Chapter 2
Scalar Field Theories

In this chapter we put the RG concept to work on the simplest QFT based on a single real scalar field. This will illustrate how the conceptually simple idea of RG actually becomes challenging to implement in practice.

In particle physics we often write down simple actions like[1]

$$S[\phi] = \int d^d x \left(-\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!}\lambda \phi^4 \right); \tag{2.1}$$

however, in the spirit of RG we should—at least to start with—allow all possible terms consistent with space-time symmetries. In the case of a scalar field this involves all powers of the field and its derivatives contracted in a Lorentz invariant way. For simplicity we shall restrict to operators even under the symmetry $\phi \to -\phi$. This is an example of using a symmetry to restrict the possible couplings, the important point being that the symmetry is respected by RG flow. Simple scaling analysis shows that a composite operator $\mathcal{O}$ containing p derivatives and $2n$ powers of the field, schematically $\partial^p \phi^{2n}$, has classical mass dimension[2]

$$d_{\mathcal{O}} = n(d-2) + p. \tag{2.2}$$

Even at the classical level we see that the number of relevant/marginal couplings—those with $d_{\mathcal{O}} \leq d$—is small. The table below classifies some of the operators as relevant, marginal and irrelevant according to the dimension of space-time.

[1] In our notation, the scalar product in Minkowski space is $a_\mu b^\mu = \eta_{\mu\nu} a^\mu b^\nu = -a^0 b^0 + \mathbf{a} \cdot \mathbf{b}$.

[2] Note that the mass dimension of the field itself is fixed by the kinetic term to be $\frac{d-2}{2}$.

T. J. Hollowood, *Renormalization Group and Fixed Points*, SpringerBriefs in Physics, DOI: 10.1007/978-3-642-36312-2_2,

$\mathcal{O}$	$d > 4$	$d = 4$	$d = 3$	$d = 2$
ϕ^2	rel	rel	rel	rel
ϕ^4	irrel	marg	rel	rel
ϕ^6	irrel	irrel	marg	rel
ϕ^{2n}	irrel	irrel	irrel	rel
$(\partial_\mu\phi)^2$	marg	marg	marg	marg
$\phi^{2n}(\partial_\mu\phi)^2$	irrel	irrel	irrel	marg

The classical scaling suggests that, at least in dimensions $d > 2$, we only need to keep track of the kinetic term along with a completely general potential energy term, that is

$$\mathscr{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - V(\phi), \tag{2.3}$$

where we take

$$V(\phi) = \sum_n \mu^{d-n(d-2)} \frac{g_{2n}}{(2n)!}\phi^{2n}. \tag{2.4}$$

In the above, we have used the powers of the cut off μ in order to have dimensionless couplings g_{2n}.

2.1 Finding the RG Flow

Now we come to crux of the problem, that of finding the RG flow. In order to do this we must apply the RG Eq. (1.4) to the Wilsonian Effective Action $S[Z(\mu)^{1/2}\varphi; \mu, g_i(\mu)]$, defined for the theory with cut off μ in such a way that the observables on momentum scales below the cut off are fixed as μ is varied.

Before we can describe how to relate the theories with different cut offs we must first settle on a particular cut-off procedure. The most basic and conceptually simple way to regularize a scalar field theory is to introduce a sharp momentum cut off on the Fourier modes after Wick rotation to Euclidean space. In Euclidean space the Lagrangian (2.3) has the form

$$\mathscr{L}_E = \frac{1}{2}(\partial_\mu\phi)^2 + V(\phi) \tag{2.5}$$

with $S_E = \int d^dx\, \mathscr{L}_E$ and the functional integral becomes $\int[d\phi]\exp(-S_E)$.[3] The momentum cut off involves Fourier transforming the field

