Chapter 2
Shell Element Formulations for General Nonlinear Analysis. Modeling Techniques

2.1 Introduction

In 1970, Ahmad, Irons and Zienkiewicz [1] presented a shell element formulation that after many years still constitutes the basis for modern finite element analysis of shell structures. The original formulation was afterwards extended to material and geometric nonlinear analysis under the constraint of the infinitesimal strains assumption [2–4].

The fundamental features of the A-I-Z shell element are,

- using isoparametric interpolation functions the displacements inside the shell element are interpolated from three displacement-d.o.f. and two rotation-d.o.f. at each node,
- the interpolated generalized displacement fields present C^0continuity,
- the element is not based on any plate/shell theory but it is a continuum element incorporating several assumptions that we list below (degenerated solid element).

The kinematic and constitutive assumptions are,

- a straight line that is initially normal to the mid-surface remains straight after the deformation,
- a straight line that is initially normal to the mid-surface is not stretched during the deformation,
- the through-the-thickness stresses are zero.

It is important to remark that the second assumption precludes the consideration of finite strain kinematics.

Although the A-I-Z shell element was a breakthrough in the field of finite element analysis of shell structures, it suffers from the locking phenomenon and much research effort has been devoted to the development of A-I-Z type elements that do not incorporate this problem [5, 6].

The MITC4 shell element [7–9] which was developed to overcome the locking problem of the A-I-Z shell elements has become, since its development in the early

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1980s, the standard shell element for many finite element codes. However, the limitation of infinitesimal strains is still present in the MITC4 formulation.

Many researchers have contributed to the development of shell elements that can model finite strain situations, among them,

- an early contribution by Rodal and Witmer for elasto-viscoplastic material models (J_2) where, at each iteration, after going through the displacements calculation, the shell element thickness is updated neglecting the elastic strains and invoking the incompressibility of the viscoplastic flow (J_2) [10],
- in 1983 Hughes and Carnoy [11] developed a finite strain shell element for the Mooney-Rivlin material model which uses a plane-stress constitutive relation for the laminae and updates afterwards the thickness via a staggered iterative formulation,
- Simo and co-workers in the period 1988–1992 developed a complete 3D non-linear shell element formulation [12–16],
- Ramm and co-workers developed 3D shell elements considering also through-the-thickness stretching [17, 18].

In 1995 Dvorkin et al. developed the MITC4-TLH element, that based on the original MITC4 formulation can model finite strain elasto-plastic (J_2) deformations. This element imposes the condition of zero transversal stresses and its computational cost was rather high [19, 20].

Later, Toscano and Dvorkin developed an element that is also based on the MITC4 formulation and can efficiently model finite strain deformations using a general 3D material model: the MITC4-3D element [21, 22].

The most relevant differences with the original MITC4 formulation are:

- for each quadrilateral element there are 22 d.o.f.: 5 generalized displacements per node plus 2 extra d.o.f. to incorporate the through-the-thickness stretching, these extra d.o.f. are condensed at the element level;
- a general 3D constitutive relation is used, instead of the original laminae plane stress constitutive relation.

2.2 The Standard A-I-Z Quadrilateral Shell Element for Linear Analysis

2.2.1 Linear Analysis Kinematics

When modeling a shell we define, on its mid-surface, *nodes* and at those nodes we define *director vectors* which are the best approximation to the shell mid-surface normal at the corresponding nodes. The A-I-Z quadrilateral element is defined using four nodes which are not necessarily coplanar.

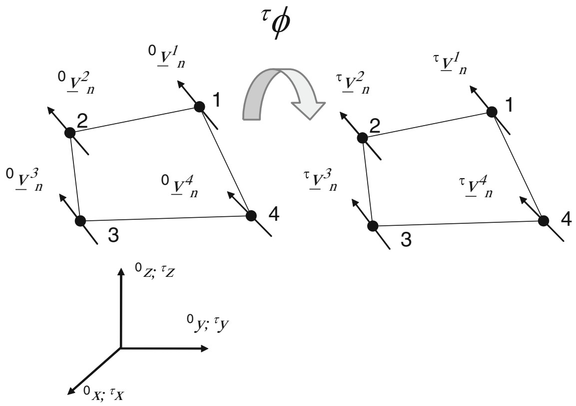

Fig. 2.1 Kinematics of the A-I-Z shell element

Under the assumption of infinitesimal strains, for a configuration at a time τ, a point inside the shell element, with natural coordinates [5] (r, s, t) (see Fig. 2.1), is defined by the Cartesian coordinates,[1]

$$^{\tau}\underline{x}(r,\, s,\, t) = h_k(r, s)\ ^{\tau}\underline{x}_k + \frac{t}{2} h_k(r,\, s)[a\,^{\tau}\underline{V}^{n}]_k. \tag{2.1}$$

