Geometrical Picture of Third-Order Tensors

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Abstract Because of its strong physical meaning, the decomposition of a symmetric second-order tensor into a deviatoric and a spheric part is heavily used in continuum mechanics. When considering higher-order continua, third-order tensors naturally appear in the formulation of the problem. Therefore researchers had proposed numerous extensions of the decomposition to third-order tensors. But, considering the actual literature, the situation seems to be a bit messy: definitions vary according to authors, improper uses of denomination flourish, and, at the end, the understanding of the physics contained in third-order tensors remains fuzzy. The aim of this paper is to clarify the situation. Using few tools from group representation theory, we will provide an unambiguous and explicit answer to that problem.

1 Introduction

In classical continuum mechanics [28, 29], only the first displacement gradient is involved and all the higher-order displacement gradients are neglected in measuring the deformations of a body. This usual kinematical framework turns out not to be rich enough to describe a variety of important mechanical and physical phenomena. In particular, the size effects and non-local behaviors due to the discrete nature of matter at a sufficiently small scale, the presence of microstructural defects or the existence of internal constraints cannot be captured by classical continuum mechanics [2, 18, 24]. The early development of higher-order (or generalized) continuum theories of elasticity was undertaken in the 1960s and marked with the major contributions of [5, 19–21, 26]. For the last two decades, the development and application of high-order continuum theories have gained an impetus, owing to a growing interest in modeling

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and simulating size effects and non-local behaviors observed in a variety of materials, such as polycrystalline materials, geomaterials, biomaterials and nanostructured materials (see, e.g., [7, 17, 22]), and in small size structures. In order to take into account size-effects, the classical continuum mechanics has to be generalized. To construct such an extension there are, at least, two options:

- Higher-order continua:
  In this approach the set of degrees of freedom is extended; a classical example is the micromorphic theory [6, 11, 20];
- Higher-grade continua:
  In this approach the mechanical state is described using higher-order gradients of the displacement field; a classical example is the strain-gradient theory [19].

In the following section the linear formulation of micromorphic and strain-gradient theory we will be detailed. The aim is to anchor the analysis that will be made on third-order tensors into a physical necessity for the understanding of those models.

2 Some Generalized Continua

2.1 Micromorphic Elasticity

Let us begin with the micromorphic approach. In this theory the set of degrees of freedom (DOF) is extended in the following way

$$\mathrm{DOF} = \{\underline{u}, \underset{\sim}{\chi}\} \quad ; \quad (\underline{u}, \underset{\sim}{\chi}) \in \mathbb{R}^3 \times \otimes^2\mathbb{R}^3,$$

where $\otimes^k\mathbb{V}$ stands for the k-th order tensorial power of $\mathbb{V}$. In this formulation the second-order tensor $\underset{\sim}{\chi}$ is generally not symmetric. This micro-deformation tensor encodes the generally incompatibility deformation of the microstructure. As a consequence, the set of primary state variables (PSV) now becomes

$$\mathrm{PSV} = \{\underline{u} \otimes \underline{\nabla}, \underset{\sim}{\chi} \otimes \underline{\nabla}\},$$

where $\underline{\nabla}$ is the classical nabla vector, i.e.

$$\underline{\nabla}^{T} = \left(\frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\ \frac{\partial}{\partial z}\right)$$

It can be observed that, despite being of higher-degree, the obtained model is still a 1st-grade continuum. The model is defined by the following set of strain measures:

- $\underset{\sim}{\varepsilon} = \varepsilon_{(ij)}$ is the strain tensor;

- $\underset{\sim}{e} = \underline{u} \otimes \underline{\nabla} - \underset{\sim}{\chi}$ is the relative strain tensor;
- $\underset{\simeq}{\kappa} = \underset{\sim}{\chi} \otimes \underline{\nabla}$ is the micro-strain gradient tensor;

where the notation $(..)$ indicates symmetry under in parentheses permutations. The first strain measure is the classical one and is, as usually, described by a symmetric second-order tensor. The relative strain tensor measures how the micro-deformation differs from the displacement gradient, this information is encoded into a non-symmetric second-order tensor. Finally, we have the third-order non-symmetric micro strain-gradient tensor. By duality the associated stress tensors can be defined:

- $\underset{\sim}{\sigma} = \sigma_{(ij)}$ is the Cauchy stress tensor;
- $\underset{\sim}{s} = s_{ij}$ is the relative stress tensor;
- $\underset{\simeq}{S} = S_{ijk}$ is the double-stress tensor.

If we suppose that the relation between strain and stress tensors is linear, the following constitutive law is obtained:

$$\begin{cases} \underset{\sim}{\sigma} = \underset{\approx}{A} : \underset{\sim}{\varepsilon} + \underset{\approx}{B} : \underset{\sim}{e} + \underset{\widetilde{\approx}}{C} \therefore \underset{\simeq}{\kappa} \\ \underset{\sim}{s} = \underset{\approx}{B}^{T} : \underset{\sim}{\varepsilon} + \underset{\approx}{D} : \underset{\sim}{e} + \underset{\widetilde{\approx}}{E} \therefore \underset{\simeq}{\kappa} \\ \underset{\simeq}{S} = \underset{\widetilde{\approx}}{C}^{T} : \underset{\sim}{\varepsilon} + \underset{\widetilde{\approx}}{E}^{T} : \underset{\sim}{e} + \underset{\underset{\sim}{\approx}}{F} \therefore \underset{\simeq}{\kappa} \end{cases}$$

The behavior is therefore defined by

- three fourth-order tensors having the following index symmetries: $A_{\underline{(ij)}\ \underline{(lm)}}$; $B_{(ij)lm}$; $D_{\underline{ij}\ \underline{lm}}$;
- two fifth-order tensors having the following index symmetries: $C_{(ij)klm}$; E_{ijklm};
- one sixth-order tensor having the following index symmetries: $F_{\underline{ijk}\ \underline{lmn}}$,

where $\underline{.}\ \underline{.}$ indicates symmetry under block permutations.

2.2 *Strain-Gradient Elasticity*

In the strain-gradient elasticity the set of degrees of freedom is the usual one, but the primary state variables are extended to take the second gradient of $\underline{u}$ into account:

$$\text{PSV} = \{\underline{u} \otimes \underline{\nabla}, \underline{u} \otimes \underline{\nabla} \otimes \underline{\nabla}\}$$

We therefore obtain a second-grade continuum defined by the following set of strain measures:

- $\underset{\sim}{\varepsilon} = \varepsilon_{(ij)}$ is the strain tensor;
- $\underset{\simeq}{\eta} = \underset{\sim}{\varepsilon} \otimes \underline{\nabla} = \eta_{(ij),k}$ is the strain-gradient tensor.

By duality, we obtain the related stress tensors:

- $\underset{\sim}{\sigma} = \sigma_{(ij)}$ is the Cauchy stress tensor;
- $\underset{\simeq}{\tau} = \tau_{(ij)k}$ is the hyper-stress tensor.

Assuming a linear relation between these two sets we obtain:

$$\begin{cases} \underset{\sim}{\boldsymbol{\sigma}} = \underset{\approx}{A} : \underset{\sim}{\varepsilon} + \underset{\approx}{\underset{\sim}{C}} \therefore \underset{\simeq}{\eta} \\ \underset{\simeq}{\boldsymbol{\tau}} = \underset{\approx}{\underset{\sim}{C}}^{T} : \underset{\sim}{\varepsilon} + \underset{\approx}{\underset{\approx}{F}} \therefore \underset{\simeq}{\eta} \end{cases}$$

The strain-gradient and hyperstress tensors are symmetric under permutation of their two first indices. The constitutive tensors verify the following index permutation symmetry properties:

$$C_{\underline{(ij)}\ \underline{(lm)}} \quad ; \quad M_{(ij)(kl)m} \quad ; \quad A_{\underline{(ij)k}\ \underline{(lm)n}}$$

2.3 Synthesis

Those two models are distinct but under the kinematic constraint $\underset{\sim}{\chi} = \underline{u} \otimes \underline{\nabla}$ strain-gradient elasticity is obtained from the micromorphic model. In the first case, the micro strain-gradient is element of:

$$\mathbb{T}_{ijk} = \{\underset{\simeq}{T} | \underset{\simeq}{T} = \sum_{i,j,k=1}^{3} T_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k\}$$

Assuming that we are in a 3D physical space, $\mathbb{T}_{ijk}$ is 27-dimensional and constructed as $\mathbb{T}_{ijk} = \otimes^3 \mathbb{R}^3$. For the strain-gradient theory, strain-gradient tensors belong to the following subspace of $\mathbb{T}_{ijk}$:

$$\mathbb{T}_{(ij)k} = \{\underset{\simeq}{T} | \underset{\simeq}{T} = \sum_{i,j,k=1}^{3} T_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k, \quad T_{ijk} = T_{jik}\}$$

which is 18-dimensional and constructed as[1] $\mathbb{T}_{(ij)k} = (\mathbb{R}^3 \otimes^S \mathbb{R}^3) \otimes \mathbb{R}^3$. Therefore, as it can be seen, the structure of the third-order tensors changes according to the considered theory.

