Chapter 2
Method of Guiding Functions in Finite-Dimensional Spaces

2.1 Periodic Problem for a Differential Inclusion

In this section we present the guiding functions method for studying the periodic problem for a differential inclusion in a finite-dimensional space.

We start considering a differential inclusion in a finite-dimensional space $\mathbf{R}^n$ of the following form:

$$x'(t) \in F(t, x(t)), \quad a.e. \quad t \in [0, T] \tag{2.1}$$

where $F : [0, T] \times \mathbf{R}^n \to Kv(\mathbf{R}^n)$ is an L^1-upper Carathéodory multimap.

By a *solution* of inclusion (2.1) we mean an absolutely continuous function $x : [0, T] \to \mathbf{R}^n$ satisfying (2.1) for a.e. $t \in [0, T]$. It is well known (see, e.g., [25, 80]) that the L^1-upper Carathéodory condition implies the existence of *a local solution* to the Cauchy problem corresponding to (2.1), i.e., a solution defined on some interval $[0, h]$, $0 < h \leq T$ and satisfying the initial condition

$$x(0) = x_0 \in \mathbf{R}^n. \tag{2.2}$$

To guarantee the existence of a *global solution* ($h = T$) it is sufficient to strengthen the condition posed on the multimap F supposing that F is an L^1-upper Carathéordory multimap with α-sublinear growth. More precisely, the following assertion holds (see, e.g., [24, 39, 64, 80])

Proposition 2.1. *If $F : [0, T] \times \mathbf{R}^n \to Kv(\mathbf{R}^n)$ is an L^1-upper Carathéodory multimap with α-sublinear growth, then for each $x_0 \in \mathbf{R}^n$, the solution set $\Pi_F(x_0)$ of the Cauchy problem*

$$x'(t) \in F(t, x(t)) \quad a.e. \quad t \in [0, T] \tag{2.3}$$

$$x(0) = x_0 \tag{2.4}$$

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is an R_δ-set in the space $C([0,T];\mathbf{R}^n)$ endowed with the usual norm of uniform convergence.

Moreover, the multimap $\Pi_F : \mathbf{R}^n \to K(C([0,T];\mathbf{R}^n))$, $x \multimap \Pi_F(x)$ is u.s.c.

Now, we say that a solution x to differential inclusion (2.1) is T*-periodic* if it satisfies the following boundary value condition of periodicity

$$x(0) = x(T)\,. \tag{2.5}$$

It is clear that such function can be extended to a T-periodic solution defined on $\mathbf{R}$ provided that F is T-periodic, i.e., the multimap $F : \mathbf{R} \times \mathbf{R}^n \to Kv(\mathbf{R}^n)$ satisfies $F(t+T,\cdot) = F(t,\cdot)$ for all $t \in [0,T]$.

In order to study periodic problem (2.1), (2.5) we can introduce the *translation multioperator* along the trajectories of (2.1), (2.5) in the following way. For any $t \in [0,T]$ let $\theta_t : C([0,T];\mathbf{R}^n) \to \mathbf{R}^n$ be the *evaluation map* defined as

$$\theta_t(y) = y(t)\ .$$

Then the multioperator $\mathbf{P}_t^F : \mathbf{R}^n \multimap \mathbf{R}^n$ given as the composition

$$\mathbf{P}_t^F(x) = \theta_t \circ \Pi_F(x)\ ,$$

is called the translation multioperator along the trajectories of problem (2.1), (2.5), or simply, the translation multioperator.
The following assertion is evident

Proposition 2.2. *Periodic problem (2.1), (2.5) has a solution if and only if the corresponding translation multioperator $\mathbf{P}_T^F$ has a fixed point $x_* \in \mathbf{R}^n$, $x_* \in \mathbf{P}_T^F(x_*)$.*

As a direct consequence of Propositions 2.1 and 2.2 we get the following general existence result for problem (2.1), (2.5).

Theorem 2.1. *Let $U \subset \mathbf{R}^n$ be an open bounded subset. Then, $\mathbf{P}_T^F = (\theta_T \circ \Pi_T^F) \in CJ(\overline{U},\mathbf{R}^n)$. Moreover, if $x \notin \mathbf{P}_T^F(x)$ for all $x \in \partial U$ and $deg(i - \mathbf{P}_T^F, \overline{U}) \neq 0$, then periodic problem (2.1), (2.5) has a solution.*

Notice that the properties of the translation multioperator and its applications to the periodic problem for differential inclusions of various types are described, e.g., in monographs [64, 80].

Our next target is to reduce periodic problem (2.1), (2.5) to a fixed point problem for an integral multioperator in an appropriate functional space. This method also allows to use the topological tools for solving the periodic problem and it can be considered as the base for the construction of the guiding function method.

In the sequel we assume that F is an L^1-upper Carathéodory multimap with α-sublinear growth. Let us recall (see Sect. 1.1.2) that, in this situation the superposition multioperator $\mathscr{P}_F : C([0,T];\mathbf{R}^n) \to P(L^1([0,T];\mathbf{R}^n))$ is well defined.

Let us consider the integral operator $j : L^1([0,T];\mathbf{R}^n) \longrightarrow C([0,T];\mathbf{R}^n)$,

$$j(f)(t) = \int_0^t f(s)\,ds.$$

It is an easy exercise to verify, by applying Proposition 1.5 and the classical Ascoli–Arzelá theorem, that the composition $j \circ \mathscr{P}_F : C([0,T];\mathbf{R}^n) \to P(C([0,T];\mathbf{R}^n))$ is closed and transfers each bounded subset $\Omega \subset C([0,T];\mathbf{R}^n)$ onto a relatively compact set $j \circ \mathscr{P}_F(\Omega)$.

As the consequence of the above result and Proposition 1.5, we have the following assertion

Corollary 2.1. *The composition $j \circ \mathscr{P}_F : C([0,T];\mathbf{R}^n) \longrightarrow Kv(C([0,T];\mathbf{R}^n))$ is completely u.s.c. multimap.*

Denote $\mathscr{C} = C\left([0,T];\mathbf{R}^n\right)$. The simplest integral multioperator that can be used to search for T-periodic solutions seems to be the following one:

$$J_T : \mathscr{C} \to Kv\left(\mathscr{C}\right),$$
$$J_T\left(x\right) = x\left(T\right) + j \circ \mathscr{P}_F\left(x\right).$$

The next assertion can be easily verified.

Theorem 2.2. *Fixed points of the multioperator J_T coincide with solutions of periodic problem (2.1), (2.5).*

From the properties of the composition $j \circ \mathscr{P}_F\left(x\right)$, mentioned above, it clearly follows that the multioperator J_T is completely u.s.c. .

So, the topological degree theory can be applied to this multioperator and we can formulate the following general principle.

Theorem 2.3. *Let $U \subset \mathscr{C}$ be a bounded open set. If $deg\left(i - J_T, \overline{U}\right) \neq 0$, then periodic problem (2.1), (2.5) has a solution in U.*

This result is one of the cornerstones on which *the method of guiding functions* can be built in its classical version. Let us outline briefly its main features.

First of all, let us describe the a priori boundedness property of solutions to Cauchy problem (2.1), (2.2). We need the following slightly modified assertion on integral inequalities known as the Gronwall lemma (see, e.g., [72], Sect. III.1.1)

Lemma 2.1 (Gronwall Lemma). *Let $u, v : [a,b] \to \mathbf{R}$ be nonnegative functions, u be summable, and v be continuous; $C \geq 0$ be a constant such that*

$$v\left(t\right) \leq C + \int_a^t u\left(s\right) v\left(s\right) ds, \quad a \leq t \leq b.$$

Then

$$v\left(t\right) \leq Ce^{\int_a^t u(s)ds}, \qquad a \leq t \leq b.$$

Lemma 2.2. *The set of solutions to Cauchy problem (2.1), (2.2) is a priori bounded.*

Proof. Each solution to problem (2.1), (2.2) has the form

$$x(t) = x_0 + \int_0^t f(s)\,ds,$$

where $f \in \mathscr{P}_F(x)$. Then we have the following estimate for the continuous function $v(t) = |x(t)|$, where $|\cdot|$ denotes the norm in $\mathbf{R}^n$:

$$v(t) \leq |x_0| + \int_0^t |f(s)|ds \leq |x_0| + \int_0^t \alpha(s)(1 + |x(s)|)\,ds \leq$$

$$\leq |x_0| + \int_0^T \alpha(s)\,ds + \int_0^t \alpha(s)\,v(s)\,ds.$$

Applying Lemma 2.1 we obtain

$$|x(t)| \leq \; Ce^{\int_0^T \alpha(s)ds}, \quad t \in [0, T],$$

where $C = |x_0| + \int_0^T \alpha(s)\,ds$. □

Let us introduce the following notion.

Definition 2.1 (cf. [90]). A point $x_0 \in R^n$ is called a *T-non-recurrence point* of trajectories of differential inclusion (2.1) if for each solution x emanating from x_0 the following condition holds:

$$x(t) \neq x_0, \quad \forall t \in (0, T]. \tag{2.6}$$

The following assertion plays a key role in the justification of the method of guiding functions. For simplicity, we restrict ourselves to the case when the right-hand side of inclusion (2.1) is u.s.c.

Theorem 2.4. *Let $U \subset \mathbf{R}^n$ be a bounded open set such that each point $x \in \partial U$ is a T-non-recurrence point of trajectories of differential inclusion (2.1). Let $F : [0, T] \times \mathbf{R}^n \to Kv(\mathbf{R}^n)$ be a u.s.c. multimap with α-sublinear growth. If the multifield $R_0 : \overline{U} \to Kv(\mathbf{R}^n)$,*

$$R_0(x) = -F(0, x),$$

does not have singular points on ∂U then

$$deg\left(\Phi, \overline{\Omega}\right) = deg\left(R_0, \overline{U}\right),$$

where $\Phi = i - J_T$ is a multifield generated by the integral multioperator J_T, and Ω is a certain bounded open set in the space $\mathscr{C}$.

Proof. From Lemma 2.2 it follows that the set of all solutions of inclusion (2.1) emanating from $\overline{U}$ is bounded. Let $m > 0$ be a number such that the norm of each solution from this set is less than m. Define an open set Ω in the space $\mathscr{C}$ by the following conditions:

$$\Omega = \{x \in \mathscr{C} \mid x(0) \in U, \; \|x\| < m\}.$$

Consider the family of multimaps

$$F_\lambda(t, x) = F(\lambda t, x), \;\; \lambda \in [0, 1]$$

and the family of multifields $\Psi : \bar{\Omega} \times [0, 1] \to Kv(\mathscr{C})$ defined in the following way:

$$\Psi(x, \lambda) = \Big\{ z \mid z(t) = x(t) - x(T) - \lambda \int_0^t f(s)\, ds - (1 - \lambda)$$

$$\int_0^T f(s)\, ds \colon f \in \mathscr{P}_{F_\lambda}(x) \Big\}$$

It is easy to verify that the family of multifields Ψ is completely u.s.c. Let us show that this family is non-singular on $\partial\Omega \times [0, 1]$.

Suppose the contrary, i.e., let there exist a function $x_0 \in \partial\Omega$ and a number $\lambda_0 \in [0, 1]$ such that $0 \in \Psi(x_0, \lambda_0)$. It means that there exists a summable selection $f(s) \in F(\lambda_0 s, x_0(s))$ which satisfies the following equality:

$$x_0(t) = x_0(T) + \lambda_0 \int_0^t f(s)\, ds + (1 - \lambda_0) \int_0^T f(s)\, ds \tag{2.7}$$

for each $t \in [0, T]$.

For $t = 0$ we have

$$x_0(0) = x_0(T) + (1 - \lambda_0) \int_0^T f(s)\, ds,$$

while for $t = T$ we obtain

$$\int_0^T f(s)\, ds = 0. \tag{2.8}$$

Whence, $x_0(0) = x_0(T)$.

By taking the derivative in t in both sides of equality (2.7) we get

$$x_0'(t) = \lambda_0 f(t) \in \lambda_0 F(\lambda_0 t, x_0(t))$$

for a.e. $t \in [0, T]$.

So, x_0 is a solution of the differential inclusion

$$x'(t) \in \lambda_0 F(\lambda_0 t, x(t)).$$

Notice, that by construction of the set Ω, its boundary consists of functions of the following two types:

1. $x(0) \in \partial U$;
2. $x(0) \in U$, $\|x\| = m$.

Consider two cases:

(a) $\lambda_0 = 0$,
(b) $\lambda_0 \neq 0$.

Case (a). Let $\lambda_0 = 0$, then $x_0(t) \equiv x_0$ for each $t \in [0, T]$, $f(t) \in F(0, x_0)$ for a.e. $t \in [0, T]$ and from (2.8) we obtain $0 \in F(0, x_0)$.

The function x_0, being a constant, can not be a function of the first type since, by condition, the multifield R_0 has no singular points on ∂U.

On the other hand, the function x_0 can not be a second type function, since $\|x_0\| < m$ by construction of the set Ω.

Case (b). Now, let $\lambda_0 \neq 0$. Consider the function $z_0(t) = x_0\left(\frac{t}{\lambda_0}\right)$. Then, for a.e. $t \in [0, \lambda_0 T]$ we have

$$z_0'(t) = \frac{1}{\lambda_0} x_0'\left(\frac{t}{\lambda_0}\right) = f\left(\frac{t}{\lambda_0}\right) \in F\left(t, x_0\left(\frac{t}{\lambda_0}\right)\right) = F(t, z_0(t)).$$

So, the function z_0 is a solution of differential inclusion (2.1) on the interval $[0, \lambda_0 T]$. According to the global existence theorem, we can extend it on the whole interval $[0, T]$.

Notice that the function x_0 can not be the first type function. Indeed, from

$$x_0(0) = z_0(0) = x_0(T) = z_0(\lambda_0 T),$$

it follows that inclusion (2.1) has a solution z_0 such that $z_0(0) \in \partial U$ and $z_0(0) = z_0(\lambda_0 T)$, contrary to the assumption that trajectories emanating from ∂U are T-non-recurrenting.