In the above equation,

h_k	2D isoparametric interpolation function corresponding to the k-node [5];
$^{\tau}\underline{x}_k$	position vector of the mid-surface k-node at time τ;
$a\vert_k$	shell thickness at the k-node (assumed as invariant during the deformation);
$^{\tau}\underline{V}^{n}\vert_k$	director vector at the k-node at time τ $\left(\left\|{}^{\tau}\underline{V}_n\vert_k\right\| = 1\right)$;

while the natural coordinates $(r,\, s)$ are defined on the element mid-surface $(t = 0)$ the natural coordinate t is measured at any point along the corresponding director vector direction. The second term on the r.h.s. in Eq. (2.1) shows that at any point on the element mid-surface the unit director vector times the thickness is interpolated from the nodal values.

The geometry interpolation in Eq. (2.1) presents C^0 continuity.

For describing the kinematics of the A-I-Z element the two main assumptions are:

- the element thickness remains constant due to the assumed infinitesimal strains deformation;
- the director vectors remain straight during the deformation.

The covariant base vectors of the (r, s, t) system are determined deriving Eq. (2.1),

[1] We use Einstein's notation: $a_k b_k \equiv \sum_k a_k b_k$, that is to say repeated indices indicate a summation.

$$^{\tau}\underline{g}_i = \frac{\partial\, ^{\tau}\underline{x}}{\partial r_i} \tag{2.2}$$

and the contravariant base vectors need to fulfill the relation,

$$^{\tau}\underline{g}^i \cdot {}^{\tau}\underline{g}_j = \delta^i_j \tag{2.3}$$

For linear kinematics we consider the τ-configuration to be coincident with the *0*-(reference) configuration.

The displacements for a time τ are,

$$^{\tau}\underline{u} = {}^{\tau}\underline{x} - {}^{0}\underline{x}. \tag{2.4}$$

Using the above kinematic assumptions we get,

$$^{\tau}\underline{u} = h_k\, {}^{\tau}\underline{u}_k + \frac{t}{2} h_k \left[a\left({}^{\tau}\underline{V}_n - {}^{0}\underline{V}_n\right)\right]_k. \tag{2.5}$$

Let us now define in the *0*-configuration at the node k two vectors that with the nodal director vector form the ortho-normal basis $({}^{0}\underline{V}_1,\ {}^{0}\underline{V}_2,\ {}^{0}\underline{V}_n)$. We can write, for infinitesimal rotations [23, 24],

$$\begin{aligned} {}^{\tau}\underline{V}^k_n &= {}^{0}\underline{V}^k_n + {}^{\tau}\underline{\theta}_k \times {}^{0}\underline{V}^k_n \\ {}^{\tau}\underline{\theta}_k &= \alpha_k {}^{0}\underline{V}^k_1 + \beta_k {}^{0}\underline{V}^k_2 \\ {}^{\tau}\underline{V}^k_n &= {}^{0}\underline{V}^k_n + \beta_k {}^{0}\underline{V}^k_1 - \alpha_k {}^{0}\underline{V}^k_2. \end{aligned} \tag{2.6}$$

Therefore,

$$^{\tau}\underline{u} = h_k\, {}^{\tau}\underline{u}_k + \frac{t}{2} h_k \left[a\ (-\alpha\, {}^{0}\underline{V}_2 + \beta\, {}^{0}\underline{V}_1)\right]_k. \tag{2.7}$$

It is apparent from Eq. (2.7) that this element formulation introduces 5 d.o.f. per node.

At any point inside the shell we can write the infinitesimal strain tensor in terms of its covariant components ($\tilde{\varepsilon}_{lm}$) in the (r, s, t) curvilinear system and the corresponding contravariant base vectors,

$$\underline{\underline{\varepsilon}} = \widetilde{\varepsilon}_{ij}\, {}^{o}\underline{g}^i\, {}^{o}\underline{g}^j \tag{2.8}$$

where we use the notation $({}^{o}\underline{g}^i\ {}^{o}\underline{g}^j)$, to indicate the dyadic (tensorial) product between the two contravariant base vectors.[2] In Eq. (2.8), $\widetilde{\varepsilon}_{tt} = 0$ because the thickness is constant.

