Chapter 2
Perturbation of a Linear Conservative System by Periodic Parametric Excitation

2.1 Motivation

Usually vibrations of machines and structures are defined about a certain state of operation. If the structure operates as planned, the deviations from this reference state are small, so that a linearized analysis of the corresponding equations of motion is meaningful and sufficient from a technical point of view. Usually one has rather good information about mass and stiffness properties of engineering structures, whereas other phenomena, such as contact conditions and damping, are hard to describe. The wide success of mechanical modeling relies on the fact that usually mass and stiffness properties dominate the vibrational behavior of mechanical structures. Mathematically this can be captured in terms of a perturbation problem

$$(\boldsymbol{M} + \Delta\boldsymbol{M}(t))\ddot{\boldsymbol{q}} + \Delta\boldsymbol{D}(t)\dot{\boldsymbol{q}} + (\boldsymbol{K} + \Delta\boldsymbol{K}(t))\boldsymbol{q} = \boldsymbol{F}(t), \tag{2.1}$$

where the mass matrix $\boldsymbol{M}$ and the stiffness matrix $\boldsymbol{K}$ describe the well known structural behavior of the parts with respect to their generalized coordinates, whereas the matrices $\Delta\boldsymbol{M}(t)$, $\Delta\boldsymbol{D}(t)$, $\Delta\boldsymbol{K}(t)$ and the force vector $\boldsymbol{F}(t)$ are perturbations, originating from small variable inertia, contacts, dissipation and unknown external forcing. For rigid body systems, the generalized coordinates can always be defined such that the linearized equations of motion have the form (2.1). In many cases, the perturbation matrices will either be constant or have periodic coefficients, as is the case for example in rotors with spatially fixed contacts. Since the perturbations originate from physical processes which are hard to model and to parameterize, they are impossible to predict accurately. Therefore a design goal must be to lay out the structure in such a way that the perturbations, which are only known to a little extend, do not cause significant vibrations. We will show in this chapter that it is a good strategy to design the system such that no multiple eigenfrequencies occur in the unperturbed problem. Usually for the consideration of (2.1) one divides the analysis into free vibrations and forced vibrations, the sum of which yield the general solution. However, the occurrence of the external forces $\boldsymbol{F}(t)$ depends on the

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system boundary of the model. If one draws the boundary such that every forcing is generated within the model, only free vibrations occur. As an example consider the one degree of freedom (dof) oscillator

$$\ddot{x} + \omega_0^2 x = f \cos \Omega t. \tag{2.2}$$

By extending the system boundary such that the forcing is generated by another oscillator

$$\ddot{y} + \Omega^2 y = 0, \tag{2.3a}$$

$$\ddot{x} + \omega_0^2 x - \varepsilon y = 0, \tag{2.3b}$$

which is asymmetrically coupled to our original oscillator, the system can be modeled as a homogeneous problem. It is clear that asymmetric couplings are always possible, for example due to frictional contacts. Equation (2.3) is a special case of (2.1) with $\boldsymbol{F} = \boldsymbol{0}$, the system we will consider throughout the chapter. This line of reasoning, to explain external forces by other dynamical systems, can be investigated in a much deeper way, as done by HERTZ, who developed a theory of mechanics which does not rely on the definition of forces [18]. If only linear equations of motion are considered the above reasoning is however a little bit artificial since limited sources of energy normally yield nonlinear equations of motion. Nevertheless the above setting can simplify the mathematical analysis of problems significantly.

2.2 Underlying Equations of Motion

We consider the differential equations of a conservative system, perturbed by linear parametric periodic excitation, which, without loss of generality, can be written as

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \Delta\boldsymbol{D}(t,\varepsilon)\dot{\boldsymbol{q}} + (\boldsymbol{K} + \Delta\boldsymbol{K}(t,\varepsilon))\boldsymbol{q} = \boldsymbol{0}, \tag{2.4a}$$

$$\Delta\boldsymbol{D}(t,\varepsilon) = \Delta\boldsymbol{D}(t+2\pi,\varepsilon), \qquad \Delta\boldsymbol{K}(t,\varepsilon) = \Delta\boldsymbol{K}(t+2\pi,\varepsilon), \tag{2.4b}$$

$$\boldsymbol{M} = \mathrm{diag}(1, 1, \ldots, 1), \qquad \boldsymbol{K} = \mathrm{diag}(\omega_1^2, \ldots, \omega_N^2) \tag{2.4c}$$

by introducing a dimensionless time based on the frequency of the parametric excitation. We assume that $\Delta\boldsymbol{D}(t,\varepsilon)$ and $\Delta\boldsymbol{K}(t,\varepsilon)$ depend smoothly on the parameter ε and can be expanded as

$$\Delta\boldsymbol{D}(t,\varepsilon) = \Delta\boldsymbol{D}_1(t)\varepsilon + \Delta\boldsymbol{D}_2(t)\varepsilon^2 + \cdots, \tag{2.5a}$$

$$\Delta\boldsymbol{K}(t,\varepsilon) = \Delta\boldsymbol{K}_1(t)\varepsilon + \Delta\boldsymbol{K}_2(t)\varepsilon^2 + \cdots. \tag{2.5b}$$