On the other hand, x_0 can not be the second type function. It follows from

$$|x_0| \leq |z_0| < m,$$

since z_0 is a solution of inclusion (2.1) emanating from the set U.

So, the family of multifields Ψ realizes the homotopy of multifields

$$\Psi_1 = \Phi = i - J_T,$$

and

$$\Psi_0 (x) = i - \Gamma_0 (x) ,$$

where the multioperator $\Gamma_0 : \overline{\Omega} \to Kv(\mathscr{C})$ is defined by the relation

$$\Gamma_0 (x) = x (T) + \int_0^T F (0, x (s))\, ds.$$

This multioperator acts into the finite dimensional subspace $C^n_{[0,T]}$ of constant functions being naturally isomorphic to $\mathbf{R}^n$. By using the restriction property of the topological degree we obtain

$$deg\, (\Psi_0; \overline{\Omega}) = deg\, (\Psi_0 \mid_{R^n}, \overline{U}).$$

It is easy to see that the multifield $\hat{\Psi}_0 = \Psi_0 \mid_{\mathbf{R}^n}$ is defined by the relations

$$\hat{\Psi}_0 (x) = -\int_0^T F (0, x)\; ds = -T \cdot F (0, x) .$$

Then we finally obtain

$$deg\left(\Phi, \overline{\Omega}\right) = deg\left(\Psi_0, \overline{\Omega}\right) = deg\left(-F (0, \cdot) , \overline{U}\right) = deg\left(R_0, \overline{U}\right) . \qquad \square$$

The following assertion on the existence of a periodic solution immediately follows from the proved result.

Corollary 2.2. *In conditions of Theorem 2.4, let*

$$deg\left(R_0, \overline{U}\right) \neq 0.$$

Then, differential inclusion (2.1) has a T-periodic solution.

Now we apply Theorem 2.4 to justify the classical version of the method of guiding functions. Let us introduce the necessary notions.

Definition 2.2. A continuously differentiable function $v : \mathbf{R}^n \to \mathbf{R}$ is called *non-degenerate potential* if its gradient is non-zero outside a certain ball centered at the origin, i.e., there exists $r_v > 0$ such that

$$grad\, v\,(x) = \left\{ \frac{\partial v\,(x)}{\partial x_1}, \frac{\partial v\,(x)}{\partial x_2}, \ldots, \frac{\partial v\,(x)}{\partial x_n} \right\} \neq 0,$$

for each $x \in \mathbf{R}^n,\ |x| \geq r_v$.

From the properties of the topological degree (see, e.g., [25, 38, 80, 89, 95]) it follows that the degree of the gradient of a non-degenerate potential

$$deg\,(grad\, v\,(x)\,, B_r)$$

on the closed ball $B_r \subset \mathbf{R}^n$ of radius $r \geq r_v$, centered at the origin, does not depend on r. This generic value of the degree is called *the index of a non-degenerate potential* and it is denoted as $ind\, v$.

As an example of potential with non-zero index we can consider a non-degenerate potential v satisfying the coercivity condition

$$\lim_{|x|\to\infty} |v\,(x)\,| \to \infty. \tag{2.9}$$

(see [95]).

Other examples of potential with non-zero index can be found in [90, 95].

Definition 2.3. A non-degenerate potential v is called a strict guiding function for differential inclusion (2.1) if

$$\left\langle grad\, v\,(x)\,, y\right\rangle\ > 0 \tag{2.10}$$

for all $y \in F\,(t, x)$, $0 \leq t \leq T$, $|x| \geq r_v$.

From this definition it follows immediately that if v is a strict guiding function of inclusion (2.1) then the field $-grad\, v$ and the multifield R_0 does not allow opposite directions on spheres S_r of the radius $r \geq r_v$, and hence, by Lemma 1.5

$$deg\,(R_0, B_r) = (-1)^n\, ind\, v. \tag{2.11}$$

(We have used the known property of the degree of single-valued fields: $deg\,(-\varphi, S) = (-1)^n\, deg\,(\varphi, S)$, see, e.g., [95]).

We can formulate now the following condition for the existence of a periodic solution.

Theorem 2.5. *Let $F : [0, T] \times \mathbf{R}^n \to Kv\,(\mathbf{R}^n)$ be an u.s.c. multimap with α-sublinear growth. If, for differential inclusion (2.1), there exists a strict guiding function v of non-zero index, then the inclusion has a T-periodic solution.*

To prove this assertion, we need the following technical result

Lemma 2.3. *Let*

$$r_0 = (r_v + \int_0^T \alpha(s)\,ds) e^{\int_0^T \alpha(s)ds} \tag{2.12}$$

where α is the function from the sublinear growth condition $(F3')$.

If x is a solution of inclusion (2.1) with initial condition $|x(0)| > r_0$, then $|x(t)| > r_v$ for all $t \in [0, T]$.

Proof. Indeed, let there exists $t_0 \in [0, T]$ such that $|x(t_0)| \le r_v$.
For $t \in [0, t_0]$, define

$$y(t) = x(t_0 - t), \quad \beta(t) = \alpha(t_0 - t), \quad G(t, x) = -F(t_0 - t, x).$$

It is clear that

$$y'(t) \in G(t, y(t)).$$

Since $\|G(t, x)\| \le \beta(t)(1 + |x|)$ for a.e. $t \in [0, t_0]$, applying Lemma 2.2, we obtain

$$|y(t)| \le \left(|y(0)| + \int_0^{t_0} \beta(s)\,ds\right) e^{\int_0^{t_0} \beta(s)ds} \le r_0$$

for all $t \in [0, t_0]$. So, $|x(0)| = |y(t_0)| \le r_0$ and we get the contradiction. □

Proof (of Theorem 2.5). Notice that, for each $r > r_0$, the sphere S_r consists of T-non-recurrence points of inclusion (2.1). Indeed, if x is a solution of (2.1) such that $x(0) \in S_r$, then from Lemma 2.3 it follows that $|x(t)| > r_v$ for all $t \in [0, T]$. Then for each $t \in (0, T]$ we have

$$v(x(t)) - v(x(0)) = \int_0^t \langle grad\, v(x(s)),\, x'(s)\rangle ds > 0, \tag{2.13}$$

and therefore relation (2.6) follows.

To conclude the proof, it remains to apply relation (2.11) and Corollary 2.2. □

This version of the method of guiding functions for differential inclusions allows extensions in various directions. Let us discuss some of them.

First of all, we extend the notion of guiding function as well as the class of differential inclusions to which the MGF can be applied.

Definition 2.4. A non-degenerate potential v is called a guiding function for the differential inclusion (2.1) if

$$\langle grad\, v(x),\ y\rangle \ge 0$$

for all $y \in F(t, x)$, $0 \le t \le T$, $|x| \ge r_v$.

Proposition 2.3. *Let $F : [0, T] \times \mathbf{R}^n \to Kv(\mathbf{R}^n)$ be an L^1-upper Carathéodory multimap with α-sublinear growth. If differential inclusion (2.1) admits a guiding function v of non-zero index, then the inclusion (2.1) has a T-periodic solution.*

Proof. STEP 1 Let us show that the assertion is true for an u.s.c. F.
For $k = 0, 1, 2, \ldots$, set

$$M_k = \sup\{|grad\, v(x)| \ : x \in B(k)\},$$

where $B(k) \subset \mathbf{R}^n$ denotes a closed ball of radius k, centered at the origin.
Define the function $\eta : \mathbf{R}^n \to \mathbf{R}$ as

$$\eta(x) = 1 + (|x| - k)M_{k+2} + (k + 1 - |x|)M_{k+1}, \quad k \leq |x| \leq k + 1.$$

It is easy to see that the function η is continuous and satisfies the condition

$$\eta(x) \geq \max\{1, |grad\, v(x)|\} \quad \text{for all} \quad x \in \mathbf{R}^n.$$

So, the map $g : \mathbf{R}^n \to \mathbf{R}$,

$$g(x) = \frac{grad\, v(x)}{\eta(x)}$$

is continuous and satisfies the condition $|g(x)| \leq 1$ for all $x \in \mathbf{R}^n$.
For any sequence $\{\epsilon_m\}$ of positive numbers, consider the corresponding sequence of auxiliary differential inclusions

$$x'(t) \in F(t, x(t)) + \epsilon_m g(x(t)) \,. \tag{2.14}$$

It is clear that the right-hand side of each inclusion (2.14) is u.s.c. with α-sublinear growth.
For each $|x| \geq r_v$ and $y \in F(t, x)$ we have

$$\langle grad\, v(x), y + \epsilon_m g(x)\rangle = \langle grad\, v(x), y\rangle + \epsilon_m \frac{\langle grad\, v(x), grad\, v(x)\rangle}{\eta(x)} > 0$$

and, by Theorem 2.5 inclusion (2.14) has a T-periodic solution x_m for each $\epsilon_m > 0$. Tending the sequence $\{\epsilon_m\}$ to zero, we obtain the desired solution of (2.1) as a limit point of the sequence $\{x_m\}$.

STEP 2 Now, assume that F is an L^1-upper Carathéodory multifunction with α-sublinear growth.
From [39], Sect. 5, it follows that we can assume, w.l.o.g., that the multimap F is bounded and, then we have the following result (see [39], Proposition 5.1)

Lemma 2.4. *For each $\epsilon > 0$ there exists a multimap $F_\epsilon : [0, T] \times \mathbf{R}^n \to Kv(\mathbf{R}^n)$ such that:*

1) $F_\epsilon(t, x) \subset F(t, x), \quad (t, x) \in [0, T] \times \mathbf{R}^n$;
2) there exists a closed subset $J_\epsilon \subset [0, T]$ with $\mu([0, T] \setminus J_\epsilon) \leq \epsilon$ such that $F_{|J_\epsilon \times \mathbf{R}^n}$ is u.s.c.

For a sequence $\{\epsilon_m\}$ of positive numbers tending to zero, let us take the corresponding sequence of multimaps $F_{\epsilon_m} : [0,T] \times \mathbf{R}^n \to Kv(\mathbf{R}^n)$ satisfying, for each ϵ_m conditions (1), (2) of the above lemma.
For each ϵ_m, let $P_m : [0,T] \multimap J_{\epsilon_m}$ be a metric projection and the multimap $\tilde{F}_{\epsilon_m} : [0,T] \to Kv(\mathbf{R}^n)$ be defined as

$$\tilde{F}_{\epsilon_m}(t,x) = \overline{co} F_{\epsilon_m}(P_m(t), x) \; .$$

Since each metric projection P_m is closed and, hence, u.s.c., from Propositions 1.7, 1.8 and 1.12 it follows that each multimap $\tilde{F}_{\epsilon_m}$ is u.s.c.

Furthermore, it easy to see that for each $\tilde{F}_{\epsilon_m}$ relation in Definition 2.4 is fulfilled and, according to Step 1, each differential inclusion

$$x'(t) \in \tilde{F}_{\epsilon_m}(t, x(t))$$

has a T-periodic solution x_m.

Tending m to infinity we obtain a T-periodic solution x as a limit point of the sequence $\{x_m\}$. □

To obtain further generalizations, let us introduce the following notion

Definition 2.5. A non-degenerate potential v is called a weak guiding function for differential inclusion (2.1) if

$$\langle grad\, v(x), y\rangle \geq 0 \text{ for at least one } y \in F(t,x) \text{ and for all } t \in [0,T], |x\,| \geq r_v \tag{2.15}$$

Now we can formulate the most general result concerning the application of the MGF to periodic problem (2.1), (2.5).

Theorem 2.6. *Let $F : [0,T] \times \mathbf{R}^n \to Kv(\mathbf{R}^n)$ be an L^1-upper Carathéodory multimap with α-sublinear growth. If differential inclusion (2.1) admits a weak guiding function of non-zero index, then the inclusion has a T-periodic solution.*

Proof. Let us define the multimap $B : \mathbf{R}^n \to Cv(\mathbf{R}^n)$ as

$$B(x) = \{y \in \mathbf{R}^n : \langle \gamma(x) grad\ v(x), y\rangle \geq 0\}$$

where

$$\gamma(x) = \begin{cases} 0, & |x\,| \leq r_v, \\ 1, & |x\,| > r_v \end{cases}$$

It is easy to verify that the multimap B is closed. So, by applying Lemma 1.9 we can see that the multimap $F^B : [0,T] \times \mathbf{R}^n \to Kv(\mathbf{R}^n)$,

$$F^B(t,x) = F(t,x) \cap B(x)$$

is well defined and is L^1-upper Carathéodory with α-sublinear growth. Moreover, for the multimap F^B condition (2.15) is fulfilled for all $|x| \geq r' > r_v$ and $y \in F^B(t,x)$, where r' is an arbitrary number. Therefore v is a guiding function for the differential inclusion

$$x'(t) \in F^B(t,x)). \tag{2.16}$$

From Proposition 2.3 it follows that inclusion (2.16) has a solution, implying the result. □

Corollary 2.3. *let $F : [0,T] \times \mathbf{R}^n \to Kv(\mathbf{R}^n)$ be an L^1-upper Carathéodory multimap with α-sublinear growth. If the differential inclusion (2.1) admits a weak guiding function v satisfying the coercivity condition (2.9), then the inclusion has a T-periodic solution.*

2.2 Non-smooth Guiding Functions

In the previous section, following the classical works on the method of guiding functions, we supposed the guiding function smooth on the whole space. This condition may be onerous, for example, in situations where the guiding potentials are different on different domains. In this case it is natural to take as the potential, defined on the whole space, the maximum of all potentials but, this new function may be non-smooth, in general. In this section we describe the extension of the notion of a guiding function to the non-smooth case. Notice, in this connection, that non-differentiable Liapunov functions are effectively used in the stability theory (see, e.g., [133]).

To deal with such potentials, let us recall some notions of non-smooth analysis (see, e.g., [34]).

Let X be a real Banach space endowed with the norm $\|\cdot\|$. The dual space is denoted by X^* and the notation $\langle\cdot,\cdot\rangle$ means the duality pairing between X^* and X.