From the kinematic relations between strain components and displacements [24] we get,

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[2] Some authors use the notation ${}^{o}\underline{g}^i \otimes {}^{o}\underline{g}^j$.

$$\widetilde{\varepsilon}_{ij} = \frac{1}{2}\left({}^{0}\underline{g}_{i} \cdot \frac{\partial \underline{u}}{\partial r_{i}} + {}^{0}\underline{g}_{j} \cdot \frac{\partial \underline{u}}{\partial r_{i}} \right). \tag{2.9}$$

Hence, using Voigt notation we can write

$$\underline{\widetilde{\varepsilon}} = \underline{\tilde{B}}\ \underline{U} \tag{2.10}$$

where $\underline{\widetilde{\varepsilon}}$ is the (5×1) column vector formed with the non-zero curvilinear components of the strain tensor, $\underline{U}$ is the (20×1) column vector with the element nodal generalized displacements and $\underline{\tilde{B}}$ is the (5×20) strain-displacement matrix, formed using Eq. (2.9) [7].

2.2.2 Stress-Strain Relations

The assumption of zero stresses through the thickness is equivalent to consider that each surface parallel to the mid-surface is in a plane stress condition. In the A-I-Z finite element discretization, with only C^0 continuity, there are two alternative ways for imposing through the shell thickness the plane stress condition,

- imposing it to the different laminae with constant t;
- imposing it at every point to the surfaces normal to the director vector.

We have chosen the second alternative.

At each point inside the shell element we define the local Cartesian system $(\underline{\widehat{e}}_1, \underline{\widehat{e}}_2, \underline{\widehat{e}}_3)$ with,

$$\begin{aligned} \underline{\widehat{e}}_1 &= \frac{{}^{0}\underline{g}_2 \times {}^{0}\underline{g}_3}{\left\| {}^{0}\underline{g}_2 \times {}^{0}\underline{g}_3 \right\|} \\ \underline{\widehat{e}}_3 &= \frac{{}^{0}\underline{g}_3}{\left\| {}^{0}\underline{g}_3 \right\|} \\ \underline{\widehat{e}}_2 &= \underline{\widehat{e}}_3 \times \underline{\widehat{e}}_1 . \end{aligned} \tag{2.11}$$

In this local Cartesian system we formulate the different plane stress constitutive relations in the plane $(\underline{\widehat{e}}_1, \underline{\widehat{e}}_2)$.

There is an obvious contradiction since the above defined plane stress state, due to the imposed kinematic constraint, is also a plane strain state; this can only be possible in a very specific orthotropic material model. We overlook this contradiction as the price that we pay for degenerating the solid into a shell element.

The constitutive tensor can be described as,

$$\underline{\underline{\underline{\underline{C}}}} = C_{ijkl}\ \underline{\widehat{e}}_i\ \underline{\widehat{e}}_j\ \underline{\widehat{e}}_k\ \underline{\widehat{e}}_l = \widetilde{C}^{pqrs}\ {}^{o}\underline{g}_p\ {}^{o}\underline{g}_q\ {}^{o}\underline{g}_r\ {}^{o}\underline{g}_s \tag{2.12}$$

and from the above we can get the curvilinear components $\widetilde{C}^{pqrs}$.

Then the element stiffness matrix can be calculated as [7],

$$\underline{K} = \int_V \underline{\tilde{B}}^T \, \underline{\tilde{C}} \, \underline{\tilde{B}} \, dV. \tag{2.13}$$

The (5 × 5) matrix $\underline{\tilde{C}}$ collects the curvilinear components of the constitutive tensor and it is symmetric for hyperelastic materials models and elasto-plastic material models when an associated plasticity model is used [24].

2.2.3 *The Locking Problem*

The locking problem has been very much analyzed in the literature [5, 6]; in the present section we just present a couple of very simple examples to illustrate it.

2.2.3.1 Shear Locking

Using a 4-node element to model the cantilever under constant moment in Fig. 2.2 we notice that,

- the u_2 displacement interpolation is linear along the coordinate x_1 with a zero at node 1,
- the θ rotation interpolation is linear with a zero at node 1.

The shear deformation $\gamma = \frac{du_2}{dx_1} - \theta$ has to be zero everywhere and the condition is imposed more strongly when the thickness tends to zero [20, 25].

It is evident that considering the order of the interpolation functions and the boundary condition, the only solution is $u_2 = const = 0$.

