(F^*)^\times$, so that $(Q^{-1})^*\psi = \frac{1}{\det Q}\psi$. We obtain

$$\rho(x)\psi = \rho(x\psi)1 = \rho(x\psi x^{-1})\rho(x)1 = \frac{\chi(x)}{\det Q}\psi.$$

Pairing with $\rho(x)1 = \chi(x)$, and using the invariance property of the bilinear form, we find that

$$\mathsf{N}(x)\psi = (\rho(x).1, \rho(x)\psi)_{\mathsf{S}} = \frac{\chi(x)^2}{\det Q}\psi,$$

hence $\chi(x)^2 = \mathsf{N}(x)\det(Q)$. □

Let $\mathbb{K}_\chi$ denote the $\Gamma(V)_F$-representation on $\mathbb{K}$, with $x \in \Gamma(V)_F$ acting as multiplication by $\chi(x)$.

Proposition 3.17 *The fiber of the associated line bundle*

$$\Gamma(V) \times_{\Gamma(V)_F} \mathbb{K}_\chi \to \mathrm{Lag}(V)$$

at $L \in \mathrm{Lag}(V)$ *is the pure spinor line* $l_L \subseteq \mathsf{S}_F$.

Proof Since $x \in \Gamma(V)_F$ acts as $\chi(x)$ on $\mathsf{l}_F \subseteq \mathsf{S}_F = \mathrm{Cl}(V)/\mathrm{Cl}(V)F$, we have $x = \chi(x) \bmod \mathrm{Cl}(V)F$ for all $x \in \Gamma(V)_F$. Thus

$$\chi(x)x^{-1} = 1 \mod \mathrm{Cl}(V)F.$$

Now let $(z, t) \in \Gamma(V) \times \mathbb{K}_\chi$, and put $L = A_z(F)$. The map

$$\mathrm{Cl}(V) \to \mathrm{Cl}(V),\ y \mapsto tyz$$

takes $\mathrm{Cl}(V)L$ to $\mathrm{Cl}(V)F$, hence it descends to an element of $\mathrm{Hom}_{\mathrm{Cl}(V)}(\mathsf{S}_L, \mathsf{S}_F) = \mathsf{l}_L$. If $(z', t') = (zx^{-1}, \chi(x)t)$ with $x \in \Gamma(V)_F$, then

$$t'yz' = \chi(x)tyzx^{-1} = tyz \mod \mathrm{Cl}(V)F,$$

thus (z', t') defines the same homomorphism as the element (z, t). $\square$

If the map $\mathrm{Pin}(V) \to \mathrm{O}(V)$ is onto (e.g., if $\mathbb{K} = \mathbb{C}$), the line bundle has the description

$$\mathrm{Pin}(V) \times_{\mathrm{Pin}(V)_F} \mathbb{K}_\chi,$$

where $\mathrm{Pin}(V)_F = \mathrm{Spin}(V)_F$ acts via the character $x \mapsto \chi(x) = \det^{1/2}(A_x|_F)$.

3.6 Cartan's triality principle

If $\dim V = 8$, one has the remarkable phenomenon of *triality*, stated (in essentially the form given below) by E. Cartan [34]. (A more geometric triality principle had been described earlier by E. Study [116]. See Porteus [108, Chapter 24] for historical background, and a discussion of Study's triality principle.)

The following discussion is based on Chevalley's exposition in [41]. As before, we assume that the bilinear form B on V is split (which is automatic if $\mathbb{K} = \mathbb{C}$). Let $\rho : \mathrm{Cl}(V) \to \mathrm{End}(\mathsf{S})$ be a spinor module, and let $\Gamma \in \mathrm{Spin}(V)$ be the chirality element, with the unique normalization for which $\rho(\Gamma)$ is the parity operator of S. Since $\Gamma, 1$ span the center of the algebra $\mathrm{Cl}^{\bar{0}}(V)$, and since the linear span of $\mathrm{Spin}(V)$ is all of $\mathrm{Cl}^{\bar{0}}(V)$, the center of the group $\mathrm{Spin}(V)$ consists of four elements

$$\mathrm{Cent}(\mathrm{Spin}(V)) = \{1, -1, \Gamma, -\Gamma\}. \tag{3.12}$$

The two half-spin representations $\mathsf{S}^{\bar{0}}, \mathsf{S}^{\bar{1}}$ of $\mathrm{Spin}(V)$ are irreducible representations of dimension 8. In addition, one has the 8-dimensional representation on V via $\pi : \mathrm{Spin}(V) \to \mathrm{SO}(V)$, $x \mapsto A_x$. These three representations are all non-isomorphic, and are distinguished by the action of the center (3.12). Indeed, the central element $-1 \in \mathrm{Spin}(V)$ acts trivially on V since $\pi(-1) = I$, while it acts as $-I$ in the half-spin representations. The triality principle, Theorem 3.8 below, shows that there is a degree 3 automorphism of $\mathrm{Spin}(V)$ relating the three representations.

Consider the direct sum

$$A = V \oplus \mathsf{S}^{\bar{0}} \oplus \mathsf{S}^{\bar{1}}.$$

Since $\dim V = 8$, Proposition 3.14 shows that the canonical bilinear form $(\cdot,\cdot)_{\mathsf{S}}$ of S is symmetric, and restricts to non-degenerate bilinear forms on both $\mathsf{S}^{\bar{0}}$ and $\mathsf{S}^{\bar{1}}$. After choice of a trivialization $K_{\mathsf{S}} \cong \mathbb{K}$, it becomes a scalar-valued symmetric bilinear form; its direct sum with the given bilinear form B on V is a non-degenerate symmetric bilinear form B_A on A. The corresponding orthogonal group is denoted $\mathrm{O}(A)$, as usual. Let $\rho_A : S\Gamma(V) \to \mathrm{Aut}(A)$ be the triagonal action on A. Then $\rho_A(\mathrm{Spin}(V)) \subseteq \mathrm{O}(A)$.

Theorem 3.8 (Triality) *There exists an orthogonal automorphism $J \in \mathrm{O}(A)$ of order 3 and a group automorphism $j \in \mathrm{Aut}(\mathrm{Spin}(V))$ of order 3 such that*

$$J(V) = \mathsf{S}^{\bar{1}}, \quad J(\mathsf{S}^{\bar{1}}) = \mathsf{S}^{\bar{0}}, \quad J(\mathsf{S}^{\bar{0}}) = V, \tag{3.13}$$

and

$$J \circ \rho_A(x) = \rho_A(j(x)) \circ J, \quad x \in \mathrm{Spin}(V). \tag{3.14}$$

One hence obtains a commutative diagram, for $x \in \Gamma(V)$,

$$\begin{array}{ccccc}
V & \xrightarrow[J]{} & \mathsf{S}^{\bar{1}} & \xrightarrow[J]{} & \mathsf{S}^{\bar{0}} \\
\big\downarrow{\scriptstyle \rho(x)} & & \big\downarrow{\scriptstyle \rho(j(x))} & & \big\downarrow{\scriptstyle \rho(j(j(x)))} \\
V & \xrightarrow[J]{} & \mathsf{S}^{\bar{1}} & \xrightarrow[J]{} & \mathsf{S}^{\bar{0}}.
\end{array}$$

A key ingredient in the proof is the following cubic form on A:

$$C_A : A \to \mathbb{K}, \; \xi = (v, \phi^{\bar{0}}, \phi^{\bar{1}}) \mapsto (\rho(v)\phi^{\bar{0}}, \phi^{\bar{1}})_{\mathsf{S}}. \tag{3.15}$$

Lemma 3.2 *The cubic form C_A satisfies $C_A(\rho_A(x)\xi) = \mathsf{N}(x)C_A(\xi)$ for all $x \in S\Gamma(V)$. Hence, $\rho_A(x)$, $x \in S\Gamma(V)$ preserves C_A precisely if $x \in \mathrm{Spin}(V)$.*

Proof For $\xi = (v, \phi^{\bar{0}}, \phi^{\bar{1}})$,

$$\begin{aligned}
C_A(\rho_A(x)\xi) &= (\rho(A_x(v))\rho(x)\phi^{\bar{0}}, \rho(x)\phi^{\bar{1}})_{\mathsf{S}} \\
&= (\rho(x)\rho(v)\phi^{\bar{0}}, \rho(x)\phi^{\bar{1}})_{\mathsf{S}} \\
&= \mathsf{N}(x)(\rho(v)\phi^{\bar{0}}, \phi^{\bar{1}})_{\mathsf{S}} = \mathsf{N}(x)C_A(\xi)
\end{aligned}$$

as claimed. □

We will construct the triality automorphism J in such way that it also preserves C_A.