[3] We take it as established fact that one can transform between the Minkowski and Euclidean versions of the theory without difficulty. In our conventions, the Wick rotation involves $\eta_{\mu\nu}a^\mu b^\nu = -a_0b_0 + \mathbf{a}\cdot\mathbf{b} \to a_\mu b_\mu = a_0b_0 + \mathbf{a}\cdot\mathbf{b}$. In Euclidean space the functional integral $\int[d\phi]e^{-S_E[\phi]}$ can be interpreted as a probability measure (when properly normalized) on the field configuration space. This is why Euclidean QFT is intimately related to systems in statistical physics. In the following

$$\phi(x) = \int \frac{d^d p}{(2\pi)^d} \tilde{\phi}(p) e^{ip\cdot x} \tag{2.6}$$

and then limiting the momentum vector by means of a sharp cut off $|p| \leq \mu$. The resulting theory is now manifestly UV finite since the momenta in loops are never taken all the way to infinity. In addition, we have a very concrete way of performing the RG transformation. Namely, we split the field ϕ defined with cut off μ' into two pieces

$$\phi = \varphi + \hat{\phi}, \tag{2.7}$$

where φ has the modes with $|p| \leq \mu$, while $\hat{\phi}$ has the modes in the interval $\mu \leq |p| \leq \mu'$:

$$\begin{aligned} \phi(x) &= \int_{|p|\leq\mu'} \frac{d^d p}{(2\pi)^d} \tilde{\phi}(p) e^{ip\cdot x} \\ &= \underbrace{\int_{|p|\leq\mu} \frac{d^d p}{(2\pi)^d} \tilde{\phi}(p) e^{ip\cdot x}}_{\varphi(x)} + \underbrace{\int_{\mu\leq|p|\leq\mu'} \frac{d^d p}{(2\pi)^d} \tilde{\phi}(p) e^{ip\cdot x}}_{\hat{\phi}(x)} . \end{aligned} \tag{2.8}$$

In order to extract the beta functions it is sufficient to consider the infinitesimal transformation with $\mu' = \mu + \delta\mu$. We can then obtain the RG flow by considering how the action changes after we integrate out the Fourier modes $\hat{\phi}$; so concretely,

$$\begin{aligned} &\exp\left(-S[Z(\mu)^{1/2}\varphi; \mu, g_{2n}(\mu)]\right) \\ &\quad = \int [d\hat{\phi}] \exp\left(-S[\varphi + \hat{\phi}; \mu + \delta\mu, g_{2n}(\mu + \delta\mu)]\right) . \end{aligned} \tag{2.9}$$

On both sides, we have the Wilsonian Effective Action defined at the scales μ and $\mu + \delta\mu$, respectively. Notice that without-loss-of-generality we have scaled the field on the right-hand side so that $Z(\mu + \delta\mu) = 1$.

Expanding the action on the right-hand side in powers of $\hat{\phi}$[4]:

$$S[\varphi + \hat{\phi}] = S[\varphi] + \int d^d x \left(\frac{1}{2}(\partial_\mu \hat{\phi})^2 + \frac{1}{2} V''(\varphi)\hat{\phi}^2 + \frac{1}{6} V'''(\varphi)\hat{\phi}^3 + \cdots\right). \tag{2.10}$$

Note that the cross term $\int d^d x\, \partial_\mu \varphi \partial_\mu \hat{\phi}$ vanishes, a fact that becomes obvious when written in terms of momentum space modes. The non-trivial part of the problem is

we shall not show the subscript E for "Euclidean" since the context will dictate whether we are working in Minkowski or Euclidean space.

[4] In writing the action we have ignored the term $\hat{\phi} V'(\varphi)$ since its presence does not affect the effective potential that we calculate below. To appreciate this note that we can effectively take φ to be constant and then shift it to lie at the minimum of $V(\varphi)$.

to integrate over the modes $\hat{\phi}$. One way to do this is to work in terms of Feynman diagrams:

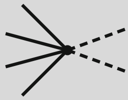
Feynman Diagram Interpretation

The terms that one gets in the effective action, after integrating out $\hat{\phi}$, can be interpreted in terms of Feynman diagrams with only $\hat{\phi}$ on internal lines contributing a propagator

$$\frac{1}{p^2 + g_2\mu^2} \quad \sim$$

and with only φ on external lines (but with no propagators). The vertices are provided by expanding the potential $V(\varphi + \hat{\phi})$ in powers of φ and $\hat{\phi}$. As an example, there is a vertex of the form

$$\frac{g_6}{4!2!}\varphi^4\hat{\phi}^2 \quad \sim$$



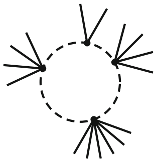
Each loop involves an integral over the momentum of $\hat{\phi}$ which lies in a shell between radii μ and μ' in momentum space:

$$\int_{\mu \le |p| \le \mu'} \frac{d^d p}{(2\pi)^d} f(p). \tag{2.11}$$

If we are only interested in an infinitesimal RG transformation $\mu' = \mu + \delta\mu$ then the integrals over loop momenta (2.11) become much simpler:

$$\int_{\mu \le |p| \le \mu + \delta\mu} \frac{d^d p}{(2\pi)^d} f(p) = \frac{\mu^{d-1}\delta\mu}{(2\pi)^d} \int d^{d-1}\hat{\Omega}\, f(\mu\hat{\Omega}), \tag{2.12}$$

where $\hat{\Omega}$ is a unit d vector to be integrated over a unit $d-1$-dimensional sphere S^{d-1}. Now comes the key part of the calculation: since each loop integral brings along a factor of $\delta\mu$, to linear order in $\delta\mu$, only one loop diagrams are needed. Since the dependence on the momentum in the loop can only be via the invariant $p^2 = \mu^2$, the integral over the solid angle $\hat{\Omega}$ just yields a constant $\mathrm{Vol}(S^{d-1})$, the volume of a $d-1$-dimensional sphere. Even with this huge simplification, we still have the non-trivial combinatorial problem of summing over an infinite set of one-loop diagrams one of which we show as an example below:



$$\sim \quad \frac{g_6 \cdot g_4 \cdot g_6 \cdot g_8}{2^5(4!)^2 6!} \frac{\mu^{\mathrm{d}-1}\delta\mu \, \mathrm{Vol}(S^{\mathrm{d}-1})}{(2\pi)^d} \frac{1}{(\mu^2 + g_2\mu^2)^4}$$

Fortunately, there is a simple way to sum all such one-loop diagrams in one go. The trick is to keep only the terms quadratic in $\hat{\phi}$ in (2.10),

$$S_2[\hat{\phi}] = \int d^d x \left(\frac{1}{2}(\partial_\mu \hat{\phi})^2 + \frac{1}{2} V''(\varphi)\hat{\phi}^2 \right) \tag{2.13}$$

and then perform the resulting Gaussian integral over $\hat{\phi}$

$$e^{-\delta S} = \int [d\hat{\phi}] e^{-S_2[\hat{\phi}]}. \tag{2.14}$$

In order to extract the RG transformation, we identify the change in the Wilsonian effective action as

$$S[\varphi; \mu, g_{2n}(\mu)] - S[\varphi; \mu + \delta\mu, g_{2n}(\mu + \delta\mu)] = \delta S. \tag{2.15}$$

In the present case there is no wave-function renormalization.[5] Since we are ultimately interested in the effective potential only, we can temporarily assume that φ is a constant. In the that case, denoting the Fourier transform of $\hat{\phi}(x)$ as $\tilde{\phi}(p)$, we have

$$S_2 = (\mu^2 + V''(\varphi)) \cdot \frac{\mu^{d-1}\delta\mu}{(2\pi)^d} \int d^{d-1}\hat{\Omega} \, \tilde{\phi}(\mu\hat{\Omega})\tilde{\phi}(\mu\hat{\Omega}) \,, \tag{2.16}$$

using the fact that $p_\nu p_\nu = \mu^2$ for the modes $\tilde{\phi}$. Hence, performing the Gaussian integral over the modes $\tilde{\phi}$ yields the result

$$e^{-\delta S} = C \left(\frac{\pi}{\mu^2 + V''(\varphi)} \right)^{\mathscr{N}/2}, \tag{2.17}$$

where $\mathscr{N}$ is the number of modes in the momentum shell. One way to count the modes is to introduce an infra-red cut off by defining the theory in a very large box of size L and imposing periodic boundary conditions on the field. This has the effect of quantizing the components of momenta as $p_\mu = 2\pi n_\mu / L$, for integers n_μ, in units $\hbar = 1$. We see that, effectively, there is one mode per volume $(2\pi)^d$ in Euclidean phase space. Therefore, assuming that L is sufficiently large, and denoting the volume of space-time as $\mathscr{V} = L^d$, the number of modes in the shell in momentum space is

[5] This is because in this theory at one-loop order the only Feynman diagram with two external legs has no external momentum running in the loop, and so cannot give rise to a term proportional to $(\partial_\mu \varphi)^2$.