Fig. 2.2 Cantilever under constant moment modeled with one 4-node element: shear locking

2.2.3.2 Shear and Membrane Locking

When using a parabolic interpolation for the generalized displacements of a curved cantilever we find the impossibility of imposing together the zero shear and zero membrane elongation conditions.

2.2.4 Solving the Locking Problem

The first remedy that was proposed for the locking problem was the use of reduced or selective integration schemes [6]; however, those schemes, even though they are very simple and produce inexpensive elements, incorporate the difficulty of the spurious rigid body modes and the oscillation in the stress predictions [5, 6].

The element MITC4 was developed by Dvorkin and Bathe as a solution for the shear locking problem that does not incorporate numerical drawbacks.

2.3 The MITC4 Quadrilateral Shell Element for Linear Analysis

This element incorporates the displacement/rotation interpolations used in the A-I-Z element; the curvilinear covariant strain components $(\widetilde{\varepsilon}_{rr},\ \widetilde{\varepsilon}_{ss},\ \widetilde{\varepsilon}_{rs})$ are directly calculated from the displacement/rotation interpolations using Eq. (2.9).

For the out-of-surface shear components we use the interpolations in Fig. 2.3, which can be written as,

$$\begin{aligned} \widetilde{\varepsilon}_{rt} &= \frac{1}{2}(1+s)\ \widetilde{\varepsilon}_{rt}\big|_A^{DI} + \frac{1}{2}(1-s)\ \widetilde{\varepsilon}_{rt}\big|_C^{DI}, \\ \widetilde{\varepsilon}_{st} &= \frac{1}{2}(1+r)\ \widetilde{\varepsilon}_{st}\big|_D^{DI} + \frac{1}{2}(1-r)\ \widetilde{\varepsilon}_{st}\big|_B^{DI}. \end{aligned} \tag{2.14}$$

In Eq. (2.14) we use the notation,

$\widetilde{\varepsilon}_{ij}\big|_P^{DI}$ covariant strain component calculated from the displacement interpolations at the *sampling point* P

The element, defined as we describe in this section, satisfies the Patch Test, does not present spurious rigid body modes and does not lock. In the literature there is abundant numerical evidence on the element robustness and accuracy.

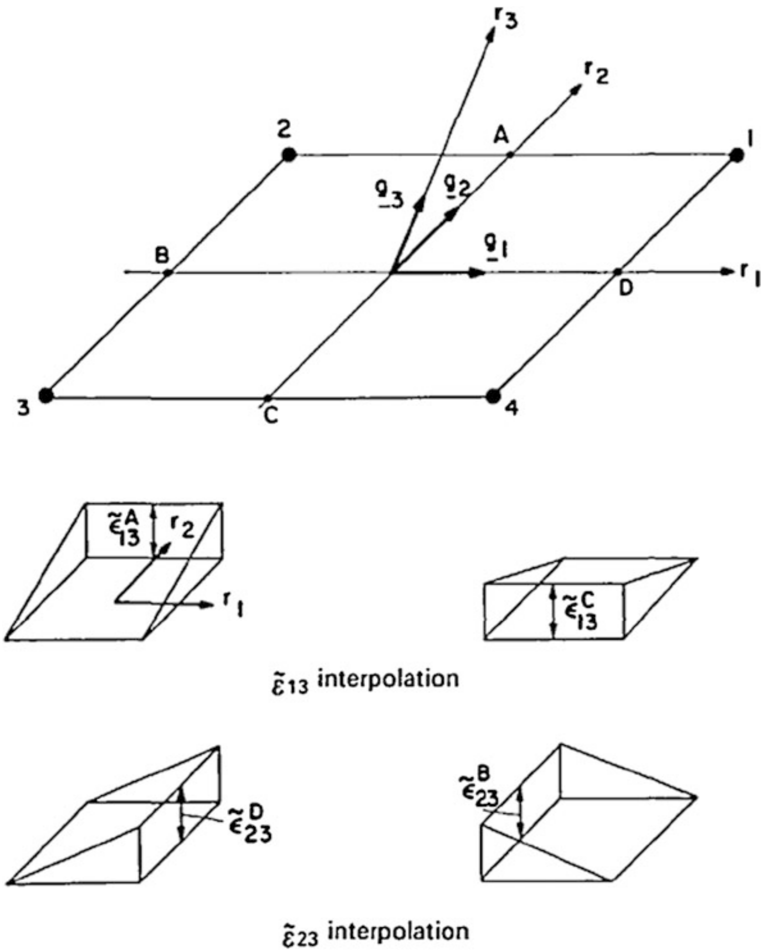

Fig. 2.3 The MITC4 out-of surface shear interpolations

2.4 Nonlinear Analysis Using the MITC4 Element

2.4.1 Infinitesimal Strains Problems: Total Lagrangean Formulation

There are a number of nonlinear structural and mechanical problems for which the infinitesimal strains approach provides acceptable results. For these cases we developed for the MITC4 element a Total Lagrangean Formulation (TLF) [5].

