Graph Metrics for Temporal Networks

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Abstract Temporal networks, i.e., networks in which the interactions among a set of elementary units change over time, can be modelled in terms of time-varying graphs, which are time-ordered sequences of graphs over a set of nodes. In such graphs, the concepts of node adjacency and reachability crucially depend on the exact temporal ordering of the links. Consequently, all the concepts and metrics proposed and used for the characterisation of static complex networks have to be redefined or appropriately extended to time-varying graphs, in order to take into account the effects of time ordering on causality. In this chapter we

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discuss how to represent temporal networks and we review the definitions of walks, paths, connectedness and connected components valid for graphs in which the links fluctuate over time. We then focus on temporal node–node distance, and we discuss how to characterise link persistence and the temporal small-world behaviour in this class of networks. Finally, we discuss the extension of classic centrality measures, including closeness, betweenness and spectral centrality, to the case of time-varying graphs, and we review the work on temporal motifs analysis and the definition of modularity for temporal graphs.

1 Introduction

Whenever a system consists of many single units interacting through a certain kind of relationship, it becomes natural to represent it as a *graph*, where each *node* of the graph stands for one of the elementary units of the system and interactions between different units are symbolised by *edges*. If two nodes are connected by an edge they are said to be *adjacent*. According to the nature of the units and to the characteristics of adjacency relationship connecting them, it is possible to construct different kind of graphs, such as friendship graphs—where nodes are people and edges connect two people who are friends, functional brain networks—where nodes are regions of the brain and edges represent the correlation or causality of their activity, communication graphs—where nodes are terminals of a communication systems, such as mobile phones or email boxes, and edges indicate the exchange of a message from a terminal to another, just to make some examples. Thanks to the availability of large data sets collected through modern digital technologies, in the last decade or so there has been an increasing interest towards the study of the structural properties of graph representations of real systems, mainly spurred by the observation that graphs constructed from different social, biological and technological systems show surprising structural similarities and are characterised by non-trivial properties. In a word, they are *complex* networks. Independently of the peculiar nature and function of the original systems, the corresponding graphs are usually *small-worlds*, i.e., they show high local cohesion and, at the same time, extremely small node–node distance [47]; the distribution of the number of neighbours of a node (its *degree*) is often heterogeneous, and decays as a power-law (i.e., they are *scale-free* [2]); they are locally organised as tightly-knit groups of nodes (called *communities*), which are in turn loosely interconnected to each other [33]. The concepts, metrics, methods, algorithms and models proposed so far to study the structure of real networks has led to the formation of theoretical framework known as *complex network theory* [1,5,31].

However, the relationships among the units of a real networked system (e.g., node adjacency) are rarely persistent over time. In many cases, the static interpretation of node adjacency is just an oversimplifying approximation: contacts among individuals in a social network last only for a finite interval and are often intermittent and recurrent [8, 13, 18]; different intellectual tasks are usually associated to different brain activity patterns [9, 45]; communication between agents in a telecommunication system are typically bursty and fluctuate over time [4, 26, 40]; transportation networks show fluctuations in their microscopic organisation, despite the stability of their global structural properties [11]. Consequently, whenever we deal with a networked system that evolves over time, the concept of adjacency needs to be appropriately redefined. The extension of node adjacency to the case of time-evolving systems has lately led to the definition of *temporal graphs* (sometimes also called *time-varying graphs* or *dynamic graphs*), in which *time* is considered as another dimension of the system and is included in the same definition of the graph. Since most of the metrics to characterise the structure of a graph, including graph connectedness and components, distance between nodes, the different definitions of centrality etc., are ultimately based on node adjacency, they need to be appropriately redefined or extended in order to take into account of the presence, frequency and duration of edges at different times. In general, the temporal dimension adds richness and complexity to the graph representation of a system, and demands for more powerful and advanced tools which allow to exploit the information on temporal correlations and causality. Recently, Holme and Saramäki have published a comprehensive review which presents the available metrics for the characterisation of temporal networks [14]. A description of some potential applications of these metrics can be found in [44].

This chapter presents the basic concepts for the analysis of time-evolving networked systems, and introduces all the fundamental metrics for the characterisation of time-varying graphs. In Sect. 2 we briefly discuss different approaches to encode some temporal information into static graphs and we introduce a formal definition of time-varying graph. In Sect. 3 we examine how reachability and connectedness are affected by time-evolving adjacency relationships and we introduce the definitions of node and graph temporal components, showing the intimate connections between the problem of finding temporal connected components and the maximal-clique problem in static graphs. In Sect. 4 we focus on the concepts of temporal distance, efficiency and temporal clustering, and we discuss the temporal small-world phenomenon. In Sect. 5 we present the extensions of betweenness, closeness and spectral centrality to time-varying graphs. Finally, in Sect. 6 we report on the definition of temporal motifs and on the extension of the modularity function to time-varying graphs.

2 Representing Temporal Networks

From a mathematical point of view a networked time-evolving system consists of a set $\mathscr{C}$ of *contacts* registered among a set of nodes $\mathscr{N} = \{1, 2, \dots, N\}$ during an *observation interval* $[0, T]$ [37, 43]. A *contact* between two nodes $i, j \in \mathscr{N}$ is represented by a quadruplet $c = (i, j, t, \delta t)$, where $0 \leq t \leq T$ is the time at which the contact started and δt is its duration, expressed in appropriate temporal units. As we stated above the relationship between i and j is usually not persistent (it could represent the co-location at a place, the transmission of a message, the temporal correlation between two areas of the brain etc.), so that in general we will observe

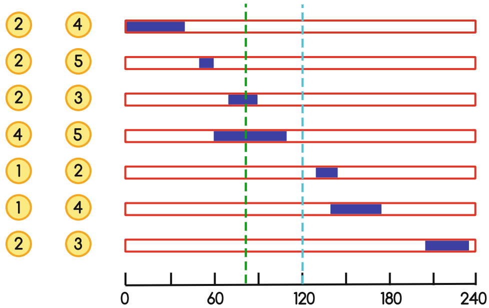

Fig. 1 The set of contacts registered among five nodes within an observation period of 4 h. *Blue bars* indicate the duration of each contact. The two *dashed lines* correspond to two instantaneous cuts of the contact set, respectively for $t = 80$ min (*green*) and $t = 120$ min (*cyan*)

more than one contact between i and j in the interval $[0, T]$. In Fig. 1 we report an example of a set of seven contacts observed between a set of $N = 5$ nodes within an interval of $T = 240$ min. The contacts in the figure are considered *symmetric*, i.e., $(i, j, t, \delta t) = (j, i, t, \delta t)$, even if in general this in not the case. Each contact is represented by a pair of nodes and a blue bar indicating the start and duration of the contact. Notice that in this example, which is indeed representative of many social and communication systems, the typical overlap between contacts is relatively small with respect to the length of the observation interval.

Before explaining how the temporal information about such a set of contacts can be represented by means of a time-varying graph, we first review some simple approaches to deal with time-evolving systems based on static graphs, and we discuss why they are inadequate for analyzing time-evolving systems.

