Chapter 2
Higher Order Ordinary Differential Equations

In this chapter, we begin by solving homogeneous linear ordinary differential equations with constant coefficients by characteristic equations. Then we solve the Euler equations and exact equations. The method of undetermined coefficients for solving inhomogeneous linear ordinary differential equations is presented, as well as the method of variation of parameters for solving second-order inhomogeneous linear ordinary differential equations. In addition, we introduce the power series method to solve variable-coefficient linear ordinary differential equations and study the Bessel equation in detail.

2.1 Basics

This section deals with homogeneous linear ordinary differential equations with constant coefficients, the Euler equations, and exact equations.

A second-order homogeneous linear ordinary differential equation with constant coefficients is of the form

$$ay'' + by' + cy = 0, \quad a, b, c \in \mathbb{R}. \tag{2.1.1}$$

To find the general solution, we assume that $y = e^{\lambda t}$ is a solution of (2.1.1), where λ is a constant to be determined. Substituting it into (2.1.1), we get

$$a\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + ce^{\lambda t} \sim a\lambda^2 + b\lambda + c = 0, \tag{2.1.2}$$

which is called the *characteristic equation* of (2.1.1). If the above equation has two distinct real roots λ_1 and λ_2, then the general solution of (2.1.1) is

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \tag{2.1.3}$$

where c_1 and c_2 are arbitrary constants. When (2.1.2) has two complex roots $r_1 \pm r_2 i$, then the real part and imaginary part of $e^{(r_1+r_2 i)t}$ are solutions of (2.1.1).

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So the general solution of (2.1.1) is

$$y = c_1 e^{r_1 t} \sin r_2 t + c_2 e^{r_1 t} \cos r_2 t. \tag{2.1.4}$$

If (2.1.2) has a repeated root r, the general solution of (2.1.1) is

$$y = (c_1 + c_2 t) e^{rt}. \tag{2.1.5}$$

Example 2.1.1 The general solution of the equation

$$y'' - 2y' - 3y = 0 \tag{2.1.6}$$

is

$$y = c_1 e^{3t} + c_2 e^{-t} \tag{2.1.7}$$

because $\lambda = 3$ and $\lambda = -1$ are real roots of the characteristic equation $\lambda^2 - 2\lambda - 3 = 0$. Moreover, the general solution of the equation

$$y'' - 4y' + 13y = 0 \tag{2.1.8}$$

is

$$y = c_1 e^{2t} \sin 3t + c_2 e^{2t} \cos 3t \tag{2.1.9}$$

because $\lambda = 2 + 3i$ and $\lambda = 2 - 3i$ are roots of the characteristic equation $\lambda^2 - 4\lambda + 13 = 0$. Furthermore, the general solution of the equation

$$y'' + 6y' + 9y = 0 \tag{2.1.10}$$

is

$$y = (c_1 + c_2 t) e^{-3t}. \tag{2.1.11}$$

In general, the algebraic equation

$$b_n \lambda^n + b_{n-1} \lambda^{n-1} + \cdots + b_0 = 0 \tag{2.1.12}$$

is called the *characteristic equation* of the differential equation

$$b_n y^{(n)} + b_{n-1} y^{(n-1)} + \cdots + b_0 y = 0, \quad b_r \in \mathbb{R}. \tag{2.1.13}$$

If (2.1.12) has a real root r with multiplicity m, then

$$\big(c_{m-1} t^{m-1} + \cdots + c_1 t + c_0\big) e^{rt} \tag{2.1.14}$$

is a solution of (2.1.13) for arbitrary $c_0, c_1, \ldots, c_{m-1} \in \mathbb{R}$. When $r_1 + r_2 i$ is a complex root of (2.1.12) with multiplicity m, then

$$\big(c_{m-1} t^{m-1} + \cdots + c_1 t + c_0\big) e^{r_1 t} \sin r_2 t \tag{2.1.15}$$

and

$$\big(a_{m-1}t^{m-1}+\cdots+a_1t+a_0\big)e^{r_1t}\cos r_2t \tag{2.1.16}$$

are solutions of (2.1.13) for arbitrary $c_r, a_r \in \mathbb{R}$. For instance, if

$$(\lambda-1)(\lambda+2)^3\big(\lambda^2-4\lambda+13\big)^2=0 \tag{2.1.17}$$

is the characteristic equation of a differential equation of the form (2.1.13), then the general solution of the differential equation is

$$\begin{aligned} y &= c_1e^t+\big(c_2t^2+c_3t+c_4\big)e^{-2t}+(c_5t+c_6)e^{2t}\sin 3t \\ &\quad +(c_7t+c_8)e^{2t}\cos 3t. \end{aligned} \tag{2.1.18}$$

An *Euler ordinary differential equation* has the general form

$$b_nt^ny^{(n)}+b_{n-1}t^{n-1}y^{(n-1)}+\cdots+b_1ty'+b_0y=0, \quad b_r\in\mathbb{R}. \tag{2.1.19}$$

We solve it by using the change of variable $x=\ln t$. In fact,

$$y'=\frac{y_x}{t}, \qquad y''=\frac{y_{xx}-y_x}{t^2}, \qquad y'''=\frac{y_{xxx}-3y_{xx}+2y_x}{t^3}. \tag{2.1.20}$$

