Chapter 6
Varieties

1 Definitions and Examples

1.1 Definitions

In this chapter we consider the schemes most closely related to projective varieties; they will be called algebraic varieties. This is exactly what we arrive at on attempting to give an intrinsic definition of algebraic variety.

Definition A *variety* over an algebraically closed field k is a reduced separated scheme of finite type over k. A *morphism* of varieties is a morphism of schemes over k. A variety X that is an affine scheme is called an *affine variety*.

We saw in Example 5.19 that every quasiprojective variety defines a scheme. This scheme is a variety, that we will also call quasiprojective.

By definition, any variety X has a finite cover $X = \bigcup U_i$, where the U_i are affine varieties. It follows from this that X is finite dimensional. If X is irreducible then all the U_i are dense in X and $\dim X = \dim U_i$. Moreover, they are all birational, since $U_i \cap U_j$ is open and dense in both U_i and U_j. Hence the function fields $k(U_i)$ are isomorphic; these fields can be identified. The resulting field is called the *function field* of X and denoted $k(X)$. The dimension of X equals the transcendence degree of $k(X)$.

A closed point of a variety X that is contained in an affine open set U is also a closed point of U, and is a point of the corresponding affine variety with coordinates in k. There are sufficiently many such points on X.

Proposition *Closed points are dense in every closed subset of X.*

Proof We note first that in an affine variety (and even in an affine scheme), every nonempty closed subset contains a closed point. Indeed, a nonempty closed subset Z of Spec A is of the form Spec B, where B is a quotient ring of A. Since every ring has a maximal ideal, Z has a closed point.

I.R. Shafarevich, *Basic Algebraic Geometry 2*, DOI 10.1007/978-3-642-38010-5_2,

If X is an arbitrary variety, $Z \subset X$ a closed subset and $z \in Z$, then it is enough to prove that $Z \cap U$ contains a closed point for any neighbourhood U of z. We can restrict to affine neighbourhoods U, since these form a basis of all open sets. For affine U, by what we have just said, $Z \cap U$ has a closed point.

But there is a trap here for the unwary—a point may be closed in U, but not in X. This actually happens, for example, in the case of the subset $U = \operatorname{Spec} \mathcal{O} \setminus \{x\}$ where $\mathcal{O}$ is a local ring of a closed point x of a curve. Fortunately, everything turns out to be all right in the case of a variety: if $z \in X$ is a closed point of some neighbourhood U of z then it is also closed in X. This follows from the fact that the closed points x of a variety are characterised by $k(x) = k$. Indeed, a point x is closed in X if and only if it is closed in all affine open sets containing it, and for affine varieties the condition $k(x) = k$ obviously characterises closed points. The field $k(x)$ depends only on the local ring of x, and hence does not change on passing from X to an open subset $U \ni x$. The proposition is proved. □

Since a variety is a reduced scheme, an element $f \in \mathcal{O}_X(U)$ is uniquely determined by its values $f(x) \in k(x)$ at all $x \in U$. By the proposition, it is determined by its values at closed points. Moreover $k = k(x)$, so that an element $f \in \mathcal{O}_X(U)$ can be interpreted as a k-valued function on the set of closed points of U.

If $\varphi \colon X \to Y$ is a morphism of varieties, $x \in X$ and $y = \varphi(x)$, then the homomorphism of local rings $\varphi^* \colon \mathcal{O}_y \to \mathcal{O}_x$ induces an inclusion of residue fields $k(y) \hookrightarrow k(x)$. If x is a closed point then $k(x) = k$, and hence also $k(y) = k$, that is, y is also closed. Therefore the image of a closed point is again closed. Thus interpreting elements $f \in \mathcal{O}_Y(U)$ as functions on closed points, the homomorphism $\psi_U \colon \mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))$ is determined by $\psi_U(f)(x) = f(\varphi(x))$. In other words, by specifying the map $\varphi \colon X \to Y$, or even its restriction to the set of closed points, we determine the morphism itself.

A variety X has of course any number of nonreduced closed subschemes. But any closed subset $Z \subset X$ can be made into a reduced scheme, or as we will say from now on, into a *closed subvariety*. If X is an affine variety, $X = \operatorname{Spec} A$ and $Z = V(\mathfrak{a})$ then we set $Z = \operatorname{Spec} A/\mathfrak{a}'$ where $\mathfrak{a}'$ is the radical of $\mathfrak{a}$, the ideal of elements $a \in A$ such that $a^r \in \mathfrak{a}$ for some r. The general case is obtained by glueing.

All this shows how close varieties are to quasiprojective varieties. Indeed, all the local notions and properties treated in Chapter 2 carry over word-for-word for algebraic varieties: nonsingular points, the theorem that the set of singular points is closed, properties of normal varieties. The same is true for properties of divisors and differential forms.

The only properties not carrying over in an obvious way to algebraic varieties are those related to the property of being projective. We now explain what condition replaces projective for the case of arbitrary varieties. Projectivity is of course very far from being an "abstract" property. But we have at our disposal one assertion, Theorem 1.11 of Section 5.2, Chapter 1, which is an intrinsic property that is characteristic of projective varieties. We take this as a definition.

Definition A variety X is *complete* if for any variety Y, the projection morphism $p \colon X \times Y \to Y$ takes closed sets to closed sets.

The main properties of projective varieties, for example, the fact that the image of a morphism is closed, and the fact that there are no everywhere regular functions except the constants (that is, $\mathcal{O}_X(X) = k$), were deduced in Section 5.2, Chapter 1 from Theorem 1.11, and therefore hold for complete varieties. Note that the proof that the image of a morphism is closed used the fact that a morphism has a closed graph. As we saw in Section 4.3, Chapter 5, this follows from the separated assumption on a variety.

Of all the properties of projective varieties proved in Chapters 1–4, there is only one that used projectivity directly, rather than via an appeal to Theorem 1.11 of Section 5.2, Chapter 1. This is the extremely important result Theorem 2.10 of Section 3.1, Chapter 2. Here we prove a generalisation of it to arbitrary complete varieties.

Theorem 6.1 *If X is a nonsingular irreducible variety and $\varphi\colon X \to Y$ a rational map to a complete variety Y, the locus of indeterminacy of φ has codimension ≥ 2.*

Proof Let $V \subset X$ be the set of points at which φ is defined, $\Gamma_\varphi \subset V \times Y$ the graph of the morphism $\varphi\colon V \to Y$ and Z its closure in $X \times Y$. The image of Z under the projection $p\colon X \times Y \to X$ is closed, since Y is complete. Since $p(Z) \supset V$, it follows that $p(Z) = X$. The restriction $p\colon Z \to p(Z)$ is a birational morphism, since it is an isomorphism of Γ_φ and V. The theorem thus follows from the next result. □

Lemma *If $p\colon Z \to X$ is a surjective birational morphism with X a nonsingular variety then the set of points of indeterminacy of the inverse rational map p^{-1} has codimension ≥ 2.*

Indeed, $\varphi = q \circ p^{-1}$, where q is the restriction to Z of the projection $p\colon X \times Y \to X$. Therefore φ is defined wherever p^{-1} is. This proves Theorem 6.1.

Proof of the Lemma Suppose that there exists a codimension 1 subvariety $T \subset X$ such that p^{-1} is not defined at any point of T. Replacing Z, X and T by affine open subsets, we can assume that they are affine and $T \subset p(Z) \subset X$. Let $Z \subset \mathbb{A}^m$, and write $u_1, \dots, u_m$ for the coordinates of $\mathbb{A}^m$ as elements of $\mathcal{O}_Z(Z)$. Consider a point $t \in T$ and represent the rational functions $(p^{-1})^*(u_i)$ in the form

$$\left(p^{-1}\right)^*(u_i) = g_i/h,$$

where $g_1, \dots, g_m, h \in \mathcal{O}_t$ and have no common factor. Then

$$h\left(p^{-1}\right)^*(u_i) = g_i, \quad \text{so that} \quad p^*(h)u_i = p^*(g_i).$$

Hence $g_i(\tau) = 0$ for every point $\tau \in T$ at which $h(\tau) = 0$, and this contradicts the assumption that $g_1, \dots, g_m, h \in \mathcal{O}_t$ have no common factor. The lemma is proved. □

The complete varieties just introduced turn out to have properties so close to those of quasiprojective varieties that the question arises as to whether the two notions might not coincide. We will see a little later in Section 2.3 that this is not the case; there exist varieties that cannot be embedded in any projective space. However, what is much more important is that the intrinsic, invariant nature of the notion of variety makes it into a much more flexible tool. Many constructions can be performed very simply and naturally within the framework of this notion. It may sometimes turn out a posteori that we have not actually left the framework of projective or quasiprojective varieties, but this is often already of secondary importance. In Sections 1.2–1.4 we give some important examples of this kind of constructions.

A very simple example is provided by the definition of the product of varieties. The definition in the framework of varieties is extremely simple: the arguments of Section 4.1, Chapter 5 simplify substantially if we use the fact that the set of closed points of the variety $X \times Y$ is the set of pairs of the form (x, y), where $x \in X$ and $y \in Y$ are closed points (see Exercises 1 and 2). But we spent quite a lot of effort on this definition in Section 5, Chapter 1, since there we needed to be sure that the product of quasiprojective varieties was again a quasiprojective variety.

Another example that we now consider is the notion of normalisation of a variety. Let X be an irreducible variety, K a finite field extension of the function field $k(X)$. We show that there exist a normal irreducible variety X_K^ν and a morphism $\nu_K \colon X_K^\nu \to X$ with the properties that $k(X_K^\nu) = K$ and the induced map $\nu_K^* \colon k(X) \to k(X_K^\nu) = K$ is the given field extension. Such a variety is unique: for any two normalisations X_K^ν and $\widetilde{X_K^\nu}$ there exists an isomorphism $f \colon X_K^\nu \to \widetilde{X_K^\nu}$ such that the diagram

$$\begin{tikzcd} X_K^\nu \arrow[rr, "f"] \arrow[dr, "\nu_K"'] & & \widetilde{X_K^\nu} \arrow[dl, "\widetilde{\nu_K}"] \\ & X & \end{tikzcd}$$

is commutative. X_K^ν is called the *normalisation* of X in K. The uniqueness of the normalisation X_K^ν is proved exactly as in Section 5.2, Chapter 2, where we considered the case $K = k(X)$. To prove the existence, consider an affine cover $X = \bigcup U_i$. The integral closure A_i^ν of $k[U_i]$ in K is a finitely generated algebra over k, as we saw in Sect 5.2, Chapter 2. Hence the normalisation $U_{i,K}^\nu \to U_i$ in K of the affine variety U_i exists and is affine. From the uniqueness of normalisation it follows that $\nu_{i,K}^{-1}(U_i \cap U_j)$ and $\nu_{j,K}^{-1}(U_j \cap U_i)$ are isomorphic. This allows us to glue the $U_{i,K}^\nu$ together into a single scheme X_K^ν, which is obviously a reduced irreducible scheme of finite type over k.

We prove that X_K^ν is separated (Section 4.3, Chapter 5). It is enough to prove that the diagonal in $X_K^\nu \times X_K^\nu$ is closed, and for this it is enough to show that it is closed in the neighbourhood of any point $\xi \in X_K^\nu \times X_K^\nu$. Suppose that the morphism $\nu \times \nu \colon X_K^\nu \times X_K^\nu \to X \times X$ takes ξ into $\eta \in X \times X$, and let U' be an affine neighbourhood of η such that $(\nu \times \nu)^{-1}(U') = V'$ is affine. The existence of U' follows from the existence of the normalisation in the affine case. Since X is a separated scheme, if we write $\Delta \subset X \times X$ for the diagonal then the scheme $U = \Delta \cap U'$ is closed in U', and hence is affine. It follows that the scheme $(\nu \times \nu)^{-1}(U)$ is affine, and hence also its irreducible component V containing ξ. Write $\delta^\nu \colon X_K^\nu \to X_K^\nu \times X_K^\nu$ and $\delta \colon X \to X \times X$ for the diagonal morphisms, and set $W = (\delta^\nu)^{-1}(V) = \nu^{-1}(U)$. We obtain the commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\delta^\nu} & V \\ & {\scriptstyle \delta\circ\nu}\searrow \quad \swarrow{\scriptstyle \nu\times\nu} & \\ & U & \end{array}$$

in which δ^ν corresponds to a finite regular map of affine varieties. This holds a fortiori for the morphism $\delta \circ \nu$ (because a finite module over a ring is a fortiori finite over a bigger ring). Applying Theorem 1.13 of Section 5.3, Chapter 1, we get that $\delta^\nu(W) = V$, which means that the diagonal is closed in the neighbourhood V' of ξ.

Thus the scheme X_K^ν is an irreducible variety, and a trivial verification shows that it is the required normalisation.

We see that in the framework of arbitrary varieties, the construction of the normalisation is quite trivial. It remains to consider the question of whether the normalisation of a quasiprojective variety is again quasiprojective. This is true, but we do not give the proof, which is based, naturally enough, on purely projective considerations; it can be found, for example, in Lang [55, Proposition 4 of Section 4, Chapter V] or Hartshorne [37, Ex. 5.7 of Chapter III]. In the case of curves, we can repeat the proofs of Theorems 2.22–2.23 of Section 5.3, Chapter 2. These results imply that the normalisation of any curve is quasiprojective (in the case $K = k(X)$), and that of a complete curve is projective. In particular, it follows from this that a nonsingular curve is quasiprojective. In fact this is true for any curve, but the proof is more complicated, and we omit it here.

1.2 Vector Bundles

The idea of a vector bundle is one of the most important constructions of algebraic varieties, and is typically "abstract" or "nonprojective" in nature. We recall that the general notion of *fibration* is nothing other than a morphism of varieties $p\colon X \to S$, that is, a variety over S. We are interested in fibrations whose fibres are vector spaces. In formulating this notion we must bear in mind that an n-dimensional vector space over a field k has a natural structure of algebraic variety isomorphic to $\mathbb{A}^n$.

Definition A *family of vector spaces* over X is a fibration $p\colon E \to X$ such that each fibre $E_x = p^{-1}(x)$ for $x \in X$ is a vector space over $k(x)$, and the structure of algebraic variety of E_x as a vector space coincides with that of $E_x \subset E$ as the inverse image of x under p.

A *morphism* of a family of vector space $p\colon E \to X$ into another family $q\colon F \to X$ is a morphism $f\colon E \to F$ for which the diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & X & \end{array}$$

commutes (so that in particular f maps E_x to F_x for all $x \in X$), and the map $f_x\colon E_x \to F_x$ is linear over $k(x)$. It's obvious how to define an isomorphism of families.

The simplest example of a family is the direct product $E = X \times V$, where V is a vector space over k, and p the first projection of $X \times V \to X$. A family of this type, or isomorphic to it, is said to be *trivial.*

Example 6.1 Let V and W be two vector spaces of dimension m and n. We determine the general form of a morphism $f: X \times V \to X \times W$ between two trivial families. We let $e_1, \dots, e_m$ and $u_1, \dots, u_n$ be bases of V and W, and write $\xi_1, \dots, \xi_m$ and $\eta_1, \dots, \eta_n$ for the corresponding coordinates. The projections $p: X \times V \to V$ and $q: X \times W \to W$ define elements $x_i = p^*(\xi_i) \in \mathcal{O}_{X\times V}(X \times V)$ and $y_j = q^*(\eta_i) \in \mathcal{O}_{X\times W}(X \times W)$. Obviously, closed points $\alpha \in X \times V$ and $\beta \in X \times W$ are uniquely determined by the values of $x_i(\alpha)$ and $y_j(\beta) \in k$. Therefore the morphism f is uniquely determined by specifying the elements $f^*(y_j) \in \mathcal{O}_{X\times V}(X \times V)$.

The composite of the isomorphism $X \to X \times e_i$ and the embedding $X \times e_i \to X \times V$ defines a morphism $\varphi_i: X \to X \times V$. Set $a_{ij} = \varphi_i^*(f^*(y_j)) \in \mathcal{O}_X(X)$. Then

$$f^*(y_j) = \sum a_{ij} x_i. \tag{6.1}$$

Indeed, it is enough to check this equality at all closed points $\alpha \in X \times V$, and there it follows at once from the definition of morphism of family of vector spaces (because $f_x: E_x \to F_x$ is linear).

Conversely, any matrix (a_{ij}) with $a_{ij} \in \mathcal{O}_X(X)$ defines a morphism $f: X \times V \to X \times W$ by means of formula (6.1). Obviously we get an isomorphism if and only if $m = n$ and the determinant $\det|a_{ij}|$ is an invertible element of $\mathcal{O}_X(X)$.

If $p: E \to X$ is a family of vector spaces and $U \subset X$ any open set, the fibration $p^{-1}(U) \to U$ is a family of vector spaces over U. It is called the *restriction* of E to U and denoted $E_{|U}$.

Definition A family of vector spaces $p: E \to X$ is a *vector bundle* if every point $x \in X$ has a neighbourhood U such that the restriction $E_{|U}$ is trivial.

The dimension of the fibre E_x of a vector bundle is obviously a locally constant function on X, and, in particular, is constant if X is connected. In this case the number $\dim E_x$ is called the *rank* of E, and denoted by $\operatorname{rank} E$.

Example 6.2 Let V be an $(n+1)$-dimensional vector space and $\mathbb{P}^n$ the vector space of lines $l \subset V$ through 0. Write l_x for the line corresponding to a point $x \in \mathbb{P}^n$. Consider the subset $E \subset \mathbb{P}^n \times V$ of pairs (x, v) such that $x \in \mathbb{P}^n$ and $v \in V$ are closed points, with $v \in l_x$. Obviously E is the set of closed points of some quasiprojective subvariety of $\mathbb{P}^n \times V$, which we continue to denote by E. The projection $\mathbb{P}^n \times V \to \mathbb{P}^n$ defines a morphism $p: E \to \mathbb{P}^n$. We prove that $p: E \to \mathbb{P}^n$ is a vector bundle. In V, we introduce a coordinate system $(x_0, \dots, x_n)$. The restriction of E to the open set U_α given by $x_\alpha \neq 0$ consists of points

$$\xi = (t_1, \dots, t_n; y_0, \dots, y_n) \quad \text{such that} \quad y_i = t_i y_\alpha,$$

where $t_i = x_i/x_\alpha$, and the map $\xi \mapsto ((t_1, \dots, t_n), y_\alpha)$ defines an isomorphism of $E_{|U_\alpha}$ with $U_\alpha \times k$.

Therefore E is a vector bundle of rank 1. The projection $\mathbb{P}^n \times V \to V$ defines a morphism $q\colon E \to V$. The reader can easily check that q coincides with the blowup of the origin $0 = (0, \dots, 0) \in V$, and $q^{-1}(0) = \mathbb{P}^n \times 0$.

Consider a vector bundle $p\colon E \to X$ and a morphism $f\colon X' \to X$. The fibre product $E' = E \times_X X'$ over X has a morphism $p'\colon E' \to X'$. This morphism defines a vector bundle. Indeed, if $E_{|U} \cong U \times V$ with $U \subset X$ then writing $U' = f^{-1}(U)$, we get $E'_{|U'} = E \times_U U' \cong U' \times V$. This vector bundle is called the pullback of E, and denoted by $f^*(E)$. Obviously rank $f^*(E) = \operatorname{rank} E$.

Example 6.3 Let X be a projective variety and $f\colon X \hookrightarrow \mathbb{P}^1$ a closed embedding to projective space. Let $p\colon E \to \mathbb{P}^n$ be the vector bundle of Example 6.2. Then $f^*(E)$ is a vector bundle over X of rank 1. It depends in general on the embedding f, and is a very important invariant of f.

Example 6.4 Let $X = \operatorname{Grass}(r, n)$ be the Grassmannian of r-dimensional vector subspaces of an n-dimensional vector space with basis $e_1, \dots, e_n$ (Example 1.24 of Section 4.1, Chapter 1). Consider in $X \times V$ the subvariety E consisting of points (x, v) such that $v \in L_x$, where we write L_x for the r-dimensional vector subspace corresponding to $x \in \operatorname{Grass}(r, n)$. Obviously the projection $p\colon X \times V \to X$ gives E the structure of a family of vector spaces. Let us prove that it is locally trivial. Consider the open subset $U_{k_1 \dots k_r} \subset \operatorname{Grass}(r, n)$ defined by $p_{k_1 \dots k_r} \neq 0$; then for $x \in U_{k_1 \dots k_r}$, the vector subspace $L_x = p^{-1}(x)$ has a basis

$$\Big\{ e_i - \sum_{j \neq k_1 \dots k_r} a_{ij} e_j \Big\} \quad \text{where } a_{ij} = \frac{p_{k_1 \dots \widehat{k_i} j \dots k_r}}{p_{k_1 \dots k_r}}.$$

This determines an isomorphism $p^{-1}(U_{k_1 \dots k_r}) \to U_{k_1 \dots k_r} \times L$, where $L = k^r$.

Since a vector bundle is locally trivial, it can be obtained by glueing together trivial bundles over a number of open sets. This leads to an effective method of constructing vector bundles.

Let $X = \bigcup U_\alpha$ be a cover such that the bundle $p\colon E \to X$ is trivial on each U_α. For each U_α, we fix an isomorphism

$$\varphi_\alpha\colon p^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times V.$$

Over the intersection $U_\alpha \cap U_\beta$ we have two isomorphisms of $p^{-1}(U_\alpha \cap U_\beta)$ with $(U_\alpha \cap U_\beta) \times V$, namely $\varphi_{\alpha|p^{-1}(U_\alpha \cap U_\beta)}$ and $\varphi_{\beta|p^{-1}(U_\alpha \cap U_\beta)}$. Hence $\varphi_\beta \circ \varphi_\alpha^{-1}$ defines an automorphism of the trivial vector bundle $(U_\alpha \cap U_\beta) \times V$ over $U_\alpha \cap U_\beta$.

We now use the result of Example 6.1. We choose a basis of V, and write the automorphism $\varphi_\beta \circ \varphi_\alpha^{-1}$ as an $n \times n$ matrix $C_{\alpha\beta} = (a_{ij})_{\alpha\beta}$ with entries in the ring $\mathcal{O}_X(U_\alpha \cap U_\beta)$. These matrixes obviously satisfy the glueing conditions

$$\begin{gathered} C_{\alpha\alpha} = \mathrm{id}, \quad \text{and} \\ C_{\alpha\gamma} = C_{\alpha\beta} C_{\beta\gamma} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma. \end{gathered} \tag{6.2}$$

Conversely, specifying matrixes $C_{\alpha\beta}$ with entries in $\mathcal{O}_X(U_\alpha \cap U_\beta)$ defines a vector bundle, provided the $C_{\alpha\beta}$ satisfy (6.2).