Facing this kind of non-conventional model, a natural question is to ask what kind of information is encoded in these higher-order strain measures. In classical elasticity the physical content of symmetric second-order tensors is well-known through the

[1] The notation $\otimes^S$ indicates the symmetric tensor product.

physical meaning of its decomposition into a deviatoric (distorsion) and a spheric (dilatation) part. But the same result for third-order tensors is not so well-known, and its physical content has to be investigated. In the literature some results concerning the strain-gradient tensors can be found, but the situation seems to be fuzzy. In mechanics,[2] third-order tensor orthogonal decomposition was first investigated in the context of strain-gradient plasticity. According to the authors and the modeling assumptions the number of components varies from 2 to 4. In the appendices of [25] the authors introduced a first decomposition of the strain-gradient tensors under an incompressibility assumption, and expressed the decomposition into the sum of 3 mutually orthogonal parts. This decomposition was then used in [7, 8]. In [17] the situation is analyzed more in depth, and a decomposition into four parts is proposed. In some other works, it is said that strain-gradient can be divided into two parts. Therefore the following questions are raised:

- What is the right generalization of the decomposition of a tensor into deviatoric parts ?
- In how many orthogonal parts a third-order tensor can be split in a irreducible way ?
- Is this decomposition canonical ?

The aim of this paper is to answer these questions. These points will be investigated using the geometrical language of group action.

3 Harmonic Space Decomposition

To study the orthogonal decomposition of third-order tensors, and following the seemingly work of Georges Backus [3], an extensive use of harmonic tensors will be made. This section is thus devoted to formally introduce the concept of harmonic decomposition. After a theoretical introduction, the space of third-order tensors identified in the first section will be decomposed into a sum of harmonic tensor spaces. This $\mathrm{O}(3)$-irreducible[3] decomposition is the higher-order generalization of the well-known decomposition of $\mathbb{T}_{(ij)}$ into a deviatoric ($\mathbb{H}^2$) and spherical ($\mathbb{H}^0$) spaces.

3.1 The Basic Idea

Before studying decomposition of third-order tensors, let us get back for a while on the case of second-order symmetric ones. It is well known that any $\mathrm{T}_{(ij)} \in \mathbb{T}_{(ij)}$ admits the following decomposition:

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[2] In field of condensed matter physics this decomposition is known since, at least, the 70' [15].

[3] $\mathrm{O}(3)$: the orthogonal group, i.e. the group of all isometries of $\mathbb{R}^3$ i.e. if $\mathrm{Q} \in \mathrm{O}(3)$ $\det(\mathrm{Q}) \pm 1$ and $\mathrm{Q}^{-1} = \mathrm{Q}^{\mathrm{T}}$.

$$T_{(ij)} = H^2_{(ij)} + \frac{1}{3}H^0\delta_{ij} = \phi(H^2_{(ij)}, H^0),$$

where $H^2 \in \mathbb{H}^2$ and $H^0 \in \mathbb{H}^0$ are, respectively, the 5-D deviatoric and 1-D spheric part of $T_{(ij)}$ and are defined by the following formula:

$$H^0 = T_{ii} \quad ; \quad H^2_{(ij)} = T_{(ij)} - \frac{1}{3}H^0\delta_{ij}$$

In fact ϕ, defined by the expression (3.1), is an isomorphism between $\mathbb{T}_{(ij)}$ and the direct sum of $\mathbb{H}^2$ and $\mathbb{H}^0$

$$\mathbb{T}_{(ij)} \cong \mathbb{H}^2 \oplus \mathbb{H}^0$$

The main property of this decomposition is to be $O(3)$-invariant, or expressed in another way the components $(H^0, \underset{\sim}{H}^2)$ are covariant with $\underset{\sim}{T}$ under $O(3)$-action, i.e.

$$\forall \underset{\sim}{Q} \in O(3),\ \forall \underset{\sim}{T} \in \mathbb{T}_{(ij)}, \quad \underset{\sim}{Q}\underset{\sim}{T}\underset{\sim}{Q}^T = \phi(\underset{\sim}{Q}\underset{\sim}{H}^2\underset{\sim}{Q}^T, H^0)$$

Irreducible tensors satisfying this property are called harmonic. By irreducible we mean that those tensors can not be split into other tensors satisfying this property. In a certain way harmonic tensors are the elementary gears of the complete tensor. Let now give a more precise and general definition of this decomposition.

3.2 Harmonic Decomposition

The $O(3)$-irreducible decomposition of a tensor is known as its harmonic decomposition. Such a decomposition is well-known in group representation theory. It allows to decompose any finite order tensor into a sum of irreducible ones [3, 14, 30]. Consider a n-th order tensor T belonging to $\mathbb{T}$ then its decomposition can be written [14]:

$$T = \sum_{k,\tau} H^{k,\tau},$$

where the tensors $H^{k,\tau}$ are components[4] of the irreducible decomposition, k denotes the order of the harmonic tensor embedded in H and τ separates the same order terms. This decomposition defines an isomorphism between $\mathbb{T}$ and a direct sum of harmonic tensor spaces $\mathbb{H}^k$ [10] as

[4] To be more precise, $H^{k,\tau}$ is the embedding of the τth irreducible component of order k into a n-th order tensor.

$$\mathbb{T} \cong \bigoplus_{k,\tau} \mathbb{H}^{k,\tau}$$

but, as explained in [12], this decomposition is not unique. Alternatively, the O(3)-isotypic decomposition, where same order spaces are grouped, is unique:

$$\mathbb{T} \cong \bigoplus_{k=0}^{n} \alpha_k \mathbb{H}^k,$$

where α_k is the multiplicity of $\mathbb{H}^k$ in the decomposition, i.e. the number of copies of the space $\mathbb{H}^k$ in the decomposition. Harmonic tensors are totally symmetric and traceless. In $\mathbb{R}^3$, the dimension of their vector space $\dim \mathbb{H}^k = 2k + 1$. For $k = 0$ we obtain the space of scalars, $k = 1$ we obtain the space of vectors, $k = 2$ we obtain the space of deviators, and for $k > 2$ we obtain spaces of k-th order deviators. The family $\{\alpha_k\}$ is a function of the tensor space order and the index symmetries. Various methods exist to compute this family [1, 14, 30]. In $\mathbb{R}^3$ a very simple method based on the Clebsch-Gordan decomposition can be used.

In the next section this construction is introduced. It worths noting that we obtain the harmonic structure of the space under investigation modulo an unknown isomorphism. The construction of an isomorphism making this decomposition explicit is an ulterior step of the process. Furthermore, according to the nature of the sought information, the explicit knowledge of the isomorphism might by unnecessary. As an example, the determination of the set of symmetry classes of a constitutive tensor space does not require such a knowledge[5] [16, 23].