The parameter ε can be regarded as a norm of the perturbation. In the following we work with the first order system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} \tag{2.6}$$
$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{K} - \varepsilon\Delta\boldsymbol{K}_1(t) + \cdots & -\varepsilon\Delta\boldsymbol{D}_1(t) + \cdots \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{bmatrix}$$

equivalent to (2.4a). It is well known from FLOQUET theory that the stability of the trivial solution depends on the eigenstructure of the monodromy matrix, which we expand in terms of the perturbation parameter ε

$$\boldsymbol{X}(2\pi, \varepsilon) = \boldsymbol{X}(2\pi, 0) + \left.\frac{\partial \boldsymbol{X}(2\pi, 0)}{\partial \varepsilon}\right|_{\varepsilon=0} \varepsilon + \cdots . \tag{2.7}$$

From the perturbation theory developed in [67, 69] it is known that the derivative of the monodromy matrix can be calculated as

$$\left.\frac{\partial \boldsymbol{X}(2\pi, \varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} = \boldsymbol{X}(2\pi, 0)\boldsymbol{H}, \tag{2.8a}$$

with

$$\boldsymbol{H} = \int_0^{2\pi} \boldsymbol{Y}(t, 0)^T \left.\frac{\partial \boldsymbol{A}(t, \varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} \boldsymbol{X}(t, 0)\mathrm{d}t, \tag{2.8b}$$

an expression which depends only on the derivative of the system matrix of the corresponding first-order system with respect to ε, the monodromy matrix of the unperturbed problem $\boldsymbol{X}$ and the monodromy matrix $\boldsymbol{Y}$ of its adjoint system

$$\dot{\boldsymbol{y}} = -\boldsymbol{A}^T \boldsymbol{y}. \tag{2.9}$$

Since in our case the unperturbed problem has constant coefficients and is decoupled, we have very simple analytic expressions for $\boldsymbol{X}(t, 0)$ and $\boldsymbol{Y}(t, 0)$ namely

$$\boldsymbol{X} = \begin{bmatrix} \cos\omega_1 t & 0 & \cdots & 0 & \frac{1}{\omega_1}\sin\omega_1 t & 0 & \cdots & 0 \\ 0 & \cos\omega_2 t & \cdots & 0 & 0 & \frac{1}{\omega_2}\sin\omega_2 t \cdots & & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & 0 & & \cos\omega_N t & 0 & 0 & & \frac{1}{\omega_N}\sin\omega_N t \\ -\omega_1\sin\omega_1 t & 0 & \cdots & 0 & \cos\omega_1 t & 0 & \cdots & 0 \\ 0 & -\omega_2\sin\omega_2 t & \cdots & 0 & 0 & \cos\omega_2 t & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & 0 & & -\omega_N\sin\omega_N t & 0 & 0 & & \cos\omega_N t \end{bmatrix} \tag{2.10}$$

and the matrix $\boldsymbol{Y} = \boldsymbol{X}^{-1}$ simply reads

$$
\boldsymbol{Y} = \begin{bmatrix}
\cos\omega_1 t & 0 & \cdots & 0 & \omega_1 \sin\omega_1 t & 0 & \cdots & 0 \\
0 & \cos\omega_2 t & \cdots & 0 & 0 & \omega_2 \sin\omega_2 t & \cdots & 0 \\
\vdots & & & \vdots & \vdots & & & \vdots \\
0 & 0 & & \cos\omega_N t & 0 & 0 & & \omega_N \sin\omega_N t \\
-\frac{1}{\omega_1}\sin\omega_1 t & 0 & \cdots & 0 & \cos\omega_1 t & 0 & \cdots & 0 \\
0 & -\frac{1}{\omega_2}\sin\omega_2 t & \cdots & 0 & 0 & \cos\omega_2 t & \cdots & 0 \\
\vdots & & & \vdots & \vdots & & & \vdots \\
0 & 0 & & -\frac{1}{\omega_N}\sin\omega_N t & 0 & 0 & & \cos\omega_N t
\end{bmatrix}. \tag{2.11}
$$

The spectrum of the unperturbed problem can be either simple or semi-simple. Semi-simple eigenvalues can occur due to the symmetry of the structure, i.e. $\omega_i = \omega_j$, or due to internal or combination resonances. According to [89] simple and semi-simple eigenvalues of the monodromy matrix can be expanded in terms of ε as

$$
\rho = \rho_0 + \left.\frac{\partial \rho}{\partial \varepsilon}\right|_{\varepsilon=0} \varepsilon + \cdots, \tag{2.12}
$$

where the derivative of the FLOQUET multiplier can be calculated analytically [66]. The derivative of the modulus can then be obtained [67, 69] from

$$
\frac{\partial|\rho|}{\partial \varepsilon} = \frac{1}{|\rho_0|}\mathrm{Re}\left(\bar{\rho}_0 \frac{\partial \rho}{\partial \varepsilon}\right). \tag{2.13}
$$

