Preface

Pluripotential theory is a very powerful tool in geometry, complex analysis and dynamics. The principal subjects of investigation in pluripotential theory are plurisubharmonic functions, namely, those functions which remain subharmonic under holomorphic changes of coordinates. Plurisubharmonic functions are objects rather easy to handle and to be constructed; therefore, they are very useful and important tools in complex analysis, geometry (such as geometry of Kähler–Einstein manifolds, hyperbolicity, Green–Griffiths conjecture) and holomorphic dynamics. Among those, maximal plurisubharmonic functions and their associated Monge–Ampère equations play a fundamental role in modern mathematics. Many problems related to manifolds endowed with particular geometric structure, such as symplectic, Kählerian, iperkählerian, quaternionial-Kählerian, algebraic spinorial and Calabi–Yau manifolds and their generalizations, can be rephrased in terms of several types of complex Monge–Ampère equations about the existence of metric with constant curvature on algebraic manifolds. In particular, pluripotential theory plays a very basic role in the study of the equations associated with the existence of Einstein metrics of constant scalar curvature and extremal “a là Calabi”, both in the static version and in the parabolic one (Ricci’s flows and Calabi) which recently allowed to solve the Poincaré and Thurston’s conjectures.

A complete and deep theory has been developed in order to characterize maximal plurisubhharmonic functions by Bedford, Taylor, Demailly, Kiselman, Siciak, Błocki and others. Indeed, maximal plurisubharmonic functions are essentially solutions of homogeneous complex Monge–Ampère equations. Special solutions to such equations are the pluricomplex Green function and the pluri-complex Poisson measures, introduced and used in reproducing formulas for plurisubharmonic functions by Klimek, Demailly, Lempert and others. Such a pluricomplex Green function turned out also to be strictly related to the Kobayashi distance and to hyperconvexity and other geometrical properties of domains in $\mathbb{C}^n$. Pluripotential theory has also a number of very important applications in algebraic geometry, in particular related to jets bundles and the solution of some of the leading conjectures in the area such as Green–Griffiths and Kobayashis conjectures. Other applications to complex dynamics in higher dimensions are also available, both in the realm of discrete dynamics and in that of holomorphic foliations. On another side, pluripotential theory and complex Monge–Ampère equations are used to characterize complex manifolds (the so-called parabolic complex manifolds and Grauert tubes).

The CIME session in Cetraro on Pluripotential theory was a great and unique occasion to present a few courses on topics of high interest in the area and to join both experts and young mathematicians in a nice environment. The school, from which these notes are taken, was aimed to provide courses on pluripotential theory and Monge–Ampère equation and applications to algebraic geometry, complex dynamics and differential geometry. The program with its wide range of topics brought together mathematicians and young researchers with different background: complex analysis and geometry, differential geometry, dynamics, and differential equations.

The courses and the notes taken from them which constitute the chapters of this volume are briefly described hereafter.

In his lectures, *François Berteloot* gives a synthetic and self-contained exposition of the theory of bifurcation currents in holomorphic families of rational maps, giving applications of pluripotential theory to complex dynamics. He constructs the Green measure (the maximal entropy measure) for a fixed rational map and discusses its dynamical properties and proves an approximation formula for its Lyapunov exponent. He presents some concrete holomorphic families of examples and proves the Branner–Hubbard result about the compactness of the connectedness locus and introduces the hypersurfaces $Per_n(w)$ whose distribution turns out to shape the bifurcation locus. He also describes the moduli space Mod_2 of degree two rational maps. Next, he studies the bifurcation current T_{bif} and gives a proof of DeMarco's fundamental results which precisely relates the Lyapunov exponent to the Green function evaluated on the critical points. He then studies how the asymptotic distribution of dynamically defined hypersurfaces is governed by the bifurcation current. Finally, he examines the higher exterior powers T_{bif}^k of the bifurcation current. He concentrates on the highest power and shows that the support of such a measure is the seat of the strongest bifurcations.