The matrixes $C_{\alpha\beta}$ are called *transition matrixes* of the vector bundle. For example, if $\mathcal{L}$ is the rank 1 vector bundle over $\mathbb{P}^n$ introduced in Example 6.2, the maps φ_α are of the form $\varphi_\alpha(x, y) = (x, y_\alpha)$, so that the transition matrix $C_{\alpha\beta}$ is the 1×1 matrix $x_\beta x_\alpha^{-1}$.

It is easy to determine how the matrixes $C_{\alpha\beta}$ depend on the choice of the isomorphisms φ_α. Any other isomorphism φ'_α is of the form $\varphi'_\alpha = f_\alpha \varphi_\alpha$ where f_α is an automorphism of the trivial bundle $U_\alpha \times V$. By Example 6.1 again, f_α can be expressed as a matrix B_α with entries in $\mathcal{O}_X(U_\alpha)$ having an inverse matrix of the same form. We thus arrive at new matrixes

$$C'_{\alpha\beta} = B_\beta C_{\alpha\beta} B_\alpha^{-1}.$$

Conversely making any such change of the matrixes $C_{\alpha\beta}$ leads to an isomorphic vector bundle.

1.3 Vector Bundles and Sheaves

A vector bundle is a generalisation of a vector space. We now introduce the analogue of a point of a vector space.

Definition A *section* of a vector bundle $p \colon E \to X$ is a morphism $s \colon X \to E$ such that $p \circ s = 1$ on X.

In particular $s(x) = 0_x$ (the zero vector in E_x) is a section, called the *zero section* of E. The set of sections of a vector bundle E is written $\mathcal{L}(E)$.

Example 6.5 A section f of the trivial rank 1 bundle $X \times k$ is simply a morphism of X to $\mathbb{A}^1$, that is, an element $f \in \mathcal{O}_X(X)$. Thus $\mathcal{L}(X \times k) = \mathcal{O}_X(X)$. In particular $\mathcal{L}(\mathbb{P}^n \times k) = k$; similarly, $\mathcal{L}(\mathbb{P}^n \times V) = V$.

Consider the vector bundle E of Example 6.2. Every section $s \colon \mathbb{P}^n \to E$ determines, in particular, a section $s \colon \mathbb{P}^n \to \mathbb{P}^n \times V$, and hence by Corollary 1.2 of Section 5.2, Chapter 1 is of the form $s(x) = (x, v)$ for some fixed $v \in V$. But since $s(x) \in E$, it follows that $v \in l_x$ for every $x \in \mathbb{P}^n$, and hence $v = 0$. Thus $\mathcal{L}(E) = 0$. This proves in particular that E is not a trivial bundle.

In terms of transition functions, a section s is given by sending each set U_α to a vector $s_\alpha = (f_{\alpha,1}, \dots, f_{\alpha,n})$ with $f_{\alpha,i} \in \mathcal{O}_X(U_\alpha)$, such that $s_\beta = C_{\alpha\beta}s_\alpha$ over $U_\alpha \cap U_\beta$.

It is easy to check from the definition of vector bundle that if s_1 and s_2 are sections of E then there exists a section $s_1 + s_2$ such that

$$(s_1 + s_2)(x) = s_1(x) + s_2(x)$$

for any point $x \in X$. The sum on the right-hand side is meaningful, since $s_1(x)$ and $s_2(x) \in E_x$, and E_x is a vector space. In a similar way the equality

$$(fs)(x) = f(x)s(x)$$

determines a multiplication of a section s by an element $f \in \mathcal{O}_X(X)$.

Thus the set $\mathcal{L}(E)$ is a module over the ring $\mathcal{O}_X(X)$. We associate with any open set $U \subset X$ the set $\mathcal{L}(E, U)$ of sections of the bundle E restricted to U. An obvious check shows that we obtain a sheaf. We denote it by $\mathcal{L}_E$; it is a sheaf of Abelian groups, but has an extra structure, which we now define in a general form.

Definition Let X be a topological space, and suppose given on X a sheaf of rings $\mathcal{G}$, a sheaf of Abelian groups $\mathcal{F}$, and in addition, for each $U \subset X$, a $\mathcal{G}(U)$-module structure on $\mathcal{F}(U)$. In this situation we say that $\mathcal{F}$ is *sheaf of $\mathcal{G}$-modules* if the multiplication map $\mathcal{F}(U) \otimes \mathcal{G}(U) \to \mathcal{F}(U)$ is compatible with the restriction homomorphisms ρ_U^V; that is, the diagram

$$\begin{tikzcd}
\mathcal{F}(V) \otimes \mathcal{G}(V) \arrow[r] \arrow[d, "\rho^V_{U,\mathcal{F}} \otimes \rho^V_{U,\mathcal{G}}"'] & \mathcal{F}(V) \arrow[d, "\rho^V_{U,\mathcal{F}}"] \\
\mathcal{F}(U) \otimes \mathcal{G}(U) \arrow[r] & sF(U)
\end{tikzcd}$$

is commutative for each $U \subset V$. Under these circumstances, each stalk $\mathcal{F}_x$ of $\mathcal{F}$ is a module over the stalk $\mathcal{G}_x$ of $\mathcal{G}$.

A *homomorphism* $\mathcal{F} \to \mathcal{F}'$ of sheaves of $\mathcal{G}$-modules is a system of homomorphisms $\varphi_U : \mathcal{F}(U) \to \mathcal{F}'(U)$ of $\mathcal{G}(U)$-modules such that the diagram

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r, "\varphi_V"] \arrow[d, "\rho^V_{U,\mathcal{F}}"'] & \mathcal{F}'(V) \arrow[d, "\rho^V_{U,\mathcal{F}'}"] \\
\mathcal{F}(U) \arrow[r, "\varphi_U"'] & \mathcal{F}'(U)
\end{tikzcd}$$

is commutative for all $U \subset V$.

Obviously the sheaf $\mathcal{L}_E$ corresponding to a vector bundle is a sheaf of modules over the structure sheaf $\mathcal{O}_X$.

Every operation on modules that can be defined intrinsically can be carried over to sheaves of modules. In particular, for any modules over a ring A, the operations

$$M \oplus M', \quad M \otimes_A M', \quad M^* = \operatorname{Hom}(M, A), \quad \bigwedge\nolimits_A^p M$$

are defined. Applying these to the modules $\mathcal{F}(U)$ and $\mathcal{F}'(U)$ over the ring $\mathcal{G}(U)$, we arrive at sheaves $\mathcal{F} \oplus \mathcal{F}'$, $\mathcal{F} \otimes_{\mathcal{G}} \mathcal{F}'$, $\mathcal{F}^*$ and $\bigwedge_{\mathcal{G}}^p \mathcal{F}$, that we call the *direct sum, tensor product, dual sheaf and exterior power.*

The sheaf of a trivial bundle of rank n is determined by $\mathcal{L}_E(U) = \mathcal{O}_X(U)^n$; that is, $\mathcal{L}_E$ is the direct sum of n copies of $\mathcal{O}_X$. This sheaf is called the *free sheaf* of rank n. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. If every point has a neighbourhood U such that $\mathcal{F}_{|U}$ is free and of finite rank then we say that $\mathcal{F}$ is a *locally free sheaf* of finite rank. If $\mathcal{F}$ is a locally free sheaf then obviously every stalk $\mathcal{F}_x$ is a free $\mathcal{O}_x$-module. The sheaf $\mathcal{L}_E$ corresponding to any vector bundle E is locally free of finite rank, since E is locally isomorphic to a trivial bundle.

Theorem 6.2 *The correspondence $E \mapsto \mathcal{L}_E$ establishes a one-to-one correspondence between vector bundles and locally free sheaves of finite rank* (*here objects of either type are considered up to isomorphism*).

Proof We show how to recover a vector bundle from a locally free sheaf $\mathcal{F}$. We can obviously assume that X is connected. Suppose that $X = \bigcup U_\alpha$ is a cover such that $\mathcal{F}_{|U_\alpha}$ is a free sheaf, and let $\varphi_\alpha \colon \mathcal{F}_{|U_\alpha} \xrightarrow{\sim} \mathcal{O}_{U_\alpha}^{n_\alpha}$ be the corresponding isomorphism. Then

$$\varphi_\beta \circ \varphi_\alpha^{-1} \colon \mathcal{O}_{U_\alpha \cap U_\beta}^{n_\alpha} \to \mathcal{O}_{U_\alpha \cap U_\beta}^{n_\beta} \tag{6.3}$$

is an isomorphism of sheaves of modules. Since X is connected, it follows that all the numbers n_α are equal; set $n_\alpha = n$. Any endomorphism of the sheaf of modules $\mathcal{O}_U^n$ is given by a matrix $C = (c_{ij})$ with $c_{ij} \in \mathcal{O}_X(U)$. Thus the isomorphism (6.3) defines a matrix $C_{\alpha\beta}$ and obviously these matrixes satisfy the relations (6.2). Hence they define some vector bundle E. A trivial verification, which we omit, shows that $\mathcal{L}_E = \mathcal{F}$. The theorem is proved. □

One checks easily that the correspondence $E \mapsto \mathcal{L}_E$ between vector bundles and locally free sheaves allows us to associate a homomorphism of sheaves of $\mathcal{O}_X$-modules to any homomorphism of bundles. In other words, the correspondence is an equivalence of the two categories.

We should point out that the fibre of a vector bundle and the stalk of the corresponding sheaf are entirely different objects. For example, if $E = X \times k$ then $\mathcal{L}_E = \mathcal{O}_X$, so that $E_x = k$, whereas $(\mathcal{L}_E)_x = \mathcal{O}_x$. In the general case the fibre E_x can be recovered from the stalk $(\mathcal{L}_E)_x$ using the relation

$$E_x = (\mathcal{L}_E)_x / \mathfrak{m}_x (\mathcal{L}_E)_x, \tag{6.4}$$

where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_x$. It is enough to verify this locally; then we can write $E = U \times k^n$ and $\mathcal{L}_E = \mathcal{O}_U^n$, and (6.4) is obvious.

Whereas the notion of vector bundle was introduced here in a set-theoretic way, that of a locally free sheaf is adapted for the more general situation, and is meaningful for arbitrary schemes. It gives a natural analogue of the language of vector bundles. Moreover, the description in terms of transition matrixes also carries over: the matrixes $C_{\alpha\beta}$ must have entries belonging to the ring $\mathcal{O}_X(U_\alpha \cap U_\beta)$, and their determinants must be invertible elements of this ring.

We can also define a vector bundle over an arbitrary scheme X as a scheme locally isomorphic to $U \times \mathbb{A}^n$, with $U_\alpha \times \mathbb{A}^n$ and $U_\beta \times \mathbb{A}^n$ glued together by transition matrixes $C_{\alpha\beta}$. Then the operations that determine the vector space structure in the fibres are defined invariantly (because the matrixes $C_{\alpha\beta}$ define linear maps). The sheaf of sections $\mathcal{L}_E$ of a vector bundle E is defined just as before, and the correspondence $E \mapsto \mathcal{L}_E$ is described by Theorem 6.2.

But even for varieties, Theorem 6.2 is convenient because it gives a method of constructing new vector bundles.

Example 6.6 Let E and F be vector bundles, and $\mathcal{L}_E$, $\mathcal{L}_F$ the corresponding locally free sheaves. It is obvious that the sheaves $\mathcal{L}_E \oplus \mathcal{L}_F$, $\mathcal{L}_E \otimes \mathcal{L}_F$, $\mathcal{L}_E^*$, $\bigwedge_{\mathcal{O}}^p \mathcal{L}_E$ are all locally free. The corresponding vector bundles are denoted by $E \oplus F$, $E \otimes F$, E^*, $\bigwedge^p E$. In case $p = \operatorname{rank} E$ we write $\bigwedge^p E = \det E$; this is a rank 1 vector bundle, called the *determinant line bundle* of E.

If $X = \bigcup U_\alpha$ is a cover in which E and F are defined by transition matrixes $C_{\alpha\beta}$ and $D_{\alpha\beta}$ then in the same cover, $E \oplus F$, $E \otimes F$, E^*, $\bigwedge^p E$ are defined by the transition matrixes

$$\begin{pmatrix} C_{\alpha\beta} & 0 \\ 0 & D_{\alpha\beta} \end{pmatrix}, \quad C_{\alpha\beta} \otimes D_{\alpha\beta}, \quad \left({}^t C_{\alpha\beta}\right)^{-1}, \quad \bigwedge^p C_{\alpha\beta} \tag{6.5}$$

where ${}^t C$ denotes the transpose matrix. Setting $p = \operatorname{rank} E$, we see that the bundle $\det E$ is defined by the 1×1 matrixes $\det C_{\alpha\beta}$.

Corollary *For any bundle E, the dual bundle E^* has* $\det E^* = (\det E)^{-1}$.

It follows from (6.4) that where these operations are performed on vector bundles, the corresponding operations on vector spaces are performed on each fibre.

Example 6.7 Let X be a nonsingular variety. Taking an open set U to the group $\Omega^p[U]$ of differential p-forms regular on U defines in an obvious way a sheaf of $\mathcal{O}_X$-modules. It is called the *sheaf of differential p-forms*.

Theorem 3.18 of Section 5.3, Chapter 3 asserts that this sheaf is locally free. Hence by Theorem 6.2 it defines a vector bundle, denoted by Ω^p. In particular, Ω^1 is called the *cotangent bundle*.

The stalk of the sheaf $\mathcal{F}$ at a point $x \in X$ is of the form $\mathcal{F}_x = \mathcal{O}_x \mathrm{d}t_1 + \cdots + \mathcal{O}_x \mathrm{d}t_n$, where $t_1, \ldots, t_n$ are local parameters at x, and the sum is a direct sum. The homomorphism $\mathcal{F}_x \to \mathcal{F}_x / \mathfrak{m}_x \mathcal{F}_x$ can be written in the form

$$u_1 \mathrm{d}t_1 + \cdots + u_n \mathrm{d}t_n \mapsto u_1(x) \mathrm{d}t_1 + \cdots + u_n(x) \mathrm{d}t_n,$$

and hence by (6.4) it follows that

$$\Omega_x^1 \cong \mathcal{F}_x / \mathfrak{m}_x \mathcal{F}_x \cong \mathfrak{m}_x / \mathfrak{m}_x^2. \tag{6.6}$$

Obviously $\bigwedge^p \Omega^1 = \Omega^p$ and $\det \Omega^1 = \Omega^n$, where $n = \dim X$.

Example 6.8 The vector bundle dual to the cotangent bundle is called the *tangent bundle*, and is denoted by Θ. By (6.6), for any point $x \in X$ we have

$$\Theta_x = \left(\mathfrak{m}_x/\mathfrak{m}_x^2\right)^*,$$

that is, it is the tangent space at x. By Remark of Section 5.2, Chapter 3 it follows that for an affine subset $U \subset X$ with $U = \operatorname{Spec} A$ we have $\mathcal{O}_X(U) = \operatorname{Der}_k(A, A)$.

The final general question we want to discuss in connection with vector bundles is the notion of subbundle and quotient bundle.

Definition A morphism of vector bundles $\varphi\colon F \to E$ which is a closed embedding of varieties is an *embedding of vector bundles*. In this case the image $\varphi(F)$ is called a *subbundle* of E.

Proposition *A subbundle $F \subset E$ of a vector bundle is locally a direct summand.*

Proof The assertion means that for any point $x \in X$ there exists a neighbourhood U of x and a subbundle G of the restriction $E_{|U}$ such that

$$E_{|U} = F_{|U} \oplus G. \tag{6.7}$$

By Theorem 6.2, this equality is equivalent to the same equality for sheaves of modules, or simply for modules over $\mathcal{O}_X(U)$. As always, the local assertion can be reformulated in terms of local rings, but for this we must first translate the assumption that $\varphi\colon F \to E$ is a closed embedding in terms of the sheaves $\mathcal{L}_E$ and $\mathcal{L}_F$. Obviously, in this case, for any closed point $x \in X$ the homomorphism $\varphi_x\colon F_x \to E_x$ is an embedding. This means that if $\mathcal{L}_{F|U} = \mathcal{O}^r$ and $\mathcal{L}_{E|U} = \mathcal{O}^n$, and $\varphi\colon \mathcal{O}^r \to \mathcal{O}^n$ is the sheaf homomorphism corresponding to the homomorphism of vector bundles, then a free basis $e_1, \dots, e_r$ of $\mathcal{O}^r$ goes into a system of elements $\varphi(e_1), \dots, \varphi(e_r) \in \mathcal{O}^n$ that are linearly independent at each point. Thus we must show that if $\mathcal{O}$ is a local ring with maximal ideal $\mathfrak{m}$, and $\varphi\colon \mathcal{O}^r \to \mathcal{O}^n$ a homomorphism, and $e_1, \dots, e_r$ a free basis of $\mathcal{O}^r$ such that $\varphi(e_1), \dots, \varphi(e_r) \in \mathcal{O}^n$ are linearly independent modulo $\mathfrak{m}\mathcal{O}^n$ then φ is an embedding and $\mathcal{O}^n$ is a direct sum of $\varphi(\mathcal{O}^r)$ and a submodule isomorphic to $\mathcal{O}^{n-r}$. Indeed, set $\overline{e}_i = \varphi(e_i)$. Since $\dim(\mathcal{O}^n/\mathfrak{m}\mathcal{O}^n) = n$, the images of the elements $\overline{e}_i$ can be lifted to a basis of $\mathcal{O}^n/\mathfrak{m}\mathcal{O}^n$. Then by Nakayama's lemma (Proposition A.11 of Appendix to Volume 1) the elements $\overline{e}_1, \dots, \overline{e}_r$ can be extended to a system of generators $\overline{e}_1, \dots, \overline{e}_n$ of $\mathcal{O}^n$.

It is easy to see that this system is a free basis of $\mathcal{O}^n$: this does not even depend on $\mathcal{O}$ being a local ring. Indeed, if $f_1, \dots, f_n$ is some free basis of $\mathcal{O}^n$ then $\overline{e}_i = \sum a_{ij} f_j$ and $f_i = \sum b_{ij} e_j$ with a_{ij} and $b_{ij} \in \mathcal{O}$. From the fact that $f_1, \dots, f_n$ is a free basis, it follows that the matrixes $A = (a_{ij})$ and $B = (b_{ij})$ satisfy $BA = 1$. But then also $AB = 1$, which means that $\overline{e}_1, \dots, \overline{e}_n$ is a free basis of $\mathcal{O}^n$. From the fact that $\overline{e}_1, \dots, \overline{e}_r$ are linearly independent over $\mathcal{O}$ if follows that $\mathcal{O}^n = \varphi(\mathcal{O}^r) \oplus N$, where N is the module generated by $\overline{e}_{r+1}, \dots, \overline{e}_n$. The proposition is proved. $\square$

Now we can define the *quotient bundle* E/F of a vector bundle E by a subbundle $F \subset E$. As a set, of course,

$$E/F = \bigcup_{x \in X} E_x/F_x.$$

To give it a structure of variety, consider an open set U for which (6.7) holds, and identify $\bigcup_{x \in X} E_x/F_x$ with the algebraic variety G. It is easy to see that these structures are compatible on different open sets U and determine E/F as a vector bundle.

The proof of the proposition obviously remains valid for vector bundles over an arbitrary scheme X and leads to the definition of quotient bundle in this case.

The translation into the language of transition matrixes is obvious. If we choose a cover $X = \bigcup U_\alpha$ such that (6.7) holds for all U_α, the matrixes $C_{\alpha\beta}$ defining E can be expressed in the form

$$C_{\alpha\beta} = \begin{pmatrix} D_{\alpha\beta} & 0 \\ * & D'_{\alpha\beta} \end{pmatrix},$$

where $D_{\alpha\beta}$ defines the vector bundle F and $D'_{\alpha\beta}$ the vector bundle E/F. It follows at once from this that

$$\det E = \det F \otimes \det E/F. \tag{6.8}$$

Example 6.9 (The normal bundle $N_{X/Y}$) Let X be a nonsingular variety and $Y \subset X$ a nonsingular closed subvariety. We define the *normal bundle* $N_{X/Y}$ to Y in X. The definition used in differential geometry is not applicable in the algebraic situation, since it is based on the notion of the orthogonal complement $W^\perp$ of a vector subspace $W \subset V$. However, as a vector space, $W^\perp$ is determined by the fact that it is isomorphic to V/W. This is what we exploit.

Write Θ'_X for the restriction to Y of the tangent bundle Θ_X. It is defined as the pullback $j^*\Theta_X$, where $j: Y \hookrightarrow X$ is the closed embedding. The vector bundle Θ_Y is a subbundle of Θ'_X. Indeed, by definition $\Theta'_X = j^*\Theta_X = j^*((\Omega^1_X)^*) = (j^*\Omega^1_X)^*$. The restriction of differential forms from X to Y defines a surjective homomorphism $\varphi: j^*\Omega^1_X \to \Omega^1_Y$ and its dual $\varphi^*: \Theta_Y = (\Omega^1_Y)^* \to (j^*\Omega^1_X)^* = \Theta'_X$. By definition

$$N_{X/Y} = \Theta'_X/\Theta_Y.$$

We compute the transition matrix of the normal bundle. The homomorphism $\Theta'_X \to N_{X/Y}$ defines a dual homomorphism

$$\psi: N^*_{X/Y} \to j^*\Omega^1_X$$

of the dual vector bundles. It is easy to see that ψ defines a closed embedding, so that we can view $N^*_{X/Y}$ as a subbundle of $j^*\Omega^1_X$ and Ω^1_Y as the quotient bundle $(j^*\Omega^1_X)/N^*_{X/Y}$. It is enough to check these assertions on open sets on which our vector bundles are trivial, where they are obvious.