The eigenvalues ρ_{0j}, the eigenvectors $\boldsymbol{u}_j$ of the monodromy matrix $X(2\pi, 0)$ of the unperturbed problem and the eigenvectors $\boldsymbol{v}_j$ of its adjoint $Y(2\pi, 0)$ have a particularly simple form given by

$$
\rho_{0j} = \cos 2\pi\omega_j + \mathrm{i}\sin 2\pi\omega_j \tag{2.14}
$$

and

$$
\boldsymbol{u}_j = \frac{1}{\sqrt{2}}\left(0, \cdots, 0, \frac{\mathrm{i}}{\omega_j}, 0, \cdots, 0, 1, 0, \cdots, 0\right)^T, \tag{2.15a}
$$

$$
\boldsymbol{v}_j = \frac{1}{\sqrt{2}}\left(0, \cdots, 0, -\mathrm{i}\omega_j, 0, \cdots, 0, 1, 0, \cdots, 0\right)^T, \tag{2.15b}
$$

where the nonzero entries appear in the j-th and $N + j$-th position. The eigenvectors are normalized such that $\boldsymbol{v}_j^T \boldsymbol{u}_j = 1$.

2.3 Perturbation of Simple Eigenvalues

For a simple FLOQUET multiplier ρ_j, the derivative $\left.\frac{\partial \rho_j}{\partial \varepsilon}\right|_{\varepsilon=0}$, in the following for simplicity denoted by $\frac{\partial \rho_j}{\partial \varepsilon}$, is given by [68]

$$\frac{\partial \rho_j}{\partial \varepsilon} = \frac{\boldsymbol{v}_j^T \left.\frac{\partial \boldsymbol{X}(2\pi,0)}{\partial \varepsilon}\right|_{\varepsilon=0} \boldsymbol{u}_j}{\boldsymbol{v}_j^T \boldsymbol{u}_j} = \frac{\rho_{0j} \boldsymbol{v}_j^T \boldsymbol{H} \boldsymbol{u}_j}{\boldsymbol{v}_j^T \boldsymbol{u}_j}. \tag{2.16}$$

Due to the simple structure of $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{u}_j$ and $\boldsymbol{v}_j$ and using

$$\begin{aligned} &\boldsymbol{v}_j^T \int_0^{2\pi} \boldsymbol{Y}(t,0)^T \left.\frac{\partial \boldsymbol{A}(t,\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} \boldsymbol{X}(t,0)\mathrm{d}t\, \boldsymbol{u}_k \\ &\quad = \int_0^{2\pi} \boldsymbol{v}_j^T \boldsymbol{Y}(t,0)^T \left.\frac{\partial \boldsymbol{A}(t,\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} \boldsymbol{X}(t,0)\boldsymbol{u}_k \mathrm{d}t, \end{aligned} \tag{2.17}$$

the derivative of ρ_j with respect to ε can be calculated as

$$\frac{\partial \rho_j}{\partial \varepsilon} = \frac{1}{2}\left(-\frac{\mathrm{i}}{\omega_j}\int_0^{2\pi} \Delta k_{jj}\mathrm{d}t - \int_0^{2\pi} \Delta d_{jj}\mathrm{d}t\right)\rho_{0j}, \tag{2.18}$$

where Δk_{jj}, Δd_{jj} are the matrix entries on the diagonal of $\Delta \boldsymbol{K}_1(t)$, $\Delta \boldsymbol{D}_1(t)$. It is interesting to note that on the main diagonal of the matrix $\int_0^{2\pi} \boldsymbol{V}^T \boldsymbol{Y}(t,0)^T \left.\frac{\partial \boldsymbol{A}(t,\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} \boldsymbol{X}(t,0)\boldsymbol{U}\mathrm{d}t$, where $\boldsymbol{U}$ ($\boldsymbol{V}$) is build up form the eigenvectors of the (adjoint) unperturbed problem, the entries are proportional to the mean value of the perturbation matrices. Since

$$\frac{\partial |\rho_j|}{\partial \varepsilon} = \frac{1}{|\rho_{0j}|}\mathrm{Re}\left(\bar{\rho}_{0j}\frac{\partial \rho_j}{\partial \varepsilon}\right) = -\frac{1}{2}\int_0^{2\pi} \Delta d_{jj}\mathrm{d}t, \tag{2.19}$$

we see that in the first approximation for simple eigenvalues the modulus of the eigenvalues is not influenced by restoring terms Δk_{jj} and is decreased by dissipative forces Δd_{jj}.