2.1 Aggregated Static Graphs

The classic approach to represent networked systems evolving over time consists in constructing a single *aggregated static graph*, in which all the contacts between each pair of nodes are flattened in a single edge. An aggregated graph can be represented by an *adjacency matrix* $A = \{a_{ij}\} \in \mathbb{R}^{N \times N}$, in which the entry $a_{ij} = 1$ if at least one contact $(i, j, \cdot, \cdot)$ has been registered in $[0, T]$ between i and j, and $a_{ij} = 0$ otherwise. If the relationship between any pair of nodes i and j is symmetric, such as in the case of co-location or collaboration graphs, also the corresponding adjacency matrix is symmetric, i.e., $a_{ij} = a_{ji} \quad \forall i, j \in \mathcal{N}$. Conversely, whenever a directionality is implied, for instance when the contact is a phone call from i to j or represents goods transferred from i to j, the adjacency matrix is in general non-symmetric.

Representing a time-evolving system by means of an adjacency matrix, i.e., an unweighted graph, is usually a severe oversimplification: the information about the number, frequency and duration of contacts between two nodes i and j is flattened down into a binary digit (i.e., $a_{ij} = 1$ if there is at least one contact, of any duration, between i and j, while $a_{ij} = 0$ otherwise). In general, a binary adjacency information does not take into account the heterogeneity observed in real

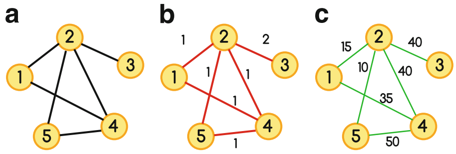

Fig. 2 Three different aggregated static graphs obtained from the set of contacts in Fig. 1. (**a**) The unweighted aggregated graph; (**b**) the weighted aggregated graph where each weight w_{ij} between node i and node j corresponds to the number of contacts observed; (**c**) the weighted aggregated graph where each weight w_{ij} is the total duration of all the contacts between i and j

systems. Just to make an example, both the number of phone calls made by a single node during a certain time interval and the duration of a call between two nodes exhibit large fluctuations, and their distribution is well approximated by a power-law [4,16,26]. This means that assigning the same weight to all the relationships can lead to misleading conclusions. This problem can be partially solved by constructing a *weighted aggregated graph*, in which the edge connecting i to j is assigned a weight w_{ij} proportional to the number of contacts observed, their duration, their frequency or a combination of the these three dimensions.

In Fig. 2 we show three different aggregated static graphs corresponding to the same set of contacts reported in Fig. 1. The leftmost graph (Panel a) is the unweighted aggregated graph; in the middle graph (Panel b) the weights correspond to the number of contacts observed between the nodes; in the rightmost one (Panel c) the weight of each edge w_{ij} is equal to the sum of the duration of all the contacts between i and j. However, both unweighted and weighted aggregated graphs fail to capture the temporal characteristics of the original system. In fact, by considering all the links as always available and persistent over time, the number of walks and paths between two nodes is overestimated, while the effective distance between two nodes is instead systematically underestimated. For instance, in all the three aggregated representations, node 2 and node 4 are connected by an edge, but their interaction is limited just to the beginning of the observation period, so that these nodes cannot directly communicate for most of the time.

Despite not being powerful enough to represent networks in which the temporal aspects are crucial, static aggregated graphs and the metrics proposed for their analysis still constitute an invaluable framework to investigate the structure and function of systems in which the topological characteristics are more relevant than the temporal ones. After all, most of the classic examples of complex networks, including the graph of the Internet at Autonomous Systems level [46], co-authorship networks [29, 30], the graph of the World Wide Web [2, 6] and functional brain networks [7] have been obtained so far by aggregating all the contacts observed among a certain number of nodes within a given temporal interval, and the analysis of their structure has provided new insights about the organisation of different complex systems.

2.2 Time-Varying Graphs

The natural way to work out a graph representation that can properly take into account all the temporal correlations of a set of contacts consists into including time as an additional dimension of the graph [17]. We notice that a set of contacts implicitly defines a graph for each instant t, made by the set of edges corresponding to all the contacts $(\cdot, \cdot, t_i, \delta t_i)$ such that $t_i \le t \le t_i + \delta t_i$. In the example shown in Fig. 1, if we consider $t = 80$ min, corresponding to the dashed green line in the figure, the graph constructed from contacts active at that time contains only two edges, namely $(2, 3)$ and $(4, 5)$. However, we notice that the graph corresponding to $t = 120$ min (the dashed cyan line in the figure) is an empty graph, since no contact is active at that time. For practical reasons, and especially when contacts are instantaneous, i.e., $\delta t \to 0$, it is convenient to consider a finite *time-window* $[t, t + \Delta t]$ and to construct a graph by creating an edge between all pairs of nodes for which there is at least a contact which overlaps with the interval $[t, t + \Delta t]$. A generic contact $(\cdot, \cdot, \tau_i, \delta\tau_i)$ overlaps with $[t, t + \Delta t]$ if it satisfies at least one of the three following conditions:

$$t \le \tau_i < t + \Delta t \tag{1}$$

$$t \le \tau_i + \delta\tau_i < t + \Delta t \tag{2}$$

$$\tau_i < t \quad \wedge \quad \tau_i + \delta\tau_i > t + \Delta t \tag{3}$$

A graph G_t obtained by aggregating all the contacts appearing in a given interval $[t, t + \Delta t]$ represents the state of the system in that interval, i.e., it is a *snapshot* which captures the configuration of the links at that particular time interval. If we consider a sequence of successive non-overlapping time-windows $\{[t_1, t_1 + \Delta t_1], [t_2, t_2 + \Delta t_2], [t_3, t_3 + \Delta t_3], \ldots, [t_M, t_M + \Delta t_M]\}$ then we obtain a *time-varying graph*, which is the simplest graph representation of a set of contacts that takes into account their duration and their temporal ordering [19,43]. A time-varying graph is an ordered sequence of M graphs $\{G_1, G_2, \ldots, G_M\}$ defined over N nodes, where each graph G_m in the sequence represents the state of the network, i.e., the configuration of links, in the time-window $[t_m, t_m + \Delta t_m], \quad m = 1, \ldots, M$. In this notation, the quantity $t_M + \Delta t_M - t_1$ is the temporal length of the observation period. In general the graphs in the sequence can correspond to any ordered sequence of times such that $t_1 < t_1 + \Delta t_1 = t_2 < t_2 + \Delta t_2 = t3 < \ldots < t_M + \Delta t_M$ [12]. In the following we assume, without loss of generality, that $t_1 = 0$ and $t_M = T$ and, at the same time, that the sequence of snapshots is uniformly distributed over time, i.e., $t_{m+1} = t_m + \Delta t, \ \forall m = 1, \ldots, M - 1$ [43]. In compact notation, we denote the graph sequence forming a time-varying graph as $\mathcal{G} \equiv \mathcal{G}_{[0,T]}$. Each graph in the sequence can be either undirected or directed, according to the kind of relationship represented by contacts. Consequently, the time-varying graph $\mathcal{G}$ is fully described by means of a time-dependent adjacency matrix $A(t_m), \quad m = 1, \ldots, M$, where

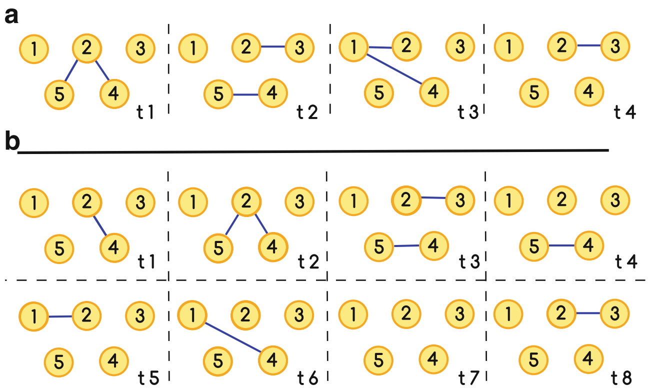

Fig. 3 Two time-varying graphs corresponding to the set of contacts in Fig. 1. Panel (**a**) the four snapshot obtained setting $\Delta t = 60$ min; Panel (**b**) the eight snapshots of the time-varying graph constructed using $\Delta t = 30$ min. The smaller the size of the time-window, the higher the probability that a snapshot contains no edges (this happens for the snapshot $t7$ in panel (**b**))

$a_{ij}(t_m)$ are the entries of the adjacency matrix of the graph at time t_m, which is in general a non-symmetric matrix.