3.3 Computation of the Decomposition

The principle is based on the tensorial product of group representations. More details can be found in [1, 14]. The computation rule is simple. Consider two harmonic tensor spaces $\mathbb{H}^i$ and $\mathbb{H}^j$, whose product space is noted $\mathbb{G}^{i+k} := \mathbb{H}^i \otimes \mathbb{H}^j$. This space, which is GL(3)-invariant, admits the following O(3)-invariant decomposition:

$$\mathbb{G}^{i+j} = \bigoplus_{k=|i-j|}^{i+j} \mathbb{H}^k$$

For example, consider $\mathbb{H}^1_a$ and $\mathbb{H}^1_b$ two different first-order harmonic spaces. Elements of such spaces are vectors. According the above formula the O(3)-invariant decomposition of $\mathbb{G}^2$ is:

[5] Even if some authors explicitly construct this isomorphism [10, 13] this step is useless.

$$\mathbb{G}^2 = \mathbb{H}^1_a \otimes \mathbb{H}^1_b = \mathbb{H}^2 \oplus \mathbb{H}^{\sharp\,1} \oplus \mathbb{H}^0$$

In this decomposition, the space indicated with the $\sharp$ superscript contains *pseudo-tensors*, also known as *axial*-tensors i.e. tensors which change sign if the space orientation is reversed. Other elements are true tensors, also known as *polar*, and transform according to the usual rules.

As an example, the tensorial product of two spaces of vectors generates a second-order tensor space. The resulting structure is composed of a scalar ($\mathbb{H}^0$), a vector ($\mathbb{H}^{\sharp 1}$) and a deviator ($\mathbb{H}^2$). The vector part corresponds to the pseudo-vector associated with the matrix antisymmetric part. This computation rule has to be completed by the following properties [14]:

Property 2.1. The decomposition of an even-order (resp. odd-order) completely symmetric tensor, i.e. invariant under any index permutation, only contains even-order (resp. odd-order) harmonic spaces.

Property 2.2. In the decomposition of an even-order (resp. odd-order) even-order (resp. odd-order) components are polar and odd-order axial (resp. even order).

3.4 *Structure of Third-Order Strain Measures of Generalized Continua*

These techniques can now be applied to the third-order tensors involved in the micromorphic and the strain-gradient elasticity model.

Micromorphic Elasticity

Let us begin with the space $\mathbb{T}_{ijk}$ used in the micromorphic theory to model the micro-strain gradient $\underset{\simeq}{\kappa}$. As $\mathbb{T}_{ijk} \cong \otimes^3\mathbb{R}^3$, we have $\mathbb{T}_{ijk} \cong \mathbb{H}^1 \otimes \mathbb{H}^1 \otimes \mathbb{H}^1$. Using the Clebsch-Gordan rule:

$$\begin{aligned}\mathbb{T}_{ijk} &\cong \mathbb{H}^1 \otimes \mathbb{H}^1 \otimes \mathbb{H}^1\\ &\cong (\mathbb{H}^2 \oplus \mathbb{H}^{\sharp 1} \oplus \mathbb{H}^0) \otimes \mathbb{H}^1\\ &\cong \mathbb{H}^3 \oplus 2\mathbb{H}^{\sharp 2} \oplus 3\mathbb{H}^1 \oplus \mathbb{H}^{\sharp 0}\end{aligned}$$

Therefore $\mathbb{T}_{ijk}$ decompose into:

Name	$\mathbb{H}^3$: 3rd-order deviator	$\mathbb{H}^{\sharp 2}$: Pseudo-deviator	$\mathbb{H}^1$: Vector	$\mathbb{H}^{\sharp 0}$: Pseudo-scalar
Dimension	7	5	3	1
Multiplicity	1	2	3	1
Total	7	10	9	1

And if we sum the dimension of all irreducible spaces the 27-D of $\mathbb{T}_{ijk}$ is retrieved.

Strain-Gradient Elasticity

Now consider the space $\mathbb{T}_{(ij)k}$ used in strain-gradient theory to model the strain-gradient $\underset{\simeq}{\eta}$. As $\mathbb{T}_{(ij)k} \cong (\mathbb{R}^3 \otimes^S \mathbb{R}^3) \otimes \mathbb{R}^3$, we have $\mathbb{T}_{(ij)k} \cong (\mathbb{H}^{\sharp 2} \oplus \mathbb{H}^0) \otimes \mathbb{H}^1$. Using the Clebsch-Gordan rule:

$$\mathbb{T}_{(ij)k} \cong \mathbb{H}^3 \oplus \mathbb{H}^{\sharp 2} \oplus 2\mathbb{H}^1$$

Therefore $\mathbb{T}_{(ij)k}$ decompose into:

Name	$\mathbb{H}^3$: 3rd-order deviator	$\mathbb{H}^{\sharp 2}$: Pseudo-deviator	$\mathbb{H}^1$: Vector	$\mathbb{H}^{\sharp 0}$: Pseudo-scalar
Dimension	7	5	3	1
Multiplicity	1	1	2	0
Total	7	5	6	0

And if we sum the dimension of all irreducible spaces the 18-D of $\mathbb{T}_{(ij)k}$ is retrieved.

Analysis

Therefore, and despite what can be read in the literature, there is no spherical part in the decomposition of an element of $\mathbb{T}_{(ij)k}$. This worths being emphasized because in the micromorphic approach tensors do have such a component. Therefore, in order to avoid any misunderstanding, it is important to use the vocabulary in an appropriate way. Furthermore the use of a correct generalization of the harmonic decomposition to higher-order tensors provides useful information on the associated constitutive law. For example:

- the number of isotropic moduli associated to the isotropic related constitutive tensor (with great symmetry);
- the number and the dimension of eigenspaces of the related isotropic related constitutive tensor;
- the structure of anisotropy classes of the associated constitutive law [23];
- etc.

For the dimension of the isotropic symmetric constitutive law[6]

Theorem 2.1. *If* $\mathbb{T} \cong \bigoplus_{k=0}^{n} \alpha_k \mathbb{H}^k$ *then* $\dim(\mathrm{End}_S^{O(3)}(\mathbb{T})) = \sum_{k=0}^{n} \frac{\alpha_k(\alpha_k+1)}{2}$,

[6] The demonstration of theses theorems will be provided in a paper currently under redaction.

where $\mathrm{End}_S^{O(3)}(\mathbb{T})$ means the space of self-adjoint isotropic endomorphism of $\mathbb{T}$. For the next property we need to introduce the following definition.

Definition 2.1. Let L be a self-adjoint endomorphism of $\mathbb{T}$. The eigensignature of L, noted $\mathcal{ES}(L)$, is defined as the concatenation of the dimension of the eigenspaces of L.

For example, if we consider $\underset{\approx}{C}$ an isotropic elasticity tensor we have:

$$\mathcal{ES}(\underset{\approx}{C}) = \{51\}$$

as an isotropic elasticity tensor possesses two eigenspaces: one 5-dimensional and a unidimensional. The eigensignature of an operator contains both the number of its eigenspaces and theirs dimension.

Theorem 2.2. *If* $\mathbb{T} \cong \bigoplus_{k=0}^{n} \alpha_k \mathbb{H}^k$ *then for almost all* $L \in \mathrm{End}_S^{O(3)}(\mathbb{T})$ *;* $\mathcal{ES}(L) = \complement_{k=0}^{n}\{\alpha_k \complement\{2k+1\}\}$

in which $\complement$ indicates the concatenation operator, and the notation $\alpha\complement\{x\}$ indicates that α copies of x should be concatenated. The direct application of these results to our concern gives:

Theory	Third-order tensor decomposition	Number of isotropic moduli of the associated the sixth-order tensor	$\mathcal{ES}$
Micromorphic	$\mathbb{H}^3 \oplus 2\mathbb{H}^{\sharp 2} \oplus 3\mathbb{H}^1 \oplus \mathbb{H}^{\sharp 0}$	11	$\{75^2 3^3 1\}$
Strain-gradient	$\mathbb{H}^3 \oplus \mathbb{H}^{\sharp 2} \oplus 2\mathbb{H}^1$	5	$\{753^2\}$

Now the questions are (from a practical point of view):

1. How explicitly construct an associated isomorphism ?
2. Is this isomorphism canonical ?
3. Is there any mechanical meaning of that decomposition ?

In the following section, attention will restricted to the space of strain-gradient tensors.