2.4 Perturbation of Semi-Simple Eigenvalues

For two semi-simple FLOQUET multipliers ρ_j and ρ_k, the first term in (2.12) is calculated according to [67, 69, 72] from

$$\det\begin{bmatrix} \rho_{0j}\boldsymbol{v}_j^T \boldsymbol{H}\boldsymbol{u}_j - \frac{\partial \rho_j}{\partial \varepsilon} & \rho_{0j}\boldsymbol{v}_j^T \boldsymbol{H}\boldsymbol{u}_k \\ \rho_{0k}\boldsymbol{v}_j^T \boldsymbol{H}\boldsymbol{u}_j & \rho_{0k}\boldsymbol{v}_k^T \boldsymbol{H}\boldsymbol{u}_k - \frac{\partial \rho_j}{\partial \varepsilon} \end{bmatrix} = 0, \tag{2.20}$$

using again the adjoint problem. Semi-simple eigenvalues can arise due to multiple eigenfrequencies, combination resonances or internal resonances. Let us first consider the case of two equal eigenfrequencies of the unperturbed problem.

For a system with $\omega_i = \omega_j = \omega$ due to (2.20) only the matrix entries k, l with $k, l \in \{i, j\}$ enter the first derivative of the FLOQUET multiplier. The derivative of a semi-simple FLOQUET multiplier can therefore be calculated from an equivalent two by two system of the type (2.4a).

We now consider the case of such a two by two system with three parameters κ, δ, γ where

$$\Delta \boldsymbol{D}(t, \varepsilon) = \delta(\varepsilon)\boldsymbol{D}(t) + \boldsymbol{G}(t), \qquad \Delta \boldsymbol{K}(t, \varepsilon) = \kappa(\varepsilon)\boldsymbol{K}(t) + \gamma(\varepsilon)\boldsymbol{N}(t), \tag{2.21}$$

and $\boldsymbol{D}(t)$, $\boldsymbol{K}(t)$ are symmetric, periodic, positive semi-definite and $\boldsymbol{G}(t)$, $\boldsymbol{N}(t)$ are skew-symmetric and periodic. In our applications the matrix $\boldsymbol{G}(t)$ originates from small gyroscopic terms, which in some rotating systems arise through CORIOLIS effects when the equations of motion are set up with respect to a rotating frame. Therefore the matrix $\boldsymbol{G}(t)$ will most often be constant, and for moderate angular velocities $\Omega \ll \omega_1$ is small, so that it can be regarded as perturbation. According to the assumptions made in the context (2.4a), the parameters can be expanded in terms of ε as

$$\kappa(\varepsilon) = \kappa_1\varepsilon + \kappa_2\varepsilon^2 + \cdots \tag{2.22a}$$

$$\delta(\varepsilon) = \delta_1\varepsilon + \delta_2\varepsilon^2 + \cdots \tag{2.22b}$$

$$\gamma(\varepsilon) = \gamma_1\varepsilon + \gamma_2\varepsilon^2 + \cdots . \tag{2.22c}$$

The derivative of the FLOQUET multiplier can be calculated from

$$\det\left(\int_0^{2\pi} \frac{-\rho_0}{2}\left(\delta\boldsymbol{D}(t) + \boldsymbol{G}(t) + \frac{\mathrm{i}}{\omega}\Big(\kappa\boldsymbol{K}(t) + \gamma\boldsymbol{N}(t)\Big)\right)\mathrm{d}t - \frac{\partial\rho}{\partial\varepsilon}\boldsymbol{I}\right) = 0. \tag{2.23}$$

Assuming that $\int_0^{2\pi} \boldsymbol{D}(t)\mathrm{d}t$ is positive definite, which is most often the case due to material damping, the number of parameters in (2.23) can be reduced by performing an orthogonal transformation

$$\begin{aligned}
&\det\left(\boldsymbol{Q}^T\left(\int_0^{2\pi} \frac{-\rho_0}{2}\left(\delta\boldsymbol{D}(t) + \boldsymbol{G}(t) + \frac{\mathrm{i}}{\omega}\Big(\kappa\boldsymbol{K}(t) + \gamma\boldsymbol{N}(t)\Big)\right)\mathrm{d}t\right)\boldsymbol{Q} - \frac{\partial\rho}{\partial\varepsilon}\boldsymbol{I}\right)\\
&= \det\left(\frac{-\rho_0}{2}\left(\begin{bmatrix}\delta d & g\\ -g & \delta(d+\Delta d)\end{bmatrix} + \frac{\mathrm{i}}{\omega}\begin{bmatrix}\kappa k & \gamma n\\ -\gamma n & \kappa(k+\Delta k)\end{bmatrix}\right) - \frac{\partial\rho}{\partial\varepsilon}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)
\end{aligned} \tag{2.24a}$$

where $\boldsymbol{Q}$ consists of the eigenvectors, which simultaneously diagonalize $\int_0^{2\pi} \boldsymbol{D}(t)\mathrm{d}t$ and $\int_0^{2\pi} \boldsymbol{K}(t)\mathrm{d}t$. The derivative of the FLOQUET multiplier can thus be calculated as