$$\mathscr{N} = \frac{\mathrm{Vol}(S^{d-1})}{(2\pi)^d}\, \mu^{d-1}\, \delta\mu\, \mathscr{V}. \tag{2.18}$$

In this case, we can write the contribution in (2.17), up to an unimportant overall constant, as

$$\exp\left(-\delta S\right) = \exp\left(-a\mu^{d-1}\delta\mu \int d^d x\, \log(\mu^2 + V''(\varphi))\right), \tag{2.19}$$

where we have defined $a = \mathrm{Vol}(S^{d-1})/(2(2\pi)^d) = 2^{-d}\pi^{-d/2}/\Gamma(\frac{d}{2})$. Note that we have replaced the volume factor $\mathscr{V}$ by the integral over space-time which allows us to remove the temporary assumption that φ is constant. Hence, we can identify the change in the Wilsonian effective action as

$$\begin{aligned} &S[\varphi; \mu, g_{2n}(\mu)] - S[\varphi; \mu + \delta\mu, g_{2n}(\mu + \delta\mu)] \\ &\quad = a\mu^{d-1} \int d^d x\, \log(\mu^2 + V''(\varphi))\, \delta\mu. \end{aligned} \tag{2.20}$$

Expanding the right-hand side in powers of φ, allows us to extract the beta functions of the couplings directly:

$$\mu \frac{dg_{2n}}{d\mu} = (n(d-2) - d)\, g_{2n} - a\mu^{n(d-2)} \frac{d^{2n}}{d\varphi^{2n}} \log(\mu^2 + V''(\varphi))\Big|_{\varphi=0}. \tag{2.21}$$

If we expand in powers of the coupling constants, the contributions on the right-hand side can be identified with individual one-loop diagrams like the one shown previously.

From (2.21), the first few beta functions in the hierarchy are

$$\begin{aligned} \mu\frac{dg_2}{d\mu} &= -2g_2 - \frac{ag_4}{1+g_2}, \\ \mu\frac{dg_4}{d\mu} &= (d-4)g_4 + \frac{3ag_4^2}{(1+g_2)^2} - \frac{ag_6}{1+g_2}, \\ \mu\frac{dg_6}{d\mu} &= (2d-6)g_6 - \frac{30ag_4^3}{(1+g_2)^3} + \frac{15ag_4g_6}{(1+g_2)^2} - \frac{ag_8}{1+g_2}. \end{aligned} \tag{2.22}$$

Notice that the quantum contributions involve inverse powers of the factor $1 + g_2$ which physically is $m^2/\mu^2 + 1$, where m is the mass of the field. These factors arise from the propagators of the modes in the loop. So when $m \gg \mu$, the quantum terms are suppressed as one would expect on the basis of decoupling.[6]

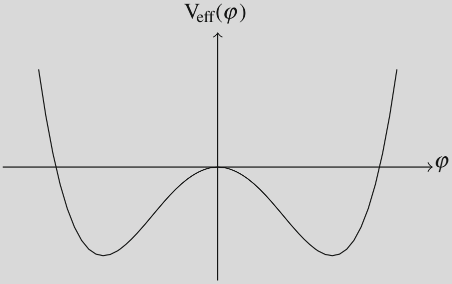
[6] Decoupling expresses the intuition that a particle of mass m cannot directly affect the physics on distance scales $\gg m^{-1}$. For instance, the potential due to the exchange of massive particle in four dimensions is $\sim e^{-mr}/r$. This is exponentially suppressed on distances scales $\gg m^{-1}$. In Chap. 4 we will make the notion of decoupling more precise.