Lemma 3.3 *Any $f \in \mathrm{O}(A)$ preserving each of the subspaces $V, \mathsf{S}^{\bar{0}}, \mathsf{S}^{\bar{1}}$ and preserving the cubic form C_A is of the form $\rho_A(x)$, for a unique $x \in \mathrm{Spin}(V)$.*

Proof For all $\xi = (v, \phi^{\bar{0}}, \phi^{\bar{1}})$ we find

$$(\rho(f(v))f(\phi^{\bar{0}}), f(\phi^{\bar{1}}))_{\mathsf{S}} = (\rho(v)\phi^{\bar{0}}, \phi^{\bar{1}})_{\mathsf{S}} = (f(\rho(v)\phi^{\bar{0}}), f(\phi^{\bar{1}}))_{\mathsf{S}},$$

where the first equality used invariance of C_A and the second equality used the invariance of B_A. Consequently,

$$f(\rho(v)\phi) = \rho(f(v))f(\phi)$$

for all even spinors ϕ. On the other hand, any odd spinor can be written $\psi = \rho(v)\phi$, hence the identity gives

$$\begin{aligned} f(\psi) &= \rho(f(v))f(\rho(v)^{-1}\psi) \\ &= \rho(f(v))B(v,v)^{-1}f(\rho(v)\psi) \\ &= \rho(f(v))^{-1}f(\rho(v)\psi). \end{aligned}$$

That is, $f(\rho(v)\psi) = \rho(f(v))f(\psi)$ for all odd spinors. This shows that

$$\rho(f(v)) = (f|_{\mathsf{S}}) \circ \rho(v) \circ (f|_{\mathsf{S}})^{-1}. \tag{3.16}$$

Since $f|_{\mathsf{S}}$ is an even endomorphism of S, it is of the form $\rho(x)$ for some $x \in \mathrm{Cl}^{\bar{0}}(V)$. Equation (3.16) shows that $\rho(f(v)) = \rho(xvx^{-1}) = \rho(A_x(v))$, hence $f(v) = A_x(v)$ and in particular $x \in S\Gamma(V)$. We have shown that $f = \rho_A(x)$. Using again the invariance of C_A and the previous lemma, we obtain $\mathsf{N}(x) = 1$, so that $x \in \mathrm{Spin}(V)$. □

We are now in a position to construct the triality isomorphism.

Proof of Theorem 3.8 Pick $n \in V$ and $q \in \mathsf{S}^{\bar{0}}$ with $B(n,n) = 1$, $(q,q)_{\mathsf{S}} = 1$. Let R_n and R_q be the corresponding reflections in $V, \mathsf{S}^{\bar{0}}$. The map

$$V \to \mathsf{S}^{\bar{1}} : \; v \mapsto \rho(v)q$$

is an isometry (see Remark 3.7); let $T_q : \; \mathsf{S}^{\bar{1}} \to V$ be the inverse map. Define orthogonal involutions $\mu, \tau \in \mathrm{O}(A)$ by

$$\begin{aligned} \mu(v, \phi^{\bar{0}}, \phi^{\bar{1}}) &= (R_n(v), \; \rho(n)\phi^{\bar{1}}, \; \rho(n)\phi^{\bar{0}}), \\ \tau(v, \phi^{\bar{0}}, \phi^{\bar{1}}) &= (T_q(\phi^{\bar{1}}), R_q(\phi^{\bar{0}}), \rho(v)q). \end{aligned}$$

Note that μ preserves V and exchanges the spaces $\mathsf{S}^{\bar{0}}, \mathsf{S}^{\bar{1}}$, while τ preserves $\mathsf{S}^{\bar{0}}$ and exchanges the spaces $V, \mathsf{S}^{\bar{1}}$. Hence the composition

$$J = \tau \circ \mu \in \mathrm{O}(A)$$

satisfies (3.13). Let us verify that $J^3 = \mathrm{id}_A$. It suffices to check on elements $v \in V \subseteq A$. We have:

$$\begin{aligned}
\mu(v) &= R_n(v) = -nvn,\\
\tau\mu(v) &= -\rho(nvn)q,\\
\mu\tau\mu(v) &= -\rho(vn)q,\\
\tau\mu\tau\mu(v) &= -\rho(vn)q + 2(\rho(vn)q, q)_{\mathsf{S}}\, q\\
&= -\rho(vn)q + 2B(v,n)q\\
&= \rho(nv)q,\\
\mu\tau\mu\tau\mu(v) &= \rho(v)q,\\
\tau\mu\tau\mu\tau\mu(v) &= v.
\end{aligned}$$

Hence $J^3 v = v$ as claimed. We next show that the cubic form C_A changes sign under μ, τ, and hence is invariant under J. For $\xi = (v, \phi^{\bar{0}}, \phi^{\bar{1}})$, we have

$$C_A(\mu(\xi)) = (\rho(R_n(v)n)\phi^{\bar{1}}, \rho(n)\phi^{\bar{0}})_{\mathsf{S}} = -(\rho(v)\phi^{\bar{1}}, \phi^{\bar{0}})_{\mathsf{S}} = -C_A(\xi).$$

The computation for τ is a bit more involved. Let $w = T_q(\phi^{\bar{1}})$, so that $\rho(w)q = \phi^{\bar{1}}$. Then

$$\begin{aligned}
C_A(\tau(\xi)) &= (\rho(w)R_q(\phi^{\bar{0}}), \rho(v)q)_{\mathsf{S}}\\
&= (R_q(\phi^{\bar{0}}), \rho(wv)q)_{\mathsf{S}}\\
&= (\phi^{\bar{0}}, \rho(wv)q)_{\mathsf{S}} - 2(\phi^{\bar{0}}, q)_{\mathsf{S}}(q, \rho(wv)q)_{\mathsf{S}}\\
&= 2B(v,w)(\phi^{\bar{0}}, q)_{\mathsf{S}} - (\phi^{\bar{0}}, \rho(v)\phi^{\bar{1}})_{\mathsf{S}} - 2(\phi^{\bar{0}}, q)_{\mathsf{S}}(\rho(w)q, \rho(v)q)_{\mathsf{S}}\\
&= -(\phi^{\bar{0}}, \rho(v)\phi^{\bar{1}})_{\mathsf{S}}\\
&= -(\rho(v)\phi^{\bar{0}}, \phi^{\bar{1}})_{\mathsf{S}} = -C_A(\xi).
\end{aligned}$$

Hence $C_A(J(\xi)) = C_A(\xi)$. Suppose now that $x \in \mathrm{Spin}(V)$. Then

$$J \circ \rho_A(x) \circ J^{-1}$$

preserves B_A, C_A and the three subspaces $V, \mathsf{S}^{\bar{0}}, \mathsf{S}^{\bar{1}}$. By Lemma 3.3 we may write this composition as $\rho_A(j(x))$ for a unique element $j(x) \in \mathrm{Spin}(V)$. Using the uniqueness part of Lemma 3.3, we find $j(x_1)j(x_2) = j(x_1x_2)$ and $j(j(j(x))) = x$. □

The theory described here goes much further. Using polarization, the cubic form C_A defines a symmetric trilinear form $T_A \in S^3(A^*)$, with $T_A(\xi,\xi,\xi) = C_A(\xi)$. This form defines a "triality": I.e., $T_A(\xi_1,\xi_2,\xi_3)$ is a non-degenerate bilinear form in ξ_1, ξ_2 for arbitrary fixed non-zero ξ_3. In turn, this triality can be used to construct an interesting non-associative product on V, making V into the algebra of *octonions*. A beautiful discussion of this theory may be found in the paper [20] by Baez.