$$\frac{\partial\rho}{\partial\varepsilon}\Big|_{\varepsilon=0} = \rho_0\Bigg(-\frac{d\delta}{2}-\frac{\delta\Delta d}{4}-\mathrm{i}\left(\frac{k\kappa}{2\omega}+\frac{\Delta k\kappa}{4\omega}\right) \pm\sqrt{-\frac{g^2}{4}+\frac{n^2\gamma^2}{4\omega^2}+\frac{\delta^2\Delta \mathrm{d}^2}{16}-\frac{\Delta \mathrm{k}^2\kappa^2}{16\omega^2}+\mathrm{i}\left(\frac{\delta\Delta \mathrm{d}\Delta \mathrm{k}\kappa}{8\omega}-\frac{gn\gamma}{2\omega}\right)}\Bigg) \tag{2.25}$$

The first approximation to the stability boundary described by $\frac{\partial|\rho|}{\partial\varepsilon}=0$ with equation (2.13) reads

$$0 = -\delta\left(\frac{d}{2}+\frac{\Delta d}{4}\right)+\mathrm{Re}\left(\sqrt{-\frac{g^2}{4}+\frac{n^2\gamma^2}{4\omega^2}+\frac{\delta^2\Delta \mathrm{d}^2}{16}-\frac{\Delta \mathrm{k}^2\kappa^2}{16\omega^2}+\mathrm{i}\left(\frac{\delta\Delta \mathrm{d}\Delta \mathrm{k}\kappa}{8\omega}-\frac{gn\gamma}{2\omega}\right)}\right) \tag{2.26a}$$

$$= -\delta\left(\frac{d}{2}+\frac{\Delta d}{4}\right)+|z|\cos\left(\frac{1}{2}\arg(z)\right) \tag{2.26b}$$

with

$$z = -\frac{g^2}{4}+\frac{n^2\gamma^2}{4\omega^2}+\frac{\delta^2\Delta \mathrm{d}^2}{16}-\frac{\Delta \mathrm{k}^2\kappa^2}{16\omega^2}+\mathrm{i}\left(\frac{\delta\Delta \mathrm{d}\Delta \mathrm{k}\kappa}{8\omega}-\frac{gn\gamma}{2\omega}\right), \tag{2.26c}$$

$$\arg(\mathrm{z}) = \arctan\left(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right). \tag{2.26d}$$

The corresponding graph is shown in Fig. 2.1 for $\omega = 0.6$, $d = \pi$, $n = \pi$, $k = \pi$, $\Delta k = \pi$, $\Delta d = 0.3\pi$. At first we consider the case $g = 0$, which means that no gyroscopic perturbations are present. Assuming additionally $\Delta d = 0$, the expression for the stability boundary simplifies substantially and can be written as

$$\gamma n = \sqrt{\frac{\kappa^2\Delta k^2}{4}+\delta^2 d^2\omega^2}. \tag{2.27}$$

The corresponding stability boundary is given in Fig. 2.2 and coincides with the numerical example

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}\ddot{\boldsymbol{q}}+\delta(\varepsilon)\cos^2 t\begin{bmatrix}1&0\\0&1\end{bmatrix}\dot{\boldsymbol{q}}+\left(\begin{bmatrix}\omega^2&0\\0&\omega^2+\kappa(\varepsilon)\end{bmatrix}+\gamma(\varepsilon)\cos^2 t\begin{bmatrix}0&-1\\1&0\end{bmatrix}\right)\boldsymbol{q}=\boldsymbol{0} \tag{2.28}$$

studied in [72] (i.e. $k = 0$, $\Delta k = 2\pi$, $d = \pi$, $\Delta d = 0$, $n = \pi$) in a three parameter setting.

By comparison of Figs. 2.1 and 2.2 it is seen that they are qualitatively equivalent. In Fig. 2.3 cuts through the stability domain at $\kappa_1 = 0$ and $\kappa_1 = 0.13$ ($\frac{\omega_2}{\omega_1} \approx 1.15$)

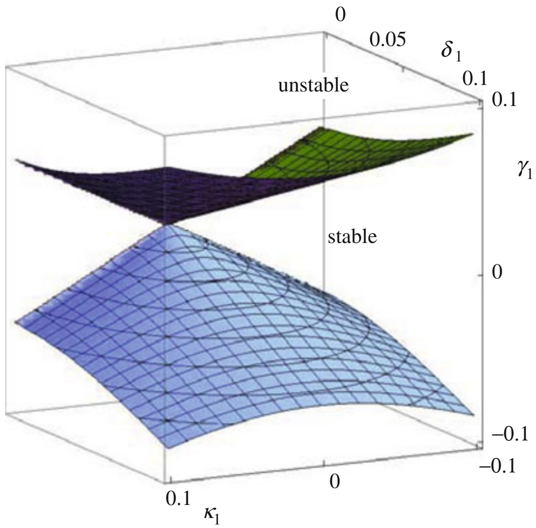


Fig. 2.1 Approximation to the stability domain for $\Delta k = \pi$, $\Delta d = 0.3\pi$