Example 2.1.2 Solve the equation

$$t^2y''-3ty'+5y=0. \tag{2.1.21}$$

Solution. Changing the variable $x=\ln t$, we get

$$y_{xx}-y_x-3y_x+5y=0\sim y_{xx}-4y_x+5y=0, \tag{2.1.22}$$

whose characteristic equation is $\lambda^2-4\lambda+5=0$. The roots are $\lambda=2\pm i$. So the general solution is

$$y=c_1e^{2x}\sin x+c_2e^{2x}\cos x=t^2(c_1\sin\ln t+c_2\cos\ln t). \tag{2.1.23}$$

Example 2.1.3 Solve the Euler equation

$$t^3y'''-t^2y''-2ty'-4y=0. \tag{2.1.24}$$

Solution. Using the change of variable $x=\ln t$, we get

$$\begin{aligned} &y_{xxx}-3y_{xx}+2y_x-(y_{xx}-y_x)-2y_x-4y \\ &\quad =0\sim y_{xxx}-4y_{xx}+y_x-4y=0, \end{aligned} \tag{2.1.25}$$

whose characteristic equation is

$$\lambda^3-4\lambda^2+\lambda-4=(\lambda-4)\big(\lambda^2+1\big)=0. \tag{2.1.26}$$

Thus the general solution is

$$y = c_1 e^{4x} + c_2 \sin x + c_3 \cos x = c_1 t^4 + c_2 \sin \ln t + c_3 \cos \ln t. \quad (2.1.27)$$

An nth-order ordinary differential equation is called an *exact equation* if the equation can be rewritten as

$$\frac{d\Phi(t, y, y', \ldots, y^{(n-1)})}{dt} = 0. \quad (2.1.28)$$

We try to find Φ term by term.

Example 2.1.4 Solve the equation

$$tyy'' + ty'^2 + yy' = 0. \quad (2.1.29)$$

Solution. Note that $\Phi = tyy'$. Thus (2.1.29) can rewritten as $(tyy')' = 0$. Thus

$$2tyy' = c_1 \sim t\big(y^2\big)' = c_1 \quad \Longrightarrow \quad y^2 = c_1 \ln t + c_2. \quad (2.1.30)$$

Example 2.1.5 Solve the equation

$$\big(1 + t + t^2\big)y''' + (3 + 6t)y'' + 6y' = 6t. \quad (2.1.31)$$

Solution. We rewrite (2.1.31) as

$$\big(1 + t + t^2\big)y''' + (1 + 2t)y'' + (2 + 4t)y'' + 6y' - 6t = 0 \quad (2.1.32)$$

$$\Longrightarrow \quad \big[\big(1 + t + t^2\big)y''\big]' + (2 + 4t)y'' + 4y' + 2y' - 6t = 0 \quad (2.1.33)$$

$$\Longrightarrow \quad \big[\big(1 + t + t^2\big)y''\big]' + \big[(2 + 4t)y'\big]' + 2y' - 6t = 0 \quad (2.1.34)$$

$$\Longrightarrow \quad \big[\big(1 + t + t^2\big)y''\big]' + \big[(2 + 4t)y'\big]' + (2y)' - \big(3t^2\big)' = 0 \quad (2.1.35)$$

$$\Longrightarrow \quad \big[\big(1 + t + t^2\big)y'' + (2 + 4t)y' + 2y - 3t^2\big]' = 0 \quad (2.1.36)$$

$$\Longrightarrow \quad \big(1 + t + t^2\big)y'' + (2 + 4t)y' + 2y - 3t^2 = 2c_1 \quad (2.1.37)$$

$$\Longrightarrow \quad \big(1 + t + t^2\big)y'' + (1 + 2t)y' + (1 + 2t)y' + 2y - 3t^2 = 2c_1 \quad (2.1.38)$$

$$\Longrightarrow \quad \big[\big(1 + t + t^2\big)y' + (1 + 2t)y - t^3\big]' = 2c_1 \quad (2.1.39)$$

$$\Longrightarrow \quad \big(1 + t + t^2\big)y' + (1 + 2t)y - t^3 = 2c_1 t + c_2 \quad (2.1.40)$$

$$\Longrightarrow \quad \big[\big(1 + t + t^2\big)y\big]' - t^3 = 2c_1 t + c_2 \quad (2.1.41)$$

$$\Longrightarrow \quad \big(1 + t + t^2\big)y - \frac{t^4}{4} = c_1 t^2 + c_2 t + c_3. \quad (2.1.42)$$

Exercises 2.1

1. Find the general solution of the equation

$$y'' - y' - 6y = 0.$$

2. Find the general solution of the equation

$$y'' + 6y' + 13y = 0.$$

3. Find the general solution of the equation

$$y^{(4)} + 8y'' + 16y = 0.$$

4. Solve the Euler equation

$$t^3 y''' + 3t^2 y'' - 2ty' + 2y = 0.$$

5. Solve the equation

$$tyy''' + 3ty'y'' + 2yy'' + 2y'^2 = 2\cos t - t\sin t.$$

2.2 Method of Undetermined Coefficients

In this section, we present the method of undetermined coefficients for solving inhomogeneous linear ordinary differential equations.