If $K = \mathbb{C}$, it is possible to arrange that the triality automorphism of $\mathrm{Spin}(8,\mathbb{C})$ preserves the compact group $\mathrm{Spin}(8) = \mathrm{Spin}(8,\mathbb{R})$. See the discussion in Section 3.7.6. Using a computation of roots, one finds that the fixed point set of this automorphism is the exceptional compact Lie group G_2.

3.7 The Clifford algebra $\mathbb{C}l(V)$

Throughout this section we denote by V a vector space over $\mathbb{K} = \mathbb{R}$, with a *positive definite* symmetric bilinear form B. The complexification of the Clifford algebra of V coincides with the Clifford algebra of the complexification of V, and will be denoted by $\mathbb{C}l(V)$. It has the additional structure of an involution, coming from the complex conjugation operation on $V^{\mathbb{C}}$, and one can consider *unitary* Clifford modules compatible with this involution. In this section, we will develop the theory of such unitary Clifford modules, and present a number of applications to the theory of compact Lie groups.

3.7.1 The Clifford algebra $\mathbb{C}l(V)$

Let $V^{\mathbb{C}}$ be the complexification of V. For $v \in V^{\mathbb{C}}$ we denote by v^c its complex conjugate. The Hermitian inner product of $V^{\mathbb{C}}$ will be denoted by $\langle \cdot, \cdot \rangle$, while the extension of B to a complex bilinear form will still be denoted by B. Thus $\langle v, w \rangle = B(v^c, w)$ for $v, w \in V^{\mathbb{C}}$. We put

$$\mathbb{C}l(V) := \mathrm{Cl}(V^{\mathbb{C}}) = \mathrm{Cl}(V)^{\mathbb{C}}.$$

The complex conjugation mapping $v \mapsto v^c$ on $V^{\mathbb{C}}$ extends to a conjugate linear algebra automorphism $x \mapsto x^c$ of the complex Clifford algebra $\mathbb{C}l(V)$. As in 2.2.6, define a conjugate linear anti-automorphism

$$x \mapsto x^* = (x^c)^{\top}.$$

Thus $(xy)^* = y^* x^*$ and $(ux)^* = u^c x^*$ for $u \in \mathbb{C}$.

Definition 3.7 A unitary Clifford module over $\mathbb{C}l(V)$ is a Hermitian super vector space E together with a morphism of super $*$-algebras $\mathbb{C}l(V) \to \mathrm{End}(E)$.

Thus, for a unitary Clifford module the action map ρ satisfies $\rho(x^*) = \rho(x)^*$ for all $x \in \mathbb{C}l(V)$. Equivalently, the elements of $v \in V \subseteq \mathbb{C}l(V)$ act as self-adjoint operators. Note that for a unitary Clifford module, the representations of $\mathrm{Spin}(V)$, $\mathrm{Pin}(V)$ preserve the Hermitian inner product. They are thus unitary representations.

Example 3.6 The Clifford algebra $\mathbb{C}l(V)$ itself has a Hermitian inner product, $\langle x, y \rangle = \mathrm{tr}(x^* y)$. Let $\rho : \mathbb{C}l(V) \to \mathrm{End}(\mathbb{C}l(V))$ be the action by left-multiplication. For $v \in V \subseteq V^{\mathbb{C}}$ we have $v^* = v$, hence $\langle x, vy \rangle = \langle vx, y \rangle$ for all $x \in \mathbb{C}l(V)$. This shows that $\mathbb{C}l(V)$ is a unitary $\mathbb{C}l(V)$-module. The quantization map intertwines the Hermitian inner product on $\wedge V^{\mathbb{C}}$, given by $\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle = \det \langle v_i, w_j \rangle$, with the Hermitian inner product $\langle x, y \rangle = \mathrm{tr}(x^* y)$ on $\mathbb{C}l(V)$.

Since $\mathbb{C}l(V)$ has a faithful unitary representation, such as in the previous example, it follows that $\mathbb{C}l(V)$ is a C^*-algebra. That is, it admits a unique norm $\|\cdot\|$ relative to which it is a Banach algebra, and such that the C^*-identity $\|x^*x\| = \|x\|^2$ is satisfied. This norm is equal to the operator norm in any such presentation, and is explicitly given in terms of the trace by the formula

$$\|a\| = \lim_{n\to\infty} \left(\mathrm{tr}(a^*a)^n\right)^{\frac{1}{2n}}.$$

Note that this C^*-norm is different from the Hilbert space norm $\mathrm{tr}(a^*a)^{1/2}$.

3.7.2 *The groups* $\mathrm{Spin}_c(V)$ *and* $\mathrm{Pin}_c(V)$

Suppose $x \in \Gamma(V^{\mathbb{C}})$, defining a complex transformation $A_x(v) = (-1)^{|x|}xvx^{-1} \in \mathrm{O}(V^{\mathbb{C}})$ as before.

Lemma 3.4 *The element $x \in \Gamma(V^{\mathbb{C}})$ satisfies $A_x(v)^* = A_x(v^*)$ for all $v \in V^{\mathbb{C}}$, if and only if x^*x is a positive real number.*

Proof For all $x \in \Gamma(V^{\mathbb{C}})$ and all $v \in V^{\mathbb{C}}$, we have

$$A_x(v)^* = (-1)^{|x|}(x^{-1})^*v^*x^* = A_{(x^{-1})^*}(v^*).$$

This coincides with $A_x(v^*)$ for all v if and only if $x = \lambda(x^{-1})^*$ for some $\lambda \in \mathbb{C}^\times$, i.e., if and only if $x^*x \in \mathbb{C}^\times$. Since x^*x is a positive element, this condition is equivalent to $x^*x \in \mathbb{R}_{>0}$. □

Definition 3.8 We define

$$\begin{aligned} \Gamma_c(V) &= \{x \in \Gamma(V^{\mathbb{C}}) |\ x^*x \in \mathbb{R}_{>0}\}, \\ \mathrm{Pin}_c(V) &= \{x \in \Gamma(V^{\mathbb{C}}) |\ x^*x = 1\}, \\ \mathrm{Spin}_c(V) &= \mathrm{Pin}_c(V) \cap S\Gamma(V^{\mathbb{C}}). \end{aligned}$$

If $V = \mathbb{R}^n$ with the standard bilinear form, we write $\Gamma_c(n)$, $\mathrm{Pin}_c(n)$, $\mathrm{Spin}_c(n)$.