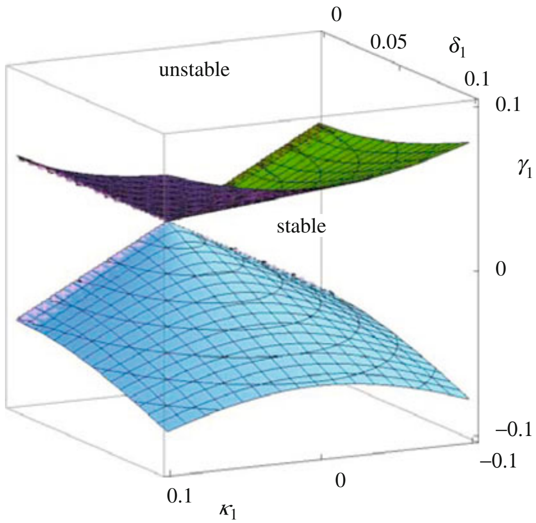


Fig. 2.2 Approximation to the stability boundary for $\Delta d = 0$

are shown in comparison with a numerical evaluation of the FLOQUET multipliers. Figure 2.4 shows cuts through the stability region at $\delta_1 = 0$ and $\delta_1 = 0.3$.

It is seen, that for small perturbations the approximation of the stability boundary is in very good agreement with the numerical results, especially in the technically relevant range with small damping and a moderate splitting of the frequencies up to 10 percent.

Now we consider the case of $g \neq 0$, i.e. the occurrence of small gyroscopic terms. Considering the same parameter set as before, $d = \pi, n = \pi, k = \pi, \Delta k = \pi$,

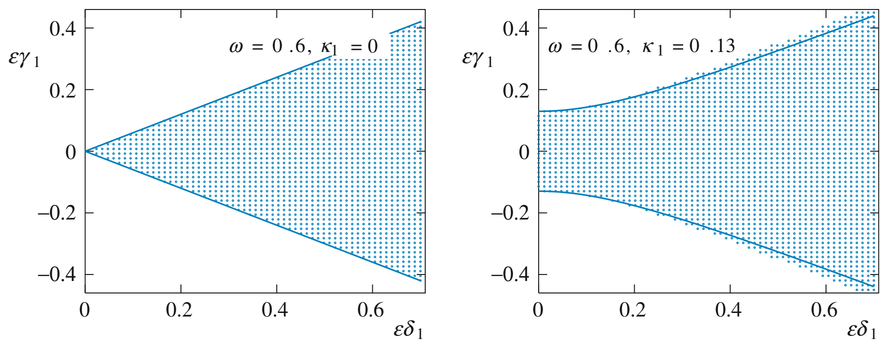


Fig. 2.3 Stable regions for a symmetric and an asymmetric system (*dot* stable)

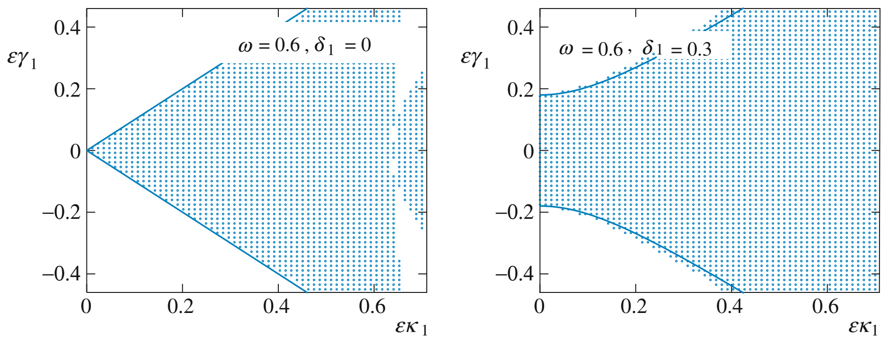


Fig. 2.4 Stable regions for a symmetric and an asymmetric system (*dot* stable)