4 Construction of the Isomorphism

As shown in the previous section:

$$\mathbb{T}_{(ij)k} \cong \mathbb{H}^3 \oplus \mathbb{H}^{\sharp 2} \oplus \mathbb{H}^{1,a} \oplus \mathbb{H}^{1,b}$$

It can be observed that any strain-gradient tensor contains 2 vectors in its decomposition. This fact is important since if the composition contains at least two harmonic components of the same order the isomorphism is not uniquely defined [12].

This indeterminacy will only concern the vector components since H^3 and $H^{\sharp 2}$ are uniquely defined. Therefore there is a degree of freedom in the definition of the vectors contained in the decomposition.

In fact, this situation also occurs in classical elasticity. The vector space of elasticity tensors can be decomposed as follows [3, 4, 10]

$$\mathbb{E}\mathrm{la} \simeq \mathbb{H}^4 \oplus \mathbb{H}^2_a \oplus \mathbb{H}^2_b \oplus \mathbb{H}^0_a \oplus \mathbb{H}^0_b$$

In this decomposition the two scalar parts are the elastic isotropic coefficients and therefore the isotropic moduli are not uniquely defined. This results in multiple ways to choose those coefficients: Young modulus & Poisson's ratio, Lamé constants, shear modulus & bulk modulus, and so on....

Therefore any construction is possible, but among them at least two are more natural since they give a physical meaning to the harmonic decomposition. The first one consists in splitting $T_{(ij)k}$ into a fully symmetric part and a remainder one before proceeding to the harmonic decomposition.

4.1 1st Decomposition: Stretch- and Rotation-Gradient

This approach is summed-up by the following diagram:

$$\begin{tikzcd}
 & \mathbb{T}_{(ij)k} \arrow[dl, "\mathrm{Sym}"'] \arrow[dr, "\mathrm{Asym{=}Id{-}Sym}"] & \\
\mathbb{S}_{(ijk)} \arrow[d, "\mathcal{H}"] & & \mathbb{R}_{ij} \arrow[d, "\mathcal{H}"] \\
\mathbb{H}^3 \oplus \mathbb{H}^1_a & & \mathbb{H}^{\sharp 2} \oplus \mathbb{H}^1_b
\end{tikzcd}$$

where Sym, Asym and $\mathcal{H}$ respectively stand for the symmetrization, anti-symmetrization and the harmonic decomposition processes. $T_{(ij)k}$ is first split into a full symmetric tensor and an asymmetric one:

$$T_{(ij)k} = S_{ijk} + \frac{1}{3}\left(\epsilon_{jkl}R_{li} + \epsilon_{ikl}R_{lj}\right)$$

The space of full symmetric third-order tensors is 10-dimensional meanwhile the space of asymmetric one is 8-dimensional, those spaces are in direct sum. In the strain-gradient literature [20] the complete symmetric part $S_{(ijk)}$, defined:

$$S_{(ijk)} = \frac{1}{3}(T_{(ij)k} + T_{(ki)j} + T_{(jk)i})$$

is the *stretch-gradient* part of $T_{(ij)k}$. Meanwhile the remaining traceless non-symmetric part R_{ij}:

$$R_{ij} = \epsilon_{ipq} T_{(jp)q}$$

is the *the rotation-gradient* part of $T_{(ij)k}$. In the couple-stress model, which is a reduced formulation of the strain-gradient model, only this tensor is taken into account in the mechanical formulation.

In terms of group action, it is important to note that this decomposition[7] is $GL(3)$-invariant,[8] and that each component is $GL(3)$-irreducible. In other terms, this decomposition of the strain-gradient into two "mechanisms" (stretch-gradient and rotation-gradient) is preserved under any invertible transformation. Under $O(3)$-action each part can further be decomposed in irreducible components by removing their different traces:

- $\mathbb{S}_{(ijk)}$ splits into a third-order deviator ($\dim \mathbb{H}^3 = 7$) and a vector ($\dim \mathbb{H}^1_a = 3$);
- $\mathbb{R}_{ij}$ splits into a pseudo-deviator ($\dim \mathbb{H}^{\sharp 2} = 5$) and a vector ($\dim \mathbb{H}^1_b = 3$).

Stretch-gradient tensors:
The space $\mathbb{S}_{(ijk)}$ is isomorphic to $\mathbb{H}^3 \oplus \mathbb{H}^1_{\nabla \mathrm{str}}$. The structure of this decomposition shows that this isomorphism is unique. Doing some algebra we obtain

$$S_{(ijk)} = H_{(ijk)} + \frac{1}{5}\left(V_i^{\nabla \mathrm{str}} \delta_{(jk)} + V_j^{\nabla \mathrm{str}} \delta_{(ik)} + V_k^{\nabla \mathrm{str}} \delta_{(ij)}\right)$$

with

$$V_i^{\nabla \mathrm{str}} = S_{(pp)i} = \frac{1}{3}\left(T_{ppi} + 2T_{ipp}\right);$$

$$H_{(ijk)} = S_{(ijk)} - \frac{1}{5}\left(V_i^{\nabla \mathrm{str}} \delta_{(jk)} + V_j^{\nabla \mathrm{str}} \delta_{(ik)} + V_k^{\nabla \mathrm{str}} \delta_{(ij)}\right)$$

In this formulation $V^{\nabla \mathrm{str}}$ is the vector part of *the stretch gradient tensor*.
Rotation-gradient tensors:
The space $\mathbb{R}_{ij}$ is isomorphic to $\mathbb{H}^{\sharp 2} \oplus \mathbb{H}^1_{\nabla \mathrm{rot}}$. The structure of this decomposition shows that this isomorphism is unique. Doing some algebra we obtain

$$R_{ij} = H_{(ij)} + \epsilon_{ijp} V_p^{\nabla \mathrm{rot}}$$

with

$$V_i^{\nabla \mathrm{rot}} = \frac{1}{2}\epsilon_{ipq}\left(R_{pq} - R_{qp}\right) = \frac{1}{2}\left(T_{ppi} - T_{ipp}\right);$$

$$H_{(ij)} = R_{ij} - \frac{1}{2}\epsilon_{ijp} V_p^{\nabla \mathrm{rot}} = \frac{1}{2}\left(R_{pq} + R_{qp}\right)$$

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7 This decomposition is sometimes known as the Schur decomposition.

8 $GL(3)$ is the group of all the invertible transformations of $\mathbb{R}^3$, i.e. if $F \in GL(3)$ then $\det(F) \neq 0$.

In this formulation $V^{\nabla\text{rot}}$ is the vector part of *the rotation gradient tensor*, and is embedded in the third-order tensor in the following way:

$$\mathsf{T}(\underline{V}^{\nabla\text{rot}})_{ijk} = \frac{1}{3}\left(-V_i^{\nabla\text{rot}}\delta_{(jk)} - V_j^{\nabla\text{rot}}\delta_{(ik)} + 2V_k^{\nabla\text{str}}\delta_{(ij)}\right)$$

Synthesis:
This decomposition can be summed-up in the following Matryoshka doll fashion[9]:

$$\mathbb{T}_{(ij)k} = \left(\mathbb{H}^3 \oplus \mathbb{H}^1_s\right)_{|\mathrm{GL}(3)} \oplus \left(\mathbb{H}^{\sharp 2} \oplus \mathbb{H}^1_r\right)_{|\mathrm{GL}(3)}$$

The decomposition into the in-parenthesis terms is preserved under any invertible transformation, and if this transformation is isometric the harmonic components are further more preserved. For a strain-gradient tensor this decomposition has the following meaning. Strain-gradient tensor encodes two orthogonal effects: stretch-gradient and rotation-gradient. These effects are canonically defined and preserved under invertible changes of variables. The harmonic decompositions of these elementary effects correspond to their decomposition in spherical harmonics. This construction has a meaning for any elements of $\mathbb{T}_{(ij)k}$.