Definition 2.6. A function $V : X \to \mathbf{R}$ is called locally Lipschitz if for each point $x \in X$ there exist a neighborhood U of x and a constant $C > 0$ such that

$$|V(y) - V(z)| \leq C\,\|y - z\|, \quad \forall y, z \in U.$$

Remark 2.1. A convex and continuous function $V : X \to \mathbf{R}$ is locally Lipschitz. More generally, a convex function $V : X \to \mathbf{R}$ which is bounded above on a neighborhood of some point is locally Lipschitz (see [34]).

Definition 2.7. Let $V : X \to \mathbf{R}$ be a locally Lipschitz function. For $x \in X$ and $\nu \in X$, the generalized derivative $V^0(x;\nu)$ of V at a point x in the direction ν is given by the formula

$$V^0(x;\nu) = \varlimsup_{\substack{z\to x\\ t\to 0+}} \frac{V(z+t\nu) - V(z)}{t}. \tag{2.17}$$

Definition 2.8. A locally Lipschitz function $V : X \to \mathbf{R}$ is called regular at a point $x \in X$ if the usual directional derivative

$$V'(x; v) = \lim_{t \to 0+} \frac{V(x + tv) - V(x)}{t}$$

exists for each $v \in X$ and is equal to $V^0(x; v)$.

Remark 2.2. A convex and continuous function $V : X \to \mathbf{R}$ is regular.

Now we can introduce the following notion.

Definition 2.9. The Clarke's generalized gradient of a locally Lipschitz function $V : X \to \mathbf{R}$ at a point $x \in X$ is defined as the set $\partial V(x) \subset X^*$ in the following way:

$$\partial V(x) = \left\{ z \in X^* : \langle z, v \rangle \le V^0(x; v) \quad \text{for all} \;\; v \in X \right\}. \tag{2.18}$$

Notice that from the classic Hahn–Banach theorem it follows that $\partial V(x) \neq \emptyset$.

Remark 2.3. If a function $V : X \to \mathbf{R}$ is continuously differentiable, then $\partial V(x) = V'(x)$ for all $x \in X$, where $V'(x)$ denotes the Fréchet derivative of V at x.

Remark 2.4. If $X = \mathbf{R}^n$ and a function $V : \mathbf{R}^n \to \mathbf{R}$ is convex and continuous, then the Clarke's generalized gradient $\partial V(x)$ coincides with the subdifferential of V at x in the sense of convex analysis, i.e.

$$\partial V(x) = \{ z \in \mathbf{R}^n \;: < z, y - x > \le V(y) - V(x), \quad \forall y \in \mathbf{R}^n \}.$$

Lemma 2.5 (see [34]). *If $X = \mathbf{R}^n$, then, for a given locally Lipschitz function $V : \mathbf{R}^n \to \mathbf{R}$, the multimap $\partial V : \mathbf{R}^n \to P(\mathbf{R}^n)$ has compact convex values and is u.s.c.*

In the sequel, we will use the following result (see [40])

Lemma 2.6. *Let a function $V : \mathbf{R}^n \to \mathbf{R}$ be regular, $x : [a, b] \to \mathbf{R}^n$ an absolutely continuous function. Then, the function $t \to V(x(t))$, $t \in [a, b]$ is absolutely continuous and*

$$V(x(t'')) - V(x(t')) = \int_{t'}^{t''} V^0(x(s), x'(s))\, ds \quad \textit{for each} \quad t', t'' \in [a, b]$$

We start with the following notions

Definition 2.10. A regular function $V : \mathbf{R}^n \to \mathbf{R}$ is called a non-degenerate non-smooth potential if there exists $r_V > 0$ such that

$$0 \notin \partial V(x) \quad \text{for all} \quad \|x\| \ge r_V.$$

Definition 2.11. A regular function $V : \mathbf{R}^n \to \mathbf{R}$ is called a direct potential if there exists $r_V > 0$ such that

$$< v, v' > > 0 \quad \text{for all } v, v' \in \partial V(x), \; \|x\| \ge r_V.$$

It is obvious that each direct potential is non-degenerate, but the converse is not true in general.

Analogously to the classical case, for a non-degenerate non-smooth potential V, the topological degree of the multifield ∂V, $deg(\partial V, B_r)$ on each ball centered at the origin $B_r = B(0,r) \subset \mathbf{R}^n$ with $r \geq r_V$ is constant and is called *the index* of V and denoted by *ind* V.
The following statement gives an example of a non-smooth potential with non-zero index (see [21]).

Proposition 2.4. *If* $V : \mathbf{R}^n \to \mathbf{R}$ *is a direct potential, satisfying the coercivity condition* $\lim_{|x|\to\infty} V(x) = +\infty$*, then ind* $V = 1$.

Now, our target is to extend the method of guiding functions to the case of non-smooth potentials.

Definition 2.12. A non-degenerate non-smooth potential $V : \mathbf{R}^n \to \mathbf{R}$ is called a strict non-smooth guiding function for differential inclusion (2.1) if

$$< v, y > > 0 \quad \text{for all} \quad v \in \partial V(x),\ y \in F(t,x), \quad t \in [0,T], \tag{2.19}$$

where $\|x\| \geq r_V$.

Definition 2.13. A direct potential $V : \mathbf{R}^n \to \mathbf{R}$ is called a non-smooth guiding function for differential inclusion (2.1) if

$$< v, y > \geq 0 \quad \text{for all } y \in F(t,x), \quad v \in \partial V(x), t \in [0,T], |x| \geq r_V \tag{2.20}$$

The main result in this section is the following assertion.

Theorem 2.7. *If the right-hand part of differential inclusion (2.1) is* L^1*-upper Carathéodory with* α*-sublinear growth and the inclusion admits a non-smooth guiding function of a non-zero index, then periodic problem (2.1), (2.5) has a solution.*

Proof. STEP 1 Let us suppose F is u.s.c. and the guiding function V is strict. Observing that condition (2.19) implies that the multifields $R_0(x) = -F(0,x)$ and $\partial V(x)$ does not allow opposite directions on spheres S_r, $r \geq r_V$, we conclude, by Lemma 1.5 that

$$deg(R_0, B_r) = (-1)^n indV \neq 0 .$$

To apply Corollary 2.2, it is sufficient to verify that each sphere S_r, $r > r_0$, where r_0 is defined by (2.12), consists of T-non-recurrence points of inclusion (2.1). Indeed, let x be a solution of inclusion (2.1) such that $x(0) \in S_r$ and hence $|x(t)| > r_V$ for all $t \in [0,T]$ (see Lemma 1.3). Notice that the multifunction $t \multimap \partial V(x(t))$, $t \in [0,T]$ is u.s.c. and hence measurable. So, by Proposition 1.14(c) it has a measurable selection $z(t) \in \partial V(x(t))$ for a.e. $t \in [0,T]$.
Then, applying Lemma 2.6 we have, for each $t \in [0,T]$.

$$V(x(t)) - V(x(0)) = \int_0^t V^0(x(s), x'(s))\, ds \geq \int_0^t \langle z(s),\ x'(s) \rangle\, ds > 0$$

implying the result.

STEP 2 Now, let F be u.s.c. and V a non-smooth guiding function, i.e. it is direct and satisfies condition (2.20).

Taking $|\partial V(x)| = \sup\{|v| : v \in \partial V(x)\}$ and defining $M_k = \sup\{|\partial V(x)| : x \in B_k\}$, $k = 0, 1, \ldots$ we obtain, similarly to STEP 1 in the proof of Proposition 2.3, a continuous function $\eta : \mathbf{R}^n \to \mathbf{R}$ satisfying $\eta(x) \geq \max\{1, |\partial V(x)|\}$. So, the multifunction $G : \mathbf{R}^n \to Kv(\mathbf{R}^n)$ defined by

$$G(x) = \frac{1}{\eta(x)} \partial V(x)$$

is u.s.c. and satisfies $|G(x)| \leq 1,\ \forall x \in \mathbf{R}^n$.

For a sequence of positive numbers $\{\epsilon_m\}$ tending to zero, let us define the corresponding sequence of differential inclusions

$$x'(t) \in F_m(t, x(t)) := F(t, x(t)) + \epsilon_m G(x(t)) \tag{2.21}$$

It is clear that each F_m is u.s.c. with the α-sublinear growth.

Further, each $z \in F_m(t, x)$ has the form $z = y + \frac{\epsilon_m}{\eta(x)} v'$, where $v' \in \partial V(x)$. So, for each $v \in \partial V(x)$ we have

$$\langle v,\ z \rangle = \langle v,\ y \rangle + \frac{\epsilon_n}{\eta(x)} \langle v,\ v' \rangle > 0$$

since the potential V is direct.

By Step 1, for each m, differential inclusion (2.21) has a solution x_m and we obtain the solution x as a limit point of the sequence $\{x_m\}$.

STEP 3 The approach to the case when F is L^1-upper Carathéodory can be made as in Step 2 of the proof of Proposition 2.3. □

Corollary 2.4. *If inclusion (2.1) admits a non-smooth coercive guiding function, then periodic problem (2.1), (2.5) has a solution.*

2.3 Integral Guiding Functions

It is easy to see that the direct application of the MGF in its classical interpretation to the periodic problem for functional differential equations and inclusions meets difficulties. To avoid them, in this section we consider the notion of integral guiding function, first introduced by A. Fonda (see [55]) and then developed in the works [84–86].

Let us start with the necessary preliminaries.

Given $\tau > 0$, let us denote the space of continuous functions $C([-\tau, 0]; \mathbf{R}^n)$ by the symbol $\mathscr{C}$. For a function $x(\cdot) \in C([-\tau, T]; \mathbf{R}^n)$, $T > 0$ the symbol $x_t \in \mathscr{C}$, $t \in [0, T]$ denotes the function defined as $x_t(\theta) = x(t + \theta)$, $\theta \in [-\tau, 0]$.

We consider the periodic problem for a functional differential inclusion of the following form:

$$x'(t) \in F(t, x_t) \quad \text{a.e. } t \in [0, T], \tag{2.22}$$

$$x(0) = x(T), \tag{2.23}$$

assuming that $F : \mathbf{R} \times \mathscr{C} \to Kv(\mathbf{R}^n)$ is a T-periodic L^1-upper Carathéodory multimap satisfying the α-sublinear growth condition.

As earlier, by a solution of problem (2.22), (2.23) we mean an absolutely continuous function $x(\cdot)$, satisfying the periodicity condition (2.23) and the inclusion (2.22) a.e. on $[0, T]$.

Denote by C_T the space of continuous T-periodic functions $x : \mathbf{R} \to \mathbf{R}^n$ with the norm $\|x\|_C = \sup\limits_{t \in [0,T]} |x(t)|$ and by L^1_T the space of summable T-periodic functions $f : \mathbf{R} \to \mathbf{R}^n$ with the norm $\|f\|_{L^1} = \frac{1}{T} \int_0^T |f(t)|\, dt$.

For each $x \in C_T$ we will also consider the norm $\|x\|_2 = \left(\int_0^T |x(t)|^2 dt\right)^{1/2}$.

Now we can introduce the following notion.

Definition 2.14. A regular function $V : \mathbf{R}^n \to \mathbf{R}$ is said to be a non-smooth integral guiding function for problem (2.22)–(2.23) if there exists $N > 0$ such that for each absolutely continuous function $x \in C_T$ with $\|x\|_2 \geq N$ and $|x'(t)| \leq \|F(t, x_t)\|$ for a.e. $t \in [0, T]$ we have

$$\int_0^T \big\langle v(s), f(s) \big\rangle\, ds > 0$$

for all summable selections $v(s) \in \partial V(x(s))$ and $f(s) \in F(s, x_s)$.

Remark 2.5. As we know, the generalized gradient ∂V is a u.s.c. multimap and so, the multifunction $s \to \partial V(x(s))$ is u.s.c. and, hence, bounded and measurable. So the multifunction $\partial V(x(s))$ admits a measurable selection.

The following assertion holds.

Theorem 2.8. *Let $V : \mathbf{R}^n \to \mathbf{R}$ be a non-smooth integral guiding function of problem (2.22)–(2.23). If V is a non-degenerate potential and* ind $V \neq 0$, *then problem (2.22)–(2.23) has a solution.*

Proof. We use the coincidence degree theory (see Sect. 1.3). Consider the following operators:

$$L : \operatorname{dom} L := \{x \in C_T : x \textit{ is absolutely continuous}\} \subset C_T \to L^1_T,$$

$$Lx = x',$$

the superposition multioperator $G = \mathscr{P}_F : C_T \to P(L^1_T)$ and the projection $\Pi : L^1_T \to R^n$, given as

$$\Pi f = \frac{1}{T} \int\limits_0^T f(s)\, ds.$$

It is easy to verify (see Proposition 1.17) that multioperators ΠG and $K_{P,Q} G$ are compact and upper semicontinuous.

Let us mention that periodic problem (2.22)–(2.23) is reduced to the existence of a coincidence point $x \in dom\, L$ for the pair (L, G) :

$$Lx \in G(x).$$

Moreover, for $\lambda \in (0, 1]$, an arbitrary solution $x \in dom\, L$ of the inclusion

$$Lx \in \lambda G(x)$$

satisfies the relations

$$x'(t) \in \lambda F(t, x_t),$$

$$x(0) = x(T).$$

It means that $x(\cdot)$ is an absolutely continuous function such that $x'(t) = \lambda f(t)$ for a.e. $t \in [0, T]$, where $f \in \mathscr{P}_F(x)$.

Then, applying Lemma 2.6 we obtain, for each summable selection $v(s) \in \partial V(x(s))$

$$\begin{aligned}\int_0^T \langle v(s), f(s)\rangle\, ds &= \frac{1}{\lambda} \int_0^T \langle v(s), x'(s)\rangle\, ds \\ &\le \frac{1}{\lambda} \int_0^T V^0(x(s), x'(s))\, ds = \frac{1}{\lambda}\left(V(x(T)) - V(x(0))\right) = 0\ ,\end{aligned}$$

and hence

$$\|x\|_2 < N.$$

Then, from α-sublinear growth condition of F it follows that there exists a constant $M' > 0$ such that $\|x'\|_2 < M'$. So, there exists also $M > 0$ such that

$$\|x\|_C < M$$

Denote by U the ball $B_r \subset C_T$ centered at the origin with the radius $r = \max\{r_V, M, NT^{-1/2}\}$. Then we have

$$Lx \notin \lambda G(x)$$

for all $x \in \partial U, \lambda \in (0, 1]$.