As we saw in Theorem 3.17, Corollary of Section 5.1, Chapter 3, forms $\mathrm{d}u_1, \dots, \mathrm{d}u_n$ are a basis of the $\mathcal{O}_X(U)$-module $\Omega^1_X[U]$ provided that the functions

$u_1, \dots, u_n$ define local parameters at each point $x \in U$. This basis defines a basis $\eta_1, \dots, \eta_n$ of the $\mathcal{O}_Y(U \cap Y)$-module of sections over $U \cap Y$ of the sheaf corresponding to the vector bundle $j^* \Omega^1_X$. Here $\varphi(\eta_i)$ is the restriction to Y of the form $\mathrm{d}u_i$.

Suppose that $n = \dim X$ and $m = \operatorname{codim}(Y \subset X)$. By Theorem 2.14 of Section 3.2, Chapter 2 we can choose the functions $u_1, \dots, u_n$ such that $u_1 = \dots = u_m = 0$ are the local equations of Y in U. By the same theorem, together with Theorem 3.17, Corollary of Section 5.1, Chapter 3, the restrictions of the forms $\mathrm{d}u_{m+1}, \dots, \mathrm{d}u_n$ define a basis of $\Omega^1_Y[U \cap Y]$, and hence $\eta_1, \dots, \eta_m$ is a basis of the $\mathcal{O}_Y(U \cap Y)$-module $N^*_{X/Y}(U \cap Y)$.

Suppose that U_α and U_β are two open sets in which $u_{\alpha,1}, \dots, u_{\alpha,n}$ and $u_{\beta,1}, \dots, u_{\beta,n}$ are systems of local parameters chosen as described. The transition matrix for the vector bundle Ω^1_X is determined by the expression

$$\mathrm{d}u_{\alpha,i} = \sum_{j=1}^{n} h_{ij} \mathrm{d}u_{\beta,j} \quad \text{for } i = 1, \dots, n, \tag{6.9}$$

where $h_{ij} \in \mathcal{O}_X(U)$ are the entries of the Jacobian matrix, that is, $h_{ij} = \partial u_{\alpha,i} / \partial u_{\beta,j}$, and the transition matrix of $j^* \Omega^1_X$ in the basis $\eta_1, \dots, \eta_n$ is obtained by restricting the entries of this matrix to $U \cap Y$.

Since $u_{\alpha,i} \in (u_{\beta,1}, \dots, u_{\beta,m})$ on $U_\alpha \cap U_\beta$ for $i = 1, \dots, m = \operatorname{codim} Y$, we have

$$u_{\alpha,i} = \sum_{j=1}^{m} f_{ij} u_{\beta,j} \quad \text{for } i = 1, \dots, m,$$

with $f_{ij} \in \mathcal{O}_X(U_\alpha \cap U_\beta)$. Hence for $i = 1, \dots, m$ we have

$$\mathrm{d}u_{\alpha,i} = \sum_{j=1}^{m} f_{ij} \mathrm{d}u_{\beta,j} + \sum_{j=1}^{m} u_{\beta,j} \mathrm{d}f_{ij}. \tag{6.10}$$

To reconcile this formula with (6.9) we would have to express the $\mathrm{d}f_{ij}$ in terms of $\mathrm{d}u_1, \dots, \mathrm{d}u_n$. But we are only interested in formulas for the η_i, which are obtained by restricting to Y all the functions occurring in it. Since $u_{\beta,j} = 0$ on Y for $j = 1, \dots, m$, the second group of terms in (6.10) vanishes. Thus

$$\eta_{\alpha,i} = \sum_{j=1}^{m} \overline{f}_{ij} \eta_{\beta,j} \quad \text{for } i = 1, \dots, m,$$

where $\overline{f}_{ij}$ is the restriction of f_{ij} to $U_\alpha \cap U_\beta \cap Y$. As we have seen, these are the transition matrixes of the vector bundle $N^*_{X/Y}$. Those for $N_{X/Y}$ are obtained on transposing and taking the inverse; taking the inverse is equivalent to interchanging α and β. We finally arrive at the simple formulas

$$C_{\alpha\beta} = (h_{ij\,|Y}), \quad \text{for } i, j = 1, \dots, m \tag{6.11}$$

where $u_{\beta,j} = \sum h_{ij} u_{\alpha,i}$ in $U_\alpha \cap U_\beta$.

An important factor in practically all the constructions of this section is the possibility of specifying a vector bundle in an abstract way, without reference to an embedding into projective space. It can however be proved that a vector bundle over a quasiprojective variety is itself quasiprojective; we omit the proof.

1.4 Divisors and Line Bundles

In what follows, we do not assume that X is nonsingular, and consider locally principal divisors D (Section 1.2, Chapter 3). Corresponding to each divisor D on an irreducible variety X we have a vector space $\mathcal{L}(D)$ (Section 1.2, Chapter 3). This correspondence gives rise to a sheaf on X. To see this, note that the divisor D on X also defines a divisor on any open subset $U \subset X$, by restricting to U the local equations of D. We write D_U for the divisor thus obtained and set

$$\mathcal{L}_D(U) = \mathcal{L}(U, D_U),$$

where $\mathcal{L}(U, D_U)$ is the vector space corresponding to the divisor D_U on U. Obviously $\mathcal{L}_D(U) \subset k(X)$, and $\mathcal{L}_D(V) \subset \mathcal{L}_D(U)$ whenever $U \subset V$; write $\rho_U^V : \mathcal{L}_D(V) \hookrightarrow \mathcal{L}_D(U)$ for the inclusion map. The system $\{\mathcal{L}_D(U), \rho_U^V\}$ is a presheaf, and it is easy to see that it is a sheaf. We denote it by $\mathcal{L}_D$.

Multiplying elements $f \in \mathcal{L}_D(U)$ by $h \in \mathcal{O}_X(U)$ makes $\mathcal{L}_D$ into a sheaf of $\mathcal{O}_X$-modules. This sheaf is locally free. Indeed, if D is defined in an open set U_α by a local equation f_α then the elements $g \in \mathcal{L}_D(U_\alpha)$ are characterised by the condition $gf_\alpha \in \mathcal{O}_X(U_\alpha)$. This shows that the map $g \mapsto gf_\alpha$ defines an isomorphism

$$\varphi_\alpha : \mathcal{L}_{D|U_\alpha} \xrightarrow{\sim} \mathcal{O}_{X|U_\alpha}. \tag{6.12}$$

We saw in Section 1.3 that such a sheaf determines a vector bundle E_D; it follows from (6.12) that rank $E_D = 1$. Vector bundles of rank 1 are called *line bundles* since their fibres are lines. We write out the transition functions for E_D. Since the isomorphism over U_α in (6.12) is given by multiplication by f_α, the automorphism $\varphi_\beta \circ \varphi_\alpha^{-1}$ over $U_\alpha \cap U_\beta$ is given by multiplication by f_β / f_α. Note that $f_\beta / f_\alpha \in \mathcal{O}_X(U_\alpha \cap U_\beta)$ by the compatibility of the f_α. Similarly $(f_\beta / f_\alpha)^{-1} = f_\alpha / f_\beta \in \mathcal{O}_X(U_\alpha \cap U_\beta)$. Thus in this case the transition matrix is the 1×1 matrix $\varphi_{\alpha\beta}$ given by

$$\varphi_{\alpha\beta} = f_\beta / f_\alpha. \tag{6.13}$$

If we replace the divisor D by a linearly equivalent divisor $D' = D + \operatorname{div} f$ with $f \in k(X)$ then multiplication by f defines an isomorphism of modules $\mathcal{L}(U, D_U) \to \mathcal{L}(U, D'_U)$. We verified this in Theorem 3.3 of Section 1.5, Chapter 3. In this way we obviously get an isomorphism of sheaves $\mathcal{L}_D \xrightarrow{\sim} \mathcal{L}_{D'}$. The two line bundles E_D and $E_{D'}$ actually have identical transition functions. Thus the sheaf $\mathcal{L}_D$ and the line bundle E_D both correspond to a whole divisor class.

Theorem 6.3 *The map $D \mapsto \mathcal{L}_D \to E_D$ defines a one-to-one correspondence between* (1) *linear equivalence classes of divisors*, (2) *isomorphism classes of sheaves of $\mathcal{O}_X$-modules locally isomorphic to $\mathcal{O}_X$, and* (3) *isomorphism classes of rank* 1 *vector bundles*.

Proof The correspondence between the sets (2) and (3) was established in Theorem 6.2. Thus we need only prove that $D \mapsto E_D$ defines a one-to-one correspondence between the sets (1) and (3). To do this we construct the inverse map.

Suppose that E is a line bundle defined in a cover $X = \bigcup U_\alpha$ by 1×1 transition matrixes $\varphi_{\alpha\beta}$, with $\varphi_{\alpha\beta}$ and $\varphi_{\alpha\beta}^{-1} \in \mathcal{O}_X(U_\alpha \cap U_\beta)$. It follows from the glueing conditions (6.2) that $\varphi_{\beta\alpha} = \varphi_{\alpha\beta}^{-1}$ and

$$\varphi_{\alpha\beta} = \varphi_{\gamma\alpha}^{-1}\varphi_{\gamma\beta} \quad \text{over } U_\alpha \cap U_\beta \cap U_\gamma. \tag{6.14}$$

The inclusion $\mathcal{O}_X(U_\alpha \cap U_\beta) \hookrightarrow k(X)$ allows us to consider the $\varphi_{\alpha\beta}$ as elements of $k(X)$, and (6.14) holds for these in the same way. Fix some subscript γ, say $\gamma = 0$. We substitute $\gamma = 0$ in (6.14) and set $f_\alpha = \varphi_{0\alpha}$. Then the system of elements f_α on U_α is compatible, since

$$f_\beta / f_\alpha = \varphi_{\alpha\beta}; \tag{6.15}$$

hence they define a certain divisor D. Comparing (6.13) and (6.15) shows that $E = E_D$.

Let us prove that the linear equivalence class of the divisor D depends only on the line bundle E and not on the choice of the cover or the transition matrixes. Two systems $\{U_\alpha, \varphi_{\alpha\beta}\}$ and $\{U'_\lambda, \varphi'_{\lambda\mu}\}$ can be compared on the cover $\{U_\alpha \cap U'_\lambda\}$ by setting

$$\widetilde{\varphi}_{\alpha\beta\lambda\mu} = \varphi_{\alpha\beta} \quad \text{and} \quad \widetilde{\varphi}'_{\alpha\beta\lambda\mu} = \varphi'_{\lambda\mu} \quad \text{on } U_\alpha \cap U_\beta \cap U'_\lambda \cap U'_\mu.$$

Therefore we can assume from the start that the two covers are the same, $X = \bigcup U_\alpha$. Then as shown in Section 1.2,

$$\varphi'_{\alpha\beta} = \psi_\alpha^{-1}\varphi_{\alpha\beta}\psi_\beta \quad \text{with } \psi_\alpha \text{ and } \psi_\alpha^{-1} \in \mathcal{O}_X(U_\alpha). \tag{6.16}$$

By definition of f_α and f'_α

$$f'_\alpha = \psi_0^{-1}\varphi_{0\alpha}\psi_\alpha = \psi_0^{-1} f_\alpha \psi_\alpha,$$

so that (6.16) gives $D' = D - \operatorname{div}(\psi_0)$.

Thus we have constructed a well-defined map from the set (3) to the set (1). An obvious check shows that it is the inverse of $D \mapsto E_D$. The theorem is proved. □

For any morphism $f\colon X \to Y$ we have the relation

$$f^*(E_D) = E_{f^*(D)}; \tag{6.17}$$

we leave the obvious verification to the reader.

The divisor class corresponding to a line bundle E under Theorem 6.3 is called the *characteristic class* of E and denoted by $c(E)$.

Example 6.10 If $\dim X = n$ and Ω^n is the line bundle introduced in Section 1.2 then $c(\Omega^n) = K$ is the canonical class of X.

Example 6.11 Let $X = \mathbb{P}^n$ and let D be a hyperplane of $\mathbb{P}^n$. The line bundle E_D corresponding to D under Theorem 6.3 is denoted by $\mathcal{O}(1)$. If D is given by $x_0 = 0$ then in the open set U_α where $x_\alpha \neq 0$ it has the local equation x_0/x_α. Hence the transition matrix for E_D is of the form $c_{\alpha\beta} = x_\alpha/x_\beta$. It follows that $\mathcal{O}(1)$ is the line bundle dual to the line bundle $\mathcal{L}$ of Example 6.2.

Let us find the sections of $\mathcal{O}(1)$. In U_α these are of the form $s_\alpha = P_\alpha/x_\alpha^k$, where P_α is a form of degree k; and they are related by $s_\beta = c_{\alpha\beta}s_\alpha$. It follows that $k = 1$ and that $P_\alpha = P_\beta$ is a form of degree 1 on $\mathbb{P}^n$. Similarly, the divisor mD corresponds to the line bundle denoted by $\mathcal{O}(m)$ with the transition matrix $c_{\alpha\beta} = (x_\alpha/x_\beta)^m$. The sections of $\mathcal{O}(m)$ are homogeneous polynomials of degree m. It is easy to see that $\mathcal{O}(m) = \mathcal{O}(1)^{\otimes m}$ is the mth tensor power of $\mathcal{O}(1)$. For a subvariety $X \subset \mathbb{P}^n$ we write $\mathcal{O}_X(m)$ for the restriction to X of the line bundle (or sheaf) $\mathcal{O}(m)$ on $\mathbb{P}^n$.

Example 6.12 Let X be a nonsingular variety and $Y \subset X$ a nonsingular hypersurface. In this case the normal bundle $N_{X/Y}$ is a line bundle. We compute its characteristic class.

Suppose that Y is given in an affine cover $X = \bigcup U_\alpha$ by local equations f_α. Then $f_\beta/f_\alpha = f_{\alpha\beta}$, where $f_{\alpha\beta}$ and $f_{\alpha\beta}^{-1} \in \mathcal{O}_X(U_\alpha \cap \mathcal{O}_\beta)$. By (6.11), the transition matrixes of $N_{X/Y}$ are of the form $f_{\alpha\beta|Y} = (f_\beta/f_\alpha)_{|Y}$. But we have just seen that f_β/f_α are the transition matrixes for the line bundle E_Y. Thus we have proved the formula

$$N_{X/Y} = E_{Y|Y}.$$

By (6.17) it follows from this that

$$c(N_{X/Y}) = \rho_Y(C_Y),$$

where C_Y is the divisor class on X containing Y and $\rho_Y \colon \operatorname{Cl} X \to \operatorname{Cl} Y$ the homomorphism of restriction to Y. Recall from Section 1.2, Chapter 3 the explicit description of ρ_Y: we must replace Y by a linearly equivalent divisor Y' not containing Y as a component, then restrict Y' to Y.

Since divisor classes form a group, the correspondence established in Theorem 6.3 defines a group law on the set of line bundles or sheaves locally isomorphic to $\mathcal{O}$. From (6.13) we see that addition of divisors corresponds to multiplication of the 1×1 transition matrixes. This operation is given more intrinsically by the tensor product of line bundles or sheaves (see Theorem 6.2). Here the sheaf $\mathcal{O}$ plays the role of the multiplicative identity, and the inverse of $\mathcal{L}_D$ is $\mathcal{L}_{-D}$. Because of this, locally free sheaves of $\mathcal{O}$-modules of rank 1 are also called *invertible sheaves*.

Although invertible sheaves and divisor classes are in one-to-one correspondence, it is often technically more convenient to use invertible sheaves. For example, the inverse image $f^*(\mathcal{F})$ can be defined in a natural way for any morphism f and any sheaf $\mathcal{F}$ (see for example Hartshorne [37, Section 5, Chapter II]). It is easy to check that if $\mathcal{F}$ is invertible then so is $f^*(\mathcal{F})$. The corresponding operation for divisor classes requires arguments concerned with moving the support of a divisor.

These technical advantages of invertible sheaves are related to matters of principle. In a closely related situation, in the theory of complex manifolds, the notions of invertible sheaf and divisor class are no longer equivalent, and there, invertible sheaves provide more information and lead to a more natural statement of the problems. For this, compare Exercises 6–8 of Section 2, Chapter 8.

For an arbitrary scheme X, a sheaf locally isomorphic to $\mathcal{O}_X$ is a natural analogue of a divisor class. Such sheaves form a group: multiplication is defined as tensor product, and the inverse of a sheaf $\mathcal{F}$ is its dual $\mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$. This group is again denoted by $\operatorname{Pic} X$. In our case, the transition matrixes are invertible elements of the ring $\mathcal{O}_X(U_\alpha \cap U_\beta)$, the multiplication and inverse operations reduce to the same operations on transition matrixes (in our case, transition functions).

As an application of the ideas treated here we deduce the genus formula stated and used repeatedly in Section 2.3, Chapter 4.

Theorem 6.4 (Adjunction formula) *The genus g_Y of a nonsingular curve Y on a complete nonsingular surface X is given by the formula*

$$g_Y = \frac{1}{2} Y(Y+K) + 1; \tag{6.18}$$

where K is the canonical class of X.

Proof Let X be a nonsingular variety and $Y \subset X$ an arbitrary nonsingular closed subvariety. By the definition of the normal bundle $N_{X/Y}$ and (6.8), we obtain

$$\rho_Y(\det \Theta_X) = \det \Theta'_X = \det \Theta_Y \otimes \det N_{X/Y}.$$

Since Θ_X is the dual of Ω^1_X and Θ_Y that of Ω^1_Y, we can apply the Corollary of Example 6.6, formula (6.5) to obtain

$$\rho_Y\bigl(c\bigl(\Omega^n_X\bigr)\bigr) = c\bigl(\Omega^m_Y\bigr) \cdot c(\det N_{X/Y})^{-1}.$$

It follows from (6.6) that $\det(E^*) = (\det E)^{-1}$ for any vector bundle. Since $\det \Omega^1_X = \bigwedge^n \Omega^1_X = \Omega^n_X$, we get

$$\rho_Y\bigl(c\bigl(\Omega^n_X\bigr)\bigr) = c\bigl(\Omega^m_Y\bigr) \cdot c(\det N_{X/Y})^{-1}, \tag{6.19}$$

with $\dim X = n$ and $\dim Y = m$. This formula holds for a nonsingular subvariety $Y \subset X$ of any dimension, and is usually called the *adjunction formula*.

Now suppose that $m = n - 1$. We apply the results obtained in Examples 6.5–6.7. We arrive at the relation

$$\rho_Y(K_X) = K_Y - \rho_Y(C_Y). \tag{6.20}$$

Finally if $n = 2$ and $m = 1$, we deduce that the divisors on either side of (6.20) have equal degrees.

Note that in our case, the restriction of any divisor D on X is a divisor $\rho_Y(D)$ on Y, and it has a well-defined degree, equal to $\deg \rho_Y(D) = YD$. Now by Corollary 3.1, of Section 7, Chapter 3 we have $\deg K_Y = 2g_Y - 2$ and so

$$YK_X = 2g_Y - 2 - Y^2,$$

and the theorem follows from this based on simple properties of intersection numbers. □

1.5 Exercises to Section 1

1 Let k be an algebraically closed field. Define a *pseudovariety* over k to be a ringed space such that every point has a neighbourhood isomorphic to m-Spec A, where A is a finitely generated k-algebra with no nilpotents; the topology and structure sheaf on m-Spec A are defined exactly as in Chapter 5. Prove that taking a variety to its set of closed points defines an isomorphism of the category of varieties and pseudovarieties.

2 Define the product of pseudovarieties X and Y. Start by setting $X \times Y$ to be the set of pairs (x, y) with $x \in X$ and $y \in Y$, then construct an affine cover of this set based on affine covers of X and Y, using the definition of products of affine varieties given in Example 1.5 of Section 2.1, Chapter 1.

3 Prove that a variety is complete if and only if its irreducible components are complete.

4 We say that a fibration $X \to S$ is *locally trivial*, or is a *fibre bundle with fibre* F if every point $s \in S$ has a neighbourhood U such that the restriction of X over U is isomorphic to $F \times U$ as a scheme over U. Prove that if $X \to S$ is a locally trivial fibration with the base S and the fibre F both complete then X is also complete.

5 Determine the transition matrixes of the line bundle of Example 6.2, which corresponds to the cover of $\mathbb{P}^n$ by the sets $\mathbb{A}^n_i$ given by $x_i \neq 0$. Find the characteristic class of this line bundle.

6 Let D be an effective divisor on a variety X for which the vector space $\mathcal{L}(D)$ is finite dimensional, and $\mathcal{F} = \mathcal{F}_D$ the corresponding invertible sheaf. Let $f : X \to \mathbb{P}^n$ with $n = l(D) - 1$ be the rational map associated with $\mathcal{L}(D)$ as in Section 1.5, Chapter 3. Assume that the divisors $\operatorname{div} f$ of functions $f \in \mathcal{L}(D)$ have no common components. Prove that f is regular at a point $x \in X$ if and only if the stalk $\mathcal{F}_x$ of $\mathcal{F}$ is generated as an $\mathcal{O}_x$-module by the space $\rho_x \mathcal{L}(D)$.

7 Suppose that $X = \operatorname{Spec} A$ is a nonsingular affine variety. Prove that the A-module $\Theta_X(X)$ is isomorphic to the module of derivations of A (that is, k-linear maps $d\colon A \to A$ such that $d(xy) = d(x)y + xd(y)$ for $x, y \in A$).

8 Prove that the normal bundle to a line C in $\mathbb{P}^n$ is a direct sum of $n-1$ isomorphic line bundles E. Find $c(E)$.

9 Suppose that $n-1$ hypersurfaces $C_1, \dots, C_{n-1}$ in $\mathbb{P}^n$ of degrees $k_1, \dots, k_{n-1}$ intersect transversally in an irreducible curve X. Find the genus of X.

10 Let $f\colon E \to X$ be a vector bundle and $X = \bigcup U_\alpha$ a cover such that E is trivial over each U_α, that is, $E_{|U_\alpha} \cong U_\alpha \times k^n$. Embed k^n in $\mathbb{P}^n$ as the set of points with $x_0 \neq 0$, and glue the varieties $U_\alpha \times \mathbb{P}^n$ by means of the transition matrixes of E, now considered as matrixes of projective transformations of $\mathbb{P}^n$. Prove that in this way we obtain a variety $\overline{E}$ containing E as an open set, and $\overline{E}$ is nonsingular; moreover, $f\colon \overline{E} \to X$ is a regular map and its fibres are isomorphic to $\mathbb{P}^n$.

