Chapter 1
Introduction

This book is concerned with one of the most fundamental questions of mathematics: the relationship between algebraic formulas and geometric images.

At one of the first international mathematical congresses (in Paris in 1900), Hilbert stated a special case of this question in the form of his 16th problem (from his list of 23 problems left over from the nineteenth century as a legacy for the twentieth century).

In spite of the simplicity and importance of this problem (including its numerous applications), it remains unsolved to this day (although, as you will now see, many remarkable results have been discovered).

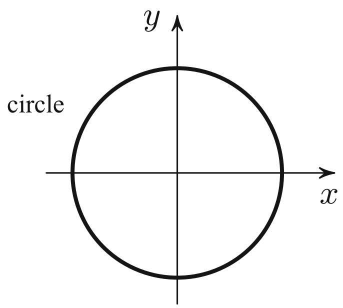
Let f be a polynomial (with real coefficients) of degree n in two variables x and y. Hilbert's question consists in investigating what topological structure an algebraic curve can have if that curve is defined in the Euclidean plane with Cartesian coordinates x and y by the equation[1]

$$f(x,y) = 0\,.$$

Example. If $n = 1$, then this equation defines a straight line, and all straight lines have the same topological structure.

If $n = 2$, then, as you know, the equation can define, for example, a circle

$x^2 + y^2 - 1 = 0$



a hyperbola

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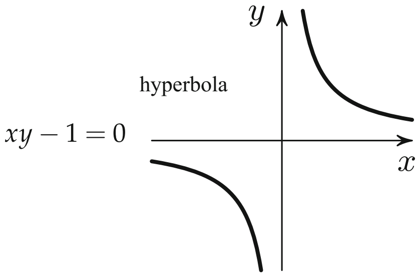
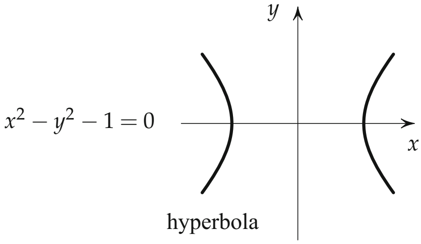
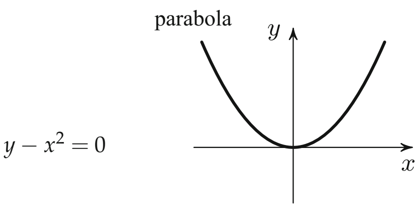


or a hyperbola in another form



The circle and hyperbola are topologically inequivalent: the circle is connected, while each hyperbola consists of two connected components (called branches) going off to infinity (along the "asymptotes" $\{x=0\}$ and $\{y=0\}$ for the first hyperbola, and $\{y=x\}$ and $\{y=-x\}$ for the second).

An equation of the second degree can also define a parabola



(which differs topologically from a circle and a hyperbola). Indeed, it is topologically equivalent to a straight line.

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