In an incremental analysis we know the τ-configuration and we seek the $(\tau + \Delta\tau)$-configuration. Using the Principle of Virtual Work we can state for the equilibrium at the configuration at $(\tau + \Delta\tau)$ [24],

$$\int_{{}^0\underline{V}} {}^{\tau+\Delta\tau}{}_0\underline{\underline{S}} : \delta^{\tau+\Delta\tau}{}_0\underline{\underline{\varepsilon}}\, {}^0dV = {}^{\tau+\Delta\tau}\delta W^{ext}. \tag{2.15}$$

In the above equation ${}^0\underline{V}$ is the volume of the reference configuration (*time* = *0*); and for the configuration at $(\tau + \Delta\tau)^{\tau+\Delta\tau}{}_0\underline{\underline{S}}$ is the second Piola-Kirchhoff stress tensor; ${}^{\tau+\Delta\tau}{}_0\underline{\underline{\varepsilon}}$ is the Green-Lagrange strain tensor (both tensors referred to the reference configuration) and ${}^{\tau+\Delta\tau}\delta W^{ext}$ is the virtual work of the external loads.

We can write for the incremental step,

$$\begin{aligned} {}^{\tau+\Delta\tau}{}_0\underline{\underline{S}} &= {}^\tau_0 S + {}_0\underline{\underline{S}}, \\ {}^{\tau+\Delta\tau}{}_0\underline{\underline{\varepsilon}} &= {}^\tau\ \varepsilon + {}_0\underline{\underline{\varepsilon}}. \end{aligned} \tag{2.16}$$

Where the tensors ${}_0\underline{\underline{S}}$; ${}_0\underline{\underline{\varepsilon}}$ are increments referred to (*time* = *0)*.

We decompose the strain increment into a linear (${}_0\underline{\underline{e}}$) and nonlinear part (${}_0\underline{\underline{\eta}}$) in terms of the incremental displacements and relate the incremental strain tensor with the incremental stress tensor using and incremental fourth order constitutive tensor ${}_0\underline{\underline{\underline{\underline{C}}}}$.

After linearizing we get for the incremental step [24],

$$\int_{{}^0\underline{V}} \delta_0\underline{\underline{e}} :\ {}_0\underline{\underline{\underline{\underline{C}}}} :\ {}_0\underline{\underline{e}}\, {}^0dV + \int_{{}^0\underline{V}} {}^\tau_0\underline{\underline{S}} :\ \delta_0\underline{\underline{\eta}}\, {}^0dV = {}^{\tau+\Delta\tau}\delta W^{ext} - \int_{{}^o\underline{V}} {}^\tau_0\underline{\underline{S}} :\ \delta_0\underline{\underline{e}}\, {}^0dV. \tag{2.17}$$

As it is well known, the $(\tau + \Delta\tau)$-configuration is determined iterating on Eq. (2.15) until equilibrium is fulfilled [5].

In the nonlinear problem, we need to handle finite rotations; hence, we can write [23],

$$ {}^{t+\Delta\tau}\underline{V}_n^k = {}^{\tau+\Delta\tau}_0\underline{\underline{R}} \cdot {}^0\underline{V}_n^k. \tag{2.18}$$

Any rotation matrix can be written as [23],

$$ {}^\tau_0\underline{\underline{R}} = \underline{\underline{I}}_3 + \frac{\sin\theta_k}{\theta_k}\underline{\underline{\Theta}}_k + \frac{1}{2}\left(\frac{\sin\left(\theta_k/2\right)}{\left(\theta_k/2\right)}\right)^2 \underline{\underline{\Theta}}_k^2 \tag{2.19}$$

where,

$$\begin{aligned} \theta_k &= \left(\alpha_k^2 + \beta_k^2\right)^{1/2} \\ [\Theta_k] &= \begin{bmatrix} 0 & 0 & \beta_k \\ 0 & 0 & -\alpha_k \\ -\beta_k & \alpha_k & 0 \end{bmatrix}. \end{aligned} \tag{2.20}$$

As in the infinitesimal rotations case, we have 5 d.o.f./node.