11 In the notation of Exercise 10, suppose that $X = \mathbb{P}^1$, and for $n \geq 0$ let E_n be the vector bundle of rank 1 corresponding to the divisor nx_∞ on $\mathbb{P}^1$. Prove that $\overline{E}_n \setminus E_n = C_\infty$ is a curve mapped isomorphically to $\mathbb{P}^1$ by f. Let C_0 be the zero section of E_n, which is obviously also contained in $\overline{E}_n$, and write F for the fibre of $\overline{E}_n \to \mathbb{P}^1$. Prove that $C_0 - C_\infty \sim nF$ on the surface $\overline{E}_n$, and determine C_0^2 and C_∞^2.

12 In the notation of Exercise 11, prove that the restriction of divisors $D \in \operatorname{Div} \overline{E}_n$ to a general fibre defines a homomorphism $\operatorname{Cl} \overline{E}_n \to \mathbb{Z}$ whose kernel is $\mathbb{Z} \cdot F$. Prove that $\operatorname{Cl} \overline{E}_n$ is a free Abelian group with the two generators C_0 and F.

13 In the notation of Exercises 11–12, find the canonical class of the surface $\overline{E}_n$.

14 Prove that the surfaces $\overline{E}_n$ corresponding to distinct $n \geq 0$ are not isomorphic. [Hint: Prove that $\overline{E}_n$ contains a unique irreducible curve with negative selfintersection, and this selfintersection is $-n$.]

15 Let X be a nonsingular affine variety and $A = k[X]$ its affine coordinate ring. Prove that the module $\Theta(X)$ is isomorphic to $\operatorname{Der}_k(A, A)$ (often written simply as $\operatorname{Der}_k(A)$. For the definition of $\operatorname{Der}_k(A, A)$, see Exercise 24, Section 1.6, Chapter 2; compare Exercise 12, Section 5.5, Chapter 3.

2 Abstract and Quasiprojective Varieties

2.1 Chow's Lemma

We prove a result that sheds some light on the relation between complete and projective varieties. Of course, every irreducible variety is birational to a projective variety, for example, the projective closure of any affine open subset. However, one can prove considerably more in this direction.

Theorem (Chow's lemma) *For any complete irreducible variety X, there exists a projective variety $\overline{X}$ and a surjective birational morphism $f\colon \overline{X} \to X$.*

Proof The idea of the proof is the same as that used to construct the projective embedding of the normalisation of a curve (Theorem 2.23 of Section 5.3, Chapter 2).

Let $X = \bigcup U_i$ be a finite affine cover. For each affine variety $U_i \subset \mathbb{A}^{n_i}$, write Y_i for the closure of U_i in the projective space $\mathbb{P}^{n_i} \supset \mathbb{A}^{n_i}$. The variety $Y = \prod Y_i$ is obviously projective.

Set $U = \bigcap U_i$. The inclusions $\psi\colon U \hookrightarrow X$ and $\psi_i\colon U \hookrightarrow U_i \hookrightarrow Y_i$ define a morphism

$$\varphi\colon U \to X \times Y, \quad \text{with } \varphi = \psi \times \prod \psi_i.$$

Write $\overline{X}$ for the closure of $\varphi(U)$ in $X \times Y$. The first projection $p_X\colon X \times Y \to X$ defines a morphism $f\colon \overline{X} \to X$. We prove that it is birational. For this it is enough to check that

$$f^{-1}(U) = \varphi(U). \tag{6.21}$$

Indeed, $p_X \circ \varphi = 1$ on U, and in view of (6.21), f coincides on $f^{-1}(U)$ with the isomorphism φ^{-1}. Now (6.21) is equivalent to

$$(U \times Y) \cap \overline{X} = \varphi(U), \tag{6.22}$$

that is, to $\varphi(U)$ closed in $U \times Y$. But this is obvious, since $\varphi(U)$ in $U \times Y$ is just the graph of the morphism $\prod \psi_i$. The morphism f is surjective, since $f(\overline{X}) \supset U$, and U is dense in X.

It remains to prove that $\overline{X}$ is projective. For this, we use the second projection $g\colon X \times Y \to Y$, and prove that its restriction $\overline{g}\colon \overline{X} \to Y$ is a closed embedding. Since to be a closed embedding is a local property, it is enough to find open sets $V_i \subset Y$ such that $\bigcup g^{-1}(V_i) \supset \overline{X}$ and $\overline{g}\colon \overline{X} \cap g^{-1}(V_i) \to V_i$ is a closed embedding. We set

$$V_i = p_i^{-1}(U_i),$$

where $p_i\colon Y \to Y_i$ is the projection. First of all, the $g^{-1}(V_i)$ cover $\overline{X}$. For this it is enough to prove that

$$g^{-1}(V_i) = f^{-1}(U_i), \tag{6.23}$$

since $\bigcup U_i = X$ and $\bigcup f^{-1}(U_i) = \overline{X}$. In turn, (6.23) will follow from

$$f = p_i \circ g \quad \text{on } f^{-1}(U). \tag{6.24}$$

But it is enough to prove (6.24) on some open subset $W \subset f^{-1}(U_i)$. We can in particular take $W = f^{-1}(U) = \varphi(U)$ (according to (6.21)), and then (6.24) is obvious.

Thus it remains to prove that

$$\overline{g}\colon \overline{X} \cap g^{-1}(V_i) \to V_i$$

defines a closed embedding. Now recall that

$$V_i = p_i^{-1}(U_i) = U_i \times \widehat{Y}_i, \quad \text{where } \widehat{Y}_i = \prod_{j \neq i} Y_j;$$

we get that

$$g^{-1}(V_i) = X \times U_i \times \widehat{Y}_i.$$

Write Z_i for the graph of the morphism $U_i \times \widehat{Y}_i \to X$, which is the composite of the projection to U_i and the embedding $U_i \hookrightarrow X$. The set Z_i is closed in $X \times U_i \times \widehat{Y}_i = g^{-1}(V_i)$, and its projection to $U_i \times \widehat{Y}_i = V_i$ is an isomorphism. On the other hand, $\varphi(U) \subset Z_i$, and since Z_i is closed, $\overline{X} \cap g^{-1}(V_i)$ is closed in Z_i. Hence the restriction of the projection to this set is a closed embedding. Chow's lemma is proved. $\square$

Similar arguments prove the analogous statement for an arbitrary variety, when $\overline{X}$ is quasiprojective (see Exercise 7).

2.2 *Blowup Along a Subvariety*

Chow's lemma shows that arbitrary varieties are rather close to projective varieties. Nevertheless, the two notions do not always coincide. We construct simple examples of non-quasiprojective varieties in the following section. The construction uses a generalisation of the notion of blowup defined in Section 4.2, Chapter 2. The difference is that here we construct a morphism $\sigma : X' \to X$ such that the rational map σ^{-1} blows up a whole nonsingular subvariety $Y \subset X$ rather than just one point $x_0 \in X$. The construction follows closely that of Sections 4.1–4.3, Chapter 2.

(a) The Local Construction According to Theorem 2.14 of Volume 1, for any closed point of a nonsingular subvariety $Y \subset X$ of a nonsingular variety X, there exists a neighbourhood U and functions $u_1, \dots, u_m \in \mathcal{O}_X(U)$, where $m = \operatorname{codim}_X Y$, such that the ideal $\mathfrak{a}_Y \subset \mathcal{O}_X(U)$ is given by $\mathfrak{a}_Y = (u_1, \dots, u_m)$, and such that $\mathrm{d}_x u_1, \dots, \mathrm{d}_x u_m$ are linearly independent at every closed point $x \in U$. The final condition means that $u_1, \dots, u_m$ can be included in a system of local parameters at $x \in U$. If these conditions are satisfied, we say that $u_1, \dots, u_m$ are local parameters for Y in U.

Suppose that X is affine and $u_1, \dots, u_m$ are local parameters for Y everywhere in X. Consider the product $X \times \mathbb{P}^{m-1}$ and the closed subvariety $X' \subset X \times \mathbb{P}^{m-1}$ defined by the equations

$$t_i u_j(x) = t_j u_i(x) \quad \text{for } i, j = 1, \dots, m,$$

where $t_1, \dots, t_m$ are the homogeneous coordinates in $\mathbb{P}^{m-1}$. The projection $X \times \mathbb{P}^{m-1} \to X$ defines a morphism $\sigma\colon X' \to X$. Clearly, now $\sigma^{-1}(Y) = Y \times \mathbb{P}^{m-1}$, and σ defines an isomorphism

$$X' \setminus \left(Y \times \mathbb{P}^{m-1}\right) \xrightarrow{\sim} X \setminus Y.$$

Let $x' = (y, t)$ be a closed point of X', with $y \in X$ and $t \in \mathbb{P}^{m-1}$; suppose that $t = (t_1 : \cdots : t_m)$ with $t_i \neq 0$. Then in a neighbourhood of x', we have $u_j = u_i s_j$, where $s_j = t_j/t_i$. Let $v_1, \dots, v_{n-m}, u_1, \dots, u_m$ be a local parameter system at $y \in X$. Then the maximal ideal of $x' \in X'$ is of the form

$$\begin{aligned}\mathfrak{m}_{x'} &= \left(v_1, \dots, v_{n-m}, u_1, \dots, u_m, s_1 - s_1\left(x'\right), \dots, s_m - s_m\left(x'\right)\right)\\ &= \left(v_1, \dots, v_{n-m}, s_1 - s_1\left(x'\right), \dots, \widehat{s_i - s_i\left(x'\right)}, u_i, \dots, s_m - s_m\left(x'\right)\right).\end{aligned}$$

It follows from this, as in Section 4.2, Chapter 2, that X' is nonsingular, n-dimensional and irreducible. As there, the following result holds.

Lemma *If $\tau\colon \overline{X} \to X$ is a blowup of the same subvariety $Y \subset X$ defined by a different local system of parameters $v_1, \dots, v_m$ of Y then there is an isomorphism $\varphi\colon X' \to \overline{X}$ for which the diagram*

$$\begin{tikzcd} X' \arrow[rr, "\varphi"] \arrow[dr, "\sigma"'] & & \overline{X} \arrow[dl, "\tau"] \\ & X & \end{tikzcd}$$

commutes. The isomorphism φ is unique.

We have $\varphi = \tau^{-1} \circ \sigma$ on the open sets $X' \setminus \sigma^{-1}(Y)$ and $\overline{X} \setminus \tau^{-1}(Y)$, and the uniqueness of φ follows from this. By definition, in these sets

$$\begin{aligned}\varphi(x; t_1 : \cdots : t_m) &= \left(x; v_1(x) : \cdots : v_m(x)\right),\\ \psi\left(x; t_1' : \cdots : t_m'\right) &= \left(x; u_1(x) : \cdots : u_m(x)\right),\end{aligned}$$

where $\psi = \varphi^{-1}$.

By assumption,

$$v_k = \sum_j h_{kj} u_j \quad \text{with } h_{kj} \in k[X]. \tag{6.25}$$

In the open set given by $t_i \neq 0$, we set $s_j = t_j/t_i$ and rewrite (6.25) in the form

$$v_k = u_i g_k \quad \text{with } g_k = \sum_j \sigma^*(h_{kj}) s_j. \tag{6.26}$$

Then define

$$\varphi(x; t_1 : \cdots : t_m) = (x; g_1 : \cdots : g_m). \tag{6.27}$$

The same simple verification as in the proof of the analogous lemma in Section 4.2, Chapter 2 shows that φ is a morphism, which is equal to that already constructed on $X' \setminus \sigma^{-1}(Y)$. The construction of ψ is similar.

(b) The Global Construction Let $X = \bigcup U_\alpha$ be an affine cover such that Y is defined in U_α by local parameters $u_{\alpha,1}, \dots, u_{\alpha,m}$. Over U_α we apply the construction of (a) to $Y \cap U_\alpha$; we get a system of varieties X'_α and morphisms $\sigma_\alpha \colon X'_\alpha \to U_\alpha$. Consider the subset $\sigma_\alpha^{-1}(U_\alpha \cap U_\beta) \subset X'_\alpha$ for all α and β; then by the lemma, there exist uniquely determined isomorphisms

$$\varphi_{\alpha\beta} \colon \sigma_\alpha^{-1}(U_\alpha \cap U_\beta) \to \sigma_\beta^{-1}(U_\alpha \cap U_\beta).$$

It is easy to check that these satisfy the glueing conditions and define a variety X' and a morphism $\sigma \colon X' \to X$. The morphism σ we have constructed is called the *blowup* of Y, or the blowup of X with centre in Y. It follows in an obvious way from the lemma that X' and σ are both independent of the cover $X = \bigcup U_\alpha$ and of the system of parameters $u_{\alpha,i}$.

(c) The Exceptional Locus The subvariety $\sigma^{-1}(Y)$ is known locally:

$$\sigma^{-1}(Y \cap U_\alpha) = (Y \cap U_\alpha) \times \mathbb{P}^{m-1}. \tag{6.28}$$

Globally, we are dealing with a fibre bundle of a new type: the fibre $\sigma^{-1}(y)$ over each $y \in Y$ is a projective space $\mathbb{P}^{m-1}$. Equation (6.28) shows the sense in which our fibre bundle is locally trivial.

With every vector bundle $p \colon E \to X$ we can associate a fibre bundle $\varphi \colon \mathbb{P}(E) \to X$ of this type. For this, we define $\mathbb{P}(E)$ as the set

$$\mathbb{P}(E) = \bigcup_{x \in X} \mathbb{P}(E_x),$$

where $\mathbb{P}(E_x)$ is the projective space of lines through 0 in the vector space E_x. To give $\mathbb{P}(E)$ the structure of an algebraic variety, consider a cover $X = \bigcup U_\alpha$ in which E is given by transition matrixes $C_{\alpha\beta}$. By fixing an isomorphism $p^{-1}(U_\alpha) \cong U_\alpha \times V$, where V is a vector space, we thus get a map

$$\bigcup_{x \in U_\alpha} \mathbb{P}(E_x) \to U_\alpha \times \mathbb{P}(V),$$

which allows us to give this set the structure of an algebraic variety. All the structures of this type are obviously compatible, and define a structure of algebraic variety on the whole of $\mathbb{P}(E)$. This variety is called the *projectivisation* of E.

More concretely, $\mathbb{P}(E)$ is obtained by glueing together open subsets

$$\varphi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{P}(V)$$

by means of the glueing law defined by automorphisms of $(U_\alpha \cap U_\beta) \times \mathbb{P}(V)$:

$$\varphi_{\alpha\beta}(u, \xi) = \bigl(u, \mathbb{P}(C_{\alpha\beta})\xi\bigr), \tag{6.29}$$

where $u \in U_\alpha \cap U_\beta$, $\xi \in \mathbb{P}(V)$ and $\mathbb{P}(C_{\alpha\beta})$ is the projective transformation with matrix $C_{\alpha\beta}$.

We return to the variety $\sigma^{-1}(Y)$ arising as the exceptional locus of a blowup $\sigma\colon X' \to X$. It is obtained by glueing together the open sets $(Y \cap U_\alpha) \times \mathbb{P}^{m-1}$, with the glueing law given by (6.25). This law is precisely of the type (6.29) if we take $C_{\alpha\beta}$ to be the matrix

$$C_{\alpha\beta} = \bigl((h_{ij})_{|Y}\bigr).$$

Here the functions h_{ij} are determined from (6.25), and a glance at the transition matrix of the normal bundle in (6.11) shows that $C_{\alpha\beta}$ corresponds to the vector bundles $N_{X/Y}$. Thus the result of our argument can be expressed by the simple formula

$$\sigma^{-1}(Y) \cong \mathbb{P}(N_{X/Y}).$$

(d) The Behaviour of Subvarieties Under a Blowup

Proposition *Let $Z \subset X$ be a closed irreducible nonsingular subvariety of X that is transversal to Y at every point of $Y \cap Z$, and let $\sigma\colon X' \to X$ be the blowup of Y. Then the subvariety $\sigma^{-1}(Z)$ consists of two irreducible components,*

$$\sigma^{-1}(Z) = \sigma^{-1}(Y \cap Z) \cup Z',$$

and $\sigma\colon Z' \to Z$ defines the blowup of Z with centre in $Y \cap Z$.

The subvariety $Z' \subset X'$ is called the *birational transform* of $Z \subset X$ under the blowup.

Proof The proof follows closely the arguments of Section 4.3, Chapter 2. The question is local, so that we can assume that $Y \subset X$ is defined by the local equations $u_1 = \cdots = u_a = u_{a+1} = \cdots = u_b = 0$, and $Z \subset X$ by the local equations $u_{a+1} = \cdots = u_b = u_{b+1} \cdots = u_c = 0$, so that the intersection $Y \cap Z$ is defined by $u_1 = \cdots = u_a = \cdots = u_b = \cdots = u_c = 0$; here $0 \le a < b < c \le d = \dim X$, and $u_1, \dots, u_d$ is a system of local parameters on X. Then X' is defined in $X \times \mathbb{P}^{b-1}$ by the equations

$$t_i u_j = t_j u_i \quad \text{for } i, j = 1, \dots, b. \tag{6.30}$$

Write $\overline{Z}$ for the closure of $\sigma^{-1}(Z \setminus (Y \cap Z))$. Then obviously, $\sigma^{-1}(Z) = \sigma^{-1}(Y \cap Z) \cup \overline{Z}$. Every point of $\sigma^{-1}(Z \setminus (Y \cap Z))$ has $u_{a+1} = \cdots = u_c = 0$ and at least one of $u_1, \dots, u_a \ne 0$; therefore

$$t_{a+1} = \cdots = t_c = 0 \quad \text{on } \overline{Z}.$$

Hence

$$\overline{Z} \subset Z \times \mathbb{P}^{a-1},$$

where $t_1, \dots, t_a$ are homogeneous coordinates of $\mathbb{P}^{a-1}$, and the relations

$$t_i u_j = t_j u_i \quad \text{for } i, j = 1, \dots, a.$$

hold on $\overline{Z}$. But these are just the equations defining the blowup $\sigma\colon Z' \to Z$ of Z with centre in $Y \cap Z$. We see that $\overline{Z} \subset Z'$, and therefore $\overline{Z} = Z'$, since both varieties have the same dimension and Z' is irreducible. The proposition is proved. □

We conclude this section with some remarks on the notion of blowup.

Remark 6.1 It can be shown that blowing up a quasiprojective variety does not take us outside the class of quasiprojective varieties; the proof is omitted.

Remark 6.2 The existence of blowups whose centres are not points creates a whole series of new difficulties in the theory of birational maps of varieties of dimension ≥ 3. In this connection, is not understood to what extent the results we obtained for birational maps of surfaces in Section 3.4, Chapter 4 can be carried over to higher dimensions. It is known that not every birational morphism $X \to Y$ can be expressed as a composite of blowups; the counterexample is due to Hironaka. It remains an open question whether every birational map is a composite of blowups and their inverses. On the other hand, the theorem on resolving the locus of indeterminacy of a rational map by blowups holds in any dimension, if the ground field k has characteristic 0; this is also a theorem of Hironaka.

2.3 Example of Non-quasiprojective Variety

The variety that we now construct to give an example of a non-quasiprojective variety will be complete. If a complete variety is isomorphic to a quasiprojective variety, then by the theorem on the closure of the image, it would be projective. Thus it is enough to construct an example of a complete nonprojective variety.

The proof of nonprojectivity will be based on the fact that intersection numbers on a projective variety has a specific property. We therefore start with some general remarks on intersection numbers.

We use notions which are a very special case of the cycle class ring mentioned in Section 6.2, Chapter 4. In our particular case, we can easily give the definitions from first principles. Let X be a complete nonsingular 3-fold, $C \subset X$ an irreducible curve and D a divisor on X. Suppose that $C \not\subset \operatorname{Supp} D$. Then the restriction $\rho_C(D)$ defines a locally principal divisor on C (we do not assume that C is nonsingular), for which the intersection number is defined (see the remark in connection with the definition of intersection number in Section 1.1, Chapter 4). In this case the intersection number is denoted by $\deg \rho_C(D)$ and is also called the *intersection number* of the curve C and the divisor D:

$$CD = \deg \rho_C(D).$$

The arguments of Sections 1.2–1.3, Chapter 4 show that this intersection number is additive as a function of D and invariant under linear equivalence. In particular, the intersection number $C\varDelta$ is defined, where $\varDelta$ is the divisor class containing D. In any case, we only require this for the case C a nonsingular curve, when both these properties are obvious.

Consider the free Abelian group A^1 generated by all curves $C \subset X$. The intersection number $a\varDelta$ is defined for $a \in A^1$ and $\varDelta \in \operatorname{Cl} X$ by additivity. We introduce on A^1 the equivalence relation

$$a \equiv b \quad \Longleftrightarrow \quad a\varDelta = b\varDelta \quad \text{for all } \varDelta \in \operatorname{Cl} X.$$

If this holds we say that a and b are *numerically equivalent*.

We consider an example which is basic for what follows. Suppose that $a = \sum n_i C_i$ and $a' = \sum n'_j C'_j$, where all the curves C_i and C'_j lie on a nonsingular surface $Y \subset X$, and $a \sim b$ are linearly equivalent as divisors on Y; then $a \approx b$. Indeed, for any divisor D on X the operation $\rho^X_{C_i}(D)$ of restriction to C_i can be carried out in two steps:

$$\rho^X_{C_i} = \rho^Y_{C_i} \circ \rho^X_Y,$$

and hence for $a \in \operatorname{Div} Y$

$$(aD)_X = \bigl(a\rho^X_Y(D)\bigr)_Y.$$

Therefore our assertion follows from the fact that intersection numbers of divisors on Y are invariant under linear equivalence of divisors.

