Preface

Given a symmetric bilinear form B on a vector space V, one defines the Clifford algebra $\mathrm{Cl}(V; B)$ to be the associative algebra generated by the elements of V, with relations

$$v_1 v_2 + v_2 v_1 = 2B(v_1, v_2), \quad v_1, v_2 \in V.$$

If $B = 0$ this is just the exterior algebra $\wedge(V)$, and for general B there is an isomorphism (the *quantization map*)

$$q: \wedge(V) \to \mathrm{Cl}(V; B).$$

Hence, the Clifford algebra may be regarded as $\wedge(V)$ with a new, "deformed" product.

Clifford algebras enter the world of Lie groups and Lie algebras from multiple directions. For example, they are used to give constructions of the spin groups $\mathrm{Spin}(n)$, the simply connected double coverings of $\mathrm{SO}(n)$ for $n \geq 3$. Going a little further, one then obtains the spin representations of $\mathrm{Spin}(n)$, which is an irreducible representation if n is odd and breaks up into two inequivalent irreducible representations if n is even. The "accidental" isomorphisms of Lie groups in low dimensions, such as $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$, all find natural explanations using the spin representation. Furthermore, there are explicit constructions of the exceptional groups E_6, E_7, E_8, F_4, G_2, using special features of the spin groups and the spin representation. There are many remarkable aspects of these constructions, see e.g., Adams' book [1] or the survey article by Baez [20].

Further relationships between Lie groups and Clifford algebras come from the theory of Dirac operators on homogeneous spaces. Recall that by the Borel–Weil Theorem, any irreducible representation of a compact Lie group G is realized as a space of holomorphic sections of a line bundle over an appropriate coadjoint orbit. These sections may be regarded as solutions of the Dolbeault–Dirac operator for the standard complex structure on the coadjoint orbit. For non-compact Lie groups, there are many constructions of representations using more general types of Dirac operators, beginning with the work of Atiyah–Schmid [16] and Parthasarathy [106]. See the book by Huang–Pandzic [66] for further references and more recent developments.

Given a Lie algebra $\mathfrak{g}$ with a non-degenerate invariant symmetric bilinear form B (e.g., $\mathfrak{g}$ semisimple, with B the Killing form), it is also of interest to consider the Clifford algebra of $\mathfrak{g}$ itself. Let $\phi \in \wedge^3\mathfrak{g}$ be the structure constant tensor of $\mathfrak{g}$, defined using the metric. By a beautiful observation of Kostant and Sternberg [90], the quantized element $q(\phi) \in \mathrm{Cl}(\mathfrak{g})$ squares to a scalar. If $\mathfrak{g}$ is complex semisimple, the cubic element is one of the basis elements

$$\phi_i \in \wedge^{2m_i+1}(\mathfrak{g}), \quad i = 1, \ldots, \mathrm{rank}(\mathfrak{g}),$$

of the *primitive subspace* $P(\mathfrak{g}) \subseteq \wedge(\mathfrak{g})^{\mathfrak{g}}$ of the invariant subalgebra of the exterior algebra. According to the Hopf–Koszul–Samelson Theorem, $\wedge(\mathfrak{g})^{\mathfrak{g}}$ is itself an exterior algebra $\wedge P(\mathfrak{g})$ with generators $\phi_1, \ldots, \phi_l$. In [86], Kostant proved the marvelous result that, similarly, the invariant subalgebra of the Clifford algebra $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}$ is itself a Clifford algebra $\mathrm{Cl}(P(\mathfrak{g}))$ over the quantized generators $q(\phi_i)$. In particular, the elements $q(\phi_i)$ all square to scalars—a surprising fact given that the ϕ_i can have very high degree.

My own involvement with this subject goes back to work with Anton Alekseev on group-valued moment maps [11]. We had developed a non-standard non-commutative equivariant de Rham theory tailored towards these applications, and were looking for a natural framework in order to understand its relation to the standard equivariant de Rham theory. Inspired by [86, 90], we were led to consider a *quantum Weil algebra*

$$\mathscr{W}(\mathfrak{g}) = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g}),$$

where $U(\mathfrak{g})$ is the enveloping algebra of $\mathfrak{g}$, and equipped with a differential similar to the usual Weil differential on the standard Weil algebra $W(\mathfrak{g}) = S(\mathfrak{g}^*) \otimes \wedge(\mathfrak{g}^*)$. The differential was explicitly given as a graded commutation with a cubic element

$$\mathscr{D}_{\mathfrak{g}} = \sum_i e^i \otimes e_i + 1 \otimes q(\phi) \in \mathscr{W}(\mathfrak{g}),$$

where e_i is a basis of $\mathfrak{g}$, e^i the dual basis, and $\phi \in \wedge^3\mathfrak{g}$ as above. The fact that $[\mathscr{D}_{\mathfrak{g}}, \cdot]$ squares to zero is due to the fact that $\mathscr{D}_{\mathfrak{g}}^2$ lies in the center of $\mathscr{W}(\mathfrak{g})$—in fact one finds that