By definition, an element x of the Clifford group lies in $\Gamma_c(V)$ if and only if the automorphism $A_x \in \mathrm{O}(V^{\mathbb{C}})$ preserves the real subspace V. That is, $\Gamma_c(V) \subseteq \Gamma(V^{\mathbb{C}})$ is the inverse image of $\mathrm{O}(V) \subseteq \mathrm{O}(V^{\mathbb{C}})$. The exact sequence for $\Gamma(V^{\mathbb{C}})$ restricts to exact sequences,

$$\begin{aligned} &1 \to \mathbb{C}^\times \to \Gamma_c(V) \to \mathrm{O}(V) \to 1, \\ &1 \to \mathrm{U}(1) \to \mathrm{Pin}_c(V) \to \mathrm{O}(V) \to 1, \\ &1 \to \mathrm{U}(1) \to \mathrm{Spin}_c(V) \to \mathrm{SO}(V) \to 1, \end{aligned}$$

where we have used $\mathbb{C}^\times \cap \mathrm{Pin}_c(V) = \mathbb{C}^\times \cap \mathrm{Spin}_c(V) = \mathrm{U}(1)$.

One can also directly define $\mathrm{Pin}_c(V)$, $\mathrm{Spin}_c(V)$ as the subgroups of $\Gamma(V^{\mathbb{C}})$ generated by $\mathrm{Pin}(V)$, $\mathrm{Spin}(V)$ together with $\mathrm{U}(1)$. That is, $\mathrm{Spin}_c(V)$ is the quotient of $\mathrm{Spin}(V) \times \mathrm{U}(1)$ by the relation

$$(x, \mathrm{e}^{\sqrt{-1}\psi}) \sim (-x, -\mathrm{e}^{\sqrt{-1}\psi}),$$

and similarly for $\mathrm{Pin}_c(V)$. A third viewpoint towards these groups, using the spinor module, is described in Section 3.7.3 below. The norm homomorphism for $\Gamma(V^{\mathbb{C}})$ restricts to a group homomorphism,

$$\mathsf{N}:\ \Gamma_c(V) \to \mathbb{C}^\times,\ x \mapsto x^\top x.$$

On the subgroup $\mathrm{Pin}_c(V)$ this may be written $\mathsf{N}(x) = (x^c)^{-1}x$, which evidently takes values in $\mathrm{U}(1)$.

Together with the map to $\mathrm{O}(V)$ this defines exact sequences,

$$1 \to \mathbb{Z}_2 \to \Gamma_c(V) \to \mathrm{O}(V) \times \mathbb{C}^\times \to 1,$$
$$1 \to \mathbb{Z}_2 \to \mathrm{Pin}_c(V) \to \mathrm{O}(V) \times \mathrm{U}(1) \to 1,$$
$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}_c(V) \to \mathrm{SO}(V) \times \mathrm{U}(1) \to 1.$$

One of the motivations for introducing the group $\mathrm{Spin}_c(V)$ is the following lifting property. Suppose J is an orthogonal complex structure on V, that is, $J \in \mathrm{O}(V)$ and $J^2 = -I$. Such a J exists if and only if $n = \dim V$ is even, and turns V into a vector space over $\mathbb{C}$, with scalar multiplication

$$(a + \sqrt{-1}b)x = ax + bJx.$$

Let $U_J(V) \subseteq \mathrm{SO}(V)$ be the corresponding unitary group (i.e., the elements of $\mathrm{SO}(V)$ commuting with J).

Theorem 3.9 *The inclusion* $\mathrm{U}_J(V) \hookrightarrow \mathrm{SO}(V)$ *admits a unique lift to a group homomorphism* $U_J(V) \hookrightarrow \mathrm{Spin}_c(V)$, *in such a way that its composite with the map* $\mathsf{N}:\ \mathrm{Spin}_c(V) \to \mathrm{U}(1)$ *is the complex determinant* $\mathrm{U}_J(V) \to \mathrm{U}(1)$, $A \mapsto \det_J(A)$.

Proof We have to show that the map

$$\mathrm{U}_J(V) \to \mathrm{SO}(V) \times \mathrm{U}(1),\ A \mapsto (A,\ \det_J(A))$$

lifts to the double cover. Since $U_J(V)$ is connected, if such a lift exists then it is unique. To prove existence, it suffices to check that any loop representing the generator of $\pi_1(\mathrm{U}_J(V)) \cong \mathbb{Z}$ lifts to a loop in $\mathrm{Spin}_c(V)$. The inclusion of any non-zero J-invariant subspace $V' \subseteq V$ induces an isomorphism of the fundamental groups of the unitary groups. It is hence sufficient to check for the case $V = \mathbb{R}^2$, with J the standard complex structure $Je_1 = e_2$, $Je_2 = -e_1$. Our task is to lift the map

$$\mathrm{U}(1) \to \mathrm{SO}(2) \times \mathrm{U}(1),\ \mathrm{e}^{\sqrt{-1}\theta} \mapsto (R(\theta), \mathrm{e}^{\sqrt{-1}\theta})$$

to the double cover, $\mathrm{Spin}_c(\mathbb{R}^2)$. This lift is explicitly given by the following modification of Example 3.3,

$$x(\theta) = \mathrm{e}^{\sqrt{-1}\theta/2}(\cos(\theta/2) + \sin(\theta/2)\, e_1e_2) \in \mathrm{Spin}_c(\mathbb{R}^2).$$

Indeed, $x(\theta + 2\pi) = x(\theta)$ and $\mathsf{N}(x(\theta)) = \mathrm{e}^{\sqrt{-1}\theta}$. □

Remark 3.9 The two possible square roots of $\det_J(A)$ for $A \in \mathrm{U}_J(V)$ define a double cover,

$$\tilde{\mathrm{U}}_J(V) = \{(A, z) \in \mathrm{U}_J(V) \times \mathbb{C}^\times \mid z^2 = \det_J(A)\}.$$

While the inclusion $\mathrm{U}_J(V) \hookrightarrow \mathrm{SO}(V)$ does not lift to the Spin group, the above proof shows that there exists a lift $\tilde{\mathrm{U}}_J(V) \to \mathrm{Spin}(V)$ for this double cover. Equivalently, $\tilde{\mathrm{U}}_J(V)$ is identified with the pull-back of the spin double cover.

3.7.3 Spinor modules over $\mathbb{C}l(V)$

We will now discuss special features of spinor modules over the complex Clifford algebra $\mathbb{C}l(V)$ for an even-dimensional real Euclidean vector space V.

The first point we wish to stress is that, similar to Remark 3.6, *the choice of a compatible $\mathbb{Z}_2$-grading on a spinor module S is equivalent to the choice of orientation on V*. Indeed, let $e_1, \ldots, e_{2m}$ be an oriented orthonormal basis of V, where $\dim V = 2m$. Then the chirality element is

$$\Gamma = (\sqrt{-1})^m e_1 \cdots e_{2m} \in \mathrm{Spin}_c(V).$$

The normalization of Γ is such that $\Gamma^2 = 1$. Changing the orientation changes the sign of Γ, and hence changes the parity operator $\rho(\Gamma)$.