4.2 2nd Decomposition: Distortion- and Dilatation-Gradient

Aside from this first construction, which was based on the algebra of third-order tensor, other constructions can be proposed. The following one is based on the derivation of the harmonic decomposition of a symmetric second-order tensor. As a consequence this construction has a physical meaning only for tensors constructed in this way.

$$\begin{tikzcd}
& \mathbb{T}_{(ij)} \arrow[dl, "\mathcal{H}"'] \arrow[dr, "\mathcal{H}"] & \\
\mathbb{H}^2 \arrow[d, "\otimes\underline{\nabla}"] & & \mathbb{H}^0 \arrow[d, "\otimes\underline{\nabla}"] \\
\mathbb{H}^3 \oplus \mathbb{H}^{\sharp 2} \oplus \mathbb{H}^{\sharp 1}_a & & \mathbb{H}^{\sharp 1}_b
\end{tikzcd}$$

So the first step is to decompose a second-order symmetric tensor into its deviatoric and its spherical part:

$$\mathsf{T}_{ij} = \mathsf{H}^2_{ij} + \frac{1}{3}\mathsf{H}^0\delta_{ij}$$

Such as

$$\mathsf{H}^0 = \mathsf{T}_{pp} \quad ; \quad \mathsf{H}^2_{ij} = \mathsf{T}_{ij} - \frac{1}{3}\mathsf{H}^0\delta_{ij}$$

[9] Another layer can be introduced in this decomposition if one consider also in-plane isometries.

Using the Clebsch-Gordan rule in the following way

$$\mathbb{H}^n \otimes \nabla \cong \mathbb{H}^n \otimes \mathbb{H}^1 \cong \bigoplus_{k=|n-1|}^{n+1} \mathbb{H}^k$$

we obtain

$$\mathbb{H}^2 \otimes \underline{\nabla} = \mathbb{H}^3 \oplus \mathbb{H}^{\sharp 2} \oplus \mathbb{H}^1_{\nabla dev} \quad ; \quad \mathbb{H}^0 \otimes \underline{\nabla} = \mathbb{H}^1_{\nabla sph}$$

In a certain way we have

$$\mathbb{T}_{(ij)k} = \mathbb{T}_{(ij)} \otimes \underline{\nabla} = \left(\mathbb{H}^3 \oplus \mathbb{H}^{\sharp 2} \oplus \mathbb{H}^1_{\nabla dev}\right)_{|\mathbb{H}^2 \otimes \underline{\nabla}} \oplus \left(\mathbb{H}^1_{\nabla sph}\right)_{\mathbb{H}^0 \otimes \underline{\nabla}}$$

But conversely to the decomposition (4.1) the in-parenthesis terms are not $GL(3)$-invariant. The first in parenthesis block is the *distortion-gradient* part of the strain-gradient meanwhile the last one is the *dilatation-gradient*.

As H^3 and $H^{\sharp 2}$ are uniquely defined their expressions are the same as before. Therefore attention is focused on the vector parts, doing some algebra we obtain:

$$V_i^{\nabla sph} = T_{ppi} \quad ; \quad V_i^{\nabla dev} = \frac{2}{3}\left(T_{ipp} - \frac{1}{3}T_{ppi}\right)$$

For $V^{\nabla sph}$ the result is direct, for $V^{\nabla dev}$ we have:

$$\mathrm{Sym}\,(H_{ij,k}) = S_{ijk} - \frac{1}{9}\left(\delta_{ij}T_{ppk} + \delta_{ki}T_{ppj} + \delta_{jk}T_{ppi}\right)$$

Therefore,

$$\begin{aligned} V_k^{\nabla dev} &= \mathrm{Sym}(H_{ij,k})\delta_{ij} = S_{iik} - \frac{1}{9}\left(5T_{ppk}\right) = \frac{1}{3}\left(T_{ppk} + 2T_{kpp}\right) - \frac{5}{9}\left(T_{ppk}\right) \\ &= \frac{2}{3}\left(T_{kpp} - \frac{1}{3}T_{ppk}\right) \end{aligned}$$

Those vectors are embedded into the third-order tensor in the following way:

$$\begin{aligned} T(V^{\nabla sph})_{ijk} &= \frac{1}{3}V_k^{\nabla sph}\delta_{ij}; \\ T(V^{\nabla dev})_{ijk} &= \frac{1}{5}\left(V_i^{\nabla dev}\delta_{(jk)} + V_j^{\nabla dev}\delta_{(ik)} + V_k^{\nabla dev}\delta_{(ij)}\right) \end{aligned}$$

4.3 Synthesis

If elements of $\mathbb{T}_{(\mathtt{ij})\mathtt{k}}$ are considered as gradient of a symmetric second-order tensors, their $\mathrm{O}(3)$-irreducible decompositions, can be defined in, at least, two ways. The first construction is the more general one and is based on the algebra of $\mathbb{T}_{(\mathtt{ij})\mathtt{k}}$, meanwhile the second is constructed from the algebra of $\mathbb{T}_{(\mathtt{ij})}$. Comparing the two decompositions, it appears that higher-order terms ($\mathbb{H}^3$ and $\mathbb{H}^{\sharp 2}$) are identical, whereas the vector parts are linear combination of each others. These results give an insight of the physical information encodes by H^3 and $\mathrm{H}^{\sharp 2}$
H^3:

- Its is generated by a part of the distortion gradient;
- Its elements encode a part of the stretch-gradient effect.

H^2:

- Its is generated by a part of the distortion gradient;
- Its elements encode a part of the rotation-gradient effect.

On the other hand the non uniqueness of the definition of the vector components shows that (Stretch- and rotation-gradient) and (Distortion- and Dilatation-gradient) are entangled phenomena. As, for example, the dilatation-gradient generates both stretch- and rotation-gradient components. Using this approach some physical based simplified strain-gradient elasticity models can be proposed.

Theory	Harmonic decomposition $\mathbb{T}_{\mathtt{ijk}}$	Dimension	Isotropic moduli
Strain gradient	$\mathbb{H}^3 \oplus \mathbb{H}^2 \oplus 2\mathbb{H}^1$	18	5
Distortion-gradient	$\mathbb{H}^3 \oplus \mathbb{H}^2 \oplus \mathbb{H}^1_{\nabla \mathtt{dev}}$	15	3
Stretch-gradient	$\mathbb{H}^3 \oplus \mathbb{H}^1_{\nabla \mathtt{str}}$	10	2
Rotation-gradient	$\mathbb{H}^2 \oplus \mathbb{H}^1_{\nabla \mathtt{rot}}$	8	2
Dilatation-gradient	$\mathbb{H}^1_{\nabla \mathtt{sph}}$	3	1

Therefore

$$\mathrm{V}^{\nabla \mathrm{str}} = \mathrm{V}^{\nabla \mathrm{dev}} + \frac{5}{9}\mathrm{V}^{\nabla \mathrm{sph}} \quad ; \quad \mathrm{V}^{\nabla \mathrm{rot}} = \frac{1}{3}\mathrm{V}^{\nabla \mathrm{sph}} - \frac{3}{4}\mathrm{V}^{\nabla \mathrm{dev}}$$

and conversely

$$\mathrm{V}^{\nabla \mathrm{sph}} = \frac{4}{3}\mathrm{V}^{\nabla \mathrm{rot}} + \mathrm{V}^{\nabla \mathrm{str}} \quad ; \quad \mathrm{V}^{\nabla \mathrm{dev}} = \frac{1}{9}(4\mathrm{V}^{\nabla \mathrm{str}} - \frac{20}{3}\mathrm{V}^{\nabla \mathrm{rot}})$$

The harmonic decomposition had been studied using two different but complementary constructions. In the context of strain-/stress-gradient [9] theories, these vector parts are related to differential operators acting on second-order symmetric tensors. To that aim, we consider $\underset{\sim}{\underline{\mathrm{T}}} \in \mathbb{T}_{(\mathtt{ij})\mathtt{k}}$ such that

$$\exists \underset{\sim}{D} \in \mathbb{T}_{(ij)} | \underset{\sim}{T} = \underset{\sim}{D} \otimes \underline{\nabla}$$

and we can define two vectors : $\nabla(\mathrm{tr}(\underset{\sim}{D}))$ and $\mathrm{div}(\underset{\sim}{D})$. The first vector is the same as $V^{\nabla sph}$ and second is:

$$V^{div} = \frac{3}{2}V^{\nabla dev} + \frac{1}{3}V^{\nabla sph} = \frac{1}{3}V^{\nabla str} - \frac{2}{3}V^{\nabla rot}$$

The irreducible vector parts of the harmonic can be expressed as a linear combination of these vectors. This is interesting because of their physical meaning. For strain-gradient elasticity, $\underline{V}^{\nabla sph}$ is the gradient of the infinitesimal volume variation δV, meanwhile $\underline{V}^{\nabla dev}$ is the strain divergence [27]. For the stress-gradient elasticity $\frac{1}{3}\underline{V}^{\nabla sph}$ represents the gradient of the isostatic pressure p, and $\underline{V}^{\nabla dev}$ is proportional to the volumic forces $\underline{f}$. Those vectors have, both for strain and stress gradient elasticity, a clear physical meaning.