$$\mathscr{D}_{\mathfrak{g}}^2 = \frac{1}{2}\mathrm{Cas}_g \otimes 1 + \mathrm{const.},$$

where $\mathrm{Cas}_{\mathfrak{g}} \in U(\mathfrak{g})$ is the quadratic Casimir element. In [4] and its sequel [7], we used this theory to give a new proof for the case of quadratic Lie algebras of *Duflo's Theorem*, giving an algebra isomorphism

$$\mathrm{sym} \circ \widehat{J^{1/2}} \colon S(\mathfrak{g})^{\mathfrak{g}} \to U(\mathfrak{g})^{\mathfrak{g}}$$

(cf. Section 5.6 for notation) between invariants in the symmetric and enveloping algebras of $\mathfrak{g}$.

Independently, Kostant [87] had introduced a more general cubic element

$$\mathscr{D}_{\mathfrak{g},\mathfrak{k}} \in U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}),$$

associated to a quadratic Lie subalgebra $\mathfrak{k} \subseteq \mathfrak{g}$ with orthogonal complement $\mathfrak{p} = \mathfrak{k}^{\perp}$, which he called the *cubic Dirac operator*. Taking $\mathfrak{k} = 0$, one recovers $\mathscr{D}_{\mathfrak{g},0} = \mathscr{D}_{\mathfrak{g}}$. For the case where $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of compact Lie groups G and K, with K a maximal rank subgroup of G, he used the cubic Dirac operator to obtain an algebraic version of the Borel–Weil construction [87, 88], not requiring invariant complex structures on G/K. (In the complex case, an algebraic version of the Borel–Weil method had been accomplished several decades earlier in Kostant's work [84].) Other beautiful applications include a generalization of the Weyl character formula, and the discovery of multiplets of representations due to Gross–Kostant–Ramond–Sternberg [57].

It is possible to go in the opposite direction and interpret $\mathscr{D}_{\mathfrak{g},\mathfrak{k}}$ as a geometric Dirac operator on a homogeneous space. As such, it corresponds to a particular affine connection with a non-zero torsion. Such geometric Dirac operators had been studied by Slebarski [114, 115] in the 1980s; the precise relation with Kostant's cubic Dirac operator was clarified in [2]. There are also precursors in conformal field theory and Kac–Moody theory; see in particular Kac–Todorov [73], Kazami–Suzuki [77] and Wassermann [119].

This book had its beginnings as lecture notes for a graduate course at the University of Toronto, taught in the fall of 2005. The plan for the lectures was to give an introduction to the cubic Dirac operator, covering some of the applications to Lie theory described above. To set the stage, it was necessary to develop some foundational material on Clifford algebras. In a similar graduate course in the fall of 2009, we included other aspects such as the theory of pure spinors and Petracci's new proof [107] of the Poincaré–Birkhoff–Witt Theorem. The book itself contains further topics not covered in the lectures; in particular it describes Kostant's structure theory of $\mathrm{Cl}(\mathfrak{g})$ for a complex reductive Lie algebra and surveys the recent developments concerning the properties of the Harish-Chandra projection for Clifford algebras, $\mathsf{hc}_{\mathrm{Cl}}\colon \mathrm{Cl}(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{t})$.

A number of interesting topics related to the theme of this book had to be omitted. For example, we did not include applications of the cubic Dirac operator to Kac–Moody algebras [53, 73, 79, 94, 103, 109, 119]. We also decided to omit a discussion of Dirac geometry [13, 43], group-valued moment maps [11] or generalized complex geometry [59, 60, 63], even though some of the material presented here is motivated by those topics. Furthermore, the book does not cover the applications to the classical dynamical Yang–Baxter equation [6, 51] or to the Kashiwara–Vergne conjecture [5, 8, 75].

Many of the topics in this book play a role in theoretical physics. We have already indicated some relationships with conformal field theory. The prototype of a Dirac operator (with respect to the Minkowski metric) appeared in Dirac's 1928 article on his quantum theory of the electron [46]. More general Dirac operators on pseudo-Riemannian manifolds were studied later, and became a standard tool in differential geometry with the advent of Atiyah–Singer's index theory [18, 97]. The role of representation theory in quantum mechanics was explored in Weyl's 1929 book [120]. The "multi-valued" representations of the rotation group, i.e., representations of the spin group, are of basic importance in quantum electrodynamics, corresponding to particles with possibly half-integer spin. A systematic study, introducing the group $\mathrm{Spin}(n)$, was made in 1935 in the work of Brauer–Weyl [29]. Detailed information on the numerous applications of Clifford algebras, spinors and Dirac operators in physics may be found in the books [23, 64].

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