Notice that, by tuning the size of the time-window used to construct each snapshot, it is possible to obtain different representations of the system at different temporal scales. In Fig. 3 we present two time-varying graphs obtained from the same set of contacts in Fig. 1 but using two different lengths for the time-window. The graph on the top panel is constructed by setting $\Delta t = 60$ min, and consists of four snapshots, while the graph on the bottom panel corresponds to a time-window of $\Delta t = 30$ min and has eight snapshots. It is usually preferable to set the size of the time-window to the maximum temporal resolution available. For instance, if the duration of contacts is measured with an accuracy of 1 s (such as in the case of email communications or phone calls), it makes sense to construct time-varying graphs using a time-window $\Delta t = 1$ s.

In the limit case when $\Delta t \to 0$, we obtain an infinite sequence of graphs, where each graph corresponds to the configuration of contacts at a given instant t. This sequence of graphs might include a certain number of empty graphs, corresponding to periods in which no contacts are registered. On the contrary, if we set $\Delta t = T$, the time-varying graph degenerates into the corresponding unweighted aggregated graph, where all the temporal information is lost.

3 Reachability, Connectedness and Components

In a static graph the *first neighbours* of a node i are the nodes to which i is connected by an edge, i.e., nodes j such that $a_{ij} = 1$. We say that the neighbours of i are *directly reachable* from i. If k is a neighbour of j and j is in turn a neighbour of i, then the node k is indirectly reachable from i, i.e., by following first the edge connecting i to j and then the one which connects j to k. In general, the direct and indirect reachability of nodes is important to characterise the global structure of a network and to investigate the dynamics of processes occurring over it. For instance, if node i has got a contagious disease, then there is a high probability that the disease will sooner or later be transmitted to the nodes that are directly reachable from i (its first neighbours). However, if a disease starts from a node i, also the nodes which are not directly connected to i but are still indirectly reachable from i have a finite probability to get the disease through a chain of transmissions.

The reachability between nodes is related to the concept of *walk*. In a static graph a walk is defined as an ordered sequence of ℓ edges $\{a_{i_0,i_1}, a_{i_1,i_2}, \ldots, a_{i_{\ell-1},i_\ell}\}$ such that $a_{i_k,i_{k+1}} = 1, k = 0, 1, \ldots, \ell - 1$. The *length* of a walk is equal to the number of edges traversed by the walk. We say that the node j is *reachable* from i if there exists a walk which starts at i and ends up at j. If each vertex in a walk is traversed exactly once, then the walk is called a *simple walk* or a *path*. For instance, in the graph shown in Fig. 2 the sequence of nodes $[2, 4, 5, 2, 1, 4]$ is a walk of length 5 which starts at node 2 and ends at node 4, while the sequence $[3, 2, 5]$ is a path of length 2 going from node 3 to node 5 passing by node 2. In a static graph the length of the shortest path connecting two nodes is called *geodesic distance*.

Since the definitions of walk and path depend on the adjacency of nodes, and given that node adjacency is a function of time in time-varying graphs, an appropriate extension of these concepts is necessary in order to define node reachability and components in time-varying graphs.

3.1 Time-Respecting Walks and Paths

In a time-varying graph, a *temporal walk* from node i to node j is defined as a sequence of L edges $[(n_{r_0}, n_{r_1}), (n_{r_1}, n_{r_2}), \ldots, (n_{r_{L-1}}, n_{r_L})]$, with $n_{r_0} \equiv i$, $n_{r_L} \equiv j$, and an increasing sequence of times $t_{r_1} < t_{r_2} < \ldots < t_{r_L}$ such that $a_{n_{r_{l-1}},n_{r_l}}(t_{r_l}) \neq 0$ $l = 1, \ldots, L$ [12,43]. A *path* (also called *temporal path*) of a time-varying graph is a walk for which each node is visited at most once. For instance, in the time-varying graph of Fig. 3a, the sequence of edges $[(5, 2), (2, 1)]$ together with the sequence of times t_1, t_3 is a temporal path of the graph. This path starts at node 5 at time t_1 and arrives at node 1 at time t_3. Notice that the aggregated static graph flattens down most of the information about temporal reachability. In fact, if we look at the static aggregated graph corresponding to this time-varying graph (shown in Fig. 2a), there are different paths going from node 1 to node 5 and vice versa; however, if we look at the time-varying graphs of Fig. 3 we notice that in both of them there is no temporal path connecting node 1 to node 5. The reason is that node 5 could be reached from node 1 only by passing through either node 2 or node 4, but node 1 actually is connected to both these nodes *after* they have been in contact to node 5.

3.2 Temporal Connectedness and Node Components

The concept of connectedness is fundamental in complex network theory. A message, a piece of information or a disease can be transferred from one node to all the other nodes to which it is connected, but will never be conveyed to nodes that are disconnected from it. For this reason, the study of node connectedness and node components is the very basic tool to investigate the structure of a graph.

In a static undirected graph two nodes are said to be connected if there exists a path between them. In this particular case connectedness is an equivalence relation: it is *reflexive* (i.e., i is connected to itself), *symmetric* (i.e., if i is connected to j then j is connected to i) and *transitive* (i.e., if i is connected to j and j is connected to k, then i is also connected to k). Instead, in a directed graph, due to the directionality of the edges, symmetry is broken and the existence of a path from i to j does not guarantee that a path from j to i does indeed exist. For this reason, the notions of strong and weak connectedness are introduced. In a word, two nodes i and j of a static directed graph are said to be *strongly connected* if there exist a path from i to j and a path from j to i, while they are *weakly connected* if there exists a path connecting them in the underlying undirected graph, i.e., in the static graph obtained from the original by discarding edge directions.

Starting from the definitions of temporal walk and path, it is possible to define temporal connectedness (in a weak and in a strong sense) for pairs of nodes in a time-varying graph. A node i of a time-varying graph $\mathscr{G}_{[0,T]}$ is *temporally connected* to a node j if there exists a temporal path going from i to j in $[0, T]$. Due to the temporal ordering of edges, this relation is trivially not symmetric, so that if i is temporally connected to j, in general j can be either temporally connected or disconnected to i. Two nodes i and j of a time-varying graph are *strongly connected* if i is temporally connected to j and also j is temporally connected to i.

Temporal strong connectedness is a reflexive and symmetric relation, so that if i is strongly connected to j, then j is strongly connected to i. However, strong connectedness still lacks transitivity, and, therefore, it is not an equivalence relation. In fact, if i and j are strongly connected and j and l are strongly connected, nothing can be said, in general, about the connectedness of i and l. For instance, in the time-varying graph shown in Fig. 3a, nodes 5 and 2 are strongly connected and also 2 and 1 are strongly connected, but nodes 5 and 1 are not strongly connected because, as we have already explained above, there exists no temporal path that connects node 1 to node 5.