	Strain-gradient	Stress-gradient
$\underline{V}^{\nabla sph}$	$\nabla\delta V$	$3\underline{\nabla p}$
$\underline{V}^{\nabla dev}$	$\frac{2}{3}\left(\mathrm{div}(\underset{\sim}{\varepsilon}) - \frac{1}{3}\nabla\delta V\right)$	$\frac{2}{3}\left(\underline{f} - \underline{\nabla p}\right)$
$\underline{V}^{\nabla str}$	$\frac{1}{3}\left(\nabla\delta V + 2\mathrm{div}(\underset{\sim}{\varepsilon})\right)$	$\underline{\nabla p} + \frac{2}{3}\underline{f}$
$\underline{V}^{\nabla rot}$	$\frac{1}{2}\left(\nabla\delta V - \mathrm{div}(\underset{\sim}{\varepsilon})\right)$	$\frac{1}{2}\left(3\underline{\nabla p} - \underline{f}\right)$

Appendix

In this appendix the explicit decompositions of $\underset{\simeq}{T}$ are provided.

Affine Decomposition

Let be defined the following subspace of third-order tensors

$$\mathcal{S}^3 = \{\mathbb{T} | \mathbb{T} = \sum_{i,j,k=1}^{3} T_{ijk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k,\;\; T_{ijk} = T_{jik}\} \tag{1}$$

which is an 18-dimensional vector space.

In order to express the strain gradient $\underset{\simeq}{T}$ as a second-order tensor, we consider the tensor product of the orthonormal basis vectors of second-order symmetric tensors with the one of classical vector.

$$\hat{\underset{\sim}{T}} = \psi(\underset{\simeq}{T}) = \hat{T}_{\alpha k}\hat{\mathbf{e}}_\alpha \otimes \mathbf{e}_k, \; 1 \leqslant \alpha \leqslant 6, 1 \leqslant k \leqslant 3 \tag{2}$$

with

$$\hat{\mathbf{e}}_\alpha = \left(\frac{1-\delta_{ij}}{\sqrt{2}} + \frac{\delta_{ij}}{2}\right)(\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i) \quad 1 \leqslant \alpha \leqslant 6 \tag{3}$$

With the orthonormal basis (3), the relationship between the matrix components $\hat{T}_{\alpha k}$ and T_{ijk} is specified by

$$\hat{T}_{\alpha k} = \begin{cases} T_{ijk} & \text{if } i = j, \\ \sqrt{2}T_{ijk} & \text{if } i \neq j; \end{cases} \tag{4}$$

Therefore for $\underset{\simeq}{T}$ we obtain the following matrix representation:

$$[\underset{\simeq}{T}] = \begin{pmatrix} T_{111} & T_{112} & T_{113} \\ T_{221} & T_{222} & T_{223} \\ T_{331} & T_{332} & T_{333} \\ \sqrt{2}T_{121} & \sqrt{2}T_{122} & \sqrt{2}T_{123} \\ \sqrt{2}T_{131} & \sqrt{2}T_{132} & \sqrt{2}T_{133} \\ \sqrt{2}T_{231} & \sqrt{2}T_{232} & \sqrt{2}T_{233} \end{pmatrix}$$

We can now construct the explicit matrix decomposition of $\underset{\simeq}{T}$.

• *Stretch-gradient tensor:*

$$[\underset{\simeq}{T}(\underset{\simeq}{S})] = \begin{pmatrix} S_1 & S_4 & S_5 \\ S_7 & S_2 & S_6 \\ S_8 & S_9 & S_3 \\ \sqrt{2}S_4 & \sqrt{2}S_7 & \sqrt{2}S_{10} \\ \sqrt{2}S_5 & \sqrt{2}S_{10} & \sqrt{2}S_8 \\ \sqrt{2}S_{10} & \sqrt{2}S_6 & \sqrt{2}S_9 \end{pmatrix}$$

$$= \begin{pmatrix} T_{111} & \frac{1}{3}(T_{112} + 2T_{121}) & \frac{1}{3}(T_{113} + 2T_{131}) \\ \frac{1}{3}(T_{221} + 2T_{122}) & T_{222} & \frac{1}{3}(T_{223} + 2T_{232}) \\ \frac{1}{3}(T_{331} + 2T_{133}) & \frac{1}{3}(T_{332} + 2T_{233}) & T_{333} \\ \frac{\sqrt{2}}{3}(T_{112} + 2T_{121}) & \frac{\sqrt{2}}{3}(T_{221} + 2T_{122}) & \frac{\sqrt{2}}{3}(T_{123} + T_{321} + T_{213}) \\ \frac{\sqrt{2}}{3}(T_{113} + 2T_{131}) & \frac{\sqrt{2}}{3}(T_{123} + T_{321} + T_{213}) & \frac{\sqrt{2}}{3}(T_{331} + 2T_{133}) \\ \frac{\sqrt{2}}{3}(T_{123} + T_{321} + T_{213}) & \frac{\sqrt{2}}{3}(T_{223} + 2T_{232}) & \frac{\sqrt{2}}{3}(T_{332} + 2T_{233}) \end{pmatrix}$$

• *Rotation-gradient tensor:*

As a second-order tensor:

$$[\underset{\sim}{\mathrm{R}}] = \begin{pmatrix} \mathrm{T}_{123} - \mathrm{T}_{132} & \mathrm{T}_{223} - \mathrm{T}_{232} & \mathrm{T}_{233} - \mathrm{T}_{332} \\ \mathrm{T}_{131} - \mathrm{T}_{113} & \mathrm{T}_{231} - \mathrm{T}_{123} & \mathrm{T}_{331} - \mathrm{T}_{133} \\ \mathrm{T}_{112} - \mathrm{T}_{121} & \mathrm{T}_{122} - \mathrm{T}_{221} & \mathrm{T}_{132} - \mathrm{T}_{231} \end{pmatrix}$$

and embedded into $\underset{\simeq}{\mathrm{T}}$:

$$[\underset{\simeq}{\mathrm{T}}(\underset{\sim}{\mathrm{R}})] = \begin{pmatrix} 0 & -2\mathrm{R}_3 & -2\mathrm{R}_5 \\ -2\mathrm{R}_1 & 0 & -2\mathrm{R}_6 \\ -2\mathrm{R}_2 & -2\mathrm{R}_4 & 0 \\ \sqrt{2}\mathrm{R}_3 & \sqrt{2}\mathrm{R}_1 & \sqrt{2}\mathrm{R}_7 \\ \sqrt{2}\mathrm{R}_5 & -\sqrt{2}(\mathrm{R}_7 + \mathrm{R}_8) & \sqrt{2}\mathrm{R}_2 \\ \sqrt{2}\mathrm{R}_8 & \sqrt{2}\mathrm{R}_6 & \sqrt{2}\mathrm{R}_4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\frac{2}{3}(\mathrm{T}_{121} - \mathrm{T}_{112}) & -\frac{2}{3}(\mathrm{T}_{131} - \mathrm{T}_{113}) \\ -\frac{2}{3}(\mathrm{T}_{122} - \mathrm{T}_{221}) & 0 & -\frac{2}{3}(\mathrm{T}_{232} - \mathrm{T}_{223}) \\ -\frac{2}{3}(\mathrm{T}_{133} - \mathrm{T}_{331}) & -\frac{2}{3}(\mathrm{T}_{233} - \mathrm{T}_{332}) & 0 \\ \frac{\sqrt{2}}{3}(\mathrm{T}_{121} - \mathrm{T}_{112}) & \frac{\sqrt{2}}{3}(\mathrm{T}_{122} - \mathrm{T}_{221}) & \frac{\sqrt{2}}{3}(2\mathrm{T}_{123} - \mathrm{T}_{132} - \mathrm{T}_{231}) \\ \frac{\sqrt{2}}{3}(\mathrm{T}_{131} - \mathrm{T}_{113}) & \frac{\sqrt{2}}{3}(2\mathrm{T}_{132} - \mathrm{T}_{123} - \mathrm{T}_{231}) & \frac{\sqrt{2}}{3}(\mathrm{T}_{133} - \mathrm{T}_{331}) \\ \frac{\sqrt{2}}{3}(2\mathrm{T}_{231} - \mathrm{T}_{132} - \mathrm{T}_{123}) & \frac{\sqrt{2}}{3}(\mathrm{T}_{232} - \mathrm{T}_{223}) & \frac{\sqrt{2}}{3}(\mathrm{T}_{233} - \mathrm{T}_{332}) \end{pmatrix}$$