If the spinor module is of the form S_F for a Lagrangian subspace $F \in \mathrm{Lag}(V^{\mathbb{C}})$, we also have the orientation defined by the complex structure J corresponding to F. (See Section 1.7.) These two orientations agree:

Proposition 3.18 *Let F be a Lagrangian subspace of $V^{\mathbb{C}}$, and J the corresponding orthogonal complex structure having F as its $+\sqrt{-1}$ eigenspace. Then the orientation on V defined by J coincides with that defined by the $\mathbb{Z}_2$-grading on S_F.*

Proof The orientation defined by J is given by the volume element $e_1 \wedge \cdots \wedge e_n$, where e_i is an orthonormal basis such that $Je_{2j-1} = e_{2j}$. The Lagrangian subspace F is spanned by the orthonormal (for the Hermitian metric) vectors

$$E_j = \frac{1}{\sqrt{2}}(e_{2j-1} - \sqrt{-1}e_{2j}).$$

We claim that the chirality element given by the basis e_i acts as $+1$ on $\mathsf{S}_F^{\bar{0}}$ and as -1 on $\mathsf{S}_F^{\bar{1}}$. We have

$$E_j E_j^c = \frac{1}{2}(e_{2j-1} - \sqrt{-1}e_{2j})(e_{2j-1} + \sqrt{-1}e_{2j}) = \sqrt{-1}e_{2j-1}e_{2j} + 1,$$

hence

$$\Gamma = (E_1 E_1^c - 1) \cdots (E_m E_m^c - 1).$$

For $I=\{i_1,\ldots,i_k\}$ let $E_I^c = E_{i_1}^c\wedge\cdots\wedge E_{i_k}^c \in \wedge F^c \cong \mathsf{S}_F$. The operator $\rho(E_i E_i^c - 1)$ acts as 0 on E_I^c if $i\notin I$, and as -1 if $i\in I$. Hence $\rho(\Gamma)$ acts on E_I^c as $(-1)^k$, proving the claim. $\square$

As a special case of a unitary Clifford module, we have unitary spinor modules. These are Clifford modules S with the property that $\rho:\ \mathbb{C}l(V)\to \mathrm{End}(\mathsf{S})$ is an isomorphism of super $*$-algebras. Equivalently, the $\mathbb{C}l(V)$-action on S is irreducible.

Proposition 3.19 *Any spinor module* S *admits a Hermitian metric, unique up to positive scalar, for which it becomes a unitary spinor module.*

Proof Let $F\in \mathrm{Lag}(V^{\mathbb{C}})$. Then S_F is a unitary spinor module, and the choice of an isomorphism $\mathsf{S}\cong\mathsf{S}_F$ determines a Hermitian metric on S. Conversely, this Hermitian metric is uniquely determined by its restriction to the pure spinor line l_F, since $\rho(\mathrm{Cl}(V))l_F=\mathsf{S}$. $\square$

For any two unitary spinor modules S,S', the space $\mathrm{Hom}_{\mathbb{C}l}(\mathsf{S},\mathsf{S}')$ of intertwining operators inherits a Hermitian metric from the full space of homomorphisms $\mathrm{Hom}(\mathsf{S},\mathsf{S}')$, and the map

$$\mathsf{S}\otimes \mathrm{Hom}_{\mathbb{C}l}(\mathsf{S},\mathsf{S}')\to\mathsf{S}'$$

is an isomorphism of unitary Clifford modules. In the special case $\mathsf{S}'=\mathsf{S}$, we have $\mathrm{Hom}_{\mathbb{C}l}(\mathsf{S},\mathsf{S})=\mathbb{C}$ and the group of $\mathbb{C}l(V)$-equivariant unitary automorphisms of S is the group $\mathrm{U}(1)\subseteq\mathbb{C}$.

Using unitary spinor modules, one obtains another characterization of the groups $\mathrm{Pin}_c(V)$ and $\mathrm{Spin}_c(V)$. Suppose $\dim V$ is even, and pick a unitary spinor module S. An even or odd element $U\in\mathrm{U}(\mathsf{S})$ *implements* $A\in\mathrm{O}(V)$ if

$$\rho(A(v))=\det(A)\ U\circ\rho(v)\circ U^{-1}$$

for all $v\in V$, with $\det(A)=\pm1$ depending on the parity of U.

Proposition 3.20 *Suppose* $\dim V$ *is even. For any unitary spinor module* S, *the map* $\mathrm{Pin}_c(V)\to\mathrm{U}(\mathsf{S})$ *is injective. Its image is of implementers of orthogonal transformations of* V. *Similarly* $\mathrm{Spin}_c(V)$ *is isomorphic to the group of implementers of special orthogonal transformations.*

Proof Let $A\in\mathrm{O}(V)$ be given. If $x\in\mathrm{Pin}_c(V)$ is a lift A, then $\det(A)\,xvx^{-1}=A(v)$ for all $v\in V$, and consequently $U=\rho(x)$ satisfies $U\rho(v)U^{-1}=\det(A)\rho(A(v))$. Conversely, suppose U is a unitary element of parity $\det(A)=\pm1$, implementing A. Let x be the unique element in $\mathrm{Cl}(V)$ with $\rho(x)=U$. Thus $\det(A)=(-1)^{|x|}$, and $\rho((-1)^{|x|}xvx^{-1})=\rho(A(v))$, hence $(-1)^{|x|}xvx^{-1}=A(v)$ since ρ is faithful. Similarly, $N(x)=x^*x\in\mathrm{U}(1)$. $\square$

Thus $\mathrm{Pin}_c(V)$ and $\mathrm{Spin}_c(V)$ are realized as unitary implementers in S, of orthogonal and special orthogonal transformations, respectively.

3.7.4 Classification of irreducible $\mathbb{C}l(V)$-modules

Let V be a Euclidean vector space. We have seen that if $\dim V$ is even, there are two isomorphism classes of irreducible $\mathbb{C}l(V)$-modules, related by parity reversal. We will now extend this discussion to include the case of $\dim V$ odd. Recall once again that by default, we take Clifford modules to be equipped with a $\mathbb{Z}_2$-grading. Such a module is irreducible if there is no non-trivial invariant $\mathbb{Z}_2$*-graded* subspace. As we will see, the classification of such $\mathbb{Z}_2$-graded Clifford modules is in a sense "opposite" to the classification of irreducible ungraded Clifford modules.

The orientation on V determines the chirality operator

$$\Gamma = (\sqrt{-1})^{n(n-1)/2} e_1 \cdots e_n \in \mathrm{Pin}_c(V),$$

where $n = \dim V$ and $e_1, \ldots, e_n$ is an oriented orthonormal basis; it satisfies $\Gamma^2 = 1$. For n odd, the element Γ is odd, and it is an element of the (ordinary) center of $\mathbb{C}l(V)$. For n even, the element Γ is even, and it lies in the super-center of $\mathbb{C}l(V)$. There is a canonical isomorphism of (ordinary) algebras

$$\mathbb{C}l(V) \to \mathbb{C}l^{\bar{0}}(V \oplus \mathbb{R}), \tag{3.17}$$

determined (using the universal property) by the map on generators $v \mapsto \sqrt{-1} v e$, where e is the standard basis vector for the $\mathbb{R}$ summand. Since $(\sqrt{-1} v e)^* = -\sqrt{-1} e\, v^* = \sqrt{-1} v^* e$ this map is a $*$-isomorphism. If n is odd, the isomorphism (3.17) takes the chirality element of $\mathbb{C}l(V)$ to the chirality element for $\mathbb{C}l(V \oplus \mathbb{R})$, up to a sign.

Theorem 3.10 *Let V be a Euclidean vector space of dimension n, and $\mathbb{C}l(V)$ its complexified Clifford algebra.*

(i) *Suppose n is even. Then there are*:

- *two isomorphism classes of irreducible $\mathbb{Z}_2$-graded $\mathbb{C}l(V)$-modules*,
- *a unique isomorphism class of irreducible ungraded $\mathbb{C}l(V)$-modules*,
- *two isomorphism classes of irreducible $\mathbb{C}l^{\bar{0}}(V)$-modules.*

(ii) *Suppose n is odd. Then there are*:

- *a unique isomorphism class of irreducible $\mathbb{Z}_2$-graded $\mathbb{C}l(V)$-modules*,
- *two isomorphism classes of irreducible ungraded $\mathbb{C}l(V)$-modules*,
- *a unique isomorphism class of irreducible $\mathbb{C}l^{\bar{0}}(V)$-modules.*

Note that an irreducible $\mathbb{Z}_2$-graded module may be reducible as an ungraded module: There may be invariant subspaces which are not $\mathbb{Z}_2$-graded subspaces.