Further, take an arbitrary point $u \in \partial U \cap Ker\, L$. Since u is a constant function satisfying $\|u\|_C \geq NT^{-1/2}$, from the definition of the integral guiding function we obtain

$$\int_0^T \langle v(s), f(s) \rangle \, ds > 0$$

for all measurable selections $v(s) \in \partial V(u)$, $f(s) \in F(s,u)$. Taking a constant selection $v(s) \equiv v \in \partial V(u)$ we have

$$\int_0^T \langle v, f(s) \rangle \, ds = \langle v, \int_0^T f(s) \, ds \rangle = T \langle v, \Pi f \rangle > 0$$

for all $v \in \partial V(u)$, and hence

$$\langle v, y \rangle > 0$$

for all $v \in \partial V(u)$, $y \in \Pi G(u)$.

This means that the multifields $\partial V(u)$ and $\Pi G(u)$ are homotopic on $\partial U \cap Ker\, L$ and hence

$$\deg_{Ker\, L}(\Pi G|_{\overline{U}_{\ker l}}, \overline{U}_{Ker\, L}) = \deg(\partial V, \overline{U}_{Ker\, L}) \neq 0.$$

So, all conditions of Theorem 1.2 are fulfilled and so the operators L and G have a coincidence point in the ball U and hence problem (2.22)–(2.23) has a solution in the same ball. □

As an example, we consider the periodic problem for a gradient functional differential inclusion of the following form:

$$x'(t) \in \partial G(x(t)) + F(t, x_t) \tag{2.24}$$

$$x(0) = x(T), \tag{2.25}$$

where the multimap F is T-periodic in the first argument and is L^1-upper Carathéodory with α-sublinear growth and ∂G is the generalized gradient of a regular function $G : \mathbf{R}^n \to \mathbf{R}$.

Theorem 2.9. *Suppose that the following conditions are satisfied:*

$(A1)$ *there exist constants* $\varepsilon > 0$, $K > 0$ *and* $\beta \geq 1$ *such that*

$$|g| \geq \varepsilon |u|^\beta - K$$

for all $g \in \partial G(u), u \in \mathbf{R}^n$*;*

$(A2)$

$$\lim_{\|x\|_2 \to \infty} \frac{\|\mathscr{P}_F(x)\|_2}{\|x\|_2^\beta} < \varepsilon T^{(1-\beta)/2}$$

for absolutely continuous functions $x \in C_T$*;*

$(A3)$ *the generalized gradient* ∂G *has a non-zero topological degree:*

$$\deg(\partial G, \overline{B}_N) \neq 0$$

for sufficiently large $N > 0$.

Then problem (2.24)–(2.25) has a solution.

Proof. Let us demonstrate that G is a non-smooth integral guiding function for problem (2.24)–(2.25). Notice that the embedding $L^{2\beta} \subset L^2$ gives the following estimation for each absolutely continuous function $x(\cdot) \in C_T$ and every summable selection $g(t) \in \partial G(x(t))$:

$$\|g\|_2 \geq \varepsilon \|x\|_{2\beta}^{\beta} - K\sqrt{T} \geq \varepsilon T^{(1-\beta)/2} \|x\|_2^{\beta} - K\sqrt{T}.$$

Then for each summable selections $f \in \mathscr{P}_F(x)$ and $g(t) \in \partial G(x(t))$ we have

$$\begin{aligned}
\int_0^T \big\langle g(s), g(s) + f(s) \big\rangle\, ds &\geq \|g\|_2 \big(\|g\|_2 - \|f\|_2\big) \\
&\geq \|g\|_2 \big(\|g\|_2 - \|\mathscr{P}_F(x)\|_2\big) \\
&\geq \|g\|_2 \Big(\varepsilon T^{(1-\beta)/2} - \frac{K\sqrt{T}}{\|x\|_2^{\beta}} - \frac{\|P_F(x)\|_2}{\|x\|_2^{\beta}}\Big) \|x\|_2^{\beta} > 0
\end{aligned}$$

for $\|x\|_2$ sufficiently large. □

2.4 Generalized Periodic Problems

We consider, here, the application of the MGF to some generalization of the classical periodic problem for differential inclusions. We discuss also some applications to differential games and other examples including the anti-periodic problem.

Starting from this section, in the sequel we use the symbol V for a smooth non-degenerate potential.

2.4.1 Preliminaries

Definition 2.15. Let X, Y be Banach spaces. By $J^c(X, Y)$ we denote the collection of all multimaps $F: X \multimap K(Y)$ that can be represented in the form of a composition

$$F = \Sigma_q \circ \cdots \circ \Sigma_1,$$

where $\Sigma_i \in J(X_{i-1}, X_i)$, $i = 1 \cdots q$, $X_0 = X$, $X_q = Y$, and X_i $(0 < i < q)$ are normed spaces.

Following the construction of the topological degree for CJ-multimap given in the chapter "Introduction", let us mention that if $U \subset \mathbf{R}^n$ is an open bounded subset and $F: \overline{U} \to K(\mathbf{R}^n)$ is a J^c-multimap such that $0 \notin F(x)$ for all $x \in \partial U$, then the topological degree $deg(F, \overline{U})$ is well-defined and has all usual properties of the Brouwer topological degree.

For a non-degenerate potential $V : \mathbf{R}^n \to \mathbf{R}$ define the vector field $W_V: \mathbf{R}^n \to \mathbf{R}^n$,

$$W_V(x) = \begin{cases} grad\, V(x) & \text{if} \quad |grad\, V(x)| \leq 1, \\ \frac{grad\, V(x)}{|grad\, V(x)|} & \text{if} \quad |grad\, V(x)| > 1. \end{cases}$$

In the sequel we need the following result.

Lemma 2.7 (see [64, Lemma 72.8]). . *Let $r_V > 0$ be a constant for the non-degenerate potential V. Then for every $r > r_V + a$, $a > 0$, there is $t_r \in (0, a]$ such that:*
for each solution $x: [0, a] \to \mathbf{R}^n$ of the problem

$$\begin{cases} x'(t) = W_V(x(t)) \\ |x(0)| = r \end{cases}$$

the following relations hold:

$$\big\langle x(t) - x(0), grad\, V(x(0)) \big\rangle > 0, \quad \forall t \in (0, t_r].$$

$$x(t) - x(0) \neq 0\,, \quad \forall t \in (0, a]$$

2.4.2 *The Setting of the Problem*

Consider the following generalized periodic problem

$$\begin{cases} u'(t) \in F(t, u(t)), \text{ for a.e. } t \in [0, T], \\ u(T) \in M(u(0)), \end{cases} \tag{2.26}$$

with the assumptions that:

$(A1)$ $F: [0, T] \times \mathbf{R}^n \to Kv(\mathbf{R}^n)$ is a L^1-upper Carathéodory multimap with α-sublinear growth;
$(A2)$ $M: \mathbf{R}^n \to K(\mathbf{R}^n)$ is a J^c-multimap.

By a solution to problem (2.26) we mean an absolutely continuous function $u: [0, T] \to \mathbf{R}^n$ satisfying (2.26).

Definition 2.16. A non-degenerate potential $V\colon \mathbf{R}^n \to \mathbf{R}$ is said to be a guiding function for problem (2.26) if there exists $r_* > 0$ such that for every $(t, x) \in [0, T] \times \mathbf{R}^n$, $|x| \geq r_*$, the following relations hold:

$(V1)$ $\langle grad\, V(x), y \rangle \geq 0$ for at least one point $y \in F(t, x)$;
$(V2)$ $V(x) \geq V(w)$ for all $w \in M(x)$;
$(V3)$ $\langle grad\, V(x), x - w \rangle \geq 0$ for at least one point $w \in M(x)$, if M has convex values, otherwise for all $w \in M(x)$.

2.4.3 Application to Differential Games

Let us mention that the class of problems having the form (2.26) is sufficiently wide. It is clear that in the case when M is the identity operator, i.e., $M(x) = x,\ \forall x \in \mathbf{R}^n$, problem (2.26) is the classical periodic problem. Consider some other examples of problems which can be represented in form (2.26).

Problem 2.1 (Differential game with a given goal set). Consider a differential game in which an object moves along the trajectories of the following differential inclusion

$$u'(t) \in F(t, u(t)), \tag{2.27}$$

where $F\colon [0, T] \times \mathbf{R}^n \to Kv(\mathbf{R}^n)$ is a given multimap.

It is supposed that for each initial position $u(0) = x$ a goal set $M(x) \subset \mathbf{R}^n$ is given. The game ends if, starting from a position x, the object can be moved to one of the goal positions $M(x)$ at the time T.

The game is called finite if there are an initial position x and a trajectory of (2.27) such that the game ends. Otherwise the game is called infinite.

It is clear that the finiteness of the game is equivalent to the existence of a solution of problem (2.26).

Problem 2.2 (Differential game of pursuit). In this game, two participating players A and B start moving at the same time from different initial positions along the trajectories of the differential inclusions

$$u'(t) \in G_0(t, u(t)), \tag{2.28}$$

(player A) and, respectively:

$$v'(t) \in G_1(t, v(t)), \tag{2.29}$$

(player B).
It is supposed that $G_1\colon [0, T] \times \mathbf{R}^n \to Kv(\mathbf{R}^n)$ is a L^1-upper Carathéodory multimap with α-sublinear growth.

The player A is considered as a pursuer whereas the player B is an evader. We assume that for each chosen initial position x of the pursuer A, the evader

B starts the game from the initial position $h(x)$ defined by a continuous function $h: \mathbf{R}^n \to \mathbf{R}^n$. The game ends if, at the moment T, the players A and B reach the same position, i.e., player A "catches up" player B at this time.

The game is called finite if there are an initial position x and trajectories of (2.28) and (2.29), respectively, such that the game ends.

Let us reduce the game to problem (2.26). To this aim, let us recall (see Proposition 2.1) that under the assumptions imposed on the multimap G_1, for each $y \in \mathbf{R}^n$ the Cauchy problem

$$\begin{cases} v'(t) \in G_1(t, v(t)), \text{ for a.e. } t \in [0, T], \\ v(0) = y, \end{cases}$$

has a solution. Moreover, if we denote by $\Pi_{G_1}(y)$ the set of all solutions, then the multimap

$$\Pi_{G_1}: \mathbf{R}^n \to K(\mathbf{R}^n) \tag{2.30}$$

is a J-multimap.

Now, let $x \in \mathbf{R}^n$ be an initial position of A. Then $h(x)$ is the initial position of B. For every $t \in [0, T]$ define an evaluation operator:

$$\theta_t: C([0, T]; \mathbf{R}^n) \to \mathbf{R}^n, \; \theta_t(u) = u(t), \tag{2.31}$$

and consider the following multimap:

$$M: \mathbf{R}^n \to K(\mathbf{R}^n), \; M(x) = \theta_T \circ \Pi_{G_1} \circ h(x).$$

It is easy to see that M is a J^c-multimap and the finiteness of the game is equivalent to the existence of a solution of the following problem

$$\begin{cases} u'(t) \in G_0(t, u(t)), \\ u(T) \in M(u(0)). \end{cases}$$

2.4.4 Existence Theorem, Corollaries and Example

Theorem 2.10. *Let conditions* $(A1)$–$(A2)$ *hold. In addition, assume that there exists a guiding function* V *for problem (2.26) such that* $\operatorname{ind} V \neq 0$. *Then problem (2.26) has a solution.*

Proof. Set $r = \max\{r_V, r_*\}$, where r_V is the constant for the non-degenerate potential V and r_* is the constant from Definition 2.16.

Let M be a convex-valued multimap. Define a multimap

$$B: \mathbf{R}^n \to P(\mathbf{R}^n),$$
$$B(x) = \{y \in \mathbf{R}^n : \langle x - y, \varphi(x) grad\, V(x) \rangle \geq 0\},$$

where $\varphi(x) = 0$ if $|x| \leq r$ and $\varphi(x) = 1$ if $|x| > r$.

It is easy to see that B is a closed multimap with convex values. Therefore, the multimap

$$M_B: \mathbf{R}^n \to K(\mathbf{R}^n),\ M_B(x) = M(x) \cap B(x),$$

is a J-multimap, and hence it is a J^c-multimap (see Proposition 1.9).

Moreover, for every $x \in \mathbf{R}^n$, $|x| \geq r + 1$, relation $\langle grad\, V(x),\ x - w \rangle \geq 0$ holds for all $w \in M_B(x)$.

So, we can study problem (2.26) with the assumption that

$(M)'$ $\langle grad\, V(x),\ x - w \rangle \geq 0$ for all $w \in M(x)$ provided $|x| \geq r + 1$.

Substitute existence problem (2.26) with the problem of the existence of $x \in \mathbf{R}^n$ such that

$$0 \in \theta_T \circ \Pi_F(x) - M(x),$$

where the map θ_T and the multimap Π_F are defined as in (2.31) and (2.30), respectively.

Let $M = (\Sigma_q \circ \cdots \circ \Sigma_1) \in J^c(\mathbf{R}^n, \mathbf{R}^n)$, where $\Sigma_i \in J(X_{i-1}, X_i)$, $i = 1, \cdots, q$; $X_0 = X_q = \mathbf{R}^n$, and X_i are normed spaces for all $0 < i < q$.