In order to solve the linear inhomogeneous ordinary differential equation

$$f_n(t)y^{(n)} + f_{n-1}(t)y^{(n-1)} + \cdots + f_1(t)y = g(t), \tag{2.2.1}$$

we find the general solution $\phi(t, c_1, \ldots, c_n)$ of the homogeneous equation

$$f_n(t)y^{(n)} + f_{n-1}(t)y^{(n-1)} + \cdots + f_1(t)y = 0 \tag{2.2.2}$$

and a particular solution $y_0(t)$ of (2.2.1). Then the general solution of (2.2.1) is $y = \phi(t, c_1, \ldots, c_n) + y_0(t)$. It often happens that y_0 is obtained by guessing it in a certain form with undetermined coefficients based on the form of $g(t)$.

Example 2.2.1 Find the general solution of the equation

$$y'' - \frac{2}{t^2}y = 7t^4 + 3t^3. \tag{2.2.3}$$

Solution. It is easy to see that $y = t^2$ and $y = 1/t$ are solutions of

$$y'' - \frac{2}{t^2}y = 0. \tag{2.2.4}$$

So the general solution of (2.2.4) is

$$y = c_1 t^2 + \frac{c_2}{t}. \tag{2.2.5}$$

Based on the form of (2.2.3), we guess a particular solution $y_0(t) = at^6 + bt^5$, where a and b are the constants to be determined. Note that

$$y_0' = 6at^5 + 5bt^4 \quad \Longrightarrow \quad y_0'' = 30at^4 + 20t^3. \tag{2.2.6}$$

By (2.2.3),

$$\begin{aligned} &30at^4 + 20t^3 - 2\big(at^4 + bt^3\big) = 7t^4 + 3t^3 \sim 28a = 7, \\ &18b = 3 \quad \Longrightarrow \quad a = \frac{1}{4}, \quad b = \frac{1}{6}. \end{aligned} \tag{2.2.7}$$

Thus $y_0 = t^6/4 + t^5/6$. The general solution of (2.2.3) is

$$y = c_1 t^2 + \frac{c_2}{t} + \frac{t^6}{4} + \frac{t^5}{6}. \tag{2.2.8}$$

Example 2.2.2 Solve the equation

$$y'' + 3y' + 2y = 3\sin 2t. \tag{2.2.9}$$

Solution. The general solution of $y'' + 3y' + 2y = 0$ is $y = c_1 e^{-t} + c_2 e^{-2t}$. We guess a particular solution of (2.2.9):

$$y_0 = a\sin 2t + b\cos 2t. \tag{2.2.10}$$

Then

$$y_0' = 2a\cos 2t - 2b\sin 2t, \qquad y_0'' = -4a\sin 2t - 4b\cos 2t. \tag{2.2.11}$$

By (2.2.9),

$$\begin{aligned} &-4a\sin 2t - 4b\cos 2t + 3(2a\cos 2t - 2b\sin 2t) + 2(a\sin 2t + b\cos 2t) \\ &\quad = 3\sin 2t, \end{aligned} \tag{2.2.12}$$

or equivalently,

$$-(2a + 6b)\sin 2t + (6a - 2b)\cos 2t = 3\sin 2t. \tag{2.2.13}$$

Hence

$$-(2a + 6b) = 3, \quad 6a - 2b = 0 \quad \Longrightarrow \quad a = -\frac{3}{20}, \quad b = -\frac{9}{20}. \tag{2.2.14}$$

So

$$y_0 = -\frac{3}{20}\sin 2t - \frac{9}{20}\cos 2t \tag{2.2.15}$$

and the general solution of (2.2.9) is

$$y = c_1 e^{-t} + c_2 e^{-2t} - \frac{3}{20}\sin 2t - \frac{9}{20}\cos 2t. \tag{2.2.16}$$

Example 2.2.3 Find the solution of the following problem:

$$y'' + y = 2\cos t, \qquad y(0) = 1, \qquad y'(0) = 3. \tag{2.2.17}$$

Solution. The general solution of the corresponding homogeneous equation $y'' + y = 0$ is

$$y = c_1 \cos t + c_2 \sin t. \tag{2.2.18}$$

Thus we cannot guess a particular solution $y_0 = a\cos t + b\sin t$. Instead, we guess that

$$y_0 = at\cos t + bt\sin t \tag{2.2.19}$$

is a particular solution. Then

$$y_0' = (a + bt)\cos t + (b - at)\sin t, \tag{2.2.20}$$

$$y_0'' = (2b - at)\cos t - (2a + bt)\sin t. \tag{2.2.21}$$

Substituting these equations into the equation in (2.2.17), we get

$$2b\cos t - 2a\sin t = 2\cos t. \tag{2.2.22}$$

So

$$a = 0, \quad b = 1; \qquad y_0 = t\sin t. \tag{2.2.23}$$

Thus the general solution is

$$y = c_1\cos t + (c_2 + t)\sin t. \tag{2.2.24}$$

Next

$$y' = (c_2 + t)\cos t + (1 - c_1)\sin t. \tag{2.2.25}$$

Then

$$y(0) = 1 \quad \Longrightarrow \quad c_1 = 1, \tag{2.2.26}$$

$$y'(0) = 3 \quad \Longrightarrow \quad c_2 = 3. \tag{2.2.27}$$

The final solution is

$$y = \cos t + (3 + t)\sin t. \tag{2.2.28}$$

Example 2.2.4 Find the solution of the following problem:

$$y'' - 4y' + 4y = 4\big(t^2 + e^{2t}\big). \tag{2.2.29}$$

Solution. The corresponding homogeneous equation is

$$y'' - 4y' + 4y = 0, \tag{2.2.30}$$

whose characteristic equation is

$$r^2 - 4r + 4 = 0 \quad \Longrightarrow \quad r = 2 \quad \text{is a repeated root.} \tag{2.2.31}$$