Proof We may assume $V = \mathbb{R}^n$, and let $\Gamma_n \in \mathbb{C}l(n)$ be the chirality element for the standard orientation. Note also that the third item in (i), resp. (ii), is equivalent to the second item in (ii), resp. (i), since $\mathbb{C}l(n-1) \cong \mathbb{C}l^0(n)$.

(i) Suppose n is even. We denote by S_n a spinor module of $\mathbb{C}l(n)$, with $\mathbb{Z}_2$-grading given by the orientation of $\mathbb{R}^n$. By the results of Section 3.2.5, S_n represents the unique isomorphism class of ungraded irreducible $\mathbb{C}l(n)$-modules, while S_n and $\mathsf{S}_n^{\mathrm{op}}$ represent the two isomorphism classes of irreducible $\mathbb{Z}_2$-graded spinor modules. The latter are distinguished by the action of Γ_n.

(ii) Suppose n is odd. Then

$$\mathbb{C}l(n) \cong \mathbb{C}l^{\bar{0}}(n+1) \cong \mathrm{End}^{\bar{0}}(\mathsf{S}_{n+1}) = \mathrm{End}(\mathsf{S}_{n+1}^{\bar{0}}) \oplus \mathrm{End}(\mathsf{S}_{n+1}^{\bar{1}})$$

identifies $\mathbb{C}l(n)$ as a direct sum of two matrix algebras. Hence there are two classes of irreducible ungraded $\mathbb{C}l(n)$-modules (given by $\mathsf{S}_{n+1}^{\bar{0}}$ and $\mathsf{S}_{n+1}^{\bar{1}}$). These are distinguished by the action of the chirality element Γ_n (note that the map to $\mathbb{C}l^{\bar{0}}(n+1)$ takes Γ_n to Γ_{n+1}, up to sign).

It remains to classify irreducible $\mathbb{Z}_2$-graded $\mathbb{C}l(n)$-modules $E = E^{\bar{0}} \oplus E^{\bar{1}}$, for n odd. If $n = 1$, since $\dim \mathbb{C}l(1) = 2$, the Clifford algebra $\mathbb{C}l(1)$ itself is an example. Conversely, if E is an irreducible $\mathbb{C}l(1)$-module, the choice of any non-zero element $\phi \in E^{\bar{0}}$ defines an isomorphism $\mathbb{C}l(1) \to E,\ x \mapsto \rho(x)\phi$. For general odd n, write $\mathbb{C}l(n) = \mathbb{C}l(n-1) \otimes \mathbb{C}l(1)$. If E is an irreducible $\mathbb{Z}_2$-*graded* $\mathbb{C}l(n)$-module, then $E_1 = \mathrm{Hom}_{\mathbb{C}l(n-1)}(\mathsf{S}_{n-1}, E)$ is a $\mathbb{Z}_2$-graded $\mathbb{C}l(1)$-module. This gives a decomposition

$$E \cong \mathsf{S}_{n-1} \otimes \mathrm{Hom}_{\mathbb{C}l(n-1)}(\mathsf{S}_{n-1}, E)$$

as $\mathbb{Z}_2$-graded $\mathbb{C}l(n-1) \otimes \mathbb{C}l(1) = \mathbb{C}l(n)$-modules (using graded tensor products). Since E is irreducible, the $\mathbb{Z}_2$-graded $\mathbb{C}l(1)$-module E_1 must be irreducible, hence it is isomorphic to $\mathbb{C}l(1)$. This proves that $E \cong \mathsf{S}_{n-1} \otimes \mathbb{C}l(1)$ as a $\mathbb{Z}_2$-graded module over $\mathbb{C}l(n-1) \otimes \mathbb{C}l(1) = \mathbb{C}l(n)$. □

Remark 3.10 (Restrictions) Any $\mathbb{C}l(n)$-module can be regarded as a $\mathbb{C}l(n-1)$-module by restriction. By dimension count, one verifies:

1. If n is even, then the ungraded module S_n restricts to a direct sum of the two non-isomorphic ungraded $\mathbb{C}l(n-1)$-modules (given by the even and odd part of S_n). The two $\mathbb{Z}_2$-graded modules S_n and $\mathsf{S}_n^{\mathrm{op}}$ both become isomorphic to the unique $\mathbb{Z}_2$-graded module over $\mathbb{C}l(n-1)$.
2. If n is odd, then the restrictions of the two irreducible ungraded $\mathbb{C}l(n)$-modules to $\mathbb{C}l(n-1)$ are both isomorphic to S_{n-1}, while the restriction of the irreducible $\mathbb{Z}_2$-graded $\mathbb{C}l(n)$-module is isomorphic to a direct sum $\mathsf{S}_{n-1} \oplus \mathsf{S}_{n-1}^{\mathrm{op}}$.

3.7.5 Spin representation

We saw that up to isomorphism, the algebra $\mathbb{C}l^{\bar{0}}(V)$ has two irreducible modules if $n = \dim V$ is even, and a unique module if n is odd. These restrict to representations of the group $\mathrm{Spin}(V) \subseteq \mathbb{C}l^{\bar{0}}(V)$, called the two *half-spin representations* if n is even, respectively the *spin representation* if n is odd. If $V = \mathbb{R}^n$, it is customary to denote the two half-spin representations (for n even) by $\Delta_n^\pm$, and the spin representation (for n odd) by Δ_n. Here Δ_n^+ (resp. Δ_n^-) is the half-spin representation where Γ_n acts as $+1$ (resp. as -1).

More concretely, taking $V = \mathbb{R}^{2m}$ with the spin representation defined by its standard complex structure $Je_{2j-1} = e_{2j}$, we may take $\Delta_{2m}^\pm$ to be the even and odd part of $\mathsf{S}_{2m} = \wedge\mathbb{C}^m$, and $\Delta_{2m-1} = \mathsf{S}_{2m-2} = \wedge\mathbb{C}^{m-1}$ (the spinor module over $\mathrm{Cl}(2m-2) \cong \mathrm{Cl}^{\bar{0}}(2m-1)$).

Proposition 3.21

(i) *If n is even, the two half-spin representations $\Delta_n^\pm$ of* $\mathrm{Spin}(n)$ *are irreducible, and are non-isomorphic. Their restrictions to* $\mathrm{Spin}(n-1)$ *are both isomorphic to Δ_{n-1}.*

(ii) *If n is odd, the spin representation Δ_n is irreducible. Its restriction to* $\mathrm{Spin}(n-1)$ *is isomorphic to $\Delta_{n-1} = \Delta_{n-1}^+ \oplus \Delta_{n-1}^-$.*

Proof This is immediate from the classification of irreducible $\mathrm{Cl}^{\bar{0}}(n)$-modules, since $\mathrm{Spin}(n)$ generates $\mathrm{Cl}^{\bar{0}}(n)$ as an algebra. (Note e.g., that $\mathrm{Spin}(n)$ contains the basis e_I of $\mathrm{Cl}^{\bar{0}}(n)$, where I ranges over subsets of $\{1, \ldots, n\}$ with an even number of elements.) □

We recall some terminology from the representation theory of compact Lie groups G (see e.g., [30, 48]). Let H be a Hermitian vector space with a unitary G-representation. The inner product on H will be denoted $\langle \cdot, \cdot \rangle$.