Similarly to the case of static directed graphs, it is possible to define weak connectedness among nodes. Given a time-varying graph $\mathscr{G}$, we consider the underlying undirected time-varying graph $\mathscr{G}^u$, which is obtained from $\mathscr{G}$ by discarding the directionality of the links of all the graphs $\{G_m\}$, while retaining their time ordering. Two nodes i and j of a time-varying graph are *weakly connected* if i is temporally connected to j and also j is temporally connected to i in the underlying undirected time-varying graph $\mathscr{G}^u$. Also weak connectedness is a reflexive and symmetric relation, but not transitive.

It is worth noting that strong and weak connectedness propagate over different time scales. In fact, if we consider two time-varying graphs obtained from the same set of contacts by using two different time-windows, such as for instance Δt_1 for the graph $\mathscr{G}_1$ and $\Delta t_2 > \Delta t_1$ for $\mathscr{G}_2$ (as in the two time-varying graphs of Fig. 3), then it is easy to prove that if i and j are strongly connected in $\mathscr{G}_1$ then they are also strongly connected in $\mathscr{G}_2$. The contrary is trivially not true. Thanks to this property, strong and weak connectedness in time-varying graphs are consistent with the corresponding definitions given for static graphs. In fact, as a limiting case, if two nodes are strongly (weakly) connected in a time-varying graph, then they are also strongly (weakly) connected in the corresponding aggregated static graph, which is the degenerate time-varying graph obtained by setting $\Delta t = T$.

By using reachability, strong and weak connectedness, different definitions of node components can be derived. For instance, the *temporal out-component of node i* (resp. *in-component)*, denoted as $\mathrm{OUT}_T(i)$ (resp. $\mathrm{IN}_T(i)$), is the set of vertices which can be reached from i (resp. from which i can be reached) in the time-varying graph $\mathscr{G}$. Similarly the *temporal strongly connected component of a node i* (resp. *weakly connected component*), denoted as $\mathrm{SCC}_T(i)$ (resp. $\mathrm{WCC}_T(i)$), is the set of vertices from which vertex i can be reached, and which can be reached from i, in the time-varying graph $\mathscr{G}$ (resp. in the underlying undirected time-varying graph $\mathscr{G}^u$).

In general, temporal node components have quite heterogeneous composition and sizes, and reveal interesting details about the structure of the graph. For instance, the out-component of node 3 in the two graphs of Fig. 3 contains only nodes 1, 2, 4 and node 3 itself, since there is no way for node 3 to reach node 5. Conversely, in the corresponding aggregated graphs (as shown in Fig. 2) the out-components of all the nodes are identical and contain all the nodes of the graph.

The importance of temporal node components has been pointed out in [36], which reports the results of temporal component analysis on time-varying graphs obtained from three different data sets. The authors compared the size of node temporal in- and out-components in these time-varying graphs with the size of the giant component of the corresponding aggregated graphs, and they found that in general temporal node components are much smaller than the giant component of the aggregated graph, and exhibit a high variability in time. This is another example of the fact that time-varying graphs are able to provide additional information that is not captured by aggregated graphs.

3.3 Graph Components and Affine Graphs

Differently from the case of directed static graphs, it is not possible to define the strongly (weakly) connected components of a time-varying graph starting from the definition of connectedness for pairs of nodes. Formally, this is due to the fact that strong and weak connectedness are not equivalence relations. For this reason, the following definition of strongly connected component of a time-varying graph has

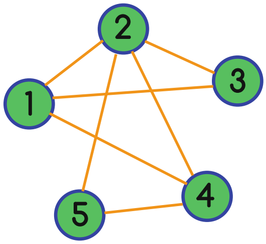

Fig. 4 Affine graph associated to the time-varying graph of Fig. 3a. Pairs of strongly connected nodes are linked by an edge, and the cliques of the affine graph correspond to the strongly connected components of the associated time-varying graph

been proposed [36]: a set of nodes of a time-varying graph $\mathscr{G}$ is a temporal strongly connected component of $\mathscr{G}$ if each node of the set is strongly connected to all the other nodes in the set. A similar definition exists for weakly connected components. These definitions enforce transitivity but have the drawback that in order to find the strongly connected components of a time-varying graph, it is necessary to check the connectedness between all pairs of nodes in the graph. In [36] it has been shown that the problem of finding the strongly connected components of a time-varying graph is equivalent to the well-known problem of finding the maximal-cliques of an opportunely constructed static graph. We define such graph as the *affine graph* corresponding to the time-varying graph. The affine graph $G_{\mathscr{G}}$ is defined as the graph having the same nodes as the time-varying graph $\mathscr{G}$, and such that two nodes i and j are linked in $G_{\mathscr{G}}$ if i and j are strongly connected in $\mathscr{G}$. In Fig. 4 we report the affine graph corresponding to the time varying graph shown in Fig. 3a. In this graph, node 1 is directly connected to nodes $\{2, 3, 4\}$, since it is temporally strongly connected to them in the time-varying graph. Similarly, node 2 is connected to nodes $\{1, 3, 4, 5\}$, node 3 is connected to $\{1, 2\}$, node 4 is connected to $\{1, 2, 5\}$ and node 5 is connected to $\{2, 4\}$. Hence, the affine graph $G_{\mathscr{G}}$ has only 7 of the 10 possible links, each link representing strong connectedness between two nodes. By construction, a clique of the affine graph $G_{\mathscr{G}}$ contains only nodes which are strongly connected to each other, so that the *maximal-cliques* of the affine graph, i.e., all the cliques which are not contained in any other clique, are temporal strongly connected components (SCC_T) of $\mathscr{G}$. Similarly, all the *maximum-cliques* of the affine graph $G_{\mathscr{G}}$, i.e., its largest maximal-cliques, are the largest temporal strongly connected components (LSCC_T) of $\mathscr{G}$.

We notice that the problem of finding a partition of $\mathscr{G}$ that contains the minimum number of disjoint strongly connected components is equivalent to the well-known problem of finding a partition of the corresponding affine graph $G_{\mathscr{G}}$ in the smallest number of disjoint maximal-cliques [15]. Unfortunately, this problem is known to be NP-complete, and in practice can be exactly solved only for small graphs. In the case of the affine graph reported in Fig. 4, it is possible to check by hand that there are only three possible partitions of $G_{\mathscr{G}}$ into maximal-cliques, namely

1. $\{1, 2, 3\} \bigcup \{4, 5\}$
2. $\{1, 2, 4\} \bigcup \{3\} \bigcup \{5\}$
3. $\{2, 4, 5\} \bigcup \{1, 3\}$

The second partition contains two isolated nodes, which are indeed degenerated maximal-cliques. Therefore, the original time-varying graph admits only two different partitions into a minimal number of non-degenerated strongly connected components, namely into two components containing at least two nodes each. This is a quite different picture from that we obtain using static aggregated graphs. In fact, all the static aggregated representations of the same time-varying graph (see Fig. 2) are composed by just one strongly connected component which includes all the nodes.