Thus the general solution is

$$y = (c_1 + c_2 t)e^{2t}. \tag{2.2.32}$$

First we want to find a particular solution of the equation

$$y'' - 4y' + 4y = 4t^2. \tag{2.2.33}$$

Let

$$y_0 = At^2 + Bt + C \tag{2.2.34}$$

be a particular solution. Then

$$y_0' = 2At + B, \qquad y_0'' = 2A. \tag{2.2.35}$$

Substitute these terms into the equation,

$$2A - 4(2At + B) + 4\big(At^2 + Bt + C\big) = 4t^2 \tag{2.2.36}$$

$$\Longrightarrow \quad 4At^2 + (4B - 8A)t + 2A - 4B + 4C = 4t^2, \tag{2.2.37}$$

$$4A = 4, \quad 4B - 8A = 0, \quad 2A - 4B + 4C = 0$$

$$\Longrightarrow \quad A = 1, \quad B = 2, \quad C = \frac{3}{2}. \tag{2.2.38}$$

So

$$y_0 = t^2 + 2t + \frac{3}{2}. \tag{2.2.39}$$

Next we want to find a particular solution of the equation

$$y'' - 4y' + 4y = 4e^{2t}. \tag{2.2.40}$$

Let

$$y_0 = At^2 e^{2t} \tag{2.2.41}$$

be a particular solution. Then

$$y_0' = 2A\big(t + t^2\big)e^{2t}, \qquad y_0'' = 2A\big(1 + 4t + 2t^2\big)e^{2t}. \tag{2.2.42}$$

Substitute them into the equation,

$$\begin{aligned} &2A\big(1 + 4t + 2t^2\big)e^{2t} - 8A\big(t + t^2\big)e^{2t} + 4At^2 e^{2t} = 4e^{2t} \\ &\implies \quad 2Ae^{2t} = 4e^{2t}. \end{aligned} \tag{2.2.43}$$

So $A = 2$ and

$$y_0 = 2t^2 e^{2t}. \tag{2.2.44}$$

The final solution is

$$y = \big(c_1 + c_2 t + 2t^2\big)e^{2t} + t^2 + 2t + \frac{3}{2}. \tag{2.2.45}$$

Exercises 2.2

1. Find the general solution of the following equation:

$$y'' + y' - 2y = 2t.$$

2. Solve the following initial value problem:

$$y'' + 2y' + 5y = 4e^{-x}\cos 2x, \qquad y(0) = 1, \qquad y'(0) = 0.$$

3. Solve the following initial value problem:

$$y'' - 2y' - 3y = \begin{cases} 3e^{-t} & \text{if } 0 \le t \le 1, \\ 2t^2 & \text{if } t > 1; \end{cases} \qquad y(0) = 0, \quad y'(0) = 1.$$

2.3 Method of Variation of Parameters

In this section, we give the method of variation of parameters for solving second-order inhomogeneous linear ordinary differential equations.

Suppose that we know the fundamental solutions $y_1(t)$ and $y_2(t)$ of the linear homogeneous equation

$$y'' + f_1(t)y' + f_0(t)y = 0, \tag{2.3.1}$$

that is, the general solution of (2.3.1) is $y = c_1y_1(t) + c_2y_2(t)$. We want to solve the linear inhomogeneous equation

$$y'' + f_1(t)y' + f_0(t)y = g(t). \tag{2.3.2}$$

Let $y = u_1(t)y_1 + u_2(t)y_2$ be a solution of (2.3.2), where $u_1(t)$ and $u_2(t)$ are functions to be determined. Note that

$$y' = u_1'y_1 + u_2'y_2 + u_1y_1' + u_2y_2'. \tag{2.3.3}$$

In order to simplify the problem, we impose the condition

$$u_1'y_1 + u_2'y_2 = 0. \tag{2.3.4}$$

Then

$$y' = u_1y_1' + u_2y_2' \quad \Longrightarrow \quad y'' = u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2'y_2'. \tag{2.3.5}$$

According to (2.3.2),

$$\begin{aligned} &u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2'y_2' + f_1\big(u_1y_1' + u_2y_2'\big) + f_0(u_1f_1 + u_2f_2)\\ &\quad = g(t) \qquad (2.3.6)\\ &\quad \Longrightarrow \quad u_1\big(y_1'' + f_1y_1 + f_0y_1\big) + u_2\big(y_2'' + f_2y_1 + f_0y_2\big) + u_1'y_1' + u_2'y_2'\\ &\quad = g(t), \qquad (2.3.7)\end{aligned}$$

or equivalently,

$$u_1'y_1' + u_2'y_2' = g(t) \tag{2.3.8}$$

because y_1 and y_2 are solutions of (2.3.1).