$\Delta d = 0.3\pi$, with $g = 0.01$ we obtain the approximation of the stability boundary shown in Fig. 2.5. Taking a closer look at the cuts for different values of κ we obtain the curves shown in Fig. 2.6. It is seen that the gyroscopic terms destroy the stability boundary from Fig. 2.4, i.e. the splitting of eigenfrequencies without introduction of damping does not have an effect. However we see that in combination with slight damping also for this case an enlargement of the stable region is achieved. For larger values of δ and γ, the role of the gyroscopic terms is almost negligible since the stability boundary calculated for $\kappa = 0.1$ gets closer to the red curve obtained for $g = 0$ and the same value of κ. For large gyroscopic terms, the effect of stabilization by splitting of eigenfrequencies is destroyed, as we see from Fig. 2.7 where the entries of the gyroscopic matrix are of the order of magnitudes of those of the stiffness matrix. If we have combination resonances i.e. $\omega_i = m\,\omega_j$ for m integer, the analysis can also be performed with (2.20), however the expressions are much lengthier than in the case $m = 1$. From numerical investigations it is seen that the system is less sensitive to self-excited vibrations at combination resonances as in the case $\omega_1 = m\,\omega_2$. This was to be expected, since the stiffness terms in the perturbation expressions are divided by the frequency. The corresponding graph is shown in Fig. 2.8 for $\omega_1 = 0.6$, $\omega_2 = 2\omega_1$,

Fig. 2.5 Approximation to the stability boundary for $g \neq 0$

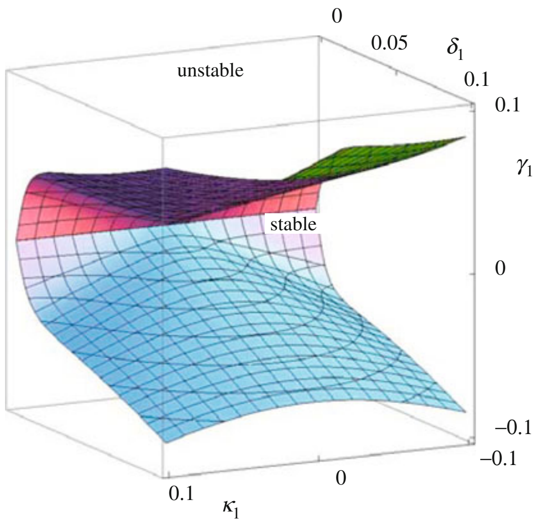
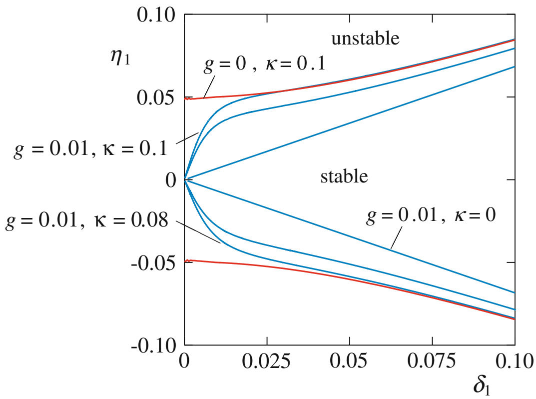


Fig. 2.6 Approximation of the stability boundary for $g = 0$ (*red*) and $g = 0.01$ for $\kappa = 0, 0.08, 0.1$



$d = \pi$, $n = \pi$, $k = \pi$, $\Delta k = \pi$, $\Delta d = 0.3\pi$, in which the unstable region is much narrower than in the corresponding case with double eigenfrequency at $\omega = 0.6$, which is shown in Fig. 2.1.

Let us now consider the case of an internal resonance for the jth eigenvalue, i.e. we have $\omega_j = \frac{k}{2}$ such that the imaginary part of the FLOQUET multiplier in (2.14) vanishes. Instead of a pair of complex conjugate FLOQUET multipliers we now have a double eigenvalue $\rho_0 = (-1)^k$ which is semi-simple. The first derivative of the FLOQUET multipliers can again be analyzed from (2.20). Examining the corresponding eigenvectors, we observe that the formulas for the first derivative of the FLOQUET multipliers coincide with the ones for a damped version of HILL's equation

$$\ddot{q} + \varepsilon \Delta d_{jj}(t)\dot{q} + (\omega_j^2 + \varepsilon \Delta k_{jj}(t))q = 0, \tag{2.29}$$

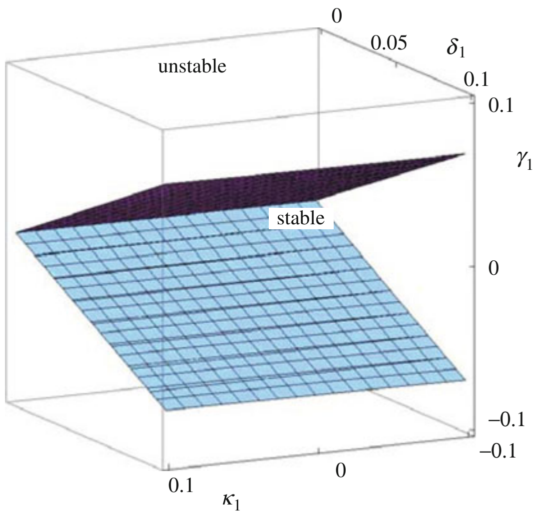


Fig. 2.7 Approximation to the stability boundary for $g = \pi$

which have been calculated in [65] for a constant damping term. For a periodic damping term, the first derivatives of the corresponding multipliers read