We notice that in general the largest temporal strongly connected components of a time-varying graph can be much smaller than the giant connected component of the corresponding aggregated graph. For instance, in [36] the authors performed temporal component analysis on time-varying graphs constructed from three different time-stamped data sets (i.e. the MIT Reality Mining project, co-location at INFOCOM 2006 and Facebook communication), and they found that despite the giant connected component of the corresponding aggregated graphs usually included almost all the nodes in the network, the maximal cliques of the affine graphs were indeed much smaller. Particularly interesting was the case of the Facebook communication data set: the giant connected components of the aggregated graphs contained from 10^4 to 10^5 nodes, while the largest temporal strongly connected components counted around 100 nodes at most. Disregarding such discrepancies could result in misleading conclusions. For example, the potential number of individuals infected by a disease which spreads through the system is in the order of tens if we correctly take into account temporal correlations, but could be erroneously estimated to be thousand times larger if one considers the aggregated graph.

4 Distance, Efficiency and Temporal Clustering

One of the most relevant properties of static complex networks is that they exhibit, on average, a surprising small geodesic distance between any pairs of nodes, where the geodesic distance between i and j is defined as the length of the shortest path connecting them. The average geodesic distance is important to characterise how fast (for example, in terms of number of hops), a message can be transmitted from a node to another in the network; therefore, it is related to the overall efficiency of communication among nodes. Having a small average geodesic distance (where the average is computed over all the pairs of connected nodes) is a desirable property when one wants to spread a message throughout the network; conversely, small geodesic distance becomes a problem if we want to control the propagation of a disease.

4.1 Temporal Distance and Efficiency

When we consider time-varying graphs, the temporal dimension is an essential element of the system, so that the concept of geodesic distance cannot be limited to the number of *hops* separating two nodes but should also take into account the temporal ordering of links. As a matter of fact, any path in a time-varying graph is characterised by two different lengths: *(a)* a *topological length*, measured as the number of edges traversed by the path, and a *(b) temporal length* or duration, measured as the time interval between the first and the last contact in the path.

Both the topological and the temporal lengths of a path usually depend on the time at which the path starts. Consider for instance two of the paths connecting node 5 to node 3 in Fig. 3a. The first path starts from 5 at snapshot t_1 which traverses the edge $(5, 2)$ at (t_1) and arrives at node 3 following the edge $(2, 3)$ at time (t_2). The second one starts from 5 at time t_2 and arrives at node 3 at time t_4, after traversing the edges $(5, 4)$ at t_2, $(4, 1)$ and $(1, 2)$ at t_3 and finally $(2, 3)$ at t_4. The first path has a topological length equal to 2 and a temporal length of two snapshots (e.g., 2 h), while the second path has a topological length equal to 4 and a temporal length equal to three snapshots (e.g., 3 h).

The *temporal shortest path* from node i to node j is defined as the temporal path connecting i to j which has minimum temporal length. Similarly, the *temporal distance* $d_{i,j}$ between i and j is the temporal length of the temporal shortest path from i to j. In the example discussed above, the temporal shortest path connecting node 5 to node 3 is the one starting at t_1 and having temporal length equal to two snapshots.

The natural extension of the average geodesic distance to the case of time-varying graphs is the *characteristic temporal path length* [42, 43], which is defined as the average temporal distance over all pairs of nodes in the graph:

$$L = \frac{1}{N(N-1)} \sum_{ij} d_{ij} \tag{4}$$

It is also possible to define the *temporal diameter* of a graph as the largest temporal distance between any pair of nodes:

$$D = max_{ij} d_{ij} \tag{5}$$

However, in real time-varying graphs it is quite common to have many pairs of temporally disconnected nodes. The problem is that if a node j is not temporally reachable from i, then $d_{ij} = \infty$, and the characteristic temporal path length diverges. In order to avoid such divergence, the *temporal global efficiency* [42, 43] of a time-varying graph has been defined as follows:

$$\mathscr{E} = \frac{1}{N(N-1)} \sum_{ij} \frac{1}{d_{ij}} \tag{6}$$

The temporal global efficiency is the straightforward generalisation of the global efficiency already defined for static graphs [22], and has been successfully employed to study and quantify the robustness of temporal graphs (see for instance [38] and [44] in this book).

4.2 Edge Persistence and Clustering

The characteristic temporal path length and the temporal efficiency provide a quantitative representation of the global structure of a graph in terms of the average temporal distance among any pair of nodes. However, in time-varying systems contacts are usually *bursty*, meaning that the distribution of the time between two contacts has a heavy tail, and *persistent*, i.e., if two nodes are connected at a time t, there is a non-negligible probability that they will still be connected at time $t + \Delta t$. This characteristic can be quantified in the following way. If we consider a node i and two adjacent snapshots of a time-varying graph, respectively starting at time t_m and $t_{m+1} = t_m + \Delta t$, we can define the *topological overlap* of the neighbourhood of i in $[t_m, t_{m+1}]$ as:

$$C_i(t_m, t_{m+1}) = \frac{\sum_j a_{ij}(t_m)a_{ij}(t_{m+1})}{\sqrt{\left[\sum_j a_{ij}(t_m)\right]\left[\sum_j a_{ij}(t_{m+1})\right]}} \tag{7}$$

and the *average topological overlap* of the neighbourhood of node i as the average of $C_i(t_m, t_{m+1})$ over all possible subsequent temporal snapshot, i.e.:

$$C_i = \frac{1}{M-1}\sum_{m=1}^{M-1} C_i(t_m, t_{m+1}) \tag{8}$$

The average topological overlap of a node i is a natural extension of the concept of local clustering coefficient which includes temporal information. In fact, while in a static graph the local clustering coefficient of a node measures the probability that its neighbours are in turn connected by an edge, the average topological overlap estimates the probability that an edge from i to one of its neighbours j persists across two consecutive time-windows. In a word, it is a measure of the *temporal clustering* of edges, i.e., of their tendency to persist across multiple windows. The average of C_i computed over all the nodes in the network, namely the quantity:

$$C = \frac{1}{N}\sum_i C_i \tag{9}$$

is called *temporal-correlation coefficient* [43], and is a measure of the overall average probability for an edge to persist across two consecutive snapshots. Notice

Table 1 Temporal-correlation, characteristic temporal path length and efficiency for brain functional networks (in four different frequency bands) [9], for the social interaction networks of INFOCOM'06 (time periods between 1 p.m. and 2:30 p.m., four different days) [39], and for messages over Facebook online social network (4 different months of year 2007) [48]

	C	C^{rand}	L	L^{rand}	E	E^{rand}
α	0.44	0.18 (0.03)	3.9	4.2	0.50	0.48
β	0.40	0.17 (0.002)	6.0	3.6	0.41	0.45
γ	0.48	0.13 (0.003)	12.2	8.7	0.39	0.37
δ	0.44	0.17 (0.003)	2.2	2.4	0.57	0.56
d1	0.80	0.44 (0.01)	8.84	6.00	0.192	0.209
d2	0.78	0.35 (0.01)	5.04	4.01	0.293	0.298
d3	0.81	0.38 (0.01)	9.06	6.76	0.134	0.141
d4	0.83	0.39 (0.01)	21.42	15.55	0.019	0.028
Mar	0.044	0.007 (0.0002)	456	451	0.000183	0.000210
Jun	0.046	0.006 (0.0002)	380	361	0.000047	0.000057
Sep	0.046	0.006 (0.0002)	414	415	0.000058	0.000074
Dec	0.049	0.006 (0.0002)	403	395	0.000047	0.000059

Results are compared with the averages measured over 1,000 time-varying graphs obtained by reshuffling the sequences of snapshots. The values in parenthesis next to C_{rand} are the respective standard deviations. The values of L and L^{rand} are computed considering only the connected pairs of nodes. Table adapted from [43]

that $C = 1$ if and only if all the snapshots of the time-varying graphs have exactly the same configuration of edges, while it is equal to zero if none of the edges is ever observed in two subsequent snapshots.