(i) H is of *real type* if it admits a G-equivariant conjugate linear endomorphism C with $C^2 = I$. In this case, H is the complexification of the real G-representation $\mathsf{H}_\mathbb{R}$ given as the fixed point set of C. Representations of real type admit a non-degenerate symmetric bilinear form $(\phi, \psi) = \langle C\phi, \psi \rangle$. Conversely, given a G-invariant non-degenerate skew-symmetric bilinear form, define a conjugate linear endomorphism T by $(\phi, \psi) = \langle T\phi, \psi \rangle$. The square $T^2 \in \mathrm{End}(\mathsf{H})$ is $\mathbb{C}$-linear and is positive definite. Hence $|T| = (T^2)^{1/2}$ commutes with T, and $C = T|T|^{-1}$ defines a real structure. We will call the bilinear form *compatible* with the Hermitian structure if $C = T$.

(ii) H is of *quaternionic type* if it admits a G-equivariant conjugate linear endomorphism C with $C^2 = -I$. In this case, C gives H the structure of a quaternionic G-representation, where scalar multiplication by the quaternions i, j, k is given by $i = \sqrt{-1}$, $j = C$, $k = ij$. From C one obtains a non-degenerate skew-symmetric bilinear form $(\phi, \psi) = \langle C\phi, \psi \rangle$. Conversely, given a G-invariant non-degenerate symmetric bilinear form, define a conjugate linear endomorphism T by $(\phi, \psi) = \langle T\phi, \psi \rangle$. Again, $|T| = (-T^2)^{1/2}$ commutes with T, and $C = T|T|^{-1}$ defines a quaternionic structure. We will call the bilinear form *compatible* with the Hermitian structure if $C = T$.

(iii) For G-representations H of real or quaternionic type, the structure map C gives an isomorphism with the dual G-representation H^*. That is, such representations are self-dual. We will call a unitary G-representation of *complex type* if it is not self-dual.

For a real or quaternionic representation, the corresponding bilinear form defines an element of $\mathrm{Hom}_G(\mathsf{H}, \mathsf{H}^*)$. If H is irreducible, then this space is 1-dimensional. Hence for irreducible representations the real and quaternionic cases are exclusive. This proves part of the following result (see [30, Chapter II.6]).

Theorem 3.11 *An irreducible unitary representation of a compact Lie group G is either of real, complex or quaternionic type.*

We now specialize to the spin representations. The canonical bilinear form on spinor modules S can be viewed as scalar-valued, after choice of a trivialization of the canonical line K_{S}.

Proposition 3.22 *Let V be a Euclidean vector space of even dimension $n = 2m$, and S a unitary spinor module over $\mathbb{C}l(V)$. Then the Hermitian metric on S and the canonical bilinear form are compatible.*

Proof We use the model $V^{\mathbb{C}} = F^* \oplus F$, $\mathsf{S} = \wedge F^*$. Let $f^1, \ldots, f^m$ be a basis of F^*, orthonormal relative to the Hermitian inner product. Then the f^I for subsets $I \subseteq \{1, \ldots, m\}$ define an orthonormal basis of $\wedge F^*$. For any subset I let I^c be the complementary subset. Use $f^1 \wedge \cdots \wedge f^m$ to trivialize $\det(F^*)$, and define signs $\varepsilon_I = \pm 1$ by

$$(f^I)^\top \wedge f^{I^c} = \varepsilon_I \, f^1 \wedge \cdots \wedge f^m.$$

Thus $(f^I, f^{I^c})_{\mathsf{S}} = \varepsilon_I$. Notice that $\varepsilon_I = \varepsilon_{I^c}$ in the symmetric case and $\varepsilon_I = -\varepsilon_{I^c}$ in the skew-symmetric case. Define C by

$$(f^I, f^J)_{\mathsf{S}} = \langle C f^I, f^J \rangle.$$

Then $Cf^I = \varepsilon_I f^{I_c}$. We note that $C^2 = I$ in the symmetric case and $C^2 = -I$ in the skew-symmetric case. $\square$

Theorem 3.12 *The types of the spin representations of* $\mathrm{Spin}(n)$ *are as follows.*

$$\begin{aligned} n = 0 \mod 8 &: \quad \Delta_n^\pm \text{ real type}, \\ n = 1, 7 \mod 8 &: \quad \Delta_n \text{ real type}, \\ n = 2, 6 \mod 8 &: \quad \Delta_n^\pm \text{ complex type}, \\ n = 3, 5 \mod 8 &: \quad \Delta_n \text{ quaternionic type}, \\ n = 4 \mod 8 &: \quad \Delta_n^\pm \text{ quaternionic type}. \end{aligned}$$

If $n = 2, 6 \mod 8$, *one has* $\Delta_n^- \cong (\Delta_n^+)^*$.

Proof We consider various subcases.

- Case 1a: $n = 2m$ with m even. The canonical bilinear form $(\cdot, \cdot)_S$ is still non-degenerate on the even and odd part of the spinor module; hence it defines a real structure if $m = 0$ mod 4 and a quaternionic structure for $m = 2$ mod 4.
- Case 1b: $n = 2m$ with m odd. Then $(\cdot, \cdot)_S$ defines a non-degenerate pairing between $\mathsf{S}_n^{\bar{0}} = \Delta_n^+$ and $\mathsf{S}_n^{\bar{1}} = \Delta_n^-$. Hence $\Delta_n^\pm \ncong \Delta_n^\mp = (\Delta_n^\pm)^*$, so that $\Delta_n^\pm$ are of complex type.
- Case 2a: $n = 2m - 1$ is odd, with m even. Recall that $\Delta_n = \mathsf{S}_{n+1}^{\bar{0}}$. The restriction of $(\cdot, \cdot)_\mathsf{S}$ gives the desired non-degenerate symmetric (if $m = 0$ mod 4) or skew-symmetric (if $m = 2$ mod 4) bilinear form.
- Case 2b: $n = 2m - 1$ with m is odd. Here the restriction of $(\cdot, \cdot)_\mathsf{S}$ to $\mathsf{S}_{n+1}^{\bar{0}}$ is zero. We instead use the form

$$(\phi, \psi)' := (\phi,\ \rho(e_{n+1})\psi)_\mathsf{S} \equiv (\rho(e_{n+1})\phi, \psi)_\mathsf{S},$$

for $\phi, \psi \in \Delta_{2m-1} = \Delta_{2m}$. This is no longer $\mathrm{Spin}(n+1)$-invariant, but is still $\mathrm{Spin}(n)$-invariant. Since

$$\begin{aligned}(\psi, \phi)' &= (\rho(e_{n+1})\psi,\ \phi)_\mathsf{S}\\ &= (-1)^{m(m-1)/2}(\phi,\ \rho(e_{n+1})\psi)_\mathsf{S}\\ &= (-1)^{m(m-1)/2}(\phi, \psi)'.\end{aligned}$$

We find that the bilinear form is symmetric for $m = 1$ mod 4 and skew-symmetric for $m = 3$ mod 4. $\square$

3.7.6 Applications to compact Lie groups

Theorem 3.12 has several important Lie-theoretic implications.

Spin(3) The spin representation Δ_3 has dimension 2, and after choice of basis defines a homomorphism $\mathrm{Spin}(3) \to \mathrm{U}(2)$. Since $\mathrm{Spin}(3)$ is semisimple, the image lies in $\mathrm{SU}(2)$, and by dimension count, the resulting map $\mathrm{Spin}(3) \to \mathrm{SU}(2)$ must be an isomorphism. Since Δ_3 is of quaternionic type, one similarly has a homomorphism $\mathrm{Spin}(3) \to \mathrm{Aut}(\mathbb{H}) = \mathrm{Sp}(1)$, which is an isomorphism by dimension count. That is, we recover $\mathrm{Spin}(3) \cong \mathrm{SU}(2) \cong \mathrm{Sp}(1)$.