The *Wronskian* of the functions $\{h_1, h_2, \ldots, h_m\}$ is the determinant

$$W(h_1, h_2, \ldots, h_m) = \begin{vmatrix} h_1 & h_2 & \ldots & h_m \\ h_1' & h_2' & \ldots & h_m' \\ \vdots & \vdots & \vdots & \vdots \\ h_1^{(m-1)} & h_2^{(m-1)} & \ldots & h_m^{(m-1)} \end{vmatrix}. \tag{2.3.9}$$

Solving the system of (2.3.4) and (2.3.8) for u_1' and u_2' by Cramer's rule, we get

$$u_1' = -\frac{g(t)y_2(t)}{W(y_1, y_2)}, \qquad u_2' = \frac{g(t)y_1(t)}{W(y_1, y_2)}. \tag{2.3.10}$$

Thus

$$u_1 = -\int \frac{g(t)y_2(t)}{W(y_1, y_2)}\,dt, \qquad u_2 = \int \frac{g(t)y_1(t)}{W(y_1, y_2)}\,dt. \tag{2.3.11}$$

The final solution is

$$y = -y_1(t)\int \frac{g(t)y_2(t)}{W(y_1, y_2)}\,dt + y_2(t)\int \frac{g(t)y_1(t)}{W(y_1, y_2)}\,dt. \tag{2.3.12}$$

This method is called the *method of variation of parameters*.

Example 2.3.1 Find the general solution of the following equation by the method of variation of parameters:

$$y'' + 4y = \frac{4}{\sin 2t}, \quad 0 < t < \frac{\pi}{4}. \tag{2.3.13}$$

Solution. The corresponding homogeneous equation is $y'' + 4y = 0$, whose fundamental solutions are $y_1 = \cos 2t$ and $y_2 = \sin 2t$. So

$$W(y_1, y_2) = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2. \tag{2.3.14}$$

Thus

$$u_1 = -\int \frac{g(t)y_2(t)}{W(y_1, y_2)}\,dt = -2\int dt = c_1 - 2t, \tag{2.3.15}$$

$$u_2 = \int \frac{g(t)y_1(t)}{W(y_1, y_2)}\,dt = \int \frac{2\cos 2t}{\sin 2t}\,dt = \ln\sin 2t + c_2. \tag{2.3.16}$$

The final solution is

$$y = (c_1 - 2t)\cos 2t + (c_2 + \ln\sin 2t)\sin 2t. \tag{2.3.17}$$

Example 2.3.2 Solve the following initial value problem by the method of variation of parameters:

$$y'' - 4y = g(t), \qquad y(0) = 1, \qquad y'(0) = -1. \tag{2.3.18}$$

Solution. First we solve the following initial value problem:

$$u'' - 4u = 0, \qquad u(0) = 1, \qquad u'(0) = -1. \tag{2.3.19}$$

The general solution of this equation is

$$u = c_1 e^{2t} + c_2 e^{-2t}.$$

So

$$u' = 2\big(c_1 e^{2t} - c_2 e^{-2t}\big),$$

$$\begin{cases} u(0) = 1, \\ u'(0) = -1 \end{cases} \implies \begin{cases} c_1 + c_2 = 1, \\ 2(c_1 - c_2) = -1 \end{cases} \implies \begin{cases} c_1 = 1/4, \\ c_2 = 3/4. \end{cases} \tag{2.3.20}$$

The solution is

$$u = \frac{1}{4}\big(e^{2t} + 3e^{-2t}\big). \tag{2.3.21}$$

Next we want to solve the following problem:

$$v'' - 4v = g(t), \qquad v(0) = 0, \qquad v'(0) = 0, \tag{2.3.22}$$

$$W\big(e^{2t}, e^{-2t}\big) = -4,$$

$$\begin{aligned} v &= -e^{2t} \int_0^t \frac{g(s)e^{-2s}}{-4}\, ds + e^{-2t} \int_0^t \frac{g(s)e^{2s}}{-4}\, ds \\ &= \frac{1}{2} \int_0^t g(s) \sinh 2(t-s)\, ds. \end{aligned} \tag{2.3.23}$$

The final solution is

$$y = u + v = \frac{1}{4}\big(e^{2t} + 3e^{-2t}\big) + \frac{1}{2} \int_0^t g(s) \sinh 2(t-s)\, ds. \tag{2.3.24}$$

If

$$v(t) = -y_1(t) \int_0^t \frac{g(s) y_2(s)}{W(y_1, y_2)(s)}\, ds + y_2(t) \int_0^t \frac{g(s) y_1(s)}{W(y_1, y_2)(s)}\, ds, \tag{2.3.25}$$

then

$$v'(t) = -y_1'(t) \int_0^t \frac{g(s) y_2(s)}{W(y_1, y_2)(s)}\, ds + y_2'(t) \int_0^t \frac{g(s) y_1(s)}{W(y_1, y_2)(s)}\, ds. \tag{2.3.26}$$

Thus we always have $v(0) = v'(0) = 0$.

Exercises 2.3

1. Solve the equation

$$y'' + 9y = \frac{9}{\cos 3t}, \quad 0 < t < \frac{\pi}{6}.$$

2. Solve the equation

$$y'' - 2y' + y = \frac{e^t}{1 + t^2}.$$

3. Let $g(t)$ be a given function. Find the solution of the following problem:

$$y'' - 3y' - 4y = g(t), \qquad y(0) = 1, \qquad y'(0) = -1.$$

2.4 Series Method and Bessel Functions

In this section, we use power series to solve certain second-order linear ordinary differential equations with variable coefficients:

$$y'' + f_1(t)y' + f_0(t)y = 0. \tag{2.4.1}$$

Suppose that f_1 and f_0 are analytic at $t = 0$. Around $t = 0$,

$$f_0 = \sum_{n=0}^{\infty} a_n t^n, \qquad f_1 = \sum_{n=0}^{\infty} b_n t^n, \quad a_n, b_n \in \mathbb{R}. \tag{2.4.2}$$