Define the following maps and multimaps:

$$\tilde{\Sigma}_1: \mathbf{R}^n \to K(\mathbf{R}^n \times X_1),\ \tilde{\Sigma}_1(x) = \{x\} \times \Sigma_1(x),$$
$$\tilde{\Sigma}_i: \mathbf{R}^n \times X_{i-1} \to K(\mathbf{R}^n \times X_i),\ \tilde{\Sigma}_i(x, y) = \{x\} \times \Sigma_i(y),\ \forall i = 2, \cdots, q.$$
$$\tilde{M}: \mathbf{R}^n \to K(\mathbf{R}^n \times \mathbf{R}^n),\ \tilde{M}(x) = \{x\} \times M(x),$$
$$\tilde{\Pi}_F: \mathbf{R}^n \times \mathbf{R}^n \to K(C([0, T]; \mathbf{R}^n) \times \mathbf{R}^n),\ \tilde{\Pi}_F(x, y) = \{\Pi_F(x)\} \times \{y\},$$
$$\tilde{\theta}_T: C([0, T]; \mathbf{R}^n) \times \mathbf{R}^n \to \mathbf{R}^n \times \mathbf{R}^n,\ \tilde{\theta}_T(u, y) = \{\theta_T(u)\} \times \{y\},$$
$$f: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n,\quad f(x, y) = x - y.$$

It is clear that $\tilde{\Sigma}_i$ $(1 \leq i \leq q)$ and $\tilde{\Pi}_F$ are J-multimaps; $\tilde{\theta}_T$, f are continuous maps and

$$\theta_T \circ \Pi_F - M = f \circ \tilde{\theta}_T \circ \tilde{\Pi}_F \circ \tilde{M} = f \circ \tilde{\theta}_T \circ \tilde{\Pi}_F \circ \tilde{\Sigma}_q \circ \cdots \circ \tilde{\Sigma}_1.$$

Therefore, $\theta_T \circ \Pi_F - M: \mathbf{R}^n \to K(\mathbf{R}^n)$ is a J^c-multimap.

Define multimaps

$$A: \mathbf{R}^n \to P(\mathbf{R}^n),\ A(x) = \{y \in \mathbf{R}^n : \langle y, \varphi(x) grad\, V(x) \rangle \geq 0\},$$

and

$$F_A(t,x) = F(t,x) \cap A(x), \quad (t,x) \in [0,T] \times \mathbf{R}^n \ ,$$

where $\varphi(x)$ is defined above.

It is easy to verify that F_A is an L^1-upper Carathéodory multimap with α-sublinear growth and

$$\langle grad\, V(x), y \rangle \geq 0, \text{ for all } y \in F_A(t,x) \text{ provided } |x| > r.$$

Following Lemmas 2.3 and 2.7 we can choose a sufficiently large number $R > r + T + 1$ such that for every $(\lambda, x) \in [0,1] \times \mathbf{R}^n$, $|x| = R$, we have:

(a) every solution $u\colon [0,T] \to \mathbf{R}^n$ of the Cauchy problem:

$$\begin{cases} u'(t) \in \Psi(t,u(t),\lambda) = \lambda W_V(u(t)) + (1-\lambda)F_A(t,u(t)), \\ u(0) = x, \end{cases} \tag{2.32}$$

satisfies the condition: $|u(t)| > r$, $\forall t \in [0,T]$;

(b) there is $t_r \in (0,T]$ such that for all $u \in \Pi_{W_V}(x)$

$$\langle u(t) - x, grad\, V(x) \rangle > 0, \quad \forall t \in (0, t_r],$$

where Π_{W_V} is defined analogously to Π_F.

Now set $x \in \partial B_{\mathbf{R}^n}(0,R)$ and $z \in \theta_{t_r} \circ \Pi_{W_V}(x) - M(x)$. Then there exist $u \in \Pi_{W_V}(x)$ and $w \in M(x)$ such that $z = u(t_r) - w$. From the choice of R it follows that

$$\langle x - w, grad\, V(x) \rangle \geq 0 \text{ for all } w \in M(x).$$

Hence,

$$\langle z, grad\, V(x) \rangle = \langle u(t_r) - x, grad\, V(x) \rangle + \langle x - w, grad\, V(x) \rangle > 0.$$

Therefore, the vector fields $\theta_{t_r} \circ \Pi_{W_V} - M$ and $grad\, V$ are homotopic on $\partial B_{\mathbf{R}^n}(0,R)$. So,

$$deg(\theta_{t_r} \circ \Pi_{W_V} - M, B_{\mathbf{R}^n}(0,R)) = ind\, V.$$

If $0 \in \theta_T \circ \Pi_{F_A}(x) - M(x)$ for some $x \in \partial B_{\mathbf{R}^n}(0,R)$, then the theorem is proved, otherwise consider the following multimap

$$\Sigma\colon B_{\mathbf{R}^n}(0,R) \times [0,1] \to K(\mathbf{R}^n),$$

$$\Sigma(x,\lambda) = \theta_{\lambda t_r + (1-\lambda)T} \circ \Pi_{\Psi_\lambda}(x) - M(x),$$

where $\Psi_\lambda(t,x) = \Psi(t,x,\lambda)$.

It is easy to verify that Σ is a J^c-multimap. Assume that there is $(x,\lambda) \in \partial B_{\mathbf{R}^n}(0,R) \times (0,1]$ such that

$$0 \in \Sigma(x,\lambda).$$

Then there is a solution $u(\cdot)$ of problem (2.32) and $w \in M(x)$ such that

$$u\big(\lambda t_r + (1-\lambda)T\big) = w.$$

From (a) it follows that $|u(t)| > r \geq r_V$ for all $t \in [0,T]$. Therefore,

$$grad\, V(u(t)) \neq 0, \ \forall t \in [0,T].$$

Hence,

$$\big\langle \lambda W_V(u(s)) + (1-\lambda)y, grad\, V(u(s))\big\rangle > 0$$

for all $s \in [0,T]$ and all $y \in F_A(s,u(s))$. So,

$$V\big(u(\lambda t_r + (1-\lambda)T)\big) - V(x) = \int_0^{\lambda t_r + (1-\lambda)T} \big\langle u'(s), grad\, V(u(s))\big\rangle ds > 0.$$

Consequently, $V(w) > V(x)$, that is a contradiction.

Thus, Σ is a homotopy connecting multimaps

$$\theta_{t_r} \circ \Pi_{W_V} - M \text{ and } \theta_T \circ \Pi_{F_A} - M.$$

Therefore,

$$deg(\theta_T \circ \Pi_{F_A} - M, B_{\mathbf{R}^n}(0,R)) = ind\, V \neq 0.$$

So, problem (2.26) has a solution. □

Corollary 2.5. *Let conditions $(A1)$–$(A2)$ hold. Assume that there is $r > 0$ such that for every $(t,x) \in [0,T] \times \mathbf{R}^n$, $|x| \geq r$, the following relations hold:*

a) $\langle x, y\rangle \geq 0$ *for at least one point $y \in F(t,x)$;*
b) $\|M(x)\| = \max\{|w| : w \in M(x)\} \leq |x|$.

Then problem (2.26) has a solution.

Corollary 2.6 (Existence of anti-periodic solutions). *Let conditions $(A1)$–$(A2)$ hold. In addition, assume that there exists $r > 0$ such that for every $(t,x) \in I \times \mathbf{R}^n$, $|x| \geq r$, there is at least one point $y \in F(t,x)$ such that*

$$\langle x, y\rangle \geq 0.$$

Then the anti-periodic problem

$$\begin{cases} u'(t) \in F(t,u(t)), \ for\ a.e.\ t \in [0,T], \\ u(T) = -u(0), \end{cases} \tag{2.33}$$

has a solution.

Proof. The conclusions of the Corollaries 2.5 and 2.6 follow immediately from Theorem 2.10 by using the guiding function $V : \mathbf{R}^n \to \mathbf{R}$, $V(x) = \frac{1}{2}|x|^2$. □

Let us mention that the necessity of studying the existence of anti-periodic solutions for differential equations and inclusions arises in the investigation of many problems of physics (see, e.g., [118, 122, 127]), wavelet theory (see, e.g., [33]) and others branches of contemporary science. Some existence theorems for anti-periodic solutions are presented in [3, 35, 36].

Example 2.1. Consider the following problem

$$\begin{cases} u'(t) \in F(u(t)), \text{ for a.e. } t \in [0,1], \\ u(1) \in \left[\frac{1}{2}u(0) + 1, \frac{1}{2}u(0) + 2\right], \end{cases} \tag{2.34}$$

where multimap $F : \mathbf{R} \to Kv(\mathbf{R})$ is defined by

$$F(x) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \end{cases}$$

and $F(0) = [-1, 1]$.

In this situation, $M(x) = \left[\frac{1}{2}x + 1, \frac{1}{2}x + 2\right]$, $x \in \mathbf{R}$. It is clear that all conditions in Corollary 2.5 hold. So, problem (2.34) has a solution.

2.5 Global Bifurcation Problems

The existence of a branch of non-trivial solutions of an operator-equation from a bifurcation point was studied first by M.A. Krasnosel'skii [89]. The global bifurcation theorem for single-valued case was proved by P. Rabinowitz [124]. The bifurcation problem for inclusions with convex-valued multimaps was studied by J.C. Alexander and P.M. Fitzpatrick [4]. The authors of this work gave sufficient conditions under which the set of all non-trivial solutions near the point $(0, 0)$ admits either a bifurcation to infinity, a bifurcation to the border of the considered domain, or a bifurcation to some trivial solution of the inclusion. After this work, in the other studies the bifurcation theory for inclusions was extended to the case when multimap takes non-convex values (see, e.g., [64, 66, 96]).

Recently, bifurcation theory has been also extended to the case of linear Fredholm inclusions (see, e.g., [59, 60, 103]). Some other results on the bifurcation theory for inclusions and differential inclusions of various types can be found, e.g., in [42, 43, 48–52, 69, 81, 99, 101, 102, 104, 115, 129] and others.

The application of topological tools is the major method for studying bifurcation problems. A *global bifurcation index* at a given point is evaluated by using topological degrees. If the global bifurcation index is non-zero, then the global structure of solutions of the considering problem can be described. However, in practice this evaluation faces several difficulties due to the problem of handling techniques related to the topological degree in functional spaces.

It turns out that the MGF can be effectively applied to the evaluation of the bifurcation index. It should be mentioned that, as far as our knowledge, the first attempt in such direction was made by W. Kryszewski [96].

In this section, we present an approach of the MGF for the evaluation of the global bifurcation index and its application to study the global structure of periodic solutions of ordinary differential inclusions in finite-dimensional spaces.

2.5.1 Abstract Result

In this section, we present the application of the bifurcation index to the description of the global structure of branches of non-trivial solutions to a family of inclusions.

Let X be a Banach space. Consider the following one-parameter family of inclusions

$$x \in \mathscr{F}(x, \mu), \tag{2.35}$$

where $\mathscr{F}\colon X \times \mathbf{R} \to Kv(X)$ is a multimap.

Assume that:

($\mathscr{F}1$) $\mathscr{F}$ is completely upper semicontinuous and $0 \in \mathscr{F}(0, \mu)$ for all $\mu \in \mathbf{R}$;
($\mathscr{F}2$) for each μ, $0 < |\mu - \mu_0| \leq \varepsilon_0$, there is $\delta_\mu > 0$ such that $x \notin \mathscr{F}(x, \mu)$ when $0 < \|x\| \leq \delta_\mu$, where μ_0, ε_0 are given numbers.

Definition 2.17. A point $(0, \mu_*)$ is said to be a bifurcation point of inclusion (2.35) if for every open subset $U \subset X \times \mathbf{R}$ containing $(0, \mu_*)$ there exists a point $(x, \mu) \in U$ such that $x \neq 0$ and $x \in \mathscr{F}(x, \mu)$.

From ($\mathscr{F}1$)–($\mathscr{F}2$) it follows that for each μ, $0 < |\mu - \mu_0| \leq \varepsilon_0$ the topological degree

$$deg\big(i - \mathscr{F}(\cdot, \mu), B_X(0, \delta_\mu)\big)$$

is well defined. Then the *bifurcation index of the multimap* $\mathscr{F}$ at $(0, \mu_0)$ can be defined as

$$Bi\big[\mathscr{F}; (0, \mu_0)\big] = \lim_{\mu \to \mu_0 +} deg\big(i - \mathscr{F}(\cdot, \mu), B_X(0, \delta_\mu)\big) - \lim_{\mu \to \mu_0^-} deg\big(i - \mathscr{F}(\cdot, \mu), B_X(0, \delta_\mu)\big).$$

Let us describe the geometric meaning of the bifurcation index. For each sufficiently small $\varepsilon \in (0, \varepsilon_0]$, where ε_0 is the constant in ($\mathscr{F}2$), consider the multifield

$$F_r\colon \overline{U}_r \to Kv(X \times \mathbf{R}),$$
$$F_r(x, \mu) = \big\{x - \mathscr{F}(x, \mu), \|x\|^2 - r^2\big\},$$

where $r \in \big(0, \min\{\delta_{\mu_0-\varepsilon}, \delta_{\mu_0+\varepsilon}\}\big)$ is taken small enough and

$$U_r = \{(x,\mu) \in X \times \mathbf{R}\colon \|x\|^2 + (\mu-\mu_0)^2 < r^2 + \varepsilon^2\}.$$

It is clear that the vector field F_r is completely upper semicontinuous.
Let us mention that it has no zeros on the boundary ∂U_r.

Indeed assume, to the contrary, that there is $(x,\mu) \in \partial U_r$ such that

$$0 \in F_r(x,\mu).$$

Then we obtain

$$\begin{cases} \|x\| = r, \\ x \in \mathscr{F}(x,\mu). \end{cases}$$

From $(x,\mu) \in \partial U_r$ it follows that $\mu = \mu_0 \pm \varepsilon$. From $(\mathscr{F}2)$ and the choice of r we obtain a contradiction.
So the topological degree $deg(F_r, \overline{U}_r)$ is well-defined and does not depend on the choice of r.

The following statement is the generalization of the Ize's lemma (for more details we refer reader to [77,78,113]).

Lemma 2.8 (see [42,69]). *For each sufficiently small $\varepsilon \in (0,\varepsilon_0]$:*

$$deg(F_r, \overline{U}_r) = -Bi[\mathscr{F}; (0,\mu_0)]\ .$$

Let us denote by $\mathscr{S}$ the set of all non-trivial solutions to inclusion (2.35), i.e.,

$$\mathscr{S} = \{(x,\mu) \in X \times \mathbf{R}\colon\ x \neq 0\ \ and\ \ x \in \mathscr{F}(x,\mu)\}.$$

The following assertion follows easily from [59,96].