In [43] the authors considered time-varying graphs constructed from three data sets, namely functional brain networks [9], the co-location at INFOCOM 2006 [39] and personal messages exchanged among Facebook users [48]. They compared the characteristic temporal path length and the temporal correlation coefficient of these temporal graphs with those obtained from the same data sets by reshuffling the sequence of snapshots. Notice that by reshuffling the snapshots one destroys all the existing temporal correlations while retaining the average connectivity of each node and the configuration of edges in the corresponding aggregated graph. They showed that the original time-varying graphs usually exhibit both a relatively smaller characteristic temporal path length and a relatively higher temporal correlation coefficient, when compared with those measured on reshuffled sequences of snapshots. This finding is the temporal analogous to the small-world effect observed in static complex networks, and has consequently been named *small-world behaviour in time-varying graphs*. Table 1 reports the results obtained for the three different data sets.

5 Betweenness, Closeness and Spectral Centrality

The structural properties of a complex network usually reveal important information about its dynamics and function. This is particularly true if we take into account the relationship between the position occupied by a node in a static graph and the *role* played by the node for the evolution of a dynamic process. For instance, not all nodes have the same impact on the transmission of a disease (or the spreading of a rumour) over a network: intuitively, the nodes having a higher number of neighbours should contribute much more to the spreading than nodes having few connections. However, if we perform a deeper analysis, we observe that not just the number of edges is important to identify good spreaders, since also the actual organisation of these edges has an impact on the speed of the spreading process. In fact, nodes mediating a large number of shortest paths are indeed those that contribute the most to the transmission of diseases and information over a network. The identification of nodes that play a central role, i.e., nodes having high *centrality*, has been a quite active research field in complex network theory. Here we review some standard centrality measures and their extension to the case of time-varying graphs.

5.1 Betweenness and Closeness Centrality

Two basic centrality measures based on shortest paths are *betweenness* centrality and *closeness* centrality. The betweenness centrality of a node i in a static graph is defined as follows:

$$C_i^B = \sum_{j \in V} \sum_{\substack{k \in V \\ k \neq j}} \frac{\sigma_{jk}(i)}{\sigma_{jk}} \tag{10}$$

where σ_{jk} is the number of shortest paths from node j to node k, while $\sigma_{jk}(i)$ is the number of such shortest paths that pass through the node i. The higher the number of shortest paths passing through i, the higher the value of C_i^B. Betweenness centrality can be also defined for single edges, by counting the fraction of shortest paths between any pair of nodes to which a given edge participate.

A simple way to extend betweenness centrality to time-varying graphs consists in counting the fraction of temporal shortest paths that traverse a given node. The formula would be exactly the same as (10), with the only difference that σ_{jk} and $\sigma_{jk}(i)$ will be, respectively, the total number of temporal shortest paths between j and k and the number of those paths which make use of node i.

Sometimes it can be important to take into account not only the number of temporal shortest paths which pass through a node, but also the length of time for which a node along the shortest path retains a message before forwarding it to the next node [41]. For example, let us consider the simple case of nodes i and j being connected by just one shortest path which consists of the two edges $(i,k)_{t_\ell}$ and $(k,j)_{t_m}$. This means that the edge connecting i to k appears at time t_ℓ, while the edge connecting k to j appears at time t_m. Since the path through k is the only way for i to temporally reach j, then we would say that k plays an important mediatory role and is "central" for communication between i and j. Nevertheless, the vulnerability of node k heavily depends on the interval $[t_\ell, t_m]$: the longer this temporal interval, the higher the probability that a message forwarded to k is lost if k is removed from the network. In order to take into account the effect of waiting times, the *temporal betweenness centrality* [41] of the node i at time t_m is defined as:

$$C_i^B(t_m) = \frac{1}{(N-1)(N-2)} \sum_{j \neq i} \sum_{\substack{k \neq j \\ k \neq i}} \frac{U(i, t_m, j, k)}{\sigma_{jk}} \tag{11}$$

where σ_{jk} is the number of temporal shortest path from j to k, and $U(i, t_m, j, k)$ is the number of temporal shortest paths from i to j in which node i is traversed from the path in the snapshot t_m or in a previous snapshot $t' < t_m$, so that the next edge of the same path will be available at a later snapshot $t'' > t_m$. The *average temporal betweenness* of node i is defined as the average of $C_i^B(t_m)$ over all the snapshots:

$$C_i^B = \frac{1}{M} \sum_m C_i^B(t_m) \tag{12}$$

The closeness centrality of a node i is a measure of how close i is to any other node in the network. It can be measured as the inverse of the average distance from i to any other node in the network:

$$C_i^C = \frac{N-1}{\sum_j d_{ij}} \tag{13}$$

where d_{ij} is the distance between i and j in a static graph. The *temporal closeness centrality* is defined in an analogous way, the only difference being that for time-varying graphs d_{ij} denotes the length of the temporal shortest path from i to j.

As shown in [41] and elsewhere in this book [44], temporal closeness and betweenness centrality have proven useful to identify key spreaders and temporal mediators in corporate communication networks. In particular, it was found that traders indeed played an important mediatory role in time-varying graphs constructed from the ENRON email communication data set, being consistently ranked among the first ones both for temporal betweenness and for temporal closeness centrality. This result is qualitatively and substantially different from the one obtained by computing betweenness and closeness centrality in the corresponding aggregated graph, where the most central nodes are the people who interacted with the most number of other people, i.e., a secretary and a managing director. This apparently unimportant discrepancy between the centrality rankings actually turns out to be fundamental for the spreading of information (or diseases) throughout the system. In fact, simulation reported in [41] confirmed that when a spreading process is initiated at the nodes having the highest temporal closeness centrality the number of other nodes reached by the spreading was higher and the time needed to reach them was smaller than in the case in which the spreading starts at nodes having higher static closeness centrality.

5.2 Spectral Centrality and Communicability

The total number of shortest paths passing through a node is not always the best way of measuring its centrality, especially because the shortest paths are not always the most relevant for a process. For instance, a disease could propagate through any path (not just through the shortest ones), and the rumours usually follow walks which are much longer (and somehow less efficient) than shortest paths. Consequently, other definitions of centrality exist which take into account walks instead of shortest paths. The classic example for static graphs is represented by the so called *Katz centrality* [28]. The basic idea is that the propensity for node i to communicate with node j can be quantified by counting how many walks of length $\ell = 1, 2, 3, \ldots$ lead from i to j. The importance of a walk of length $\ell = 1$ (i.e., the direct edge (i, j)) is higher than that of a walk of length $\ell = 2$, which in turn is higher than that of a walk of length $\ell = 3$ and so forth. For this reason, it makes sense to appropriately rescale the contribution of longer walks. The original proposal consisted into scaling walks of length ℓ by a factor α^{ℓ}, where α is an appropriately chosen real value. We notice that the element a_{ij}^{ell} of the ℓ^{th} power of the adjacency matrix corresponds to the number of existing walks of length ℓ between i and j. Consequently, the entry s_{ij} of the matrix sum $S = I + \alpha A + \alpha^2 A^2 + \ldots$ measures the propensity of i to interact with j (notice that I is the $N \times N$ identity matrix). It is possible to prove that the sum S converges to $(I - \alpha A)^{-1}$ if $\alpha < \rho(A)$, where $\rho(A)$ is the spectral radius of the adjacency matrix. In this case, the Katz centrality of node i is measured as the sum of the i^{th} row of S:

$$C_i^K = \sum_j \left[(I - \alpha A)^{-1} \right]_{ij} \tag{14}$$

Katz centrality can be extended to the case of time-varying graphs by using a similar reasoning [12]. We notice that each entry of the product of the adjacency matrices corresponding to an increasing sequence of ℓ snapshots $[t_{r_1}, t_{r_2}, \ldots, t_{r_\ell}]$ represents the number of temporal walks in which the first edge belongs to the snapshot t_{r_1}, the second edge to t_{r_2} and so on. So, in order to count all the possible temporal walks of any length we should sum over all possible products of the form:

$$\alpha^k A(t_{r_1}) A(t_{r_2}) \cdots A(t_{r_k}), \quad t_{r_1} \leq t_{r_2} \leq \cdots \leq t_{r_k} \tag{15}$$

for any value of the length k. It is possible to prove that if $\alpha < \min_{t_m} \rho(A(t_m))$ then the sum of all these products can be expressed as:

$$Q \equiv [I - \alpha A(t_0)]^{-1} [I - \alpha A(t_1)]^{-1} \cdots [I - \alpha A(t_m)]^{-1} \tag{16}$$

The matrix Q is called *communicability matrix*. Starting from this matrix one can define the *broadcast centrality*:

$$C_i^{Broad} = \sum_j Q_{ij} \tag{17}$$

which quantifies how well node i can reach all the other nodes in the time-varying graph, and the *receive communicability*:

$$C_i^{Recv} = \sum_j Q_{ji} \tag{18}$$

which is an estimation of how well node i can be reached from any other node in the network. In [12] it has been found that broadcast and receive communicability can be useful to spot the most influential spreaders in different time-evolving communication networks.

6 Meso-scale Structures

Real static networks differ from random graphs in many ways. In fact, together with heterogeneous distributions of node properties (e.g. degree and centrality) and with specific global characteristics (e.g. high average clustering and small average path length), complex networks show a non-trivial organisation of subsets of their nodes and exhibit a variety of meso-scale structures, including *motifs* and *communities*. The characterisation of the abundance of specific motifs has helped to explain why biological and technological networks are relatively resilient to failures [24, 25], while the analysis of communities has revealed that there exists a tight relationship between structure of a network and its functioning [10]. In the following we discuss how motifs analysis can be performed also in time-varying graphs and we present the extension of the modularity (a function for measuring the quality of a partition in communities, defined for static graphs by Newman in [34]) to temporal communities.

6.1 Temporal Motifs

In static graphs a *motif* is defined as a class of isomorphic subgraphs. We recall that two graphs G' and G'', having adjacency matrices A' and A'', are *isomorphic* if there exists a permutation of the labels of the nodes of A' such that, after the permutation, $A' \equiv A''$. A permutation is represented by a matrix P that has the effect of swapping the rows and columns of the matrix to which it is applied. If A' and A'' are isomorphic, then there exists a permutation matrix P such that:

$$P^{-1}A'P = A'' \tag{19}$$

If two graphs are isomorphic then they are topologically equivalent, i.e., the arrangement of the edges in the two graphs is exactly the same, up to an appropriate relabelling of the nodes. Consequently, a motif can be thought as the typical representative of a class of subgraphs sharing the same arrangement of edges. It has been shown that in real networks, especially in biological ones, motifs are not uniformly distributed and some motifs are over-represented while others are rare [24, 25].

As for all the metrics described so far, the extension of *motifs* to time-varying graphs has to take into account time in a meaningful way. In a recent paper Kovanen et al. [20] propose an extension of motifs to time-varying graphs, based on the definition of $\Delta\tau$-adjacency and $\Delta\tau$-connectedness of contacts.[1] For practical reasons, the authors made the simplifying assumption that each node can be involved in no more than one contact at a time. This assumption is in general too restrictive, but it could be valid in some cases, e.g. when the contacts represent phone calls or when the duration δt of a contact is so small that the probability for a node to have two simultaneous contacts is negligible.

We say that two contacts $c_a = (i, j, t_a, \delta t_a)$ and $c_b = (k, \ell, t_b, \delta t_b)$ are $\Delta\tau$-adjacent if they have at least one node in common and the time difference between the end of the first contact and the beginning of the second one is no longer than $\Delta\tau$. We assume, without loss of generality, that $t_a < t_b$, so that c_a and c_b are $\Delta\tau$-adjacent if $0 \leq t_b - t_a - \delta t_a \leq \Delta\tau$. We say that an ordered pair of contacts (c_a, c_b) is *feasible* if $t_a < t_b$. Notice that $\Delta\tau$-adjacency is defined only for the subset of feasible pairs of contacts. Two contacts c_a and c_b are $\Delta\tau$-connected if there exists a sequence of m contacts $S = \{c_a = c_{n_0}, c_{n_1}, c_{n_2}, \ldots, c_{n_m} = c_b\}$ such that each pair of consecutive contacts in S is feasible and $\Delta\tau$-adjacent. From $\Delta\tau$-connectedness, we derive the definition of *connected temporal subgraph*, which is a set of contacts such that all feasible pairs of contacts in the set are $\Delta\tau$-connected.

For the definition of temporal motifs, we restrict ourselves to the subset of *valid temporal subgraphs*. A temporal subgraph is considered valid if all the $\Delta\tau$-adjacent contacts of the nodes in the subgraph are consecutive. This means that if a node j appears in the pair of $\Delta\tau$-adjacent contacts $c_a = (i, j, t_a, \delta t_a)$ and $c_b = (j, k, t_b, \delta t_b)$ of the subgraph, then does not exist any contact $c_x = (j, k, t_x, \delta t_x)$

[1] In order to avoid confusion with the size Δt of the time-window used to define the temporal snapshot of a time-varying graph, here we preferred to use $\Delta\tau$ instead of the original Δt proposed by the authors of [20]. Also, notice that the authors use to call *events* what we have called here *contacts*.

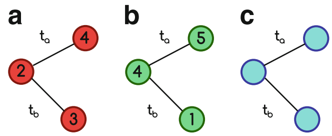

Fig. 5 Connected temporal subgraphs and motifs. The subgraph in panel (**a**) is not a valid temporal subgraph, because node 2 is involved in another contact after the contact with node 4 at time t_a and before the contact with node 3 at time t_b. Conversely, the subgraph in panel (**b**) is valid. Panel (**c**) shows the motif associated to the valid subgraph in panel (**b**)

such that $t_a < t_x < t_b$. In Fig. 5 we report two temporal subgraphs of three nodes obtained from the set of contacts in Fig. 1 considering $\Delta\tau = 30$ min. The subgraph of Panel (a) corresponds to the sequence of contacts $S_1 = \{c_1 = (2, 4, 0, 40), c_3 = (2, 3, 70, 20)\}$, and is not valid because node 2 is involved in another contact, namely $c_2 = (2, 5, 50, 10)$ after c_1 and before c_3. Conversely, the connected temporal subgraph reported in Panel (b) and corresponding to the pair of contacts $S_2 = \{c_4 = (4, 5, 60, 50), c_6 = (1, 4, 140, 35)\}$, is valid, since node 4 is not involved in any contact between c_4 and c_6.