We consider the solution of the form

$$y = \sum_{n=0}^{\infty} c_n t^n, \quad \text{where } c_n \text{ are to be determined.} \tag{2.4.3}$$

$$y' = \sum_{n=1}^{\infty} n c_n t^{n-1}, \qquad y'' = \sum_{n=2}^{\infty} n(n-1)c_n t^{n-2}. \tag{2.4.4}$$

Now (2.4.1) becomes

$$\sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2}t^n + \left(\sum_{n=0}^{\infty} b_n t^n\right)\left(\sum_{n=0}^{\infty}(n+1)c_{n+1}t^n\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right)\left(\sum_{n=0}^{\infty} c_n t^n\right) = 0, \tag{2.4.5}$$

$$(n+1)(n+2)c_{n+2} = -\sum_{r=0}^{n}\big[(r+1)b_{n-r}c_{r+1} + a_{n-r}c_r\big]. \tag{2.4.6}$$

Example 2.4.1 Solve the equation

$$y'' - ty' - y = 0.$$

Solution. Suppose that $y = \sum_{n=0}^{\infty} c_n t^n$ is a solution. Note that $a_r = -\delta_{r,0}$ and $b_r = -\delta_{r,1}$. Thus (2.4.6) becomes

$$(n+1)(n+2)c_{n+2} = (n+1)c_n \sim c_{n+2} = \frac{c_n}{n+2}. \tag{2.4.7}$$

Hence

$$y = c_0 \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!!} + c_1 \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!!}. \tag{2.4.8}$$

Suppose

$$f_0 = \sum_{n=-2}^{\infty} a_n t^n, \qquad f_1 = \sum_{n=-1}^{\infty} b_n t^n, \quad a_n, b_n \in \mathbb{R}. \tag{2.4.9}$$

Assume that $y = \sum_{n=0}^{\infty} c_n t^{n+\mu}$ is a solution of Eq. (2.4.1) with $c_0 \neq 0$. Substituting it into (2.4.1), we find that the coefficients of $t^{\mu-2}$ give

$$\mu(\mu-1) + \mu b_{-1} + a_{-2} = 0 \sim \mu^2 + (b_{-1}-1)\mu + a_{-2} = 0, \tag{2.4.10}$$

which is called the *indicial equation* of (2.4.1) with (2.4.9). If (2.4.10) has two distinct real roots μ_1 and μ_2 such that $\mu_1 - \mu_2$ is not an integer, then Eq. (2.4.1) has two linearly independent solutions of the forms

$$y_1 = t^{\mu_1} \sum_{n=0}^{\infty} c_n t^n, \qquad y_2 = t^{\mu_2} \sum_{n=0}^{\infty} d_n t^n. \tag{2.4.11}$$

When (2.4.10) has a repeated root μ, then Eq. (2.4.1) has two linearly independent solutions of the forms

$$y_1 = t^{\mu} \sum_{n=0}^{\infty} c_n t^n, \qquad y_2 = y_1 \ln t + t^{\mu} \sum_{n=0}^{\infty} d_n t^n. \tag{2.4.12}$$

If (2.4.10) has two distinct real roots μ_1 and μ_2 such that $\mu_2 - \mu_1$ is an integer, then Eq. (2.4.1) has two linearly independent solutions of the forms

$$y_1 = t^{\mu_1} \sum_{n=0}^{\infty} c_n t^n, \qquad y_2 = k y_1 \ln t + t^{\mu_2} \sum_{n=0}^{\infty} d_n t^n, \tag{2.4.13}$$

where k may be zero.

Example 2.4.2 Solve the following equation by the power series method:

$$t^2 y'' + 3t y' + (1+t) y = 0, \quad t > 0. \tag{2.4.14}$$

Solution. Note that $t = 0$ is a regular singular point. Let $y = \sum_{n=0}^{\infty} c_n t^{n+\mu}$ be a solution with $c_0 \neq 0$. Then

$$y' = \sum_{n=0}^{\infty} (n+\mu) c_n t^{n+\mu-1}, \qquad y'' = \sum_{n=0}^{\infty} (n+\mu)(n+\mu-1) c_n t^{n+\mu-2}. \tag{2.4.15}$$

Substituting these equalities into the equation, we have

$$\sum_{n=0}^{\infty}(n+\mu)(n+\mu-1)c_n t^{n+\mu} + 3\sum_{n=0}^{\infty}(n+\mu)c_n t^{n+\mu} + (1+t)\sum_{n=0}^{\infty} c_n t^{n+\mu} = 0, \tag{2.4.16}$$

or equivalently,

$$\sum_{n=0}^{\infty}(n+\mu)(n+\mu-1)c_n t^{n+\mu} + 3\sum_{n=0}^{\infty}(n+\mu)c_n t^{n+\mu} + \sum_{n=0}^{\infty} c_n t^{n+\mu} + \sum_{n=0}^{\infty} c_n t^{n+\mu+1} = 0. \tag{2.4.17}$$

So we have

$$\big[\mu(\mu-1)c_0 + 3\mu c_0 + c_0\big]t^{\mu} + \sum_{n=1}^{\infty}\big((n+\mu)(n+\mu-1)c_n + 3(n+\mu)c_n + c_n + c_{n-1}\big)t^{n+\mu} = 0. \tag{2.4.18}$$