Theorem 2.11. *Under conditions $(\mathscr{F}1)$–$(\mathscr{F}2)$, assume that $Bi\big[\mathscr{F}; (0,\mu_0)\big] \neq 0$. Then there exists a connected subset $\mathscr{R} \subset \mathscr{S}$ such that $(0,\mu_0) \in \overline{\mathscr{R}}$ and one of the following cases occurs:*

(a) *$\mathscr{R}$ is unbounded;*
(b) *$(0,\mu_*) \in \overline{\mathscr{R}}$ for some $\mu_* \neq \mu_0$.*

2.5.2 Global Bifurcation of Periodic Solutions

Here we want to use the MGF for the evaluation of the bifurcation index for a family of differential inclusions in a finite-dimensional space. Then, the abstract result of the previous section is applied to the study of the global bifurcation of periodic solutions for this family.

The Setting of the Problem

Let $I = [0, T]$. We consider the following family of inclusions

$$x'(t) \in F(t, x(t), \mu) \text{ for a.e. } t \in I, \tag{2.36}$$

$$x(0) = x(T). \tag{2.37}$$

We assume the following conditions:

($H1$) The multimap $F: \mathbf{R} \times \mathbf{R}^n \times \mathbf{R} \to Kv(\mathbf{R}^n)$ is T-periodic L^p-upper Carathéodory, $p \geq 1$.
($H2$) The multimap $F(0, \cdot, \cdot): \mathbf{R}^n \times \mathbf{R} \to Kv(\mathbf{R}^n)$ is u.s.c.
($H3$) $0 \in F(s, 0, \mu)$ for all $\mu \in \mathbf{R}$ and a.e. $s \in [0, T]$.

We know (see Sect. 1.1.2) that under condition ($H1$) the superposition multioperator $\mathscr{P}_F: C(I, \mathbf{R}^n) \times \mathbf{R} \to Cv(L^p(I, \mathbf{R}^n))$ is well-defined and closed.

By a T-periodic solution to problem (2.36)–(2.37) we mean a pair $(x, \mu) \in W_T^{1,p}(I, \mathbf{R}^n) \times \mathbf{R}$ satisfying (2.36). From ($H3$) it follows that $(0, \mu)$ is a solution to problem (2.36)–(2.37) for each $\mu \in \mathbf{R}$. These solutions are called *trivial*. Let us denote by $\mathscr{S}$ the set of all nontrivial solutions of problem (2.36)–(2.37).

Global Structure of $\mathscr{S}$ When $p = 1$

We consider the global structure of T-periodic solutions to problem (2.36)–(2.37) when $p = 1$. Notice that T-periodic solutions of problem (2.36)–(2.37) are fixed points of the following family of integral multioperators

$$J_T: C(I, \mathbf{R}^n) \times \mathbf{R} \to Kv(C(I, \mathbf{R}^n)),$$

$$J_T(x, \mu) = \left\{ u: u(t) = x(T) + \int_0^t f(s)\, ds,\ f \in \mathscr{P}_F(x, \mu) \right\}.$$

It is easy to see that the multioperator J_T is completely upper semicontinuous. Extending Definition 2.1, we say that for a fixed $\mu \in \mathbf{R}$, a point $x_0 \in \mathbf{R}^n$ is a T-non-recurrence point of trajectories of inclusion (2.36), if for every nontrivial solution x of inclusion (2.36) satisfying condition $x(0) = x_0$ we have $x(t) \neq x_0$ for all $t \in (0, T]$.

The following theorem is a basic tool for considering the application of the MGF to bifurcation problems.

Theorem 2.12. *Let conditions* ($H1$)–($H3$) *hold. Assume that for each* μ *with*

$$0 < |\mu - \mu_0| \leq \varepsilon_0, \text{ where } \mu_0, \varepsilon_0 \text{ are given numbers},$$

the following conditions hold:

$(H4)$ *there exists a sufficiently small* $\varepsilon_\mu > 0$ *such that from the fact that* (x, μ) *is a non-trivial solution of inclusion (2.36) with the initial condition* $x(0) = 0$*, it follows that* $\|x\|_C \geq \varepsilon_\mu$*;*

$(H5)$ *there is* $\delta_\mu \in (0, \varepsilon_\mu)$*, where* ε_μ *is the constant in* $(H4)$*, such that every point* $y \in B_{\mathbf{R}^n}(0, \delta_\mu) \setminus \{0\}$ *is a* T*-non-recurrence point of trajectories of inclusion (2.36);*

$(H6)$ *multifield* $Q_\mu \colon \mathbf{R}^n \to Kv(\mathbf{R}^n)$*,*

$$Q_\mu(y) = -F(0, y, \mu),$$

has no zeros on $B_{\mathbf{R}^n}(0, \delta_\mu) \setminus \{0\}$.

Then $x \notin J_T(x, \mu)$ *provided* $0 < \|x\|_C \leq \delta_\mu$ *and*

$$deg(i - J_T(\cdot, \mu), B_C(0, \delta_\mu)) = deg(Q_\mu, B_{\mathbf{R}^n}(0, \delta_\mu)).$$

Proof. Fixing μ, $0 < |\mu - \mu_0| \leq \varepsilon_0$, we consider the following family of multimaps

$$F_\mu(t, y, \lambda) = F(\lambda t, y, \mu), \ \lambda \in [0, 1]$$

and the corresponding family of multifields

$$\Psi_\mu \colon C(I, \mathbf{R}^n) \times [0, 1] \to Kv(C(I, \mathbf{R}^n))$$

$$\Psi_\mu(x, \lambda) = \Big\{u \colon u(t) = x(t) - x(T) - \lambda \int_0^t f(s)ds - (1 - \lambda) \int_0^T f(s)ds, \ f \in \mathscr{P}_{F_\mu}(x, \lambda)\Big\}.$$

It is clear that the family of multifields Ψ_μ corresponds to the completely u.s.c. family of multimaps $i - \Psi_\mu$.

Let us show that Ψ_μ has no singular points on $(B_C(0, \delta_\mu) \setminus \{0\}) \times [0, 1]$.

To the contrary, assume that there is

$$(x_*, \lambda_*) \in (B_C(0, \delta_\mu) \setminus \{0\}) \times [0, 1],$$

such that $0 \in \Psi_\mu(x_*, \lambda_*)$. It means that there is a function $f \in L^1(I, \mathbf{R}^n)$ such that

$$f(s) \in F(\lambda_* s, x_*(s), \mu) \text{ for a.e. } s \in I,$$

and

$$x_*(t) = x_*(T) + \lambda_* \int_0^t f(s)ds + (1 - \lambda_*) \int_0^T f(s)ds, \qquad (2.38)$$

for all $t \in [0, T]$.

For $t = 0$ we have

$$x_*(0) = x_*(T) + (1 - \lambda_*) \int_0^T f(s)ds,$$

Taking $t = T$, we obtain

$$\int_0^T f(s)ds = 0. \tag{2.39}$$

Therefore, $x_*(0) = x_*(T)$.

From (2.38) it follows that

$$x_*'(t) = \lambda_* f(t) \in \lambda_* F(\lambda_* t, x_*(t), \mu)$$

for a.e. $t \in [0, T]$.

Thus, x_* is a solution of the inclusion

$$x'(t) \in \lambda_* F(\lambda_*, x(t), \mu).$$

(i) If $\lambda_* = 0$, then $x_*(t) = x_0 \in B_{\mathbf{R}^n}(0, \delta_\mu) \setminus \{0\}$ for all $t \in [0, T]$. We have

$$\int_0^T F(0, x_0, \mu)ds = T \cdot F(0, x_0, \mu).$$

From (2.39) and

$$\int_0^T f(s)ds \in \int_0^T F(0, x_0, \mu)ds$$

it follows that

$$0 \in F(0, x_0, \mu),$$

that is a contradiction because of the fact that Q_μ has no zeros on $B_{\mathbf{R}^n}(0, \delta_\mu) \setminus \{0\}$.

(ii) Let $\lambda_* \neq 0$. Consider a function $z_*(t) = x_*(\frac{t}{\lambda_*})$. Then for a.e. $t \in [0, \lambda_* T]$ we have

$$z_*'(t) = \frac{1}{\lambda_*} x_*'(\frac{t}{\lambda_*}) = f(\frac{t}{\lambda_*}) \in F(t, x_*(\frac{t}{\lambda_*}), \mu) = F(t, z_*(t), \mu).$$

Thus, (z_*, μ) is a solution to inclusion (2.36) on the interval $[0, \lambda_* T]$.

The case $x_*(0) = 0$: in this situation the pair $(\tilde{z}_*, \mu)$, where

$$\tilde{z}_*(t) = \begin{cases} z_*(t) & if \quad t \in [0, \lambda_* T] \\ 0 & if \quad t \in [\lambda_* T, T] \end{cases}$$

is a solution to inclusion (2.36) with the initial condition $\tilde{z}_*(0) = 0$. On the other hand,

$$\|\tilde{z}_*\|_C \leq \delta_\mu < \varepsilon_\mu$$

giving a contradiction.

The case $\boldsymbol{x_*(0) \neq 0}$: w.l.o.g. we can assume that z_* is extended to $[0, T]$. We have $z_*(0) = x_*(0) = x_*(T) = z_*(\lambda_* T)$. Hence, inclusion (2.36) has a nontrivial solution (z_*, μ) such that $z_*(0) \in B_{\mathbf{R}^n}(0, \delta_\mu) \setminus \{0\}$ and $z_*(0) = z_*(\lambda_* T)$, that is a contradiction with the T-non-recurrence of the trajectories of inclusion (2.36).

Thus, Ψ_μ is a homotopy connecting multifields

$$\Psi_\mu^{(1)} = i - J_T(\cdot, \mu),$$

and

$$\Psi_\mu^{(0)} = i - \Gamma_\mu^{(0)}(\cdot),$$

where multioperator $\Gamma_\mu^{(0)} \colon B_C(0, \delta_\mu) \to Kv(C(I, \mathbf{R}^n))$ is defined as

$$\Gamma_\mu^{(0)}(x) = x(T) + \int_0^T F(0, x(s), \mu) ds.$$

This multioperator has its range in the space $C^n_{[0,T]}$ of constant functions which can be identified with $\mathbf{R}^n$. Then

$$deg(\Psi_\mu^{(0)}, B_C(0, \delta_\mu)) = deg(\hat{\Psi}_\mu^{(0)}, B_{\mathbf{R}^n}(0, \delta_\mu)),$$

where $\hat{\Psi}_\mu^{(0)} = \Psi_\mu^{(0)}|_{\mathbf{R}^n}$ is defined by

$$\hat{\Psi}_\mu^{(0)}(y) = -\int_0^T F(0, y, \mu) ds = -T \cdot F(0, y, \mu).$$

So we obtain

$$deg(i - J_T(\cdot, \mu), B_C(0, \delta_\mu)) = deg(Q_\mu, B_{\mathbf{R}^n}(0, \delta_\mu)). \qquad \square$$

Definition 2.18. A continuously differentiable function $V \colon \mathbf{R}^n \to \mathbf{R}$ is said to be a local non-degenerate potential if there exists a sufficiently small number $r > 0$ such that the gradient

$$grad\, V(x) = \left(\frac{\partial V(x)}{\partial x_1}, \frac{\partial V(x)}{\partial x_2}, \cdots, \frac{\partial V(x)}{\partial x_n} \right)$$

is not equal zero provided $0 < |x| \leq r$.

It is clear that the topological degree

$$deg(grad\,V, B_{\mathbf{R}^n}(0, r'))$$

is well-defined and does not depend on $r' \in (0, r)$. This number is called *local index of a non-degenerate potential* V and is denoted by $ind\,V$.

Definition 2.19. For each $\mu \in \mathbf{R}$, a continuously differentiable function $V_\mu : \mathbf{R}^n \to \mathbf{R}$ is said to be a local guiding function for inclusion (2.36), if there exists a sufficiently small number $\tau_\mu > 0$ such that for every $y \in F(t, x, \mu)$:

$$\begin{cases} \langle grad\,V_\mu(x), y \rangle > 0 \; for\; t = 0 \; and\; a.e.\; t \in (0, \tau_\mu),\; 0 < |x| < \tau_\mu, \\ \langle grad\,V_\mu(x), y \rangle \geq 0 \; for\; a.e.\; t \in [\tau_\mu, T]. \end{cases}$$

From Definition 2.19 it follows that if V_μ is a local guiding function for inclusion (2.36) then V_μ is a non-degenerate potential and vector fields $-grad\,V_\mu$ and Q_μ are homotopic on $\partial B_{\mathbf{R}^n}(0, r)$ for every $0 < r < \tau_\mu$. Therefore

$$deg(Q_\mu, B_{\mathbf{R}^n}(0, r)) = deg(-grad\,V_\mu, B_{\mathbf{R}^n}(0, r)) = (-1)^n ind\,V_\mu.$$

Theorem 2.13. *Let conditions* $(H1)$–$(H4)$ *hold for* $p = 1$. *Assume that for each* μ,

$$0 < |\mu - \mu_0| \leq \varepsilon_0, \text{ where } \varepsilon_0 \text{ and } \mu_0 \text{ are given numbers},$$

there is a local guiding function V_μ *for inclusion (2.36) such that*

$$\lim_{\mu \to \mu_0^+} ind\,V_\mu - \lim_{\mu \to \mu_0^-} ind\,V_\mu \neq 0.$$

Then there exists a connected subset $\mathscr{W} \subset \mathscr{S}$ *such that* $(0, \mu_0) \in \overline{\mathscr{W}}$ *and either* $\mathscr{W}$ *is unbounded, or* $(0, \mu_*) \in \overline{\mathscr{W}}$ *for some* $\mu_* \neq \mu_0$.

Proof. Let us show that the multioperator J_T satisfies all conditions in Theorem 2.11.

In fact, condition $(\mathscr{F}1)$ can be easily verified. In order to verify condition $(\mathscr{F}2)$ and calculate the bifurcation index $Bi\big[J_T; (0, \mu_0)\big]$ we fix μ, $0 < |\mu - \mu_0| \leq \varepsilon_0$, and choose

$$0 < \delta_\mu < \min\{\varepsilon_\mu, \tau_\mu\},$$

where $\varepsilon_\mu, \tau_\mu$ are numbers in $(H4)$ and Definition 2.19, respectively.