Spin(4) The two half-spin representations $\Delta_4^\pm$ are both 2-dimensional. By an argument similar to that for $n = 3$, we see that the representation on $\Delta_4^+ \oplus \Delta_4^-$ defines an isomorphism $\mathrm{Spin}(4) \to \mathrm{SU}(2) \times \mathrm{SU}(2)$.

Spin(5) Δ_5 is a 4-dimensional representation of quaternionic type. After choice of an orthonormal quaternionic basis, this gives a homomorphism to $\mathrm{Sp}(2) =$

$\mathrm{Aut}(\mathbb{H}^2)$, which by dimension count must be an isomorphism. This realizes the isomorphism

$$\mathrm{Spin}(5) \cong \mathrm{Sp}(2).$$

In particular, we see that Spin(5) acts transitively on the unit sphere $S^7 \subseteq \Delta_5 \cong \mathbb{C}^4$, since Sp(2) acts transitively on the quaternions of unit norm. The stabilizer of the base point $(0, 1) \in \mathbb{H}^2$ is $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ (embedded in Sp(2) as the upper left block). We hence see that

$$S^7 = \mathrm{Spin}(5)/\mathrm{SU}(2).$$

This checks with dimensions, since Spin(5) has dimension 10, while SU(2) has dimension 3.

Spin(6) The half-spin representations $\Delta_6^{\pm}$ are 4-dimensional, and define homomorphisms $\mathrm{Spin}(6) \to \mathrm{U}(4)$. Since Spin(6) is semisimple, this homomorphism must take values in SU(4), realizing the isomorphism

$$\mathrm{Spin}(6) \cong \mathrm{SU}(4).$$

In particular Spin(6) acts transitively on $S^7 \subseteq \Delta_6^{\pm} \cong \Delta_5$, extending the action of Spin(5), with stabilizers SU(3).

Spin(7) The 8-dimensional spin representation Δ_7 is of real type, hence it can be regarded as the complexification of an 8-dimensional real representation $\Delta_7^{\mathbb{R}} \cong \mathbb{R}^8$. Restricting to Spin(6), we have $\Delta_7 = \Delta_6^+ \oplus \Delta_6^-$. Under the symmetric bilinear form on Δ_7, both $\Delta_6^{\pm}$ are Lagrangian.[1] This implies that $\Delta_7^{\mathbb{R}} \cong \Delta_6^{\pm}$ as real $\mathrm{Spin}(6) \subseteq \mathrm{Spin}(7)$-representations. Since Spin(6) acts transitively on the unit sphere $S^7 \subseteq \Delta_6^{\pm}$, this shows that Spin(7) acts transitively on the unit sphere $S^7 \subseteq \Delta_7^{\mathbb{R}}$. Let H be the isotropy group at some given base point on S^7. It is a compact Lie group of dimension

$$\dim H = \dim \mathrm{Spin}(7) - \dim S^7 = 21 - 7 = 14.$$

More information is obtained using some homotopy theory. For a compact, simply connected simple Lie group G, one knows that $\pi_1(G) = \pi_2(G) = 0$, while $\pi_3(G) = \mathbb{Z}$. By the long exact sequence of homotopy groups of a fibration,

$$\cdots \to \pi_{k+1}(S^7) \to \pi_k(H) \to \pi_k(\mathrm{Spin}(7)) \to \pi_k(S^7) \to \cdots,$$

and using that $\pi_k(S^7) = 0$ for $1 < k < 7$ (Hurewicz' Theorem), we find that $\pi_1(H) = \pi_2(H) = 0$ and $\pi_3(H) = \mathbb{Z}$. It follows that H is simply connected and simple (otherwise $\pi_3(H)$ would have more summands). But in dimension 14 there is a unique such group: the exceptional Lie group G_2. This proves the following remarkable result.

[1] In more detail, recall that the bilinear form on $\Delta_7 \cong \mathsf{S}_8^{\bar{0}}$ is $\mathbb{C}l(7) \cong \mathbb{C}l^0(8)$-invariant. Restricting to $\mathbb{C}l(6) \subseteq \mathbb{C}l(7)$, we obtain a $\mathbb{C}l(6)$-invariant bilinear form on $\Delta_6^+ \oplus \Delta_6^- \cong \mathsf{S}_6$, which must agree with the canonical bilinear form up to scalar multiple. But the latter vanishes on the even and odd part of S_6.

Theorem 3.13 *There is a transitive action of* Spin(7) *on* S^7. *The stabilizer subgroups for this action are isomorphic to the exceptional Lie group* G_2. *That is,*

$$S^7 = \mathrm{Spin}(7)/G_2.$$

We remark that one can also directly identify the root system for H, avoiding the use of algebraic topology or appealing to the classification of Lie groups. This is carried out in Adams' book [1, Chapter 5].

Spin(8) The triality principle from Section 3.6 specializes to give a degree 3 automorphism j of the group $\mathrm{Spin}(8,\mathbb{C})$, along with a degree 3 automorphism J of $\mathbb{C}^8 \oplus \Delta_8^+ \oplus \Delta_8^-$ interchanging the three summands, such that the induced maps $J:\ \Delta_8^- \to \mathbb{C}^8$, etc., are equivariant relative to the automorphism j. Since $\Delta_8^\pm$ are of real type, one may hope for J to preserve the real subspace $\mathbb{R}^8 \oplus \Delta_8^{+,\mathbb{R}} \oplus \Delta_8^{-,\mathbb{R}}$, and for j to preserve Spin(8). This is accomplished by taking the vectors n, q in the construction of J, j (see Section 3.6) to lie in $\mathbb{R}^8$ and $\Delta_8^{+,\mathbb{R}}$, respectively. The trilinear form on $\mathbb{C}^8 \oplus \Delta_8^+ \oplus \Delta_8^-$ (cf. (3.15)) restricts to the real part, and can be used [34, 41] to define on $\mathbb{R}^8$ an octonion multiplication, $\mathbb{R}^8 \cong \mathbb{O}$. The exceptional group G_2 is now realized as the fixed point group for the automorphism j of the compact group Spin(8), and also as the automorphism group $\mathrm{Aut}(\mathbb{O})$ of the octonions. A beautiful survey of this theory is given in Baez's article [20]. For the history and other constructions of G_2, see Agricola [3].

The other exceptional groups E_6, E_7, E_8, F_4 are related to spin groups as well. For example, F_4 contains a copy of Spin(9), and the action of Spin(9) on $\mathfrak{f}_4/\mathfrak{o}(9)$ is isomorphic to the (real) spin representation $\Delta_9^{\mathbb{R}}$. In a similar fashion, E_8 contains a copy of $\mathrm{Spin}(16)/\mathbb{Z}_2$ (where $\mathbb{Z}_2$ is generated by the chirality element Γ_{16}), and the action of $\mathrm{Spin}(16)/\mathbb{Z}_2$ on the quotient $\mathfrak{e}_8/\mathfrak{o}(16)$ is isomorphic to the (real) spin representation of Spin(16) on $\Delta_{16}^{+,\mathbb{R}}$. (This checks with dimensions: E_8 is 248-dimensional, Spin(16) is 120-dimensional, and $\Delta_{16}^{+,\mathbb{R}}$ is $2^7 = 128$-dimensional.) Proofs, and a wealth of related results, can be found in Adams' book [1].

http://www.springer.com/978-3-642-36215-6

Clifford Algebras and Lie Theory
Meinrenken, E.
2013, XX, 321 p., Hardcover
ISBN: 978-3-642-36215-6