The preceding considerations apply to any complete variety X. The assumption that X is projective has an important consequence for X: if $a = \sum n_i C_i$ with $n_i > 0$ then $a \not\approx 0$. Indeed, when we intersect an irreducible curve C with a hyperplane section H of X we obviously have the equality

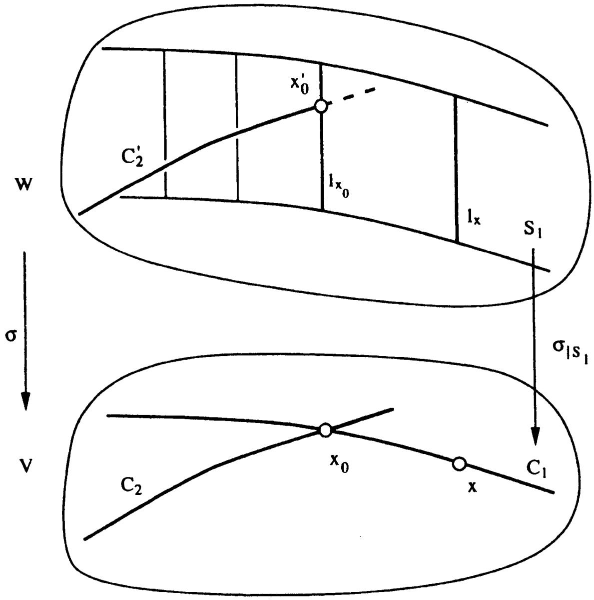
$$CH = \deg C,$$

and in particular $CH > 0$. Hence also $aH = \sum n_i C_i H > 0$.

Before we start on the construction of the example, we consider an auxiliary construction. Suppose C_1 and C_2 are two nonsingular curves in a nonsingular 3-fold V intersecting transversally at a point x_0. We assume that C_1 and C_2 are rational; our results hold independently of this assumption, but it somewhat simplifies the deduction. Let $\sigma\colon W \to V$ be the blowup of C_1 and $S_1 = \sigma^{-1}(C_1)$ the exceptional surface. The restriction

$$\sigma_{|S_1}\colon S_1 \to C_1,$$

Figure 25 The first blowup



is a $\mathbb{P}^1$-bundle by Section 2.2, (c), and we write l_x for the fibre over $x \in C_1$. By Proposition 6.2, $\sigma^{-1}(C_2)$ consists of two components:

$$\sigma^{-1}(C_2) = l_{x_0} \cup C_2';$$

here $\sigma: C_2' \to C_2$ is the blowup of C_2 with centre in x_0, and in our case is therefore an isomorphism. As a very simple exercise in the formulas defining a blowup, we leave the reader to check that S_1 and C_2' intersect in a single point x_0' with $\sigma(x_0') = x_0$, and are transversal there. We arrive at the situation of Figure 25.

Since we have assumed that the curve C_1 is rational, all its points are linearly equivalent $x_1 \sim x_2$, and hence

$$l_{x_1} \sim l_{x_2} \quad \text{on } S_1 \quad \text{for all } x_1, x_2 \in C_1.$$

Now consider a second blowup, the blowup $\tau: X \to W$ of W with centre in C_2'. The inverse image $\tau^{-1}(S_1)$ of S_1 is irreducible: by Section 2.2, (d), Proposition, $\tau^{-1}(S_1) = \tau^{-1}(x_0') \cup S_1'$, where $\tau: S_1' \to S_1$ is the blowup of S_1 centred in x_0'. It follows that $\tau^{-1}(x_0') \subset S_1'$. On S_1' we have $\tau^{-1}(l_{x_0}) = L + L'$, where $L = \tau^{-1}(x_0')$ and $\tau: L' \to l_{x_0}$ is an isomorphism. For $x \neq x_0$, the fibre $\tau^{-1}(l_x)$ is irreducible; we denote it by L_x. By what we have said, we have

$$L_x \sim L + L' \quad \text{as divisors on } S_1'. \tag{6.31}$$

Now write S_2 for the surface $\tau^{-1}(C_2')$. In this same way as S_1, it is a $\mathbb{P}^1$-bundle $S_2 \to C_2'$ with fibre L_y over $y \in C_2'$, and on S_2

$$L_{y_1} \sim L_{y_2} \quad \text{for all } y_1, y_2 \in C_2, \text{ and } L_{x_0'} = L. \tag{6.32}$$

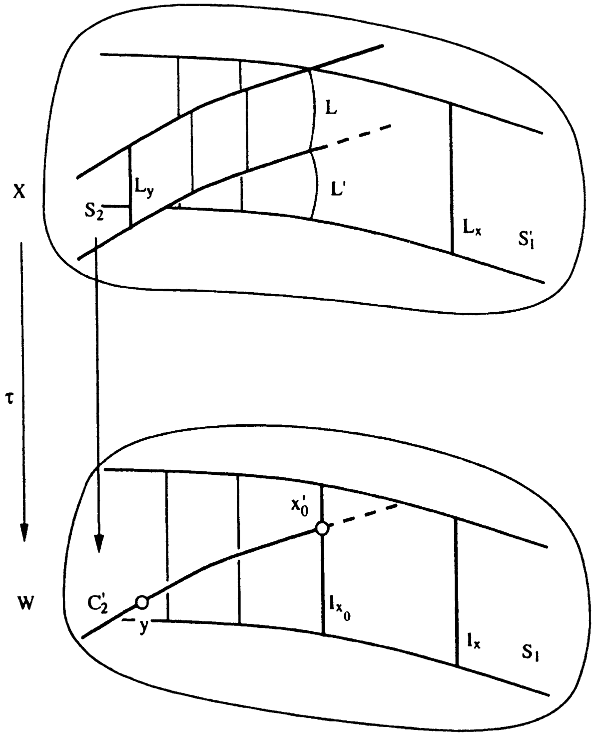


Figure 26 The second blowup

The two surfaces S_1' and S_2 intersect along the line L as shown in Figure 26.

We go over to numerical equivalence on X. Substituting (6.31) in (6.32) we get that

$$L_x \approx L + L' \approx L_y + L'. \tag{6.33}$$

The basic feature of this relation is its lack of symmetry with respect to the fibres L_x and L_y of the two ruled surfaces S_1' and S_2, arising from the order in which the blowups were performed. This is what we exploit in the example, the construction of which we now embark on.

Consider a nonsingular 3-fold V and two nonsingular rational curves $C_1, C_2 \subset V$ that intersect transversally in two points x_0 and x_1 (for example, V could contain a copy of $\mathbb{P}^2$, with C_1 a line and C_2 a conic). In the 3-fold $V_0 = V \setminus x_1$ we blow up as above first $C_1 \setminus x_1$, then the birational transform of $C_2 \setminus x_1$; we get a morphism

$$\sigma_0 \colon X_0 \to V \setminus x_1.$$

In $V_1 = V \setminus x_0$ we blow up the two curves in the opposite order, first $C_2 \setminus x_0$ then the birational transform of $C_1 \setminus x_0$; we get a morphism

$$\sigma_1 \colon X_1 \to V \setminus x_0.$$

Now the two varieties $\sigma_0^{-1}(V \setminus \{x_0, x_1\})$ and $\sigma_1^{-1}(V \setminus \{x_0, x_1\})$ are obviously isomorphic, and the morphisms σ_0 and σ_1 coincide on them. Indeed, the curve $C_1 \cup C_2 \setminus \{x_0, x_1\}$ is disconnected, and thus both $\sigma_0^{-1}(V \setminus \{x_0, x_1\})$ and $\sigma_1^{-1}(V \setminus \{x_0, x_1\})$ can be obtained by carrying out the blowup of $V \setminus \{x_0, x_1\}$ with centre $C_1 \setminus \{x_0, x_1\}$ on the open set $V \setminus C_2$ and with centre $C_2 \setminus \{x_0, x_1\}$ on the open set $V \setminus C_1$, then glueing the resulting varieties along the open set $V \setminus \{C_1 \cup C_2\}$, over which both blowups are isomorphisms.

Thus we can glue X_0 and X_1 along their open subsets $\sigma_0^{-1}(V \setminus \{x_0, x_1\})$ and $\sigma_1^{-1}(V \setminus \{x_0, x_1\})$, obtaining a 3-fold X and a morphism

$$\sigma : X \to V.$$

In X we have the relation (6.33), which we deduced using the existence of the point of intersection x_0 of C_1 and C_2. In the same way the point x_1 leads to the relation

$$L_y \approx L_1 + L_1' \approx L_x + L_1' \tag{6.34}$$

where L_1 is the irreducible curve of intersection of S_1 and S_2' over x_1 and L_1' the other component of $\sigma^{-1}(x_1)$. Adding (6.33) and (6.34) gives

$$L_x + L_y \approx L' + L_1' + L_x + L_y,$$

whence

$$L' + L_1' \approx 0. \tag{6.35}$$

To get a contradiction to X projective, it remains to prove that it is complete. For an arbitrary variety Z the projection $X \times Z \to Z$ factors as a composite of the map $\sigma \times 1 : X \times Z \to V \times Z$ and the projection $V \times Z \to Z$. Since V is projective, the image of a closed set under the second projection is closed, and we need only prove that the same holds for $\sigma \times 1$. We know that V is a union of two open sets $V \setminus x_0$ and $V \setminus x_1$, and since closed is a local property, it is enough to check that both the restrictions

$$\sigma \times 1 : (\sigma \times 1)^{-1}\bigl((V \setminus x_i) \times Z\bigr) \to (V \setminus x_i) \times Z \quad \text{for } i = 0, 1$$

take closed sets to closed sets. Now σ over $V \setminus x_i$ is just a composite of blowups, and it remains to prove that for any blowup $\sigma : U' \to U$ and any Z the morphism

$$\sigma \times 1 : U' \times Z \to U \times Z$$

takes closed sets to closed sets. Once more since the question is local, we can assume that σ is given by the local construction Section 2.2, (a), that is, $U' \subset U \times \mathbb{P}^{m-1}$ and σ is induced by the projection $U \times \mathbb{P}^{m-1} \to U$. But then our assertion follows from the fact that projective space is complete, Theorem 1.11 of Section 5.2, Chapter 1.

Thus if X were quasiprojective it would be projective, and this is a contradiction, since (6.35) is impossible in a projective variety.

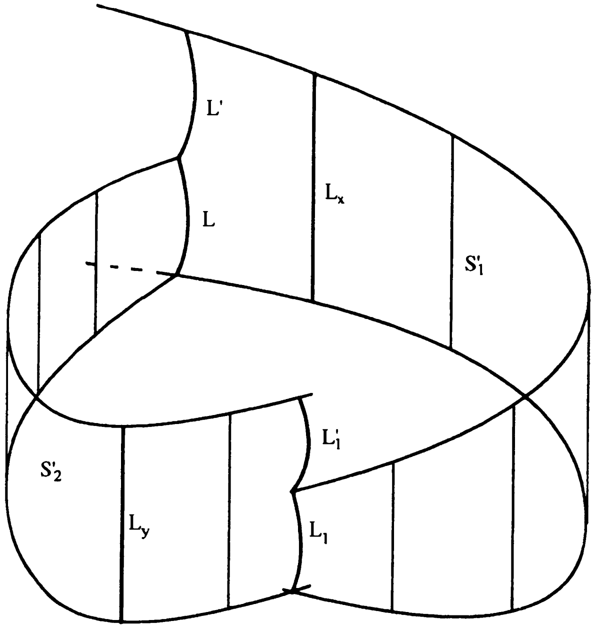


Figure 27 Hironaka's counterexample

The basic idea on which the example is built is of course the relations (6.33) and (6.34): they lead to (6.35), which cannot hold on a projective variety. These relations are perhaps clearer if we express them in a very primitive picture (Figure 27): the fibre L_{x_0} of the ruled surface S_1' is shown breaking up into two lines drawn as intervals L and L'.

Remark 6.3 It is no accident that this example has dimension 3. It can be proved that a 2-dimensional nonsingular complete variety is projective. On the other hand, there exist examples of complete nonprojective 2-dimensional varieties with singularities.

Remark 6.4 In the example we have constructed, consider an affine open set $U \subset X$. If both curves L' and L_1' in (6.35) had nonempty intersection with U, we would be able to find a divisor D such that $L'D > 0$, $L_1'D > 0$, contradicting (6.35); indeed, we could just take D to be the closure of a hyperplane section of the affine space containing U. Thus L' and L_1' are "very far apart" in X: if an affine open subset contains a point of L' then it must be disjoint from L_1'.

2.4 Criterions for Projectivity

To conclude this section, we discuss a number of criterions that characterise projective varieties among arbitrary complete varieties. We do not state them in the greatest possible generality. In particular, in the first two we assume that the varieties are nonsingular. We could avoid this assumption, but this would require some extra explanations.

1. Chevalley–Kleiman Criterion *A complete nonsingular variety X is projective if and only if every finite set of points of X is contained in an affine open subset.*

If X is a projective variety then there obviously exists a hyperplane section H not meeting any finite subset $S \subset X$, so that $S \subset X \setminus H$, and $X \setminus H$ is affine. Hence one half of the criterion is obvious. In the example of a nonprojective variety constructed at the end of Section 2.3, this criterion obviously fails (see Remark 6.4).

2. Nakai–Moishezon Criterion *A complete nonsingular variety X is projective if and only if there exists a divisor H on X such that for every irreducible subvariety $Y \subset X$,*

$$\bigl(\rho_Y(H)^m\bigr)_Y = H^m Y > 0, \quad \textit{where } m = \dim Y;$$

here $\rho_Y(H)$ is the restriction to Y of H and $(\rho_Y(H)^m)_Y$ its m-fold selfintersection number on Y.

If X is a projective variety then we can take H to be a hyperplane section. In this case

$$H^m Y = \deg Y.$$

Thus again the criterion obviously holds for projective varieties.

To state the final criterion, recall that projective space $\mathbb{P}^n$ has a line bundle $E \subset \mathbb{P}^n \times V$, where V is the vector space whose lines are represented by points of $\mathbb{P}^n$ (see Examples 6.10–6.11). Moreover, the projection $\mathbb{P}^n \times V \to V$ defines a morphism $E \to V$ that is the blowup of V centred in the origin. For this map, the unique exceptional subvariety is the zero section of E. Let $X \subset \mathbb{P}^n$ be a closed subvariety. The line bundle $E' = \rho_X(E)$, the restriction to X of E, is a closed subvariety of E, and the blowup $\sigma\colon E \to V$ defines a morphism $\sigma'\colon E' \to V$. The completeness of $\mathbb{P}^n$ implies that σ takes closed sets to closed sets. Hence $W = \sigma'(E')$ is an affine variety. In fact it is easy to see that W is the affine cone over $X \subset \mathbb{P}^n$ as in the proof of Theorem 6.7 (compare Exercise 8 of Section 4.5). Obviously the unique exceptional subvariety of σ' is the zero section of E'.

These arguments prove the "only if" part of the following criterion.

Grauert Criterion *A complete variety X is projective if and only if there exists a line bundle E over X, and a morphism $f\colon E \to V$ to an affine variety V such that f is birational, and the unique exceptional subvariety of f is the zero section of E. A shorter way of stating the condition is that the zero section of the line bundle E can be contracted to a point.*

2.5 Exercises to Section 2

1 Give an alternative proof of Theorem 6.1 using Chow's lemma and a reduction to Theorem 2.12 of Section 3.1, Chapter 2.

2 If X is a complete variety and $\sigma\colon X' \to X$ a blowup, prove that X' is also complete.

3 If E and E' are vector bundles such that $E' = E \otimes L$ for L a line bundle, and $\mathbb{P}(E)$, $\mathbb{P}(E')$ are as in Exercise 10 of Section 1.5, prove that $\mathbb{P}(E) \cong \mathbb{P}(E')$.

4 Suppose that X is a nonsingular complete variety with $\dim X = 3$, and $Y \subset X$ a nonsingular curve; let $\sigma\colon X' \to X$ be the blowup of X with centre Y, and $l = \sigma^{-1}(y_0)$ with $y_0 \in Y$. Prove that $\sigma^*(D)l = 0$, where D is any divisor on X and $\sigma^*(D)$ its pullback to X'.

5 Under the conditions of Exercise 4, set $S = \sigma^{-1}(Y)$. Prove that $Sl = -1$. [Hint: Consider a surface D on X containing Y and nonsingular at y_0, and apply the result of Exercise 4 to D. Compare the calculations of Section 3.2, Chapter 4.]

6 Prove that for any nonsingular projective 3-fold X there exists a complete non-projective variety birational to X.

7 Prove that for any irreducible variety X there exists a quasiprojective variety $\overline{X}$ and a surjective birational morphism $f\colon \overline{X} \to X$. There exists an embedding $\overline{X} \hookrightarrow \mathbb{P}^n \times X$ such that f is the restriction to $\overline{X}$ of the projection $\mathbb{P}^n \times X \to X$.

3 Coherent Sheaves

3.1 Sheaves of $\mathcal{O}_X$-Modules

Sheaves of modules over the sheaf of rings $\mathcal{O}_X$ have already appeared in Section 1.3 in connection with vector bundles. Sheaves of this type are an extraordinarily convenient tool in the study of algebraic varieties; we discuss one example of this in this section. But first we start with certain general properties of these sheaves.

Consider the most general situation: a ringed space, that is, a topological space X with a given sheaf of rings $\mathcal{O}$. In what follows we consider sheaves on X that are sheaves of modules over $\mathcal{O}$; we usually omit mention of this, speaking simply of sheaves of modules. Any sheaf of Abelian groups on a topological space X can obviously be viewed as a sheaf of modules over a sheaf of rings $\mathcal{O}$ by taking $\mathcal{O}$ to be the sheaf of locally constant $\mathbb{Z}$-valued functions.

The definition of a homomorphism $f\colon \mathcal{F} \to \mathcal{G}$ of sheaves of modules was given in Section 1.3. Recall that it is a system of $\mathcal{O}(U)$-module homomorphisms $f_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ satisfying certain compatibility requirements.

Example 6.13 Let X be a nonsingular algebraic variety over k, $\mathcal{O}_X$ the sheaf of regular functions, and Ω^1 the sheaf of regular differential 1-forms. Sending $f \in \mathcal{O}_X(U)$ to the differential $\mathrm{d}f \in \Omega^1(U)$ defines a homomorphism of sheaves

$$\mathrm{d}\colon \mathcal{O}_X \to \Omega^1.$$

This is a homomorphism of sheaves of modules over the sheaf of locally constant k-valued functions, but not over $\mathcal{O}_X$ (because, by Leibnitz' rule, d is not $\mathcal{O}_X$-linear).

Our immediate objective is to define the kernel and image of a homomorphism of sheaves of modules. The first definition is completely obvious. Let $f\colon \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves of modules. Set $\mathcal{K}(U) = \ker f_U$. By definition of a homomorphism it follows that for $U \subset V$ we have $\rho_U^V(\mathcal{K}(V)) \subset \mathcal{K}(U)$. Hence the system $\{\mathcal{K}(U), \rho_U^V\}$ is a presheaf; an easy verification shows that it is a sheaf of modules. By definition this is the *kernel* of f.

The kernel of a homomorphism is an example of a *subsheaf* of a sheaf $\mathcal{F}$. This is a sheaf of modules $\mathcal{F}'$ such that $\mathcal{F}'(U) \subset \mathcal{F}(U)$ for every open set $U \subset X$, and such that $\rho_{U,\mathcal{F}'}^V$ is the restriction of $\rho_{U,\mathcal{F}}^V$ to the submodule $\mathcal{F}'(V)$.

The image of a homomorphism $f\colon \mathcal{F} \to \mathcal{G}$ is a somewhat more complicated notion. The point is that the $\mathcal{O}(U)$-modules $\mathcal{I}(U) = \operatorname{im} f_U$, together with the homomorphisms $\rho_{U,\mathcal{G}}^V$, define a presheaf that is in general not a sheaf.

Example 6.14 Let X be a nonsingular irreducible curve, and $\mathcal{K}^*$ the constant sheaf with $\mathcal{K}^*(U) = k(X)^*$ the group of nonzero elements of $k(X)$ under multiplication; let $\mathcal{D}$ be the sheaf of local divisors, defined by $\mathcal{D}(U) = \operatorname{Div} U$, with the obvious restriction homomorphisms. The homomorphism $f\colon \mathcal{K}^* \to \mathcal{D}$ takes a function $u \in \mathcal{K}^*(U)$ into its divisor $\operatorname{div} u$ on U. Since every divisor is locally principal, for every $D \in \operatorname{Div} U$ and every point $x \in U$ there exists a neighbourhood V_x of x and a function $u \in \mathcal{K}^*(V_x)$ such that $f_{V_x}(u) = D$; in other words, $(\operatorname{im} f)(V_x) \ni \rho_{V_x}^U(D)$. However, it is not always the case that $D \in (\operatorname{im} f)(U)$. For example, if X is projective then not every divisor is principal. Thus $\operatorname{im} f$ does not satisfy condition (2) in the definition of a sheaf in Section 2.3, Chapter 5.

Thus it seems natural to define the image of a homomorphism $f\colon \mathcal{F} \to \mathcal{G}$ of sheaves of modules as follows. First define the presheaf $\mathcal{I}'$ by setting

$$\mathcal{I}'(U) = f_U\big(\mathcal{F}(U)\big) \quad \text{for } U \subset X;$$

then take the sheafication $\mathcal{I}$ of the presheaf $\mathcal{I}'$ as in Section 2.4, Chapter 5; it is called the *image* of f and is denoted by $\operatorname{im} f$.

Recalling the definition of the sheafication of a presheaf, we see that $\operatorname{im} f$ is a subsheaf of $\mathcal{G}$, and $(\operatorname{im} f)(U)$ consists of elements $a \in \mathcal{G}(U)$ such that every point $x \in U$ has a neighbourhood U_x for which

$$\rho_{U_x}^U(a) \in f_{U_x}\big(\mathcal{F}(U_x)\big).$$

Obviously, f defines a homomorphism

$$\mathcal{F} \to \operatorname{im} f.$$

It follows at once from the definition that a homomorphism $f\colon \mathcal{F} \to \mathcal{G}$ for which $\ker \mathcal{F} = 0$ and $\operatorname{im} f = \mathcal{G}$ is an isomorphism.

A sequence $\mathcal{F}_1 \xrightarrow{f_1} \mathcal{F}_2 \to \cdots \xrightarrow{f_n} \mathcal{F}_{n+1}$ of homomorphisms is called an *exact sequence* if $\operatorname{im} f_i = \ker f_{i+1}$ for $i = 1, \dots, n$. If $0 \to \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H} \to 0$ is an exact sequence then $\mathcal{F}$ can be viewed as a subsheaf of $\mathcal{G}$. Because of this,

$$(\operatorname{im} f)(U) = f\big(\mathcal{F}(U)\big);$$

that is, in constructing the image sheaf of an injective homomorphism f, passing to the sheafication is unnecessary. Hence the sequence

$$0 \to \mathcal{F}(U) \xrightarrow{f_U} \mathcal{G}(U) \xrightarrow{g_U} \mathcal{H}(U) \tag{6.36}$$

is exact for any open set U.