Harmonic Decomposition

• *Vector part of the stretch-gradient tensor:*

As a vector:

$$[\underline{\mathrm{V}}^{\nabla\mathrm{str}}] = \begin{pmatrix} \mathrm{V}_1^{\nabla\mathrm{str}} = \frac{1}{3}(3\mathrm{T}_{111} + (\mathrm{T}_{221} + 2\mathrm{T}_{122}) + (\mathrm{T}_{331} + 2\mathrm{T}_{133})) \\ \mathrm{V}_2^{\nabla\mathrm{str}} = \frac{1}{3}(3\mathrm{T}_{222} + (\mathrm{T}_{332} + 2\mathrm{T}_{233}) + (\mathrm{T}_{112} + 2\mathrm{T}_{121})) \\ \mathrm{V}_3^{\nabla\mathrm{str}} = \frac{1}{3}(3\mathrm{T}_{333} + (\mathrm{T}_{113} + 2\mathrm{T}_{131}) + (\mathrm{T}_{223} + 2\mathrm{T}_{232})) \end{pmatrix}$$

and embedded into $\underset{\simeq}{\mathrm{T}}$:

$$[\underset{\simeq}{T}(\underline{V}^{\nabla str})] = \begin{pmatrix} \frac{3}{5}V_1^{\nabla str} & \frac{3}{15}V_2^{\nabla str} & \frac{3}{15}V_3^{\nabla str} \\ \frac{3}{15}V_1^{\nabla str} & \frac{3}{5}V_2^{\nabla str} & \frac{3}{15}V_3^{\nabla str} \\ \frac{3}{15}V_1^{\nabla str} & \frac{3}{15}V_2^{\nabla str} & \frac{3}{5}V_3^{\nabla str} \\ \frac{3\sqrt{2}}{15}V_2^{\nabla str} & \frac{3\sqrt{2}}{15}V_1^{\nabla str} & 0 \\ \frac{3\sqrt{2}}{15}V_3^{\nabla str} & 0 & \frac{3\sqrt{2}}{15}V_1^{\nabla str} \\ 0 & \frac{3\sqrt{2}}{15}V_3^{\nabla str} & \frac{3\sqrt{2}}{15}V_2^{\nabla str} \end{pmatrix}$$

• *Third-order deviator of any strain gradient tensor:*

We have the following relations:

$$\begin{cases} H_{111}^3 + H_{122}^3 + H_{133}^3 = 0 \\ H_{222}^3 + H_{112}^3 + H_{233}^3 = 0 \\ H_{333}^3 + H_{223}^3 + H_{113}^3 = 0 \end{cases}$$

Therefore

$$\begin{cases} H_{133}^3 = -H_{111}^3 - H_{122}^3 \\ H_{112}^3 = -H_{222}^3 - H_{233}^3 \\ H_{223}^3 = -H_{333}^3 - H_{113}^3 \end{cases}$$

Hence we got seven independent components $H_{111}^3, H_{122}^3, H_{222}^3, H_{233}^3, H_{333}^3, H_{113}^3$ and H_{123}^3, leading to the embedding

$$[\underset{\simeq}{T}(\underset{\simeq}{H}^3)] = \begin{pmatrix} H_1^3 & -(H_2^3 + H_5^3) & H_6^3 \\ H_4^3 & H_2^3 & -(H_3^3 + H_6^3) \\ -(H_1^3 + H_4^3) & H_5^3 & H_3^3 \\ -\sqrt{2}(H_2^3 + H_5^3) & \sqrt{2}H_4^3 & \sqrt{2}H_7^3 \\ \sqrt{2}H_6^3 & \sqrt{2}H_7^3 & -\sqrt{2}(H_1^3 + H_4^3) \\ \sqrt{2}H_7^3 & -\sqrt{2}(H_3^3 + H_6^3) & \sqrt{2}H_5^3 \end{pmatrix}$$

with

$$H_1^3 = H_{(111)}^3 = \frac{1}{5}\left(2T_{111} - (T_{221} + 2T_{122}) - (T_{331} + 2T_{133})\right)$$

$$H_2^3 = H_{(222)}^3 = \frac{1}{5}\left(2T_{222} - (T_{332} + 2T_{233}) - (T_{112} + 2T_{121})\right)$$

$$
\begin{aligned}
\mathrm{H}^3_3 &= \mathrm{H}^3_{(333)} = \frac{1}{5}\left(2\mathrm{T}_{333} - (\mathrm{T}_{113} + 2\mathrm{T}_{131}) - (\mathrm{T}_{223} + 2\mathrm{T}_{232})\right)\\
\mathrm{H}^3_4 &= \mathrm{H}^3_{(122)} = \frac{1}{15}\left(-3\mathrm{T}_{111} + 4(\mathrm{T}_{221} + 2\mathrm{T}_{122}) - (\mathrm{T}_{331} + 2\mathrm{T}_{133})\right)\\
\mathrm{H}^3_5 &= \mathrm{H}^3_{(233)} = \frac{1}{15}\left(-3\mathrm{T}_{222} + 4(\mathrm{T}_{332} + 2\mathrm{T}_{233}) - (\mathrm{T}_{112} + 2\mathrm{T}_{121})\right)\\
\mathrm{H}^3_6 &= \mathrm{H}^3_{(113)} = \frac{1}{15}\left(-3\mathrm{T}_{333} + 4(\mathrm{T}_{113} + 2\mathrm{T}_{311}) - (\mathrm{T}_{223} + 2\mathrm{T}_{322})\right)\\
\mathrm{H}^3_7 &= \mathrm{H}^3_{(123)} = \frac{1}{3}\left(\mathrm{T}_{123} + \mathrm{T}_{321} + \mathrm{T}_{213}\right)
\end{aligned}
$$

• *Vector part of the rotation-gradient tensor:*

As a vector:

$$
[\underline{\mathrm{V}}^{\nabla\mathrm{rot}}] = \begin{pmatrix}
\mathrm{V}_1^{\nabla\mathrm{rot}} = \frac{1}{2}((\mathrm{T}_{221} - \mathrm{T}_{122}) + (\mathrm{T}_{331} - \mathrm{T}_{133}))\\
\mathrm{V}_2^{\nabla\mathrm{rot}} = \frac{1}{2}((\mathrm{T}_{332} - \mathrm{T}_{233}) + (\mathrm{T}_{112} - \mathrm{T}_{121}))\\
\mathrm{V}_3^{\nabla\mathrm{rot}} = \frac{1}{2}((\mathrm{T}_{113} - \mathrm{T}_{311}) + (\mathrm{T}_{223} - \mathrm{T}_{322}))
\end{pmatrix}
$$

and embedded into $\underset{\simeq}{\mathrm{T}}$:

$$
[\underset{\simeq}{\mathrm{T}}(\underline{\mathrm{V}}^{\nabla\mathrm{rot}})] = \begin{pmatrix}
0 & \frac{2}{3}\mathrm{V}_2^{\nabla\mathrm{rot}} & \frac{2}{3}\mathrm{V}_3^{\nabla\mathrm{rot}}\\
\frac{2}{3}\mathrm{V}_1^{\nabla\mathrm{rot}} & 0 & \frac{2}{3}\mathrm{V}_3^{\nabla\mathrm{rot}}\\
\frac{2}{3}\mathrm{V}_1^{\nabla\mathrm{rot}} & \frac{2}{3}\mathrm{V}_2^{\nabla\mathrm{rot}} & 0\\
-\frac{\sqrt{2}}{3}\mathrm{V}_2^{\nabla\mathrm{rot}} & -\frac{\sqrt{2}}{3}\mathrm{V}_1^{\nabla\mathrm{rot}} & 0\\
-\frac{\sqrt{2}}{3}\mathrm{V}_3^{\nabla\mathrm{rot}} & 0 & -\frac{\sqrt{2}}{3}\mathrm{V}_1^{\nabla\mathrm{rot}}\\
0 & -\frac{\sqrt{2}}{3}\mathrm{V}_3^{\nabla\mathrm{rot}} & -\frac{\sqrt{2}}{3}\mathrm{V}_2^{\nabla\mathrm{rot}}
\end{pmatrix}
$$