Zbigniew Błocki's lectures present two situations where the complex Monge–Ampère equation appears in Kähler geometry: the Calabi conjecture and geodesics in the space of Kähler metrics. In the first case the problem is to construct, in a given Kähler class, a metric with prescribed Ricci curvature. It turns out that this is equivalent to finding a metric with prescribed volume form and thus to solving nondegenerate complex Monge–Ampère equation on a manifold with no boundary. In the second case to find a geodesic in a Kähler class one has to solve a homogeneous complex Monge–Ampère equation on a manifold with boundary. In his self-contained lecture notes, Błocki discusses both the geometric aspects and the PDE part, mostly a priori estimates, starting from a very elementary introduction to Kähler geometry. He introduces the Calabi conjecture and its equivalence to complex Monge–Ampère equation. Later he gives basic properties of the Riemannian structure of the space of Kähler metrics, the Aubin–Yau functional and the Mabuchi K-energy as well as relation to constant scalar curvature metrics. The Lempert–Vivas example is also described. The notes contain also fundamental results on complex Monge–Ampère equations such as the basic uniqueness results as well as the comparison principle. Among other things, the continuity method, used to prove existence of solutions, is described and Yau's proof of the L^∞-estimate using Moser's iteration is presented.

Pluripotential theory is a powerful and strong tool also in algebraic geometry as shown in *Jean–Pierre Demailly*'s lectures note. In his lectures, he describes the main techniques involved in the proof of holomorphic Morse inequalities which relate certain curvature integrals to the asymptotic cohomology of large tensor powers of line or vector bundles bring a useful complement to the Riemann–Roch formula. He also describes their link with Monge–Ampère operators and intersection theory. Finally, he provides applications to the study of asymptotic cohomology functionals and the Green–Griffiths–Lang conjecture. The latter conjecture asserts that every entire curve drawn on a projective variety of general type should satisfy a global algebraic equation; via a probabilistic curvature calculation, holomorphic Morse inequalities imply that entire curves must at least satisfy a global algebraic differential equation.

Giorgio Patrizio's lectures in the CIME session were based on the lecture notes by himself and Andrea Spiro, included in this volume. In these notes the authors discuss the link between pluripotential theory and Monge–Ampère foliations. The latter turned out to have many applications in complex geometry, and the selection of a good candidate for the associated Monge–Ampère foliation is always the first step in the construction of well-behaved solutions of the complex homogeneous Monge–Ampère equation. After reviewing some basic notions on Monge–Ampère foliations, the authors concentrate on two main topics. They discuss the construction of (complete) modular data for a large family of complex manifolds, which carry regular pluricomplex Green functions. This class of manifolds naturally includes all smoothly bounded, strictly linearly convex domains and all smoothly bounded, strongly pseudoconvex circular domains of $\mathbb{C}^n$. Then they report on the problem of defining pluricomplex Green functions in the almost complex setting, providing sufficient conditions on almost complex structures, which ensure existence of almost complex Green pluripotentials and equality between the notions of stationary disks and of Kobayashi extremal disks, and allow extensions of known results to the case of non-integrable complex structures.

It is a real great pleasure to thank the speakers for their very interesting lectures and all the authors for the nice lectures notes they have carefully prepared for this volume. We also want to warmly thank all the participants to the CIME session for their enthusiasm and interest in the subject and for having created a very friendly environment which made possible to experience such a great scientific atmosphere.

Last but not least, we want to thank the CIME organization for giving us the opportunity to organize and financing this school and the GNSAGA of INDAM for support. Our special gratitude also to Pietro Zecca and Elvira Mascolo.

Also, a special thanks to Mrs. Ute McCrory at Springer for her assistance in preparing the volume.

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http://www.springer.com/978-3-642-36420-4

Pluripotential Theory
Cetraro, Italy 2011, Editors: Filippo Bracci, John Erik Fornæss
Patrizio, G.; Błocki, Z.; Berteloot, F.; Demailly, J.P. - Bracci, F.; Fornæss, J.E. (Eds.)
2013, IX, 319 p. 11 illus., 9 illus. in color., Softcover
ISBN: 978-3-642-36420-4