A *temporal motif* is a class of isomorphic valid temporal subgraphs, where two temporal subgraphs are considered isomorphic if they are topologically similar (i.e., the organisation of the links in the subgraph is equivalent up to an appropriate relabelling of the nodes) and represent the same temporal pattern, i.e., the order of the sequence of contacts is the same. The typical element of the temporal motif corresponding to the graph reported in Fig. 5b is shown in Fig. 5c, where the labels on the edges of the graph correspond to the ordering of contacts. In [20] the authors report also an algorithm to discover temporal motifs, and discuss the problems connected with the estimation of the significance of motifs.

6.2 Temporal Communities and Modularity

The identification of communities, i.e., groups of tightly connected nodes, has allowed to reveal the richness of static graphs and has helped to understand their organisation and function. The simplest way to partition a graph is by dividing it into a set of $\mathscr{M}$ non-overlapping groups, so that each node of the graph is assigned to one of the $\mathscr{M}$ communities. The quality of a non-overlapping partition in communities can be measured by the *modularity function*. This function, originally proposed by Newman in [34], estimates the difference between the fraction of edges among nodes belonging to the same community and the expected fraction of such edges in a null-model graph with no communities. More formally, it is defined as follows:

$$Q = \frac{1}{2K} \sum_{ij} (a_{ij} - P_{ij}) \delta(c_i, c_j) \tag{20}$$

where a_{ij} are the elements of the adjacency matrix, P_{ij} is the expected number of edges between i and j in the null-model graph, c_i is the community to which node i belongs, $\delta(c_i, c_j) = 1$ if and only if node i and node j belong to the same community and K is the total number of edges in the graph. The simplest null-model is represented by a configuration model graph, where all the nodes of the graph have the same degree as in the real graph but edges are placed at random. In this case the modularity function reads:

$$Q = \frac{1}{2K} \sum_{ij} \left(a_{ij} - \frac{k_i k_j}{2K} \right) \delta(c_i, c_j) \tag{21}$$

where k_i is the degree of node i. Different extensions of the modularity function have been proposed for directed graphs, weighted graphs and graphs with overlapping communities [3, 23, 32, 35].

The extension of modularity for time-varying graphs is based on an interesting result valid for the modularity of static graphs and presented in [21], which connects the modularity function with the dynamics of a random walk over the graph. The authors of [21] show that the modularity function can be considered a particular case of a class of functions that measure the dynamical stability of a partition $\mathscr{P}$, where the stability of $\mathscr{P}$ at time t is defined as:

$$R(t) = \sum_{\mathscr{C} \in \mathscr{P}} P(\mathscr{C}, t) - P(\mathscr{C}, \infty) \tag{22}$$

Supposing that the random walk has reached the stationary state,[2] then $P(\mathscr{C}, t)$ is the probability that a random walker which starts from a node in the community $\mathscr{C}$ is found in a node of $\mathscr{C}$ after time t. Similarly, $P(\mathscr{C}, \infty)$ is the probability that a random walker that started from a node in $\mathscr{C}$ is found in $\mathscr{C}$ after an infinite number of steps; when the walk has reached the stationary state, this corresponds to the probability that two independent random walks are found in $\mathscr{C}$ at the same time. It is possible to show that if we consider a discrete-time random walk, in which a walker jumps from a node to another at equally-spaced time-steps of length $\Delta\tau$, then the stability of a partition at one step $R(t = \Delta\tau)$ is identical to the modularity function.

In [27] Mucha et al. propose an extension of modularity to multi-slice graphs which exploits the connection between the modularity function and the stability of a random walk on the multi-slice graph. Indeed, a time-varying graph can be considered a multi-slice graph if we connect each node of a snapshot with the other instances of itself in neighbouring snapshots by means of *multi-slice edges*. For brevity, we give here the definition of the modularity function for multi-slice graphs,

[2] It is possible to prove that a random walk on a graph always converges towards a stationary state, independently of the initial condition, if the adjacency matrix of the graph is primitive, which is the case for the vast majority of real graphs.

which corresponds to the stability at one step of a random walk on the multi-slice graph, but we omit the derivation of the formula.[3] The modularity for multi-slice graphs reads:

$$Q_{multi} = \frac{1}{2\mu} \sum_{ijsr} \left[\left(a_{ij}(s) - \gamma_s \frac{k_i(s) k_j(s)}{2m_s} \right) \delta_{sr} + C_{jsr} \delta_{ij} \right] \delta(g_{is}, g_{jr}) \tag{23}$$

The indices i and j are used for nodes while the indices r and s indicate different slices. Here $a_{ij}(s)$ are the elements of the adjacency matrix of slice s, $k_i(s)$ represents the degree of node i in slice s (i.e., the number of neighbours to with i is connected in that slice) and $m_s = \frac{1}{2}\sum_i k_i(s)$ is the total number of edges in the slice s. The term C_{jsr} is the weight of the link that connects node j in slice s to itself in slice r, and γ_s is a resolution parameter. The terms δ_{ij} and δ_{rs} indicate the Kronecker function and $\delta(g_{is}, g_{jr})$ is equal to 1 only if node i in slice s and node j in slice r belong to the same community. The definition looks a bit complicated but it is essentially composed of two parts. The term in parentheses represents the standard modularity for the graph at slice s (the only difference being the resolution parameter γ_s), while the term C_{jsr} accounts for inter-slice connections. Once we have defined modularity for multi-slice networks, the search for the best partition can be performed by using one of the standard methods for modularity optimisation [10].

This definition of modularity is quite general, works well for any kind of multi-slice network, and is also applicable to assess the quality of a partition of a time-varying graph, which can be considered a multi-slice network. However, when using (23) one should take into account that in order to derive it as the stability of a random walk on the graph, the edges connecting different slices have to be undirected.[4] Consequently, this definition of modularity in invariant under inversion of the sequence of slices which, in the particular case of time-varying graphs, implies invariance under time inversion. This means that (23) gives the same result on the time-varying graph $\mathscr{G}_{[0,T]}$ and on the graph $\mathscr{G}_{[T,0]}$ in which the sequence of time-windows is given in the opposite order.

In general, invariance under time inversion is not a desirable property for a metric used to characterise the structure of time-varying graphs, because most of the interesting characteristics of time-evolving systems, including temporal correlation of links and reachability, are due to the asymmetry introduced by the *arrow of time*. Time-invariance disregards this asymmetry completely, washing out most of the richness of time-varying systems. Consequently, we believe that the definition of appropriate metrics for the evaluation of community structures in time-varying graphs is still an open field of investigation.

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[3]The interested reader can find the derivation of (23) in [27] and in the Supplemental Information of the same paper.

[4]This is required to ensure the existence of a stationary state for the Laplacian dynamics on the graph.

7 Final Remarks

The description of temporal networks in terms of time-varying graphs and the analysis of their structural properties is still in its infancy but has already produced many encouraging results, showing that complex networks theory is a quite flexible and promising framework for the characterisation of different real systems. There are still some open problems to be tackled, such as the definition of appropriate methods to detect temporal communities and the construction of analytical methods to assess how the structure of a time-varying graph can affect the dynamics of processes occurring over it, including spreading, synchronisation and evolutionary games. However, even if the community has not yet converged towards a unified notation and a fully consistent set of definitions and approaches is still lacking, the metrics and concepts devised so far for time-varying graphs constitute a valid and consistent alternative to the standard methods for the study of time-evolving systems, and will certainly represent a fundamental contribution to our understanding of complex systems in general.

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