Example 6.14 shows that the sequence

$$0 \to \mathcal{F}(U) \xrightarrow{f_U} \mathcal{G}(U) \xrightarrow{g_U} \mathcal{H}(U) \to 0$$

is in general not exact (for example, for $U = X$). This phenomenon is the reason for the existence of a nontrivial theory of sheaf cohomology.

For any subsheaf $\mathcal{F}$ of a sheaf $\mathcal{G}$ one can construct a homomorphism $f\colon \mathcal{G} \to \mathcal{H}$ such that $\ker f = \mathcal{F}$ and $\operatorname{im} f = \mathcal{H}$. To obtain this, set

$$\mathcal{H}'(U) = \mathcal{G}(U)/\mathcal{F}(U)$$

and define homomorphisms $\rho^V_{U,\mathcal{H}'}$ as the maps induced on these quotient groups by the homomorphisms $\rho^V_{U,\mathcal{G}}$. We define $\mathcal{H}$ to be the sheafication of $\mathcal{H}'$.

It is easy to check that the stalks of this sheaf satisfy

$$\mathcal{H}_x = \mathcal{G}_x/\mathcal{F}_x.$$

Hence an element $a \in \mathcal{G}(U)$ defines elements $a_x \in \mathcal{H}_x$ for all points $x \in U$. An obvious verification shows that the set of all the $\{a_x\}$ specify an element $a' \in \mathcal{H}(U)$, and $f\colon a \mapsto a'$ defines a homomorphism with the required properties. The sheaf $\mathcal{H}$ is the *quotient sheaf* of $\mathcal{G}$ by $\mathcal{F}$. Obviously the sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is exact.

Example 6.15 Let X be an irreducible algebraic variety over a field k and $\mathcal{K}^*$ the sheaf of locally constant functions with values in the multiplicative group of $k(X)$. The sheaf $\mathcal{O}^*$ is defined by setting $\mathcal{O}^*(U)$ to be the set of invertible elements of $\mathcal{O}(U)$; here $\mathcal{K}^*$ and $\mathcal{O}^*$ are viewed as sheaves of Abelian groups. It is each to check that the quotient sheaf $\mathcal{D} = \mathcal{K}^*/\mathcal{O}^*$ has $\mathcal{D}(U)$ isomorphic to the group of locally principal divisors of U.

Definition The *support* of a sheaf $\mathcal{F}$ is the set $X \setminus W$, where W is the union of all open sets $V \subset X$ with $\mathcal{F}(U) = 0$ for all nonempty open set $U \subset V$. This is a closed set, and is denoted by $\operatorname{Supp}\mathcal{F}$.

Proposition *If S is the support of a sheaf $\mathcal{F}$ and $U \subset V$ are two open sets such that $U \cap S = V \cap S$ then the restriction $\rho_U^V : \mathcal{F}(V) \to \mathcal{F}(U)$ is an isomorphism.*

Proof Let $a \in \mathcal{F}(V)$ be such that $\rho_U^V(a) = 0$. By definition of S every point $x \in V$ with $x \notin S$ has a neighbourhood V_x, which we can assume to be contained in V, for which

$$\rho_{V_x}^V(a) = 0.$$

By the assumption, U is a neighbourhood with this property for points $x \in S$. It follows from the definition of a sheaf that $a = 0$, and thus ρ_U^V is injective.

Let $a \in \mathcal{F}(U)$. Consider a cover $V = \bigcup U_\alpha$ with $U_0 = U$ and $U_\alpha \cap S = \emptyset$ for $\alpha \neq 0$ (for example, this holds if U_α for $\alpha \neq 0$ are sufficiently small neighbourhoods of points $x \in V$ with $x \notin S$). Set $a_0 = a$ and $a_\alpha = 0$ for $\alpha \neq 0$. From the assumption of the proposition it follows that

$$\rho_{U_\alpha \cap U_\beta}^{U_\alpha}(a_\alpha) = \rho_{U_\alpha \cap U_\beta}^{U_\beta}(a_\beta).$$

Hence by the definition of a sheaf there exists an element $a' \in \mathcal{F}(V)$ such that

$$\rho_{U_\alpha}^V(a') = a_\alpha,$$

and in particular, when $\alpha = 0$, we have $\rho_U^V(a') = a$. Thus ρ_U^V is surjective, and the proposition is proved. $\square$

It follows from the proposition that if S is the support of a sheaf $\mathcal{F}$ then the modules $\mathcal{F}(U)$ are canonically isomorphic for all open sets U whose intersection with S is a given subset. Therefore we can define a sheaf $\overline{\mathcal{F}}$ on S by setting

$$\overline{\mathcal{F}}(\overline{U}) = \mathcal{F}(U), \quad \text{where } U \cap S = \overline{U}$$

for open sets $\overline{U} \subset S$.

Example 6.16 Let X be a scheme and $Y \subset X$ a closed subscheme. Define a subsheaf $\mathcal{J}_Y$ of the structure sheaf $\mathcal{O}_X$ by the condition $\mathcal{J}_Y(U) = \mathfrak{a}_Y$ if U is an affine open set with $U = \operatorname{Spec} A$ and $\mathfrak{a}_Y \subset A$ is the ideal of the subscheme $Y \cap U$. Obviously if U is disjoint from Y then $\mathcal{J}_{Y|U} = \mathcal{O}_{X|U}$. Hence the sheaf $\mathcal{F} = \mathcal{O}_X/\mathcal{J}_Y$ is equal to 0 on such open sets, that is, its support is contained in Y. The corresponding sheaf $\overline{\mathcal{F}}$ on Y coincides with the structure sheaf $\mathcal{O}_Y$ on the subscheme Y.

Remark The definition of support just given is not the usually accepted one, but it is slightly more convenient for our purposes. In any case, in what follows, the two definitions coincide in the cases where they are applied.

3.2 Coherent Sheaves

Locally free sheaves have already appeared in Section 1.3 in connection with vector bundles. We now consider a class of sheaves that are to arbitrary finite modules as locally free sheaves are to finite free modules.

We now apply the notions introduced in Section 3.1 to the case that $X, \mathcal{O}_X$ is an arbitrary scheme. We start with local considerations, and suppose that $X = \operatorname{Spec} A$, where A is an arbitrary ring.

For any module M over a ring A and any multiplicative system S of elements of A we define the localisation of M with respect to S, setting

$$M_S = M \otimes_A A_S.$$

M_S can be described in the same way as the localisation A_S in Section 1.1, Chapter 5. It consists of pairs (m, s) with $m \in M$ and $s \in S$ with the same rules of identification, addition and multiplication by elements of A_S as in the case of rings; we write m/s for the pair (m, s). In particular, taking S to be the system of powers of an element $f \in A$ gives the module M_f over A_f.

The homomorphisms $A_S \to A_{S'}$ defined for $S \subset S'$ generate homomorphisms $M_S \to M_{S'}$. This allows us to associate with an A-module M a sheaf $\widetilde{M}$ on $\operatorname{Spec} A$. The definition mirrors exactly that of the sheaf $\mathcal{O}$, to which it reduces in the case $M = A$. In view of this, we omit some of the verifications, when these do not differ in the general case from those carried out in Section 2.2, Chapter 5.

For an open set of the form $U = D(f)$ with $f \in A$ we set

$$\widetilde{M}(U) = M_f.$$

For an arbitrary open set U we consider all $f \in A$ for which $D(f) \subset U$. Whenever $D(g) \supset D(f)$, we have homomorphisms

$$M_g \to M_f.$$

Using these, we can define the projective limit of the groups M_f. Set

$$\widetilde{M}(U) = \varprojlim M_f$$

where the limit runs over $f \in A$ such that $D(f) \subset U$. Then $\widetilde{M}(U)$ is a module over the ring $\mathcal{O}(U) = \varprojlim A_f$; this is a general property of projective limits. An inclusion $U \subset V$ defines a homomorphism $\rho_U^V \colon \widetilde{M}(V) \to \widetilde{M}(U)$ as in the case $M = A$. The system $\{\widetilde{M}(U), \rho_U^V\}$ defines a sheaf $\widetilde{M}$ of modules over the sheaf of rings $\mathcal{O}_X$.

A homomorphism of A-modules $\varphi \colon M \to N$ defines homomorphisms $\varphi_f \colon M_f \to N_f$ for all $f \in A$, and on passing to the limit, a homomorphism of sheaves $\widetilde{\varphi} \colon \widetilde{M} \to \widetilde{N}$. If $\varphi \colon M \to N$ and $\psi \colon N \to L$ are two such homomorphisms then

$$\widetilde{\varphi \circ \psi} = \widetilde{\varphi} \circ \widetilde{\psi}.$$

M can be recovered from $\widetilde{M}$. Namely, we have a generalisation of the relation proved in Section 2.2, Chapter 5:

$$\widetilde{M}(\operatorname{Spec} A) = M;$$

the proof is word-for-word the same. It follows that $M \mapsto \widetilde{M}$ is a one-to-one correspondence between modules M and sheaves of the form $\widetilde{M}$. Moreover, a simple check allows us to deduce that $\varphi \mapsto \widetilde{\varphi}$ is an isomorphism of groups

$$\operatorname{Hom}_A(M, N) \cong \operatorname{Hom}_{\mathcal{O}}(\widetilde{M}, \widetilde{N}),$$

from the group of A-module homomorphisms to the group of homomorphisms of sheaves of $\mathcal{O}_{\operatorname{Spec} A}$-modules.

We can now proceed to globalise these notions. Let X be a Noetherian scheme.

Definition A sheaf $\mathcal{F}$ on X is *coherent* if every point $x \in X$ has an affine neighbourhood U of the form $U = \operatorname{Spec} A$ with A a Noetherian ring, such that $\mathcal{F}_{|U}$ is isomorphic to a sheaf of the form $\widetilde{M}$ for some finite A-module M.

Proposition *If $X = \operatorname{Spec} A$ is an affine and Noetherian scheme, then any coherent sheaf $\mathcal{F}$ on X is of the form $\widetilde{M}$, where M is a finite A-module.*

Proof We set $\mathcal{F}(X) = M$ and prove that $\mathcal{F} = \widetilde{M}$.

Since open sets of the form $D(f)$ are a basis of the Zariski topology, there exist elements $f_i \in A$ such that $\bigcup D(f_i) = X$ and $\mathcal{F}$ is isomorphic over $D(f_i)$ to a sheaf $\widetilde{M}_i$, where M_i is a finite A_{f_i}-module. Since $\operatorname{Spec} A$ is compact we can assume that the f_i are finite in number.

For any $g \in A$, since $\mathcal{F}(D(g))$ is an A_g-module, the restriction homomorphism $\rho^X_{D(g)} \colon M = \mathcal{F}(X) \to \mathcal{F}(D(g))$ extends in a unique way to a homomorphism of A_g-modules.

$$\varphi_g \colon \widetilde{M}\big(D(g)\big) \to \mathcal{F}\big(D(g)\big).$$

One checks easily that this system of homomorphisms defines a homomorphism $\widetilde{M} \to \mathcal{F}$ of sheaves of modules.

Everything thus reduces to proving that the homomorphism φ_g is an isomorphism. For this, consider the sequence of homomorphisms

$$0 \to M \xrightarrow{\lambda} \bigoplus_i M_i \xrightarrow{\mu} \bigoplus_{i,j} M_{ij}, \tag{6.37}$$

where

$$M_{ij} = (M_i)_{f_j} = (M_j)_{f_i} = \mathcal{F}\big(D(f_i f_j)\big), \qquad \lambda(m) = \big(\dots, \rho^X_{D(f_i)}(m), \dots\big),$$

$$\text{and} \quad \mu(\dots, m_i, \dots, m_j, \dots) = \big(\dots, \big(\rho^{D(f_i)}_{D(f_i f_j)}(m_i) - \rho^{D(f_j)}_{D(f_i f_j)}(m_j)\big), \dots\big).$$

We view M_i and M_j as A-modules in (6.37). It follows from the definition of sheaf that (6.37) is exact. We now use a property of the functor $M \mapsto M_g$ which is important, although trivial to verify: it takes exact sequences into exact sequences. In particular,

$$0 \to M_g \xrightarrow[g]{\lambda} \bigoplus (M_i)_g \xrightarrow[g]{\mu} \bigoplus (M_{ij})_g$$

is exact. On the other hand, consider the sheaf $\mathcal{F}_{|D(g)}$. For it we have a similar exact sequence

$$0 \to \mathcal{F}\big(D(g)\big) \xrightarrow{\lambda'_g} \bigoplus_i \mathcal{F}\big(D(gf_i)\big) \xrightarrow{\mu'_g} \bigoplus_{ij} \mathcal{F}\big(D(gf_i f_j)\big).$$

But $\mathcal{F}(D(gf_i)) = (M_i)_g$ and $\mathcal{F}(D(gf_i f_j)) = (M_{ij})_g$. These isomorphisms induce an isomorphism $\varphi'_g \colon M_g \to \mathcal{F}(D(g))$. It is easy to check that $\varphi'_g = \varphi_g$ on the images of elements of M, and therefore on the whole of M_g. This proves that φ is an isomorphism and $\mathcal{F} = \widetilde{M}$.

It remains to prove that M is Noetherian; we know that the modules $M_i = M_{f_i}$ are Noetherian. Let M_n be an ascending chain of submodules of M. Then $(M_n)_{f_i} = (M_{n+1})_{f_i}$ for all f_i and for n sufficiently large. It follows from this that $M_n = M_{n+1}$. The proposition is proved. □

Example 6.17 The simplest example of a coherent sheaf is the structure sheaf $\mathcal{O}_X$. In the case $X = \operatorname{Spec} A$, this is the ring A viewed as a module over itself. A more general example is the sheaf $\mathcal{L}_E$ corresponding to a vector bundle over a scheme X as in Theorem 6.2.

Example 6.18 For any sheaf $\mathcal{F}$ on a scheme X, the *dual sheaf* $\mathcal{G} = \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ is the sheafication of the presheaf $\mathcal{G}(U) = \operatorname{Hom}(\mathcal{F}(U), \mathcal{O}_X(U))$. If $X = \operatorname{Spec} A$ and $\mathcal{F} = \widetilde{M}$ then $\mathcal{H}om(\mathcal{F}, \mathcal{O}_X) = \widetilde{N}$, where $N = \operatorname{Hom}_A(M, A)$. If A is Noetherian and $M = Am_1 + \cdots + Am_r$ is finite then a homomorphism $M \to A$ is determined by its values on the generators m_i, so that $\operatorname{Hom}_A(M, A) \subset A^r$, and is therefore again finite. It follows from this that if X is a Noetherian scheme and $\mathcal{F}$ is coherent then $\mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ is again coherent.

Example 6.19 Let X be a scheme of finite type over k. We define for X the analogue of the cotangent sheaf Ω^1_X (Example 6.7). If $X = \operatorname{Spec} A$ then we constructed in Section 5.2, Chapter 3 an A-module Ω_A that coincides with $\Omega^1_X[X]$ for a nonsingular variety X. By construction, Ω_A is a finite A-module. For any scheme X of finite type over k and any affine open $U = \operatorname{Spec} A$ we set $\Omega(U) = \Omega_A$. The sheafication Ω of this subsheaf is coherent and is called the *cotangent sheaf*. The sheaf $\Theta = \mathcal{H}om(\Omega, \mathcal{O}_X)$ is also coherent and is called the *tangent sheaf*. If $X = \operatorname{Spec} A$ then $\Theta(X) = \operatorname{Der}_k(A, A)$ is the module of derivations of A (compare Exercise 10 of Section 5.5, Chapter 3). If X is nonsingular then, as we know, both sheaves Ω and Θ are locally free, and correspond to the cotangent and tangent bundles.

Example 6.20 Let X be a Noetherian scheme, Y a closed subscheme and $\mathcal{I}_Y$ the sheaf of ideals corresponding to Y (Example 6.16). Since $\mathcal{O}_X(U)$ is Noetherian by assumption, $\mathcal{I}_Y$ is a coherent sheaf.

Example 6.21 Under the assumptions of Example 6.20, the sheaf of modules $\mathcal{I}_Y/\mathcal{I}_Y^2$ is coherent. We prove that if X and Y are nonsingular then it is locally free. This is a local assertion, and it is enough to check it in the case $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ and $B = A/I$, and we can even assume that A is the local ring of a point $x \in X$. Since X and Y are nonsingular we can assume that $I = (u_1, \dots, u_m)$, where $u_1, \dots, u_n$ (with $n > m$) is a system of local parameters of the maximal ideal of the ring A. Obviously I/I^2 is generated as B-module by $u_1, \dots, u_m$, and we need only check that they are free. This means that if $\sum u_i a_i \in I^2$ then $a_i \in I$. Suppose that $\sum u_i a_i = \sum u_i v_i$ with $v_i \in I$. Then $\sum u_i a_i' = 0$ where $a_i' = a_i - v_i$. Hence $u_i a_i' \in (u_1, \dots, \widehat{u_i}, \dots, u_m)$, and since $u_1, \dots, u_n$ is a regular sequence (see Section 1.2, Chapter 4), it follows that $a_i' \in (u_1, \dots, \widehat{u_i}, \dots, u_m) \subset I$, and hence $a_i \in I$.

Thus in this case, the sheaf $\mathcal{I}_Y/\mathcal{I}_Y^2$ corresponds to some vector bundle on Y. The transition matrixes $C_{\alpha\beta}$ of this vector bundle are of the form $C_{\alpha\beta} = (h_{ij})$, where the h_{ij} are given as follows: if $u_{\alpha,1}, \dots, u_{\alpha,m}$ are local equations of Y in U_α and $u_{\beta,1}, \dots, u_{\beta,m}$ local equations of Y in U_β and $u_{\alpha,i} = \sum f_{ij} u_{\beta,j}$ then h_{ij} is the restriction to Y of f_{ij}. As we saw in Section 1.3, this is the transition matrix of the vector bundle $N^*_{X/Y}$, which is in this case the vector bundle corresponding to the sheaf $\mathcal{I}_Y/\mathcal{I}_Y^2$. In the general case (when X and Y are not assumed to be nonsingular), $\mathcal{I}_Y/\mathcal{I}_Y^2$ is the *conormal sheaf* to Y in X. If X and Y are nonsingular then the vector bundle $N_{X/Y}$ corresponds to the sheaf $\mathcal{H}om(\mathcal{I}_Y/\mathcal{I}_Y^2, \mathcal{O}_Y)$. This sheaf is called the *normal sheaf* of the subscheme $Y \subset X$ and denoted by $\mathcal{N}_{X/Y}$.

We give an interpretation in these terms of the sequence

$$0 \to \Theta_Y \to j^*\Theta_X \to N_{X/Y} \to 0, \tag{6.38}$$

where j^* is the restriction to Y. For the corresponding sheaves and affine varieties it gives

$$0 \to \operatorname{Der}_k(B, B) \to \operatorname{Der}_k(A, B) \to \operatorname{Der}_k(I, B) \to 0, \tag{6.39}$$

where $B = A/I$, and $\operatorname{Der}_k(P, Q)$ is the module of derivations $D\colon P \to Q$. It is easy to see that $D(I^2) = 0$ for $D \in \operatorname{Der}_k(I, B)$, so that $\operatorname{Der}_k(I, B) = \operatorname{Hom}_B(I/I^2, B)$. Hence the sequences (6.38) and (6.39) coincide.

3.3 Dévissage of Coherent Sheaves

We now discuss a method that allows us to reduce arbitrary coherent sheaves to free sheaves (admittedly, only in some very coarse respects).

Proposition 6.1 *For any coherent sheaf $\mathcal{F}$ over a Noetherian reduced irreducible scheme X, there exists a dense open set W such that $\mathcal{F}_{|W}$ is free.*

Proof The assertion is local in nature, so that we can restrict to the case $X = \operatorname{Spec} A$, where A is a Noetherian ring without nilpotents and $\mathcal{F} = \widetilde{M}$ for a finite A-module M. In addition, we can obviously assume that X is irreducible. Then X reduced and irreducible implies that A has no zerodivisors.

Recall that the rank of an A-module is the maximal number of linearly independent elements of M over A. By assumption, M has finite rank. Write r for the rank, and let $x_1, \dots, x_r \in M$ be linearly independent over A; by definition, they generate a free submodule $M' \subset M$. Let $y_1, \dots, y_m$ be a system of generators of M. Then for each i there exist elements $d_i \in A$ with $d_i \neq 0$ such that

$$d_i y_i \in M'. \tag{6.40}$$

Consider the open set $W = D(d)$, where $d = d_1 \cdots d_m$. The sheaf $\mathcal{F}_{|W}$ is isomorphic to $\widetilde{M}_d$. But $M_d = M'_d$ by (6.40), and hence

$$\mathcal{F}_{|W} \cong \widetilde{M}'_d.$$

Now M'_d is a free module over the ring A_d, since M' is free. The proposition is proved. □

Proposition 6.2 *For any coherent sheaf $\mathcal{F}$ over a Noetherian reduced irreducible scheme X, there exists a coherent sheaf $\mathcal{G}$ containing a free subsheaf $\mathcal{O}^r$, and a homomorphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ such that the two sheaves* $\ker\varphi$ *and $\mathcal{G}/\mathcal{O}_r$ both have support distinct from X.*

As we will see, in the proof we construct a homomorphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ such that both $\ker\varphi$ and $\mathcal{G}/\operatorname{im}\varphi$ have support distinct from the whole of X. Since $\mathcal{G}/\mathcal{O}^r$ also has support distinct from the whole of X, Proposition 6.2 shows that any coherent sheaf is "free modulo sheaves with support distinct from the whole of X".