Let us show that $B_{\mathbf{R}^n}(0, \delta_\mu) \setminus \{0\}$ is the set consisting of T-non-recurrence points of trajectories of inclusion (2.36). Indeed, take $x_0 \in B_{\mathbf{R}^n}(0, \delta_\mu) \setminus \{0\}$ and let x be an arbitrary nontrivial solution of inclusion (2.36) with initial condition $x(0) = x_0$. Assume that there is $t_* \in (0, T]$ such that $x(t_*) = x(0)$. Since $|x_0| < \tau_\mu$, there exists $t_\mu \in (0, \tau_\mu)$ such that $t_\mu < t_*$ and $|x(t)| < \tau_\mu$ for all $t \in (0, t_\mu)$. Therefore

$$
\begin{aligned}
0 = V_\mu(x(t_*)) - V_\mu(x(0)) &= \int_0^{t_*} \big\langle grad\, V_\mu(x(s)), x'(s)\big\rangle\, ds \\
&= \int_0^{t_\mu} \big\langle grad\, V_\mu(x(s)), x'(s)\big\rangle\, ds + \int_{t_\mu}^{t_*} \big\langle grad\, V_\mu(x(s)), x'(s)\big\rangle\, ds > 0,
\end{aligned}
$$

giving the contradiction.

Notice that for every μ, $0 < |\mu - \mu_0| \le \varepsilon_0$, from the existence of the guiding function V_μ for inclusion (2.36) it follows that the vector field $Q_\mu = -F(0, y, \mu)$ has no zeros on $B_{\mathbf{R}^n}(0, \delta_\mu) \setminus \{0\}$. By Theorem 2.12 we have that $x \notin J_T(x, \mu)$ provided $0 < \|x\|_C \le \delta_\mu$ and

$$
\begin{aligned}
&\lim_{\mu\to\mu_0^+} deg(i - J_T(\cdot,\mu), B_C(0,\delta_\mu)) - \lim_{\mu\to\mu_0^-} deg(i - J_T(\cdot,\mu), B_C(0,\delta_\mu)) \\
&\quad = \lim_{\mu\to\mu_0^+} deg(Q_\mu, B_{\mathbf{R}^n}(0,\delta_\mu)) - \lim_{\mu\to\mu_0^-} deg(Q_\mu, B_{\mathbf{R}^n}(0,\delta_\mu)) \\
&\quad = (-1)^n\Big(\lim_{\mu\to\mu_0^+} ind\ V_\mu - \lim_{\mu\to\mu_0^-} ind\ V_\mu\Big).
\end{aligned}
$$

Hence,

$$
Bi\big[J_T; (0,\mu_0)\big] = (-1)^n\Big(\lim_{\mu\to\mu_0^+} ind\ V_\mu - \lim_{\mu\to\mu_0^-} ind\ V_\mu\Big) \ne 0.
$$

To conclude the proof we need only to apply Theorem 2.11. □

Global Bifurcation When $p = 2$

Now, by introducing the notion of local integral guiding functions for inclusion (2.36) we consider the global structure of the set of all T-periodic solutions of problem (2.36)–(2.37). Assume that F is a T-periodic L^2-upper Carathéodory multimap satisfying condition $(H3)$.

Define the operator ℓ: $W_T^{1,2}(I, \mathbf{R}^n) \to L^2(I, \mathbf{R}^n)$ as

$$
\ell x = x'.
$$

It is clear that ℓ is a linear Fredholm operator of index zero and

$$
Ker\,\ell \cong \mathbf{R}^n \cong Coker\,\ell.
$$

Then we can substitute problem (2.36)–(2.37) by the following family of operator inclusions

$$
\ell x \in \mathscr{P}_F(x, \mu),
$$

or by equivalently (see, Sect. 1.3)

$$x \in G(x, \mu), \tag{2.40}$$

where

$$\begin{aligned} G: C_T(I, \mathbf{R}^n) \times \mathbf{R} \to Kv(C_T(I, \mathbf{R}^n)), \\ G(x, \mu) = Px + (\Lambda\Pi + K_{P,Q})\mathscr{P}_F(x, \mu). \end{aligned} \tag{2.41}$$

Recall that

$$\Pi: L^2(I, \mathbf{R}^n) \to \mathbf{R}^n,$$

is defined by

$$\Pi f = \frac{1}{T} \int_0^T f(s)\, ds$$

and the homomorphism $\Lambda: \mathbf{R}^n \to \mathbf{R}^n$ can be treated as the identity map.

Definition 2.20. For each $\mu \in \mathbf{R}$, a continuously differentiable function $V_\mu: \mathbf{R}^n \to \mathbf{R}$ is said to be a local integral guiding function for inclusion (2.36), if there exists a sufficiently small number $\pi_\mu > 0$ such that from $x \in W_T^{1,2}(I, \mathbf{R}^n)$ with $0 < \|x\|_2 \leq \pi_\mu$ it follows that

$$\int_0^T \langle grad\, V_\mu(x(s)), f(s) \rangle\, ds > 0$$

for all $f \in \mathscr{P}_F(x, \mu)$.

Notice that the local integral guiding function V_μ is a non-degenerate potential. In fact, for every $y \in \mathbf{R}^n$ with $0 < |y| \leq \frac{\pi_\mu}{\sqrt{T}}$, considering y as a constant function we have

$$\int_0^T \langle grad\, V_\mu(y), f(s) \rangle\, ds = \langle grad\, V_\mu(y), \int_0^T f(s) ds \rangle = T \langle grad\, V_\mu(y), \Pi f \rangle > 0$$

for all $f \in \mathscr{P}_F(y, \mu)$.

Hence, $grad\, V_\mu(y) \neq 0$ provided $0 < |y| \leq \frac{\pi_\mu}{\sqrt{T}}$.

Theorem 2.14. *Let F be a T-periodic L^2-Carathéodory multimap satisfying conditions (H3). Assume that for each μ, $0 < |\mu - \mu_0| \leq \varepsilon_0$, where ε_0, μ_0 are given numbers, there exists a local integral guiding function V_μ for inclusion (2.36) such that*

$$\lim_{\mu \to \mu_0^+} ind\, V_\mu - \lim_{\mu \to \mu_0^-} ind\, V_\mu \neq 0.$$

Then there is a connected subset $\mathscr{W} \subset \mathscr{S}$ such that $(0, \mu_0) \in \overline{\mathscr{W}}$ and either $\mathscr{W}$ is unbounded or $(0, \mu_*) \in \overline{\mathscr{R}}$ for some $\mu_* \neq \mu_0$.

Proof. We show that multioperator G defined in (2.41) satisfies all conditions in Theorem 2.11. At first, the space $L^2(I,\mathbf{R}^n)$ can be decomposed by

$$L^2(I,\mathbf{R}^n) = \mathscr{L}_0 \oplus \mathscr{L}_1,$$

where $\mathscr{L}_0 = Coker\,\ell$, $\mathscr{L}_1 = Im\,\ell$. The corresponding decomposition of an element $f \in L^2(I,\mathbf{R}^n)$ is denoted by

$$f = f_0 + f_1,\ f_0 \in \mathscr{L}_0,\ f_1 \in \mathscr{L}_1.$$

STEP 1. From $(H3)$ it follows that $0 \in G(0,\mu)$ for all $\mu \in \mathbf{R}$. Let

$$\begin{aligned}\Phi\colon C_T(I,\mathbf{R}^n)\times\mathbf{R}\times[0,1] &\to Cv(L^2(I,\mathbf{R}^n)),\\ \Phi(x,\mu,\lambda) &= \chi(\mathscr{P}_F(x,\mu),\lambda),\end{aligned}$$

where

$$\chi(f_0+f_1,\lambda) = f_0 + \lambda f_1.$$

We prove that the multimap

$$\begin{aligned}\Sigma\colon C_T(I,\mathbf{R}^n)\times\mathbf{R}\times[0,1] &\to Kv(C_T(I,\mathbf{R}^n)),\\ \Sigma(x,\mu,\lambda) &= Px + (\Lambda\Pi + K_{P,Q})\Phi(x,\mu,\lambda),\end{aligned}$$

is completely u.s.c.

Indeed, from the fact that the multioperator $\mathscr{P}_F$ is closed and the operator $(\Lambda\Pi + K_{P,Q})\circ\chi$ is linear and continuous it follows that the multimap $(\Lambda\Pi + K_{P,Q})\chi\circ\mathscr{P}_F$ is closed (see Proposition 1.5). Further, for every bounded subset $U \subset C_T(I,\mathbf{R}^n)\times\mathbf{R}$ the set $\mathscr{P}_F(U)$ is bounded in $L^2(I,\mathbf{R}^n)$. Then the set $(\Lambda\Pi + K_{P,Q})\chi\circ\mathscr{P}_F(U)$ is bounded in $W_T^{1,2}(I,\mathbf{R}^n)$ and by the compact embedding property (see, e.g. [14, 41]), the set $(\Lambda\Pi + K_{P,Q})\chi\circ\mathscr{P}_F(U)$ is relatively compact in $C_T(I,\mathbf{R}^n)$. Finally, our assertion follows from the fact that the operator P is continuous and takes values in a finite dimensional space. In particular, the multimap $G = \Sigma(\cdot,\cdot,1)$ is completely u.s.c.. So condition $(\mathscr{F}1)$ holds.

STEP 2. For each μ, $0 < |\mu-\mu_0| \le \varepsilon_0$, choosing r_μ such that

$$0 < r_\mu \le \min\{\pi_\mu, \frac{\pi_\mu}{\sqrt{T}}\},$$

where π_μ is a constant from Definition 2.20, assume that (x,μ), $x \in B_C(0,r_\mu)$ is a nontrivial solution to inclusion (2.40). Then there is $f \in \mathscr{P}_F(x,\mu)$ such that $x'(t) = f(t)$ for a.e. $t \in [0,T]$.

Since $0 < \|x\|_2 \le \pi_\mu$ we have

$$\int_0^T \langle grad\, V_\mu(x(s)), f(s) \rangle\, ds = \int_0^T \langle grad\, V_\mu(x(s)), x'(s) \rangle\, ds$$
$$= V_\mu\big(x(T)\big) - V_\mu\big(x(0)\big) > 0,$$

giving a contradiction, i.e., inclusion (2.40) has no nontrivial solutions on $B_C(0, r_\mu)$. Therefore ($\mathscr{F}2$) holds.

STEP 3. Now we evaluate the bifurcation index $Bi\big[G; (0, \mu_0)\big]$. Toward this goal, we fix μ, $0 < |\mu - \mu_0| \le \varepsilon_0$, and choose r_μ as in Step 2. Consider the following family of inclusions

$$x \in \Sigma_\mu(x, \lambda), \tag{2.42}$$

where $\Sigma_\mu\colon C_T(I, \mathbf{R}^n) \times [0, 1] \to Kv(C_T(I, \mathbf{R}^n))$,

$$\Sigma_\mu(x, \lambda) = Px + (\Lambda\Pi + K_{P,Q})\Phi(x, \mu, \lambda).$$

As in Step 1 , multioperator Σ_μ is completely u.s.c.. Assume that there is a solution $(x^*, \lambda^*) \in \partial B_C(0, r_\mu) \times [0, 1]$ of inclusion (2.42). Then there exists a function $f^* \in \mathscr{P}_F(x^*, \mu)$ such that

$$x^* = Px^* + (\Lambda\Pi + K_{P,Q}) \circ \chi(f^*, \lambda^*),$$

or equivalently,

$$\begin{cases} \ell x^* = \lambda^* f_1^* \\ 0 = f_0^*, \end{cases}$$

where $f_0^* + f_1^* = f^*$, $f_0^* \in \mathscr{L}_0$ and $f_1^* \in \mathscr{L}_1$.

Since $0 < \|x^*\|_2 \le \pi_\mu$ we have

$$\int_0^T \langle grad\, V_\mu(x^*(s)), f^*(s) \rangle\, ds > 0.$$

If $\lambda^* \ne 0$, then

$$\int_0^T \langle grad\, V_\mu(x^*(s)), f^*(s) \rangle\, ds = \int_0^T \langle grad\, V_\mu(x^*(s)), \frac{1}{\lambda^*} x^{*\prime}(s) \rangle\, ds$$
$$= \frac{1}{\lambda^*}\Big(V_\mu(x^*(T)) - V_\mu(x^*(0))\Big) = 0,$$

that is a contradiction.

If $\lambda^* = 0$, then $\ell x^* = 0$, i.e., $x^* \equiv a \in \mathbf{R}^n$, $|a| = r_\mu$. For every $f \in \mathscr{P}_F(a, \mu)$ we have

$$\int_0^T \langle grad\, V_\mu(a), f(s) \rangle\, ds > 0.$$

On the other hand,

$$\int_0^T \langle grad\, V_\mu(a), f(s)\rangle\, ds = \Big\langle grad\, V_\mu(a), \int_0^T f(s)\, ds\Big\rangle = T\langle grad\, V_\mu(a), \Pi f\rangle.$$

Therefore,

$$T\langle grad\, V_\mu(a), \Pi f\rangle > 0. \tag{2.43}$$

Hence, $\Pi f \neq 0$ for all $f \in \mathscr{P}_F(a,\mu)$, in particular, $\Pi f^* \neq 0$. But $\Pi f^* = \Pi f_0^* = 0$, giving a contradiction.