Thus $\mu(\mu-1)c_0 + 3\mu c_0 + c_0 = 0$ and, for $n \geq 1$,

$$(n+\mu)(n+\mu-1)c_n + 3(n+\mu)c_n + c_n + c_{n-1} = 0 \implies (n+\mu+1)^2 c_n = -c_{n-1}, \tag{2.4.19}$$

$$c_n = -\frac{c_{n-1}}{(n+\mu+1)^2} = \frac{(-1)^n c_0}{\prod_{j=1}^{n}(j+\mu+1)^2}. \tag{2.4.20}$$

Denote

$$b_n = \frac{(-1)^n}{\prod_{j=1}^{n}(j+\mu+1)^2}. \tag{2.4.21}$$

Set

$$\varphi(\mu,t) = t^{\mu}\left(1 + \sum_{n=1}^{\infty} b_n t^n\right). \tag{2.4.22}$$

The indicial equation is

$$\mu(\mu-1) + 3\mu + 1 = 0 \sim (\mu+1)^2 = 0 \implies \mu = -1 \tag{2.4.23}$$

is a double root. Then

$$y_1 = \varphi(-1,t) = t^{-1}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{\prod_{j=1}^{n} j^2} t^n\right) = t^{-1}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} t^n\right) \tag{2.4.24}$$

is a solution of (2.4.14).

Observe

$$t^2\varphi_{tt} + 3t\varphi_t + (1+t)\varphi = t^{\mu}(\mu+1)^2 \tag{2.4.25}$$

(cf. the left-hand side of (2.4.18) with $c_0 = 1$). Taking the partial derivative of (2.4.25) with respect to μ, we get

$$t^2\varphi_{tt\mu} + 3t\varphi_{t\mu} + (1+t)\varphi_{\mu} = (\ln t)t^{\mu}(\mu+1)^2 + 2t^{\mu}(\mu+1), \tag{2.4.26}$$

or equivalently,

$$t^2\varphi_{\mu tt} + 3t\varphi_{\mu t} + (1+t)\varphi_{\mu} = \big(2 + (\mu+1)\ln t\big)t^{\mu}(\mu+1). \tag{2.4.27}$$

Taking $\mu = -1$ in the above equation, we find

$$t^2\left(\frac{d}{dt}\right)^2 \varphi_{\mu}(-1,t) + 3t\frac{d}{dt}\varphi_{\mu}(-1,t) + (1+t)\varphi_{\mu}(-1,t) = 0. \tag{2.4.28}$$

Thus $y_2 = \varphi_{\mu}(-1,t)$ is another solution. Note that for $n \geq 1$,

$$\begin{aligned} \frac{db_n}{d\mu}(-1) &= \left(\frac{(-1)^n}{\prod_{j=1}^{n}(j+\mu+1)^2}\right)'\Bigg|_{\mu=-1} \\ &= \left(\frac{2(-1)^{n+1}}{\prod_{j=1}^{n}(j+\mu+1)^2}\right)\left(\sum_{j=1}^{n}\frac{1}{j+\mu+1}\right)\Bigg|_{\mu=-1} \\ &= \frac{2(-1)^{n+1}}{(n!)^2}\left(\sum_{j=1}^{n}\frac{1}{j}\right). \end{aligned} \tag{2.4.29}$$

Thus

$$y_2(t) = \varphi_{\mu}(-1,t)|_{r=-1} = y_1(t)\ln t + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{(n!)^2}\left(\sum_{j=1}^{n}\frac{1}{j}\right)t^{n-1}. \tag{2.4.30}$$

The general solution is $y = c_1 y_1(t) + c_2 y_2(t)$.

The *Bessel equation* has the form

$$y'' + t^{-1}y' + \big(1 - \nu^2 t^{-2}\big)y = 0, \tag{2.4.31}$$

where ν is a constant called the *order*. The indicial equation is

$$\mu^2 - \nu^2 = 0 \sim \mu = \pm\nu. \tag{2.4.32}$$

We rewrite (2.4.31) as

$$t^2 y'' + t y' + \left(t^2 - \nu^2\right) y = 0. \tag{2.4.33}$$

Let $y = \sum_{n=0}^{\infty} c_n t^{n+\mu}$ be a solution of (2.4.33) with $\mu = \pm\nu$ and $c_0 \neq 0$. We have

$$t y' = \sum_{n=0}^{\infty} (n+\mu) c_n t^{n+\mu}, \qquad t^2 y'' = \sum_{n=0}^{\infty} (n+\mu)(n+\mu-1) c_n t^{n+\mu}. \tag{2.4.34}$$

Denote by $\mathbb{N}$ the set of nonnegative integers. So (2.4.33) is equivalent to

$$\begin{aligned} & c_1\left[(\mu+1)^2 - \nu^2\right] = 0, \\ & \left[(\mu+n+2)^2 - \nu^2\right] c_{n+2} + c_n = 0, \quad n \in \mathbb{N}. \end{aligned} \tag{2.4.35}$$

Thus $c_{2r+1} = 0$ for $r \in \mathbb{N}$, and

$$c_{2n} = \frac{c_0}{\prod_{r=1}^{n} [\nu^2 - (\mu+2r)^2]} = \frac{(-1)^n c_0}{n! 2^{2n} \prod_{r=1}^{n} (\mu+r)}. \tag{2.4.36}$$