Proof Let W be the open set and $f\colon \mathcal{F}_{|W} \xrightarrow{\sim} \mathcal{O}^r{}_{|W}$ the isomorphism whose existence was established in Proposition 6.1. We can assume that W is a principal open set, and will do so in what follows. Define the sheaf $\mathcal{G}$ by the condition

$$\mathcal{G}(U) = f_{U\cap W}\big(\rho^U_{U\cap W}\mathcal{F}(U)\big) + \rho^U_{U\cap W}\big(\mathcal{O}^r(U)\big). \tag{6.41}$$

Since $\rho^U_{U\cap W}(\mathcal{O}^r(U)) \subset \mathcal{O}^r(U\cap W)$ and $f_{U\cap W}(\rho^U_{U\cap W}\mathcal{F}(U)) \subset \mathcal{O}^r(U\cap W)$, both terms of the right-hand side of (6.41) are contained in the same group. We consider the sum of these subgroups, which obviously becomes an $\mathcal{O}(U)$-submodule of $\mathcal{O}^r(U\cap W)$ when we set

$$ax = \rho^U_{U\cap W}(a)x \quad \text{for } a \in \mathcal{O}(U) \text{ and } x \in \mathcal{O}^r(U\cap W).$$

Since $\mathcal{F}(U)$ and $\mathcal{O}^r(U)$ are finite $\mathcal{O}(U)$-modules, the same holds for $\mathcal{G}(U)$.

The definition of the homomorphisms $\rho^U_{V,\mathcal{G}}$ is self-explanatory. It follows at once from what we said above that the sheaf $\mathcal{G}$ we have constructed is coherent.

For the sheaf $\mathcal{O}^r$, the restriction $\rho^U_{U\cap W}$ is an inclusion. It is enough to verify this for an affine open set $U = \operatorname{Spec} A$. Consider a principal open set $D(f) \subset U \cap W$. The kernel of $\rho^U_{D(f)}$ consists of elements $x \in A$ such that $f^n x = 0$ for some $n \geq 0$; since X is irreducible, the ring A has no zerodivisors, and hence $x = 0$. A fortiori, $\ker \rho^U_{U\cap W} = 0$. Thus $\rho^U_{U\cap W}$ allows us to identify the sheaf $\mathcal{O}^r$ with a subsheaf of $\mathcal{G}$.

We define the homomorphism $\varphi \colon \mathcal{F} \to \mathcal{G}$ by the condition

$$\varphi_U = f_{U\cap W} \circ \rho^U_{U\cap W}.$$

If $U \subset W$ then

$$\mathcal{G}(U) = f_U\big(\rho^U_{U\cap W}\big(\mathcal{F}(U)\big)\big) = f_U\big(\mathcal{F}(U)\big) = \mathcal{O}^r(U) = \rho^U_{U\cap W}\big(\mathcal{O}^r(U)\big),$$

and f_U is an isomorphism. Hence φ_U is an isomorphism, and $\mathcal{G}(U) = \mathcal{O}^r(U)$. This proves that the sheaves $\ker\varphi$ and $\mathcal{G}/\mathcal{O}^r$ are both 0 on W, and hence they have supports contained in $X \setminus W$. The proposition is proved. □

Proposition 6.2 leads us to the question of the structure of coherent sheaves whose support is distinct from the whole scheme. If the support of $\mathcal{F}$ is a closed set $Y \subset X$ then by the discussion at the end of Section 3.1, there is a sheaf $\overline{\mathcal{F}}$ on Y defined by the condition

$$\overline{\mathcal{F}}(\overline{U}) = \mathcal{F}(U), \quad \text{where } U \cap Y = \overline{U}$$

for open sets $\overline{U} \subset Y$.

We consider Y as a reduced closed subscheme of $\overline{X}$. Is $\overline{\mathcal{F}}$ a coherent sheaf on Y, or even a sheaf of $\mathcal{O}$-modules? This is false in general, as shown by the following example. Suppose that $X = \operatorname{Spec} \mathbb{Z}$, and let $\mathcal{F}$ be the coherent sheaf corresponding to the module $\mathbb{Z}/p^2\mathbb{Z}$, where p is a prime. The support of $\mathcal{F}$ is the prime ideal (p), and the corresponding reduced subscheme is $\operatorname{Spec}(\mathbb{Z}/p\mathbb{Z})$. It is obviously impossible to put a $(\mathbb{Z}/p)$-module structure on $\mathbb{Z}/p^2\mathbb{Z}$.

Nevertheless, we prove that there is a weaker sense in which the sheaf $\mathcal{F}$ can be reduced to coherent sheaves on Y.

Proposition 6.3 *A coherent sheaf $\mathcal{F}$ on a Noetherian scheme X with support $Y \neq X$ has a chain of subsheaves*

$$\mathcal{F} = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots \supset \mathcal{F}_m = 0$$

such that each quotient sheaf $\overline{\mathcal{F}}_i/\overline{\mathcal{F}}_{i+1}$ is a coherent sheaf of $\mathcal{O}_Y$-modules.

Proof In Example 6.16, we gave the example of the sheaf $\mathcal{I}_Y$ of ideals of the reduced subscheme Y. Obviously $\overline{\mathcal{F}}$ is a coherent sheaf of $\mathcal{O}_Y$-modules if

$$\mathcal{I}_Y \cdot \mathcal{F} = 0. \tag{6.42}$$

Indeed, under this assumption, all the $\mathcal{O}_X(U)$-modules $\mathcal{F}(U)$ are modules over $\mathcal{O}_X(U)/\mathcal{I}_Y(U) = \mathcal{O}_Y(\overline{U})$. Thus if $\mathcal{F}$ is of the form $\widetilde{M}$ on an affine open set

$U = \operatorname{Spec} A$ then $\mathfrak{a}_Y \cdot M = 0$, and M is therefore an $(A/\mathfrak{a}_Y)$-module. Moreover, if we now view M as an $(A/\mathfrak{a}_Y)$-module then $\overline{\mathcal{F}} = \widetilde{M}$.

We show that a slightly weaker statement always holds: there exists an integer $k > 0$ such that

$$\mathcal{I}_Y^k \cdot \mathcal{F} = 0. \tag{6.43}$$

Consider an affine open set $U = \operatorname{Spec} A$ such that $\mathcal{F}_{|U}$ is of the form $\widetilde{M}$ with M a finite A-module. Let $\mathfrak{a}_Y$ be the ideal of the subset $Y \cap U$. If $f \in \mathfrak{a}_Y$ then $D(f) \subset U \setminus (U \cap Y)$, and by assumption the restriction of $\mathcal{F}$ to $D(f)$ is zero. This means that $M_f = 0$, and hence for every $m \in M$ there exists $k(m) > 0$ such that $f^{k(m)} m = 0$. Since M is a finite A-module, it follows that $f^k M = 0$ for some $k > 0$. Since this relation holds for any $f \in \mathfrak{a}_Y$ and $\mathfrak{a}_Y$ has a finite basis, it follows that

$$\mathfrak{a}_Y^l \cdot M = 0 \tag{6.44}$$

for some $l > 0$. In other words, (6.43) holds on the open set U. Choosing a finite cover of X by open sets U as above, and taking k to be the maximum of the l for which (6.44) holds on each of the U, we get (6.43) on the whole of X.

Set $\mathcal{F}_i = \mathcal{I}_Y^i \cdot \mathcal{F}$ for $i = 0, \dots, k$ and $\mathcal{F} = \mathcal{F}_0$. Obviously the support of each of the $\mathcal{F}_i$ is contained in Y. Write $\overline{\mathcal{F}}_i$ for the sheaf on Y determined by $\mathcal{F}_i$ on X. Since

$$\mathcal{I}_Y \cdot (\mathcal{F}_i / \mathcal{F}_{i+1}) = 0,$$

the sheaf $\overline{\mathcal{F}}_i / \overline{\mathcal{F}}_{i+1}$ satisfies (6.42), and so is a coherent sheaf of $\mathcal{O}_Y$-modules. This proves Proposition 6.3. □

To conclude, we show how the methods used throughout this section allow us to reduce the study of sheaves to the case of irreducible schemes.

Proposition 6.4 *Let X be a Noetherian reduced scheme with $X = \bigcup X_i$ its decomposition as a union of irreducible components, and suppose that $\mathcal{F}$ is a coherent sheaf on X. There exist coherent sheaves $\mathcal{F}_i$ on X and a homomorphism $\varphi \colon \mathcal{F} \to \bigoplus \mathcal{F}_i$ such that the support of $\mathcal{F}_i$ is contained in X_i, the sheaf $\overline{\mathcal{F}}_i$ defined on X_i by $\mathcal{F}$ is coherent, and the kernel of φ has support contained in $\bigcup_{i \neq j} X_i \cap X_j$.*

Proof Set $\mathcal{F}_i = \mathcal{F}/(\mathcal{I}_{X_i} \cdot \mathcal{F})$, and let $\varphi_i \colon \mathcal{F} \to \mathcal{F}_i$ be the natural projection and $\varphi = \bigoplus \varphi_i$. We saw in Section 3.2 that the support of $\mathcal{F}_i$ is contained in X_i, and $\overline{\mathcal{F}}_i$ is a coherent sheaf of $\mathcal{O}_{X_i}$-modules since $\mathcal{I}_{X_i} \cdot \mathcal{F}_i = 0$.

Consider the open set

$$U_i = X_i \setminus \bigcup_{i \neq j} X_i \cap X_j.$$

On U_i we have $\mathcal{I}_{X_j} = \mathcal{O}_X$ for $j \neq i$ and $\mathcal{I}_{X_i} = 0$, so that $\mathcal{F}_{j|U_i} = 0$ for $j \neq i$ and $\mathcal{F}_{i|U_i} = \mathcal{F}_{|U_i}$. Therefore on U_i we have $\varphi_j = 0$ for $j \neq i$, and $\varphi = \varphi_i$ is an isomorphism. Hence the kernel of φ equals 0 on $\bigcup U_i$, as required to prove. □

3.4 The Finiteness Theorem

Theorem *If X is a complete variety over a field k and $\mathcal{F}$ a coherent sheaf on X, the vector space $\mathcal{F}(X)$ is finite dimensional over k.*

Proof The essence of the proof is the following remark. Given a homomorphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of sheaves over X, set $\mathcal{H} = \ker \varphi$; then

$$\mathcal{H}(X) \text{ and } \mathcal{G}(X) \text{ finite dimensional} \quad \Longrightarrow \quad \mathcal{F}(X) \text{ is finite dimensional.} \tag{6.45}$$

This follows since $\mathcal{H}(X) = \ker\{\varphi_X\colon \mathcal{F}(X) \to \mathcal{G}(X)\}$, by definition of the kernel. From this, we deduce by induction that $\mathcal{F}(X)$ is finite dimensional if there exist subsheaves

$$\mathcal{F} = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots \supset \mathcal{F}_m = 0 \tag{6.46}$$

such that each vector space $\mathcal{F}_i/\mathcal{F}_{i+1}(X)$ is finite dimensional.

We prove the theorem by induction on the dimension of X. If $\dim X = 0$ then X consists of a finite number of points, and a coherent sheaf $\mathcal{F}$ on X is by definition a finite dimensional vector space over k, so that the theorem is obvious.

Suppose that the theorem holds for complete varieties of dimension less than $\dim X$. Let us prove that this implies the theorem for all sheaves $\mathcal{F}$ on X having support contained in a closed subvariety $Y \subset X$ with $\dim Y < \dim X$.

Indeed, by definition, the sheaf $\overline{\mathcal{F}}$ on Y has $\mathcal{F}(X) = \overline{\mathcal{F}}(Y)$, and we can apply the assertion of the theorem to coherent sheaves on Y. Here we run into the difficult that $\overline{\mathcal{F}}$ is not in general coherent on Y, but Proposition 6.3 saves the day. It provides a sequence

$$\overline{\mathcal{F}} = \overline{\mathcal{F}}_0 \supset \overline{\mathcal{F}}_1 \supset \cdots \supset \overline{\mathcal{F}}_m = 0$$

such that the quotient sheaves $\overline{\mathcal{F}}_i/\overline{\mathcal{F}}_{i+1}$ are coherent on Y and hence we can apply the inductive assumption to them. We get the existence of a sequence of sheaves (6.46), from which the finite dimensionality of $\overline{\mathcal{F}}(Y)$ follows, and hence also that of $\mathcal{F}(X)$.

The next step of the proof consists of reducing the assertion to the case of an irreducible variety. Suppose that $X = \bigcup X_i$ is a decomposition into irreducible components. Now we can apply Proposition 6.4. The homomorphism φ constructed there has kernel supported in the subvariety $\bigcup_{i \neq j} X_i \cap X_j$, which has dimension less than $\dim X$. Hence it is enough to prove that $(\bigoplus \mathcal{F}_i)(X)$ is finite dimensional. But

$$\Bigl(\bigoplus \mathcal{F}_i\Bigr)(X) = \bigoplus \overline{\mathcal{F}}_i(X_i),$$

and since $\overline{\mathcal{F}}_i$ is a coherent sheaf on X_i, this reduces the assertion to the case of the irreducible varieties X_i.

Finally we can proceed with the central step of the proof, assuming that X is irreducible. Here we build on the foundation of Proposition 6.2. Since X is complete,

$\mathcal{O}(X) = k$ by the discussion in Section 1.1, so that $\dim \mathcal{O}^r(X) = r$. Since the support of $\mathcal{G}/\mathcal{O}^r$ is distinct from X, the theorem holds for $\mathcal{G}/\mathcal{O}^r$, and hence for $\mathcal{G}$ we have a homomorphism $\psi\colon \mathcal{G} \to \mathcal{G}/\mathcal{O}^r$ satisfying conditions (6.45). Hence $\mathcal{G}(X)$ is finite dimensional. On the other hand, the homomorphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ constructed in Proposition 6.2 again satisfies (6.45), so that $\mathcal{F}(X)$ is finite dimensional, which is what the theorem asserts. The theorem is proved. $\square$

The theorem we have proved has many important applications. Some of these have been mentioned earlier. First of all, in Section 1.4 we associated with each divisor D on a variety X a sheaf $\mathcal{L}_D$ such that $\mathcal{L}_D(X)$ is isomorphic to the space $\mathcal{L}(D)$ introduced in Section 1.5, Chapter 3. We saw in Section 3.4 that $\mathcal{L}_D$ is locally free of rank 1, and therefore coherent. Thus our theorem is applicable to it, and we obtain the result that we have already used many times:

Corollary 6.1 *The dimension $l(D)$ of a locally principal divisor D on a complete variety is finite.*

Applying the theorem to the sheaf corresponding to the cotangent sheaf Ω^1 and its exterior powers Ω^p we get the following result.

Corollary 6.2 *On a complete nonsingular variety X, the dimension h^p of the space $\Omega^p[X]$ of regular differential p-forms is finite.*

This result was also stated in Section 6.1, Chapter 3, where we saw that it provides a series of birational invariants of varieties.

As a further example, consider the sheaf $\mathcal{T}$ corresponding to the tangent bundle. An element of $\mathcal{T}(X)$ is called a *regular vector field* on X. It can be viewed as a function taking each point $x \in X$ to a tangent vector $t_x \in \Theta_x$ at x. In this case our theorem gives the next result.

Corollary 6.3 *The space of regular vector fields on a complete nonsingular variety is finite dimensional.*

3.5 Exercises to Section 3

1 In this question, X is assumed to be irreducible. A coherent sheaf $\mathcal{F}$ is a *torsion sheaf* if $\mathcal{F}(U)$ is a torsion module over $\mathcal{O}_X(U)$ for any open set U. Prove that $\mathcal{F}$ is a torsion sheaf if and only if its support is distinct from X.

2 Find the general form of torsion sheaves on a nonsingular curve.

3 Let $E \to X$ be a vector bundle over an affine variety $X = \operatorname{Spec} A$. Prove that the set M_E of sections of E is a finite A-module.

4 Prove that the module M_E introduced in Exercise 3 is a projective A-module. (For the definition of a projective module, see for example Bourbaki [18, Section 3.2.2, Chapter II] or Matsumura [57, Appendix B].)

5 Under the assumptions of Exercise 3, prove that the modules M_E and $M_{E'}$ are isomorphic if and only if E and E' are isomorphic vector bundles.

6 Prove that every vector bundle over the affine line $\mathbb{A}^1$ is trivial.

7 Let $E \to X$ be a vector bundle over a complete variety X. Prove that the set of sections of E is a finite dimensional vector space.

8 Prove that the set of morphisms $f\colon E_1 \to E_2$ between vector bundles $E_i \to X$ (for $i = 1, 2$) over a complete variety X is a finite dimensional vector space.

9 Suppose that A is a 1-dimensional regular local ring with field of fractions K, and $X = \operatorname{Spec} A$; let $x \in X$ be the generic point and $U = \{x\}$. A sheaf $\mathcal{F}$ of $\mathcal{O}$-modules on X is given by an A-module M, a K-vector space L and a restriction map $\varphi\colon M \to L$ which is an A-module homomorphism. Express in terms of M, L and φ what it means for $\mathcal{F}$ to be a coherent sheaf. Construct an example of a subsheaf of a coherent sheaf which is not coherent.

10 Let X be an irreducible variety and $x_0 \in X$ a closed point. Define a sheaf $\mathcal{F}$ on X by setting $\mathcal{F}(U) = \mathcal{O}(U)$ if $U \not\ni x_0$ and $\mathcal{F}(U) = 0$ if $U \ni x_0$. Prove that $\mathcal{F}$ is a sheaf, that it is a subsheaf of $\mathcal{O}$, and that it is not coherent.

4 Classification of Geometric Objects and Universal Schemes

4.1 Schemes and Functors

A phenomenon that has already occurred several times is that a set of certain geometric objects depends on parameters, and more precisely, is parametrised by the points of some algebraic variety. For example, lines in the projective space $\mathbb{P}^3$ are parametrised by points of the 4-dimensional Plücker quadric (Section 4.1, Chapter 1). What is the precise meaning of this assertion? What meaning at all? We indicated the construction of the Plücker coordinates of a line, and showed that it defines a one-to-one correspondence between lines of $\mathbb{P}^3$ and points of the Plücker quadric. But there is no guarantee that this construction is unique; that is, that we might not be able to establish some other equally natural one-to-one correspondence between lines of $\mathbb{P}^3$ and points of some other variety, perhaps even of a different dimension. After all, as far as set theory goes, the set of lines has only one invariant, its cardinality. At the same time, it is obviously very important to be able to define some natural variety (or a more general notion) classifying geometric objects of a given type: its properties, such as dimension, rationality or unirationality and so on, give important characteristics of the whole set of these objects. We describe one approach that in many cases allows us to determine what precisely it means to say that a given set of objects is parametrised by the points of a given variety or scheme.

Since we are talking about geometric objects, the notion of an algebraic family of objects is usually well defined. For example, if we are talking about r-dimensional linear subspaces of a given vector space V, an algebraic family of these with base S is a vector bundle $E \to S$ of rank r which is a vector subbundle of the direct product $S \times V$. In exactly the same way, since we study objects modulo a well-defined equivalence relation, this equivalence relation carries over also to families over any base. For example, in studying the subspaces of a given space V, we naturally consider two vector subbundles $E \to S$ and $E' \to S$ in $S \times V$ as the same if they are equal as subschemes of $S \times V$. Or if we are interested, say, in the classification of nonsingular complete curves of genus g, then by a family of these curves we mean a scheme $C \to S$ all of whose (scheme-theoretic) fibres over closed points of S are nonsingular complete curves of genus g. An isomorphism between two families $C \to S$ and $C' \to S$ is an isomorphism of schemes $f\colon C \to C'$ commuting with the projection to S, that is, such that the diagram

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr] & & C' \arrow[dl] \\ & S & \end{tikzcd}$$

is commutative.

Suppose that for some type of geometric objects we have found a "natural" variety (or scheme) X classifying them. Let's try to clarify this idea of "naturality". Obviously, to each object there should correspond a definite closed point of X. Let $\varphi\colon Y \to S$ be an algebraic family of our objects over a base which is a variety. Then to each fibre $\varphi^{-1}(s)$ for $s \in S$ there corresponds some point of X, and this defines a map $f\colon S \to X$. In the notion of "naturality" it is first of all reasonable to include the requirement that this map of points be a morphism, and even to require that the same type of morphism exists for families whose base is a scheme (with certain conditions: over a field k, of finite type, and so on). Moreover, it is reasonable to suppose that two families $Y \to S$ and $Y' \to S$ determine the same morphism $f\colon S \to X$ if and only if they are equivalent in the sense of the equivalence defined for our objects. Finally, the "naturality" of X should include the requirement that every point of X corresponds to some object of our type. Then any map $f\colon S \to X$ of a variety S to X will determine over each point $s \in S$ the object which the point $f(s) \in X$ parametrises; in other words, set-theoretically, it will determine a "family" of objects parametrised by points $s \in S$. It is also reasonable to include in the notion of "naturality" the requirement that if f is a morphism then we obtain in this way an algebraic family of objects.

All of these conditions are summed up very simply in the single statement that there should exist a one-to-one correspondence between algebraic families $Y \to S$

of our objects (the base S may satisfy some restrictions such as being a Noetherian scheme), considered up to equivalence, and all morphisms $S \to X$.

We now formulate the definition we have arrived at. A scheme X is *universal* for some type of objects if for any scheme S (possibly with certain restrictions) there exists a one-to-one correspondence f_S between the set $\Phi(S)$ of all algebraic families $Y \to S$ of objects of the given type, considered up to equivalence, and the set $\mathcal{M}(S, X)$ of morphisms $S \to X$. The correspondence $f_S \colon \Phi(S) \to \mathcal{M}(S, X)$ should satisfy the following condition: for any morphism $\varphi \colon S \to S'$, the diagram

$$\begin{array}{ccc} \Phi(S') & \xrightarrow{f_{S'}} & \mathcal{M}(S', X) \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ \Phi(S) & \xrightarrow[f_S]{} & \mathcal{M}(S, X) \end{array} \tag{6.47}$$

is commutative, where g is defined by taking the inverse image of families under φ (that is, their fibre product or pullback by φ), and h by composing a morphism $S' \to X$ with the morphism $\varphi \colon S \to S'$.

In the language of categories, an operation that sends a scheme S to a set $\Phi(S)$ and a morphism $\varphi \colon S \to S'$ to a map $\Phi(\varphi) \colon \Phi(S') \to \Phi(S)$ is called a *functor* if for two morphisms $\varphi \colon S \to S'$ and $\psi \colon S' \to S''$ we have $\Phi(\psi \circ \varphi) = \Phi(\varphi) \circ \Phi(\psi)$. In particular, if $\Phi(S)$ is the set of all algebraic families of objects of our type, and for a morphism $\varphi \colon S \to S'$ the map $\Phi(\varphi) \colon \Phi(S') \to \Phi(S)$ is defined by taking inverse image of families, then Φ is a functor. A trivial example of a functor $\Psi_X(S)$ is determined by an arbitrary scheme X: here $\Psi_X(S) = \mathcal{M}(S, X)$ is the set of all morphisms $S \to X$ to X and, if $\varphi \colon S \to S'$ is a morphism, the map $\Psi_X(\varphi) \colon \Psi_X(S') \to \Psi_X(S)$ sends $f \colon S' \to X$ into the composite $f \circ \varphi \colon S \to X$. Diagram (6.47) in the definition of universal scheme means that the functor Φ is isomorphic to the functor Ψ_X for some scheme X; in the theory of categories, Φ is then called a *representable functor*. Thus the question of the existence of a universal scheme is the question of the representability of the functor Φ of families of objects of the given type.