In the case of finite rotations the linearized equilibrium equations present extra terms that were discussed in Refs. [20, 23].

2.4.2 Finite Strains

For the analysis of finite strain problems we use the following interpolation for the reference configuration geometry [21, 22],

$$ {}^{0}\underline{x}(r,s,t) = h_k(r,s)\ {}^{0}\underline{x}_k + \frac{t}{2}\,{}^{0}\underline{d}a \tag{2.21}$$

where,

$$ {}^{0}\underline{d} = \frac{h_k(r,\ s)\ {}^{0}\underline{V}^n\big|_k}{\left\| h_k(r,\ s)\ {}^{0}\underline{V}^n\big|_k \right\|}. \tag{2.22}$$

It is important to remark that we considered in Eq. (2.21) elements with uniform thickness.

For the displacement field we use,

$$ {}^{\tau}u(r,\ s,\ t) = h_k(r,\ s)\ {}^{\tau}\underline{u}_k + \frac{t}{2}\left({}^{\tau}\lambda_o + {}^{\tau}\lambda_1 t\right)\left({}^{\tau}\underline{d} - {}^{0}\underline{d}\right)a \tag{2.23}$$

where,

$$ {}^{\tau}\underline{d} = \frac{h_k(r,\ s)^{\tau}\underline{V}^n|_k}{\left\| h_k(r,\ s)^{\tau}\underline{V}^n|_k \right\|}. \tag{2.24}$$

Equations (2.22) and (2.24) are used to avoid spurious director vector stretching [26] that in this finite strains case, in which we do not neglect the stretching through the thickness, may affect the results.

In Eq. (2.23) ${}^{\tau}\lambda_o$ is the constant thickness stretching and ${}^{\tau}\lambda_1$ is the through-the-thickness stretching gradient. In our formulation both stretching d.o.f. are condensed at the element level.

The strain interpolations are the same as the ones we used in the infinitesimal strains case. However, for hyperelastic material models we interpolate the Green-Lagrange covariant strain tensor components and for the elasto-plastic material we interpolate the covariant components of the Hencky (logarithmic) strain tensor [24].

We use 3D constitutive relations; hence, the through-the-thickness stress component is not neglected. In [22] we developed the 3D constitutive equations for the elasto-plastic element based on Lee's multiplicative decomposition of the deformation gradient tensor and maximum energy dissipation [24, 27].

2.5 Modeling Considerations

In this section we discuss several considerations that need to be taken into account when modeling shell structures.

2.5.1 The Nodal Director Vectors

The nodal director vectors may be either defined by the analyst or calculated by the finite element code.

When the analyst introduces the director vectors together with the mid-surface nodes, she/he selects them at each node so as to be the best approximation to the actual normal to the shell mid-surface.

When the finite element code calculates at a given node the director vectors, the normals to the interpolated mid-surface are calculated for all the elements sharing the node; hence, the code defines at the node as many director vectors as the number of elements sharing the node (the interpolated mid-surfaces only have C^0 continuity). All the element normals sharing a node rotate together.

2.5.2 Number of d.o.f. per Node

As discussed above, in a shell model there may be nodes at which only one director vector is defined and nodes at which multiple director vectors are defined.

For the case of nodes with only one director vector, the analyst has to consider 5 d.o.f. (3 displacements and 2 rotations around the local axes ${}^{\tau}\underline{V}_1$ and ${}^{\tau}\underline{V}_2$).

For the case of nodes with multiple director vectors, the analyst has to consider 6 d.o.f. (3 displacements and 3 rotations around the global Cartesian axes).

The case of a node with multiple but very close director vectors has to be treated as a case with 5 d.o.f. collapsing the very close director vectors.

It is important to be aware of the fact that when using 6 d.o.f. the rotational boundary conditions need to be defined along the global Cartesian axes and when using 5 d.o.f. the rotational boundary conditions need to be defined along the local axes. In geometrically nonlinear analyses these local axes change for each incremental step.

When modeling a stiffened shell using shell elements and iso-beam (Timoshenko beam) elements [5],[3] at the nodes shared by a shell and a beam element 6 d.o.f. have to be used.

[3] Please notice that shell elements are not compatible with Bernoulli beam elements.

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