• *Second-order pseudo-deviator of any strain gradient tensor:*

As a second-order tensor:

$$[\underset{\sim}{H}^2] = \begin{pmatrix} \frac{1}{3}(2H_4^2 + H_5^2) & H_1^2 & H_2^2 \\ H_1^2 & \frac{1}{3}(H_5^2 + H_3^2) & H_3^2 \\ H_2^2 & H_3^2 & -\frac{1}{3}(H_4^2 + 2H_5^2) \end{pmatrix}$$

With

$$H_1^2 = \frac{1}{2}\left((T_{223} - T_{232}) + (T_{131} - T_{113})\right); H_2^2 = \frac{1}{2}\left((T_{233} - T_{332}) + (T_{112} - T_{121})\right)$$

$$H_3^2 = \frac{1}{2}\left((T_{331} - T_{313}) + (T_{122} - T_{221})\right); H_4^2 = \frac{1}{3}\left(2T_{123} - T_{132} - T_{231}\right)$$

$$H_5^2 = \frac{1}{3}\left(2T_{231} - T_{132} - T_{123}\right)$$

and embedded in $\underset{\simeq}{T}$:

$$[\underset{\simeq}{T}(\underset{\sim}{H}^2)] = \begin{pmatrix} 0 & \frac{2}{3}H_2^2 & -\frac{2}{3}H_1^2 \\ -\frac{2}{3}H_3^2 & 0 & \frac{2}{3}H_1^2 \\ \frac{2}{3}H_3^2 & -\frac{2}{3}H_2^2 & 0 \\ -\frac{\sqrt{2}}{3}H_2^2 & \frac{\sqrt{2}}{3}H_3^2 & \sqrt{2}H_4^2 \\ \frac{\sqrt{2}}{3}H_1^2 & \sqrt{2}(H_4^2 + H_5^2) & -\frac{\sqrt{2}}{3}H_3^2 \\ -\sqrt{2}H_5^2 & -\frac{\sqrt{2}}{3}H_1^2 & \frac{\sqrt{2}}{3}H_2^2 \end{pmatrix}$$

Interpretation in Terms of Gradient

• *Dilatation-gradient vector:*

As a vector:

$$[\underline{V}^{\nabla\mathrm{sph}}] = \begin{pmatrix} V_1^{\nabla\mathrm{sph}} = T_{111} + T_{221} + T_{331} \\ V_2^{\nabla\mathrm{sph}} = T_{112} + T_{222} + T_{332} \\ V_3^{\nabla\mathrm{sph}} = T_{113} + T_{223} + T_{333} \end{pmatrix}$$

and embedded in $\underset{\simeq}{T}$:

$$[\underset{\simeq}{\mathrm{T}}(\underline{V}^{\nabla\mathrm{sph}})] = \begin{pmatrix} \frac{1}{3}V_1^{\nabla\mathrm{sph}} & \frac{1}{3}V_2^{\nabla\mathrm{sph}} & \frac{1}{3}V_3^{\nabla\mathrm{sph}} \\ \frac{1}{3}V_1^{\nabla\mathrm{sph}} & \frac{1}{3}V_2^{\nabla\mathrm{sph}} & \frac{1}{3}V_3^{\nabla\mathrm{sph}} \\ \frac{1}{3}V_1^{\nabla\mathrm{sph}} & \frac{1}{3}V_2^{\nabla\mathrm{sph}} & \frac{1}{3}V_3^{\nabla\mathrm{sph}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3}(\mathrm{T}_{111}+\mathrm{T}_{221}+\mathrm{T}_{331}) & \frac{1}{3}(\mathrm{T}_{112}+\mathrm{T}_{222}+\mathrm{T}_{332}) & \frac{1}{3}(\mathrm{T}_{113}+\mathrm{T}_{223}+\mathrm{T}_{333}) \\ \frac{1}{3}(\mathrm{T}_{111}+\mathrm{T}_{221}+\mathrm{T}_{331}) & \frac{1}{3}(\mathrm{T}_{112}+\mathrm{T}_{222}+\mathrm{T}_{332}) & \frac{1}{3}(\mathrm{T}_{113}+\mathrm{T}_{223}+\mathrm{T}_{333}) \\ \frac{1}{3}(\mathrm{T}_{111}+\mathrm{T}_{221}+\mathrm{T}_{331}) & \frac{1}{3}(\mathrm{T}_{112}+\mathrm{T}_{222}+\mathrm{T}_{332}) & \frac{1}{3}(\mathrm{T}_{113}+\mathrm{T}_{223}+\mathrm{T}_{333}) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

• *Distortion-gradient vector:*
As a vector:

$$[\underline{V}^{\nabla\mathrm{dev}}] = \begin{pmatrix} V_1^{\nabla\mathrm{dev}} = \frac{2}{9}(2\mathrm{T}_{111}+(3\mathrm{T}_{122}-\mathrm{T}_{221})+(3\mathrm{T}_{133}-\mathrm{T}_{331})) \\ V_2^{\nabla\mathrm{dev}} = \frac{2}{9}(2\mathrm{T}_{222}+(3\mathrm{T}_{233}-\mathrm{T}_{332})+(3\mathrm{T}_{121}-\mathrm{T}_{112})) \\ V_3^{\nabla\mathrm{dev}} = \frac{2}{9}(2\mathrm{T}_{333}+(3\mathrm{T}_{131}-\mathrm{T}_{113})+(3\mathrm{T}_{232}-\mathrm{T}_{223})) \end{pmatrix}$$

and embedded in $\underset{\simeq}{\mathrm{T}}$:

$$[\underset{\simeq}{\mathrm{T}}(\underline{V}^{\nabla\mathrm{dev}})] = \begin{pmatrix} \frac{3}{5}V_1^{\nabla\mathrm{dev}} & -\frac{3}{10}V_2^{\nabla\mathrm{dev}} & -\frac{3}{10}V_3^{\nabla\mathrm{dev}} \\ -\frac{3}{10}V_1^{\nabla\mathrm{dev}} & \frac{3}{5}V_2^{\nabla\mathrm{dev}} & -\frac{3}{10}V_3^{\nabla\mathrm{dev}} \\ -\frac{3}{10}V_1^{\nabla\mathrm{dev}} & -\frac{3}{10}V_2^{\nabla\mathrm{dev}} & \frac{3}{5}V_3^{\nabla\mathrm{dev}} \\ \frac{9\sqrt{2}}{20}V_2^{\nabla\mathrm{dev}} & \frac{9\sqrt{2}}{20}V_1^{\nabla\mathrm{dev}} & 0 \\ \frac{9\sqrt{2}}{20}V_3^{\nabla\mathrm{dev}} & 0 & \frac{9\sqrt{2}}{20}V_1^{\nabla\mathrm{dev}} \\ 0 & \frac{9\sqrt{2}}{20}V_3^{\nabla\mathrm{dev}} & \frac{9\sqrt{2}}{20}V_2^{\nabla\mathrm{dev}} \end{pmatrix}$$

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