The part of the Wilson effective action that depends just on the field and not its derivatives in the IR limit as $\mu \to 0$ is known as the *effective potential* $V_{\rm eff}(\varphi)$. It plays an important rôle since its minima determine the possible vacuum states of the theory. In the present approximation

$$V_{\rm eff}(\varphi) = \lim_{\mu\to 0}\sum_n \mu^{d-n(d-2)}\frac{g_{2n}(\mu)}{(2n)!}\varphi^{2n}. \qquad (2.23)$$

Vacuum Expectation Values

In a scalar QFT, the field can develop a non-trivial Vacuum Expectation Value (VEV) $\langle\phi\rangle \neq 0$. This possibility is determined by finding the global minima of the effective potential $\langle\phi\rangle = \varphi$. It can be shown that the effective potential defined in terms of the Wilsonian effective action is equal to the effective potential extracted from the more familiar 1-Particle Irreducible (1-PI) effective action defined in perturbation theory—at least for a QFT with a mass gap. A VEV develops when the effective potential develops minima away from the origin as in the example



Since we started with a theory symmetric under $\varphi \to -\varphi$, there will necessarily be two possible vacuum states with opposite values of the VEV $\langle\phi\rangle$. The theory must choose one or the other and so we say that the symmetry $\varphi \to -\varphi$ is *spontaneously broken*. The reason why spontaneous symmetry breaking can occur is that the constant mode of a scalar field is not part of the variables that appear in the measure of the functional integral $\int[d\phi]$. The rôle of the constant mode is to act as a boundary condition on the scalar field at spatial infinity. However, this is only true in space-time dimensions $d > 2$: in $d = 2$ small fluctuations can change the field at spatial infinity and so for consistency one must also integrate over the zero mode as well and as a consequence, spontaneous symmetry breaking of a *continuous* symmetry cannot occur. This is the statement of the Mermin-Wagner Theorem. In our case $\varphi \to -\varphi$ is a discrete symmetry and so escapes the implications of the theorem and can still be spontaneously broken even in $d = 2$.

2.2 Mapping the Space of Flows

The beta functions allow us to map-out the RG flow on theory space. The first thing to do is to find the RG fixed points corresponding to the CFTs. The "Gaussian fixed point" is the trivial fixed point where all the couplings vanish $g_{2n} = 0$. Linearizing around this point, the beta-functions are

$$\mu \frac{dg_{2n}}{d\mu} = (n(d-2) - d)\, g_{2n} - a g_{2n+2}. \tag{2.24}$$

So the scaling dimensions at the Gaussian fixed point are simply the classical dimensions $\Delta_{2n} = d_{2n} = n(d-2)$, i.e. the anomalous dimensions vanish, although the couplings that diagonalize the matrix of scaling dimensions σ_{2n} are not precisely equal to g_{2n} due to the second term in (2.24). In particular, $\sigma_2 = g_2$ is always relevant, $\sigma_4 = g_4 + a g_2/(2-d)$ is relevant for $d < 4$, irrelevant for $d > 4$ and marginally irrelevant for $d = 4$. In this latter case we need to go beyond the linear approximation. Since g_6 is irrelevant in $d = 4$, we shall ignore it, and using $a = 1/16\pi^2$ we have

$$\mu \frac{dg_4}{d\mu} = \frac{3}{16\pi^2} g_4^2, \tag{2.25}$$

whose solution is

$$\frac{1}{g_4(\mu)} = C - \frac{3}{16\pi^2} \log \mu. \tag{2.26}$$

This shows that g_4 is actually *marginally irrelevant* at the Gaussian fixed point because it gets smaller as μ decreases. We can write the integration constant in terms of a parameter Λ with dimensions of mass as follows:

$$g_4(\mu) = \frac{16\pi^2}{3 \log(\Lambda/\mu)}, \tag{2.27}$$

with $\mu < \Lambda$. This is an example of *dimensional transmutation*, where the freedom to specify a dimensionless coupling g_4 in the action turns into a quantity Λ with unit mass dimension in the quantum theory. Notice that Λ is the momentum scale at which the running coupling $g_4(\mu)$ diverges. Clearly, perturbation theory will break down as this scale is approached as we discuss in Chap. 3.

To find other non-trivial fixed points is difficult, but one way we can make progress is to work perturbatively in the couplings. In order to do this we have to do something that looks counter-intuitive and consider the RG flow equations in arbitrary non-integer dimension d regarding $\varepsilon = 4 - d$ as a small parameter. Hopefully, what is established for small ε will be qualitatively true for in the physical cases where $\varepsilon = 1, 2$, etc. Proceeding in this way, we find a new non-trivial fixed point known as the Wilson-Fisher fixed point at

$$g_2^* = -\frac{1}{6}\varepsilon + \cdots\,, \quad g_4^* = \frac{1}{3a}\varepsilon + \cdots\,, \quad g_{2n>4}^* \sim \varepsilon^n + \cdots \tag{2.28}$$