$$\frac{\partial \rho}{\partial \varepsilon} = (-1)^k \left(-c_d k \pm \pi \sqrt{(2a_k - b_d k)^2 + (2b_k - a_d k)^2 - (2c_k)^2} \right), \qquad (2.30)$$

where

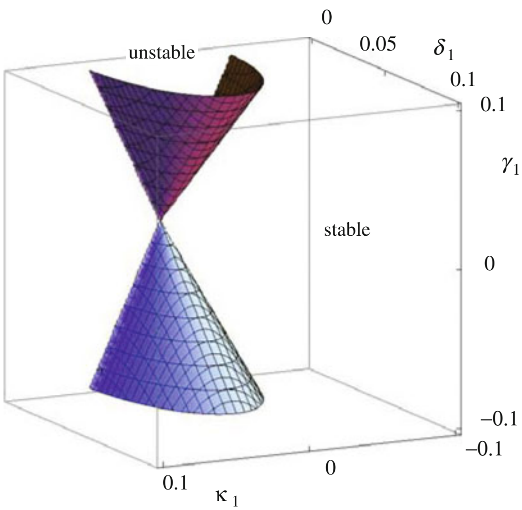


Fig. 2.8 Approximation to the stability domain for $\Delta k = \pi$, $\Delta d = 0.3\pi$

$$a_k = \frac{1}{2\pi k}\int_0^{2\pi} \sin(k\tau)\Delta k_{jj}(\tau)\mathrm{d}\tau, \qquad a_d = \frac{1}{2\pi k}\int_0^{2\pi} \sin(k\tau)\Delta d_{jj}(\tau)\mathrm{d}\tau,$$
$$b_k = \frac{1}{2\pi k}\int_0^{2\pi} \cos(k\tau)\Delta k_{jj}(\tau)\mathrm{d}\tau, \qquad b_d = \frac{1}{2\pi k}\int_0^{2\pi} \cos(k\tau)\Delta d_{jj}(\tau)\mathrm{d}\tau,$$
$$c_k = \frac{1}{2\pi k}\int_0^{2\pi} \Delta k_{jj}(\tau)\mathrm{d}\tau, \qquad c_d = \frac{1}{2\pi k}\int_0^{2\pi} \Delta d_{jj}(\tau)\mathrm{d}\tau.$$

From the truncated expansion

$$\rho_j = \rho_{0j} + \frac{\partial \rho_j}{\partial \varepsilon} + \mathcal{O}(\varepsilon) = (-1)^k \left(1 + \left(-c_d k \pm \pi\sqrt{D}\right)\right) + \mathcal{O}(\varepsilon), \tag{2.31}$$
$$D = (2a_k - b_d k)^2 + (2b_k - a_d k)^2 - (2c_k)^2,$$

of the FLOQUET multiplier we observe that the system tends to get unstable if the term under the square root is larger than $|c_d k|$. In particular, this is the case when damping is absent and describes vertices of the instability regions for the HILL and the MATHIEU equation [65, 68]. From (2.30) we see that damping has a stabilizing effect, provided

$$c_d > \sqrt{a_d^2 + b_d^2}.$$

For increasing k the damping terms dominate, since they are multiplied by k.

2.5 Discussion

The results obtained in the previous sections obviously only have a local character. They show that, in the vicinity of a semi-simple eigenvalue, a skew symmetric perturbation has a particularly strong destabilizing effect. At the same time it has to be noted that for $\frac{\partial|\rho|}{\partial\varepsilon} = 0$, ρ moves along the unit circle in the complex plane in the first approximation. The system can still be destabilized by higher order terms. In particular this is the case for constant coefficient $\boldsymbol{M}$, $\Delta\boldsymbol{G}$, $\boldsymbol{K}$, $\Delta\boldsymbol{N}$ systems with very small gyroscopic terms that are almost always unstable [24, 33, 40] although neither $\Delta\boldsymbol{G}$ nor $\Delta\boldsymbol{N}$ appear in the first derivative of the expansion for the FLOQUET multipliers, provided the spectrum of the unperturbed problem is simple.

Only in specific cases, for example for systems with reversible symmetry (i.e. invariant under time reversal), we can conclude that in the absence of damping the system has a double eigenvalue at the stability boundary [96]. In [96] p. 133 it is shown that if $\Delta\boldsymbol{D}(-t) = -\Delta\boldsymbol{D}(t)$ and $\Delta\boldsymbol{K}(-t) = \Delta\boldsymbol{K}(t)$, then the system is invariant under time reversal.

Despite of having only local character, the perturbation formulas indicate that splitting up the eigenfrequencies of the unperturbed problem tends to stabilize the system, especially when additional damping is present. In order to design a robust system against self-excited vibrations one should therefore maximize the minimal distance between two neighboring eigenfrequencies. This is the goal of structural optimization approaches of later chapters, and the mathematical structure of such an optimization problem is investigated in the next chapter.

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