Thus, Σ_μ is a homotopy connecting multimaps $\Sigma_\mu(\cdot,1) = G(\cdot,\mu)$ and $\Sigma_\mu(\cdot,0) = P + \Pi\mathscr{P}_F(\cdot,\mu)$. From the homotopy invariance property of the topological degree it follows that

$$deg\big(i - G(\cdot,\mu), B_C(0,r_\mu)\big) = deg\big(i - P - \Pi\mathscr{P}_F(\cdot,\mu), B_C(0,r_\mu)\big).$$

Multimap $P + \Pi\mathscr{P}_F(\cdot,\mu)$ takes its values in $\mathbf{R}^n$, then

$$deg\big(i - P - \Pi\mathscr{P}_F(\cdot,\mu), B_C(0,r_\mu)\big) = deg\big(i - P - \Pi\mathscr{P}_F(\cdot,\mu), B_{\mathbf{R}^n}(0,r_\mu)\big),$$

In the space $\mathbf{R}^n$ multifield $i - P - \Pi\mathscr{P}_F(\cdot,\mu)$ has the form

$$i - P - \Pi\mathscr{P}_F(\cdot,\mu) = -\Pi\mathscr{P}_F(\cdot,\mu),$$

therefore

$$deg\big(i - P - \Pi\mathscr{P}_F(\cdot,\mu), B_{\mathbf{R}^n}(0,r_\mu)\big) = deg\big(-\Pi\mathscr{P}_F(\cdot,\mu), B_{\mathbf{R}^n}(0,r_\mu)\big).$$

From (2.43) and Lemma 1.5 it follows that the vector fields $\Pi\mathscr{P}_F(\cdot,\mu)$ and $grad\, V_\mu$ are homotopic on $\partial B_{\mathbf{R}^n}(0,r_\mu)$, so

$$deg\big(-\Pi\mathscr{P}_F(\cdot,\mu), B_{\mathbf{R}^n}(0,r_\mu)\big) = deg\big(-grad\, V_\mu, B_{\mathbf{R}^n}(0,r_\mu)\big) = (-1)^n ind\, V_\mu.$$

Consequently,

$$\begin{aligned}
&\lim_{\mu\to\mu_0^+} deg(i - G(\cdot,\mu), B_C(0,r_\mu)) - \lim_{\mu\to\mu_0^-} deg(i - G(\cdot,\mu), B_C(0,r_\mu))\\
&\quad = \lim_{\mu\to\mu_0^+} deg\big(-\Pi\mathscr{P}_F(\cdot,\mu), B_{\mathbf{R}^n}(0,r_\mu)\big) - \lim_{\mu\to\mu_0^-} deg\big(-\Pi\mathscr{P}_F(\cdot,\mu), B_{\mathbf{R}^n}(0,r_\mu)\big)\\
&\quad = (-1)^n\Big(\lim_{\mu\to\mu_0^+} ind\, V_\mu - \lim_{\mu\to\mu_0^-} ind\, V_\mu\Big).
\end{aligned}$$

Hence,

$$Bi\big[G;(0,\mu_0)\big] = (-1)^n\big(\lim_{\mu\to\mu_0^+} ind\,V_\mu - \lim_{\mu\to\mu_0^-} ind\,V_\mu\big) \neq 0.$$

To conclude the proof we need only to apply Theorem 2.11. □

2.5.3 Application 1: Differential Inclusion with a Bounded Nonlinearity

Consider the following differential inclusion

$$x'(t) \in \mu x(t)\Big(a + F(t, x(t))\Big), \tag{2.44}$$

where $F\colon \mathbf{R}\times\mathbf{R} \to Kv(\mathbf{R})$ is a T-periodic upper Carathéodory multimap; $a > 0$, $\mu \in \mathbf{R}$.

Denote by $\mathscr{S}$ the set of all non-trivial T-periodic solutions of inclusion (2.44).

Theorem 2.15. *Assume that:*

(A) *there is* $0 < K < a$ *such that*

$$\|F(t,y)\| = \max\{|z|\colon z \in F(t,y)\} < K$$

for all $y \in \mathbf{R}$ *and a.e.* $t \in [0,T]$.

Then there is a connected subset $\mathscr{W} \subset \mathscr{S}$ *such that* $(0,0) \in \overline{\mathscr{W}}$ *and* $\mathscr{W}$ *is unbounded.*

Proof. It is clear that $(0,\mu)$ is a solution of inclusion (2.44) for every $\mu \in \mathbf{R}$ and $(y,0)$ is a solution of inclusion (2.44) for every constant function $y \in \mathbf{R}$. Therefore $(0,0)$ is a bifurcation point. Now let us show that $(0,0)$ is the unique bifurcation point of (2.44).

To this aim we define a multimap $\tilde{F}\colon \mathbf{R}\times\mathbf{R}\times\mathbf{R} \to Kv(\mathbf{R})$ by

$$\tilde{F}(t,y,\mu) = \mu y\big(a + F(t,y)\big).$$

It is easy to see that $\tilde{F}$ is a T-periodic L^2-upper Carathéodory multimap.

We want to show that for every $\mu \neq 0$ the function

$$V_\mu\colon \mathbf{R} \to \mathbf{R},$$

$$V_\mu(y) = \frac{1}{2}\,\mu y^2,$$

is a local integral guiding function for inclusion (2.44).

In fact, let $x \in W_T^{1,2}([0,T];\mathbf{R})$ and choose an arbitrary $f \in \mathscr{P}_F(x)$. Then

$$\tilde{f} = \mu x(a + f) \in \mathscr{P}_{\tilde{F}}(x, \mu).$$

We have

$$\begin{aligned}
\int_0^T \Big\langle grad\, V_\mu(x(s)), \tilde{f}(s)\Big\rangle ds &= \int_0^T \Big\langle \mu x(s), \mu a x(s) + \mu x(s) f(s)\Big\rangle ds \\
&\geq \mu^2\Big(a\|x\|_2^2 + \int_0^T x^2(s) f(s) ds\Big) \\
&\geq \mu^2\Big(a\|x\|_2^2 - \|x\|_2\, \|xf\|_2\Big) \\
&\geq \mu^2\|x\|_2^2\big(a - K\big) > 0, \qquad (2.45)
\end{aligned}$$

for $\|x\|_2 > 0$.

Therefore, inclusion (2.44) has no non-trivial solution provided $\mu \neq 0$. □

2.5.4 Application 2: Global Bifurcation for Functional Differential Inclusions

We use the notion of phase space given in Sect. 1.4.

Set $I = [0,T]$ and denote by $\mathscr{BC}(\mathbf{R}^n)$ the Banach space of bounded continuous functions $BC((-\infty, 0]; \mathbf{R}^n)$.

Consider a functional differential inclusion with infinite delay of the following form:

$$x'(t) \in F(t, x_t, \mu) \;\; for\, a.e \;\; t \in [0,T], \qquad (2.46)$$

where $F: \mathbf{R} \times \mathscr{BC}(\mathbf{R}^n) \times \mathbf{R} \to Kv(\mathbf{R}^n)$ is a multimap.

Assume that the multimap F satisfies the next conditions:

($H1$) F is a T-periodic upper Carathéodory multimap.

($H2$) For every bounded subset $\Omega \subset C_T(I, \mathbf{R}^n) \times \mathbf{R}$ there exists a function $\nu_\Omega \in L_+^2[0,T]$ such that for each $(\varphi, \mu) \in \Omega$

$$\|F(t, \tilde{\varphi}_t, \mu)\|_{\mathbf{R}^n} \leq \nu_\Omega(t)\, for\, a.e.\, t \in [0,T],$$

where $\tilde{x}$ denotes the T-periodic extension of x on $(-\infty, T]$.

($H3$) $0 \in F(t, 0, \mu)$ for all $\mu \in \mathbf{R}$ and a.e. $t \in [0,T]$.

Notice that under conditions ($H1$)–($H2$) the superposition multioperator

$$\mathscr{P}_F: C_T(I, \mathbf{R}^n) \times \mathbf{R} \to Cv(L^2(I, \mathbf{R}^n)),$$

$$\mathscr{P}_F(x,\mu) = \{f \in L^2(I;\mathbf{R}^n)\colon f(s) \in F(s,\tilde{x}_s,\mu)\, for\, a.e.\, t \in I\},$$

is well-defined and closed.

As earlier, we can treat the global bifurcation problem of T-periodic solutions of inclusion (2.46) as the global bifurcation problem of solutions of the following operator inclusion

$$\ell x \in \mathscr{P}_F(x,\mu), \tag{2.47}$$

where ℓ is the operator of differentiation.

From $(H3)$ it follows that problem (2.47) has a trivial solution $(0,\mu)$ for all $\mu \in \mathbf{R}$. Let us denote by $\mathscr{S}$ the set of all nontrivial solutions of (2.47).

We use the notion of local integral guiding functions as given in Definition 2.20. Following the method given in the proof of Theorem 2.14 we obtain

Theorem 2.16. *Let conditions $(H1)$–$(H3)$ hold. Assume that for each μ with*

$$0 < |\mu - \mu_0| \le \varepsilon_0, \ \textit{where } \mu_0, \varepsilon_0 \textit{ are given constants},$$

there exists a local integral guiding function V_μ to problem (2.46) such that

$$\lim_{\mu\to\mu_0^+} ind\ V_\mu - \lim_{\mu\to\mu_0^-} ind\ V_\mu \neq 0.$$

Then there is a connected subset $\mathscr{R} \subset \mathscr{S}$ such that $(0,\mu_0) \in \overline{\mathscr{R}}$ and either $\mathscr{R}$ is unbounded or $\overline{\mathscr{R}} \ni (0,\mu_*)$ for some $\mu_* \neq \mu_0$.

2.5.5 Application 3: Feedback Control System

Consider the following feedback control system with infinite delay

$$\begin{cases} x'(t) = \mu a x(t) + f(x_t, u(t), \mu) \ \ for\, a.a.\ t \in [0,T], \\ u(t) \in U(x(t)) \ \ for\, a.a.\ t \in [0,T], \\ x(0) = x(T), \end{cases} \tag{2.48}$$

where $a > 0$, $\mu \in \mathbf{R}$, a map $f\colon \mathscr{BC}(\mathbf{R}^n) \times \mathbf{R}^m \times \mathbf{R} \to \mathbf{R}^n$ is continuous; a multimap $U\colon \mathbf{R}^n \to Kv(\mathbf{R}^m)$ is u.s.c.; $n, m \in \mathbf{N}$ and n is an odd number.

We assume the following conditions:

$(f1)$ There exist $\gamma > 1$ and $b > 0$ such that

$$|f(\tilde{\varphi}_t, y, \mu)| \le b\ \|\varphi\|_2^\gamma(|\mu| + |y|)$$

for all $(\varphi, y, \mu) \in C_T(I, \mathbf{R}^n) \times \mathbf{R}^m \times \mathbf{R}$ and a.e. $t \in [0,T]$.

$(U1)$ For every $(\varphi,\mu) \in \mathscr{BC}(\mathbf{R}^n) \times \mathbf{R}$ the set $f\Big(\varphi, U\big(\varphi(0)\big), \mu\Big)$ is convex.

$(U2)$ There exists $c > 0$ such that

$$\|U(y)\|_{\mathbf{R}^m} \leq c(1 + |y|)$$

for all $y \in \mathbf{R}^n$.

Define a multimap $F\colon \mathscr{BC}(\mathbf{R}^n) \times \mathbf{R} \to Kv(\mathbf{R}^n)$ by

$$F(\varphi, \mu) = \mu a\varphi(0) + f(\varphi, U(\varphi(0)), \mu).$$

Then we treat the problem of global bifurcation of T-periodic solutions of problem (2.48) as the problem of global bifurcation of T-periodic solutions of the following differential inclusion:

$$x'(t) \in F(x_t, \mu), \quad \text{for a.e. } t \in I\ .$$

Let us denote by $\mathscr{S}$ the set of all nontrivial T-periodic solutions of (2.48).

Theorem 2.17. *Let conditions* $(f1)$ *and* $(U1) - (U2)$ *hold. Then there is a connected unbounded subset* $\mathscr{R} \subset \mathscr{S}$ *such that* $(0,0) \in \overline{\mathscr{R}}$.

Proof. It is easy to see that multimap F satisfies all conditions $(H1)$–$(H3)$ in Theorem 2.16. For each $\mu \neq 0$ consider the function

$$V_\mu\colon \mathbf{R}^n \to \mathbf{R},$$
$$V_\mu(y) = \frac{1}{2}\mu\langle y, y\rangle$$

Letting $x \in W_T^{1,2}(I, \mathbf{R}^n)$ and choosing an arbitrary $g \in \mathscr{P}_F(x, \mu)$, we obtain that there exists $u \in L^2(I, \mathbf{R}^m)$ such that $u(s) \in U(x(s))$ for a.e. $s \in I$, and

$$g(s) = \mu a x(s) + f(\tilde{x}_s, u(s), \mu)\, for\ a.e.\ s \in I.$$

We have

$$\begin{aligned}
\int_0^T \langle grad\, V_\mu(x(t)), g(t)\rangle dt &= \int_0^T \langle \mu x(t), \mu a x(t) + f(\tilde{x}_t, u(t), \mu)\rangle dt \\
&\geq a\mu^2\|x\|_2^2 - |\mu| \int_0^T |x(t)||f(\tilde{x}_t, u(t), \mu)| dt \\
&\geq a\mu^2\|x\|_2^2 - b|\mu|\|x\|_2^\gamma \int_0^T |x(t)|\big(|\mu| + |u(t)|\big) dt \\
&\geq a\mu^2\|x\|_2^2 - b|\mu|\|x\|_2^\gamma \int_0^T |x(t)|\big(|\mu| + c + c|x(t)|\big) dt \\
&\geq a\mu^2\|x\|_2^2 - b\sqrt{T}(\mu^2 + c|\mu|)\|x\|_2^{1+\gamma} - bc|\mu|\, \|x\|_2^{2+\gamma}.
\end{aligned}$$

Therefore

$$\int_0^T \left\langle grad\, V_\mu(x(t)), g(t) \right\rangle dt \geq |\mu| \|x\|_2^2 \Big(a|\mu| - b\sqrt{T}(|\mu| + c)\|x\|_2^{\gamma-1} - bc\|x\|_2^{\gamma} \Big) > 0, \tag{2.49}$$

for all $\mu \neq 0$ and sufficiently small $\|x\|_2 \neq 0$.

Thus for every $\mu \neq 0$, V_μ is a local integral guiding function for problem (2.48). From the fact that

$$\lim_{\mu \to 0+} indV_\mu - \lim_{\mu \to 0-} indV_\mu = 1 - (-1)^n = 2$$

and (2.49) it follows that $(0, 0)$ is the unique bifurcation point for problem (2.48). Applying Theorem 2.16 we obtain the conclusion of the theorem. □

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