In particular, the Wilson-Fisher fixed point is only physically acceptable if $\varepsilon > 0$, or $d < 4$, since otherwise the couplings g_{2n}^* are all negative and the potential of the theory would not be bounded from below leading to an instability. In the neighbourhood of the fixed point in the (g_2, g_4) subspace, to linear order in ε, we have

$$\mu\frac{d}{d\mu}\begin{pmatrix}\delta g_2\\ \delta g_4\end{pmatrix} = \begin{pmatrix}\varepsilon/3-2 & -b(1+\varepsilon/6)\\ 0 & \varepsilon\end{pmatrix}\begin{pmatrix}\delta g_2\\ \delta g_4\end{pmatrix} \tag{2.29}$$

with

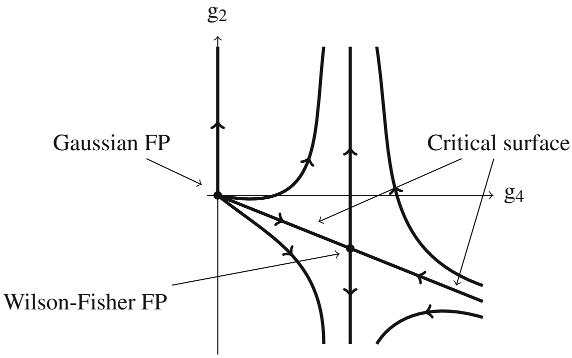
$$b = \frac{1}{16\pi^2} + \frac{\varepsilon}{32\pi^2}(1-\gamma_E+\log 4\pi) + \mathscr{O}(\varepsilon^2). \tag{2.30}$$

So the scaling dimensions of the associated operators and the associated couplings at the Wilson-Fisher fixed point are

$$\begin{aligned}\Delta_2 &= 2-\frac{2\varepsilon}{3}, \quad \sigma_2 = \delta g_2,\\ \Delta_4 &= 4, \quad \sigma_4 = 2(3+\varepsilon)\delta g_4 - b(3+\varepsilon/2)\delta g_2.\end{aligned} \tag{2.31}$$

Therefore at this fixed point only the mass coupling σ_2 is relevant.

The flows in the (g_2, g_4) subspace of scalar QFT for small $\varepsilon > 0$ are shown below:



The Gaussian and Wilson-Fisher fixed points are shown and since all the other couplings are irrelevant we only show the flows in the (g_2, g_4) subspace. It is important to realize, as is clear from (2.28), that the irrelevant couplings do not vanish on this subspace. Notice that the critical surface intersects this subspace in the line that joins the two fixed points as shown.

Although we have only proved the existence of the Wilson-Fisher fixed point for small ε, it is known to exist in both $d = 3$ and $d = 2$. In the language of statistical physics it lies in the universality class of the Ising Model.[7] What our simple analysis

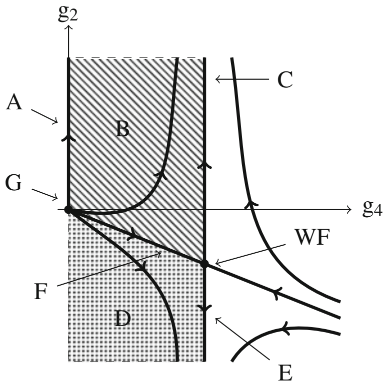
[7] The Ising Model is a statistical model defined on a square lattice with spins $\sigma_i \in \{+1, -1\}$ at each site and with an energy (which we identify with the Euclidean action)

fails to show is that in $d = 2$ there are actually an infinite sequence of additional fixed points.[8]

Now that we have a qualitative picture of the RG flows, it is possible to describe the possible continuum limits of scalar field theories:

$d = 4$: In this case, only the Gaussian fixed point exists and this fixed point only has one relevant direction; namely, the mass coupling g_2. Hence there is a single renormalized trajectory on which $g_2(\mu) = (\mu'/\mu)^2 g_2(\mu')$ while all the other couplings vanish. This renormalized trajectory describes the free massive scalar field. If we sit precisely at the fixed point we have a free massless scalar field. In particular, according to our crude analysis there is no interacting continuum theory in $d = 4$. In order to have an interaction the cut off must be kept finite. It turns out that this conclusion is backed up in more sophisticated analyses. This important fact, known as "triviality", has important implications for the Higgs sector of the standard model as we describe in Chap. 3.