The function

$$J_\mu(t) = \left(\frac{t}{2}\right)^{\mu} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n! \prod_{r=1}^{n} (\mu+r)} \left(\frac{t}{2}\right)^{2n+\mu} \tag{2.4.37}$$

is called a *Bessel function of the first kind*. If ν is not an integer, then the general solution of (2.4.31) is

$$y = c_1 J_\nu(t) + c_2 J_{-\nu}(t). \tag{2.4.38}$$

Note that

$$\begin{aligned} \frac{d}{dt}\left(t^\mu J_\mu\right) &= \mu t^\mu \left[\left(\frac{t}{2}\right)^{\mu-1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n! \prod_{r=1}^{n} (\mu+r-1)} \left(\frac{t}{2}\right)^{2n+\mu-1}\right] \\ &= \mu t^\mu J_{\mu-1} \end{aligned} \tag{2.4.39}$$

and

$$\begin{aligned} \frac{d}{dt}\left(t^{-\mu} J_\mu\right) &= \sum_{n=1}^{\infty} \frac{(-1)^n t^{-\mu}}{(\mu+1)(n-1)! \prod_{r=1}^{n-1} (\mu+r+1)} \left(\frac{t}{2}\right)^{2n+\mu-1} \\ &= -\frac{t^{-\mu} J_{\mu+1}}{\mu+1}. \end{aligned} \tag{2.4.40}$$

Thus

$$\frac{d}{tdt}\left(t^{\mu}J_{\mu}\right)=\mu t^{\mu-1}J_{\mu-1}, \qquad \frac{d}{tdt}\left(t^{-\mu}J_{\mu}\right)=-\frac{t^{-\mu-1}J_{\mu+1}}{\mu+1}. \tag{2.4.41}$$

By induction,

$$\left(\frac{d}{tdt}\right)^{m}\left(t^{\mu}J_{\mu}\right)=\left[\prod_{r=0}^{m-1}(\mu-r)\right]t^{\mu-m}J_{\mu-m} \tag{2.4.42}$$

and

$$\left(\frac{d}{tdt}\right)^{m}\left(t^{-\mu}J_{\mu}\right)=(-1)^{m}\frac{t^{-\mu-m}J_{\mu+m}}{\prod_{r=1}^{m}(\mu+r)}. \tag{2.4.43}$$

On the other hand, (2.4.39) gives

$$\mu t^{\mu-1}J_{\mu}+t^{\mu}J_{\mu}'=\mu t^{\mu}J_{\mu-1}\sim \mu J_{\mu}+tJ_{\mu}'=\mu tJ_{\mu-1} \tag{2.4.44}$$

and (2.4.40) yields

$$-\mu t^{-\mu-1}J_{\mu}+t^{-\mu}J_{\mu}'=-\frac{t^{-\mu}J_{\mu+1}}{\mu+1}\sim -\mu J_{\mu}+tJ_{\mu}'=-\frac{tJ_{\mu+1}}{\mu+1}. \tag{2.4.45}$$

Thus

$$\mu J_{\mu-1}+\frac{J_{\mu+1}}{\mu+1}=\frac{2\mu}{t}J_{\mu}, \qquad \mu J_{\mu-1}-\frac{J_{\mu+1}}{\mu+1}=2\mu J_{\mu}'. \tag{2.4.46}$$

Observe that

$$\left(\frac{d}{dt}\right)\frac{t^{n}}{n!}=\frac{t^{n-1}}{(n-1)!} \tag{2.4.47}$$

for a positive integer n. If we have a continuous analogue of $n!$, then we can simplify (2.4.42) and (2.4.43) by rescaling J_{μ}. Indeed, it is the spatial function $\Gamma(s)$.

When $\nu=n+1/2$ with $n\in\mathbb{N}$, the indicial equation has two roots: $\mu_1=n+1/2$ and $\mu_2=-n-1/2$. Moreover, $\mu_1-\mu_2=2n+1$ is an integer. However, both $J_{n+1/2}(t)$ and $J_{-n-1/2}(t)$ are well defined by (2.4.37). They form a set of fundamental solutions of the Bessel equation. Suppose that $\nu=m$ is a positive integer. The indicial equation has two roots: $\mu_1=m$ and $\mu_2=-m$. The function $J_m(t)$ is still well defined, but $J_{-m}(t)$ is not defined. If $\mu=-m$, by the second equation in (2.4.35) with $n=2m-2$, we get

$$\begin{aligned}0&=\left[(-m+2m-2+2)^2-m^2\right]c_{2m}=-c_{2m-2}=-\frac{c_0}{(m!)^2}\\ &\Longrightarrow\quad c_0=0,\end{aligned} \tag{2.4.48}$$

which contradicts the assumption $c_0 \neq 0$. Thus we do not have a solution of the form $y = \sum_{n=0}^{\infty} c_n t^{n-m}$. We look for another fundamental solution of the form

$$y = J_m(t) \ln t + \sum_{n=0}^{\infty} c_n t^{n-m}, \tag{2.4.49}$$

which is related to *Bessel functions of the second kind*.

Exercises 2.4 Solve the following equations by the power series method:

1. $(1 - t^2)y'' - ty' + 16ty = 0$.
2. $t^2 y'' + 7ty' + (9 - t)y = 0$, $t > 0$.

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