$d = 3$: Assuming the existence of the Wilson-Fisher fixed point there are two fixed points and a two-dimensional space of renormalized trajectories parametrized by the couplings g_2 and g_4 on which g_{2n}, $n > 2$ have some values fixed by g_2 and g_4. In particular, if we parametrize our continuum theories by the values of g_2 and g_4 then they are limited to the regions shown below:



In particular, the line of theories **A** is free and massive (and must have $g_2 > 0$); the theories in regions **B** and **D** are interacting and massive and in the UV becomes non-

(Footnote 7 continued)

$$\mathscr{E} = -\frac{1}{T}\sum_{(i,j)} \sigma_i \sigma_j. \tag{2.32}$$

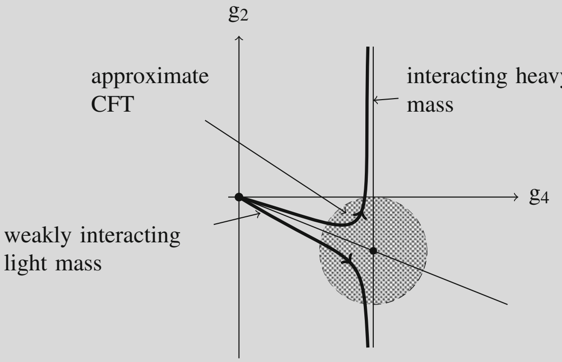
The sum is over all nearest-neighbour pairs (i, j) and T is the temperature. Notice that at low temperatures, the action/energy favours alignment of all the spins, while at high temperatures thermal fluctuations are large and the long-range order is destroyed. This can be viewed as a competition between energy and entropy. There is a 2nd order phase transition at a critical temperature $T = T_c$ at which there are long-distance power-law correlations. This critical point is in the same universality class as the Wilson-Fisher fixed point.

[8] In $d = 2$ there are powerful exact methods for analyzing CFTs because in $d = 2$ the conformal group is infinite dimensional as it consists of any holomorphic transformation $t \pm x \rightarrow f_\pm(t \pm x)$.

interacting, or free, since the trajectories originate from the Gaussian fixed point. The continuum theories are parametrized by two couplings, a mass scale and an interaction strength. In case **D**, $g_2 < 0$ and the field has a VEV; the line of theories **C** and **E** describe massive interacting theories that become the Ising model CFT in the UV. Furthermore, case **E** has $g_2 < 0$ and a VEV; the line of theories **F** describe a massless interacting theory that interpolates between a free theory in the UV and the Ising Model CFT in the IR; the point **G** is a free massless theory (the Gaussian fixed point); and the point **WF** is the Ising model CFT (the Wilson-Fisher fixed point). Any points to the right of the line **C–E** do not have continuum limits.

RG Crossover

The RG flows in $d = 3$ illustrate the notion of an RG *crossover*. Consider the two theories associated to the RG trajectories that we denoted as **B** and **D** shown again below:



These trajectories issue from the Gaussian fixed point and in the far UV the theories become free, a property more common in gauge theories known as *asymptotic freedom*, studied in Chap. 4. At intermediate energy scales the trajectories pass close to the Wilson-Fisher fixed point (the dotted region) and the spectrum consists of a light interacting scalar m with $m \ll \mu$. The theory here is approximately conformally invariant with the physics determined approximately by the Wilson-Fisher CFT. However, as μ runs down to and beyond m, the RG trajectory veers away the Wilson-Fisher fixed-point and the theory becomes interacting and massive.

Bibliographical Notes

The implementation of the Wilsonian RG in scalar field theories with a sharp momentum cut off is a well-studied problem. The approximation of the full effective action by the kinetic term and effective potential is known as the *local potential approximation* and was studied originally by Wegner and Houghton (1973). The problem is studied in Chap. 12 of Peskin and Schroeder (1995). Approaches that go beyond this are known as the "exact", "functional" or "continuous" RG and have been developed by many authors including, for example, Polchinski (1984), Morris (1996) and Polonyi (2003).

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