Note that our definition does not in any way guarantees the existence of a universal scheme: we will soon see that it does not always exist. For the moment, we assume that a universal scheme exists for objects of some type, and note some properties that support the naturality of the definition.

First of all, a universal scheme X is unique if it exists. Indeed, if Y is a second such scheme then by definition, we have isomorphisms $u \colon \mathcal{M}(X, X) \cong \Phi(X) \cong \mathcal{M}(X, Y)$ and $v \colon \mathcal{M}(Y, Y) \cong \Phi(Y) \cong \mathcal{M}(Y, X)$; and a morphism $\varphi \colon X \to Y$ gives rise to a commutative diagram

$$\begin{array}{ccc} \Phi(X) \cong \mathcal{M}(X, X) & \xrightarrow{u} & \mathcal{M}(X, Y) \\ {\scriptstyle g}\big\uparrow & & \big\uparrow{\scriptstyle h} \\ \Phi(Y) \cong \mathcal{M}(Y, X) & \xrightarrow[v]{} & \mathcal{M}(Y, Y) \end{array} \tag{6.48}$$

where $g(\xi) = \xi \circ \varphi$ and $h(\eta) = \eta \circ \varphi$. Let $1_X : X \to X$ and $1_Y : Y \to Y$ be the identity morphisms, and set $u(1_X) = \alpha$ and $v^{-1}(1_Y) = \beta$. Consider the diagram (6.48) for $\varphi = \alpha$, and apply it to $\beta \in \mathcal{M}(Y, X)$; then $u(\beta \circ \alpha) = \alpha = u(1_Y)$, and since u is a bijection, $\beta \circ \alpha = 1_Y$. Similarly one proves that $\alpha \circ \beta = 1_X$. Therefore α is an isomorphism.

But we can get even more. In view of the one-to-one correspondence $\Phi(X) \cong \mathcal{M}(X, X)$, the identity morphism $1_X \in \mathcal{M}(X, X)$ determines an element $\varepsilon_X \in \Phi(X)$ called the *universal family* over X. It follows from the definition that any family $\xi \in \Phi(S)$ not only determines a morphism $f : S \to X$, but is determined by it, as the inverse image of the universal family ε_X under f, that is, as the fibre product $\varepsilon_X \times_X S$.

Finally, suppose that all the objects and schemes are defined over an algebraically closed field k. Consider some individual object ξ, that is, a family $\xi \to \operatorname{Spec} k$. Then ξ is an element of the set $\Phi(\operatorname{Spec} k)$, which by definition is in one-to-one correspondence with the set $\mathcal{M}(\operatorname{Spec} k, X)$, that is, with the closed points of X. Therefore our object ξ determines a closed point of the scheme X, and all objects, up to equivalence, are in one-to-one correspondence with these points. Thus in this sense the objects under consideration are parametrised by points of X.

Example 6.22 Let us see that the Grassmannian $\operatorname{Grass}(r, V)$ really is a universal scheme for r-dimensional subspaces of a vector space V. We consider schemes over an algebraically closed field k. For a k-scheme S, we define $\Phi(S)$ as the set of vector bundles $E \to S$ that are vector subbundles of the direct product $S \times_k V$. For a morphism $\varphi : S' \to S$, we define $\Phi(\varphi) : \Phi(S) \to \Phi(S')$ as the inverse image map $E \mapsto E \times_S S'$. We need to determine a one-to-one correspondence $f_S : \Phi(S) \to \mathcal{M}(S, \operatorname{Grass}(r, V))$ which is functorial (that is, gives commutative diagrams (6.47)). These maps f_S are an exact analogue of writing down the Plücker coordinates (see Example 1.24 of Section 4.1, Chapter 1). Let $E \to S$ be a vector bundle of rank r, and $S = \bigcup U_\alpha$ a cover such that $E_{|U_\alpha} \cong U_\alpha \times \mathbb{A}^r$. We choose a basis $f_1, \dots, f_r$ in $\mathbb{A}^r$ and a basis $e_1, \dots, e_n$ in V. The embedding $E \hookrightarrow S \times_k V$ allows us to express the f_i as $f_i = \sum a_{ij} e_j$ with $a_{ij} \in \mathcal{O}(U_\alpha)$, and $f_1 \wedge \dots \wedge f_r$ as $\sum p_{j_1 \dots j_r} e_{j_1} \wedge \dots \wedge e_{j_r}$ with $p_{j_1 \dots j_r} \in \mathcal{O}(U_\alpha)$. This gives the morphism $U_\alpha \to \bigwedge^r V$ determined by the functions $p_{j_1 \dots j_r}$. From it we get a morphism $U_\alpha \to \mathbb{P}(\bigwedge^r V)$, which does not depend on the choice of the basis $f_1, \dots, f_r$. Obviously $p_{j_1 \dots j_r}$ satisfy the Plücker equations of the Grassmannian, so that we have a morphism $U_\alpha \to \operatorname{Grass}(r, V)$. Since these morphisms for different α are defined invariantly, they glue together to give a global morphism $S \to \operatorname{Grass}(r, V)$ that we take for $f_S(E)$. The inverse map $\mathcal{M}(S, \operatorname{Grass}(r, V)) \to \Phi(S)$ is obtained by taking the inverse image under any map $\varphi : S \to \operatorname{Grass}(r, V)$ of the universal bundle over $\operatorname{Grass}(r, V)$ (see Example 6.4). It is trivial to check that these two maps are inverse to one another.

Example 6.23 We now give an example of a situation where the universal scheme does not exist. This is an extremely important case, nonsingular curves of given genus g. The reason for nonexistence is already present most vividly in the most trivial case, curves of genus 0. We know that all such curves are isomorphic to $\mathbb{P}^1$. Therefore, if the universal scheme X exists, it must have a single closed point; that is, it would be an affine scheme $\operatorname{Spec} A$, where A is a local ring. Now consider a concrete family of curves of genus 0. For this, consider the plane $\mathbb{P}^2$ with coordinates $(x_0 : x_1 : x_2)$ and the rational map $\mathbb{P}^2 \to \mathbb{P}^1$ given by $(x_0 : x_1 : x_2) \mapsto (x_1 : x_2)$. This has a single point of indeterminacy $(1 : 0 : 0)$. Blowing up this point we get a surface V and a morphism $\varphi \colon V \to \mathbb{P}^1$ (see the example at the end of Section 3.3 of Chapter 4). The fibres of φ are all isomorphic to the projective line, so φ is precisely a family of curves of genus 0 over $\mathbb{P}^1$, that is, an element of the set $\Phi(\mathbb{P}^1)$. If a universal scheme X existed then our family would be the inverse image of the universal family over $X = \operatorname{Spec} A$ under some morphism $f \colon \mathbb{P}^1 \to X$, and f must map $\mathbb{P}^1$ to the single closed point of X. However, f then corresponds to another element of $\Phi(\mathbb{P}^1)$, the direct product $\mathbb{P}^1 \times \mathbb{P}^1$. To nail down the contradiction, it remains to see that the family $V \to \mathbb{P}^1$ is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. This follows for example from the fact that the selfintersection of any divisor on $\mathbb{P}^1 \times \mathbb{P}^1$ is even: if $C_1 = \mathbb{P}^1 \times x$ and $C_2 = y \times \mathbb{P}^1$ then any divisor D on $\mathbb{P}^1 \times \mathbb{P}^1$ is linearly equivalent to $n_1 C_1 + n_2 C_2$, so that $D^2 = 2n_1 n_2$. On the other hand, V contains the curve L obtained by blowing up $(1 : 0 : 0) \in \mathbb{P}^2$, and $L^2 = -1$ (compare Exercise 14 of Section 1.5).

The family constructed above is locally trivial: it is easy to see that if $U_1 = \mathbb{P}^1 \setminus \infty$ and $U_2 = \mathbb{P}^1 \setminus 0$ then $\varphi^{-1}(U_1) \cong U_1 \times \mathbb{P}^1$ and $\varphi^{-1}(U_2) \cong U_2 \times \mathbb{P}^1$. But this is not necessarily the case: the family in $\mathbb{P}^2 \times \mathbb{A}^2$ given by $\xi_0^2 = u\xi_1^2 + v\xi_2^2$, where $\mathbb{A}^2$ has coordinates u, v and $\mathbb{P}^2$ has coordinates $(\xi_0 : \xi_1 : \xi_2)$ is not isomorphic to a trivial family over any open subset $U \subset \mathbb{A}^2$. This follows from the fact that it has no rational section: there do not exist polynomials $p_0, p_1, p_2 \in k[u, v]$ such that $p_0^2 = up_1^2 + vp_2^2$. Indeed, we can suppose that p_0, p_1, p_2 do not have any common factors. Setting $u = 0$ we get $p_0(0, v)^2 = vp_2(0, v)^2$, which is only possible if $p_0(0, v) = p_2(0, v) = 0$, that is, both p_0 and p_2 are divisible by u. Then p_1 would also be divisible by u.

Of course, similar examples can be constructed for curves of genus $g > 0$. Nevertheless, the notion of universal scheme can be modified in such a way that it does exist for curves of any genus. This can be done in two different ways. One can either drop the requirement in the definition of universal scheme X that the correspondence between families over S and morphisms $S \to X$ be one-to-one, and require only that every family defines a morphism: then the universal object will exist as a variety. Or one can insist on having a one-to-one correspondence, but allow the universal object to be something more general than a scheme, a so-called *topology* or *algebraic stack*. See Mumford and Fogarty [64] and Mumford [63].

The interpretation of a scheme as a functor has already appeared in a slightly different context. In Section 3.4, Chapter 5 we showed that if $x \in X$ is any closed point of a scheme X over a field k, we can describe the tangent space $\mathcal{O}_{X,x}$ as the set of morphisms $\mathcal{M}_x(\operatorname{Spec} D, X)$, where $D = k[\varepsilon]/(\varepsilon^2)$, and we allow in $\mathcal{M}_x$ only the morphisms that map the closed point of $\operatorname{Spec} D$ to the point $x \in X$. This interpretation of the tangent space gives a convenient method of describing it if the scheme

X itself is a universal scheme for some type of objects. Putting together Proposition of Section 3.4, Chapter 5 with the definition of a universal scheme shows that in this case $\Theta_{X,x}$ coincides with the set $\mathcal{M}_x(\operatorname{Spec} D, X) \cong \Phi_x(\operatorname{Spec} D)$ of families over $\operatorname{Spec} D$ with given fibre over the point 0. This provides grounds for the intuition that the tangent vector to a universal scheme is a first order infinitesimal deformation of a given object.

Example 6.24 (The tangent space to the Grassmannian $\operatorname{Grass}(r, V)$) Suppose that a point $x \in \operatorname{Grass}(r, V)$ corresponds to a vector subspace E with basis $e_1, \dots, e_r$. By what we said above, Θ_x is isomorphic to the set of vector bundles over $\operatorname{Spec} D$ that are vector subbundles of $\operatorname{Spec} D \times V$ with fibre over 0 equal to E. Passing to the corresponding sheaves, we see that a vector bundle over $\operatorname{Spec} D$ is a module over D that is locally free, hence free. Hence the vector bundle is trivial and has basis $e_1 + \varepsilon u_1, \dots, e_r + \varepsilon u_r$. It remains to determine when two bases of this form give the same vector subbundle of $\operatorname{Spec} D \times V$. If the second basis is $e_1 + \varepsilon v_1, \dots, e_r + \varepsilon v_r$ then this will happen if and only if

$$e_i + \varepsilon v_i = \sum_j (c_{ij} + \varepsilon d_{ij})(e_j + \varepsilon u_j)$$

for $i = 1, \dots, r$. This implies $e_i = \sum c_{ij} e_j$, so that (c_{ij}) is the identity matrix. Next, $v_i = \sum c_{ij} u_j + \sum d_{ij} e_j = u_i + w_i$, where $w_i = \sum d_{ij} e_j$ is an arbitrary vector in E. Thus the vector subbundle of $\operatorname{Spec} D \times V$ is uniquely determined by the vectors u_i in V/E. Setting $\varphi(e_i) = u_i \bmod E$, we see that the required vector subbundles are uniquely specified by homomorphisms $\varphi\colon E \to V/E$, so that $\Theta_x \cong \operatorname{Hom}(E, V/E)$.

Example 6.25 (The scheme of associative algebras) (See Example 2.5 of Section 4.1, Chapter 1 and Example 5.20.) A closed point of this scheme is a multiplication $E \times E \to E$; if E has basis $e_1, \dots, e_n$, the multiplication is given by $e_i e_j = \sum c_{ij}^m e_m$. Tautologically, the scheme is universal for multiplication laws in $S \times_k E$, where now S is an arbitrary scheme and $c_{ij}^m \in \mathcal{O}(S)$. Hence if x is the closed point of this scheme corresponding to the structure constants $\{c_{ij}^m\}$, the tangent space Θ_x is isomorphic to the set of multiplication laws on $D \times E$ of the form $e_i e_j = \sum (c_{ij}^m + \varepsilon d_{ij}^m) e_m$ where $d_{ij}^m \in k$ are any elements for which this multiplication is associative. The associativity condition can be written out at once by comparing the coefficient of ε in $(e_i e_j) e_k$ and $e_i(e_j e_k)$:

$$\sum_m c_{ij}^m d_{mk}^l + \sum_m d_{ij}^m c_{mk}^l = \sum_m c_{jk}^m d_{im}^l + \sum_m c_{jk}^m d_{im}^l$$

for all i, j, k, l. These are the same equations as we obtained in Example 2.5 of Section 1.3, Chapter 2 by differentiating the associativity relation; but now they have acquired a transparent meaning, as the first order infinitesimal deformations of the structure constants.

4.2 The Hilbert Polynomial

The remainder of this section will be taken up with the description of the universal scheme for an extremely important type of object: closed subvarieties, and even subschemes of projective space $\mathbb{P}^N$. For the case of linear subspaces we already know the universal scheme, the Grassmannian.

Already in the example of linear subspaces we see that, rather than considering all subvarieties at the same time, we get the natural answer by breaking up the subvarieties into classes, and then considering these separately. In the case of the Grassmannian, we fixed the dimension r of the subspace and its degree 1. We now describe similar discrete invariants of projective schemes that one has to fix in order to arrive at the natural universal schemes; these are the so-called Hilbert polynomials.

With each projective subscheme $X \subset \mathbb{P}^N$ we associate an infinite sequence $a_r(X)$ of integers: $a_r(X)$ is the number of forms of degree r in the homogeneous coordinates of $\mathbb{P}^N$ that are linearly independent on X. To give a more formal definition, consider the homogeneous ideal $\mathfrak{a}_X$ of a projective scheme $X \subset \mathbb{P}^N$ (Section 3.3, Chapter 5, and compare Section 4.1, Chapter 1), and write $\mathfrak{a}_X^{(r)}$ for its homogeneous piece of degree r, that is, the space of forms of degree r in $\mathfrak{a}_X$. Write $S^{(r)}$ for the space of forms of degree r in the homogeneous coordinates of $\mathbb{P}^N$. Now set $a_r(X) = \dim_k S^{(r)}/\mathfrak{a}_X^{(r)}$. These numbers depend, of course, on the embedding $X \hookrightarrow \mathbb{P}^N$, and in this respect they are analogous to the degree.

The infinite sequence of numbers just constructed can be described in finite terms.

Theorem 6.5 *There exists a polynomial $P_X(T) \in \mathbb{Q}[T]$ such that $a_r(X) = P_X(r)$ for all sufficiently large integers r.*

The polynomial $P_X(T)$ whose existence is established in the theorem is obviously uniquely determined. It is called the *Hilbert polynomial* of X.

Proof The theorem is proved by induction on the dimension N, and, as often happens, it is convenient to prove a more general assertion. Consider a finite *graded module* M over the polynomial ring $S = k[\xi_0, \dots, \xi_N]$. This means that M is a module over S, with a fixed decomposition $M = \bigoplus M^{(r)}$ as a direct sum of k-vector subspaces such that

$$x \in M^{(r)} \quad \text{and} \quad f \in S^{(l)} \quad \Longrightarrow \quad fx \in M^{(r+l)}.$$

The subspaces $M^{(r)}$ are called the *homogeneous pieces* of M of degree r. Each subspace $M^{(r)}$ is finite dimensional over k: indeed, as a k-vector space, $M^{(r)} \cong M'_r = (\bigoplus_{i\geq r} M^{(i)})/(\bigoplus_{i>r} M^{(i)})$, where $\xi_i M'_r = 0$ for each i, so that M'_r is a finite module over k. We set $a_r(M) = \dim_k M^{(r)}$ and prove that the statement of the theorem holds for $a_r(M)$. The theorem itself is obtained by setting $M = S/\mathfrak{a}_X$.

We can set $S = k$ for $N = -1$ and assume that in this case a graded module over S is of the form $M = M_0$, with M_0 a finite dimensional graded k-vector space. From this point on, the theorem is proved by induction on N. Consider the homomorphism $\xi_N : M \to M$ consisting of multiplication by the variable ξ_N. Then $\xi_N M^{(r)} \subset M^{(r+1)}$, which implies that the kernel K and cokernel $C = M/\xi_N M$ are both graded modules: $K = \bigoplus K^{(r)}$ and $C = \bigoplus C^{(r)}$, where $K^{(r)} = M^{(r)} \cap K$ and $C^{(r)} = M^{(r)}/\xi_N M^{(r-1)}$. We have an exact sequence

$$0 \to K^{(r)} \to M^{(r)} \xrightarrow{\xi_N} M^{(r+1)} \to C^{(r+1)} \to 0. \tag{6.49}$$

Now by construction K and C are graded S-modules on which ξ_N acts by 0, so that we can view them as modules over $k[\xi_0, \dots, \xi_{N-1}]$ and assume by induction that the assertion holds for them. Write P_K and P_C for the polynomials corresponding to them. Then the exact sequence (6.49) implies that

$$a_{r+1}(M) - a_r(M) = P_C(r+1) - P_K(r).$$

for all sufficiently large r. Now it follows from very simple properties of polynomials (see Section 2, Appendix) that a sequence of integers satisfying this condition is given for all sufficiently large r as the values of some polynomial $P_M(T) \in \mathbb{Q}[T]$, that is, $a_r(M) = P_M(T)$, as asserted. □

Example 6.26 Let $X \subset \mathbb{P}^N$ be a 0-dimensional subscheme. Suppose that the underlying set X_{red} does not intersect the hyperplane $\xi_0 = 0$. Taking a homogeneous polynomial $F \in S^{(r)}$ to the polynomial $f = F/\xi_0^r \in k[x_1, \dots, x_N] = k[\mathbb{A}^N]$ where $x_i = \xi_i/\xi_0$, we see that $S^{(r)}/\mathfrak{a}^{(r)} \cong V^{(r)}/(V^{(r)} \cap I)$, where $V^{(r)} \subset k[x_1, \dots, x_N]$ is the space of polynomials of degree $\le r$ and I the ideal defining the subscheme $X \subset \mathbb{A}^N$. Since $\dim V^{(r)}/(V^{(r)} \cap I) \le \dim V^{(r+1)}/(V^{(r+1)} \cap I)$, the sequence of numbers $a_r(X)$ stabilises from some r onwards. It follows that $P_X(T) = \text{const.} = \dim k[\mathbb{A}^N]/I$. In other words,

$$X = \operatorname{Spec} A, \quad A = k\bigl[\mathbb{A}^N\bigr]/I \quad \text{and} \quad P_X(T) = \text{const.} = \dim_k A.$$

Since $S^{(r)} \ne \mathfrak{a}_X^{(r)}$ for any r (assuming X nonempty), the Hilbert polynomial cannot be identically zero. We now determine how it reflects two of the simplest invariants of a scheme X, the dimension and the degree. We only carry through the proof in the case that X is a nonsingular variety (possibly irreducible). The same result holds for arbitrary closed subschemes $X \subset \mathbb{P}^N$, but to prove it requires a little more commutative algebra (see Hartshorne [37, Section 7, Chapter I] or Fulton [29, Example 2.5.2]).

Theorem 6.6 *The Hilbert polynomial P_X of a nonsingular variety X has degree equal to the dimension of X. If X has dimension n and degree d then the leading term of P_X is $(d/n!)T^n$.*

Proof The proof is based on the same arguments as that of Theorem 6.5. We use induction on $n = \dim X$. If $n = 0$, the result follows from Example 6.26: X consists of d distinct points, and $A = k[\mathbb{A}^N]/I$ is a direct sum of d copies of k, and obviously $a_r(X) = d$ for $r \gg 0$. This proves the theorem in this case.

In the general case, choose a coordinate system such that the hyperplane $\xi_N = 0$ is transversal to X at all points of intersection. The fact that this is possible follows easily from the usual dimension count that we have used many times. Write $\mathbb{P}^*$ for the projective space of hyperplanes of $\mathbb{P}^N$. In $X \times \mathbb{P}^*$ we need to consider the subvariety $Z = \{(x, \lambda) \mid x \in X, \lambda \in \mathbb{P}^* \text{ and } \lambda \supset \Theta_{X,x}\}$. Considering the projection $Z \to X$ shows that $\dim Z \le N - 1$, and hence the image of the projection of Z to $\mathbb{P}^*$ is not the whole of $\mathbb{P}^*$. Hence there exists a hyperplane transversal to X at every point of intersection, and we can take this to be $\xi_N = 0$.

We now apply the argument from the proof of Theorem 6.5 to the module $M = S/\mathfrak{a}_X$ and determine K and C in this case. We prove that $K = 0$. Suppose that $F \in K^{(r)}$, that is, $\xi_N F = 0$ on X. Then for any $i < N$ the function $f = F/\xi_i^r$ satisfies $(\xi_N/\xi_i)f = 0$ on X. But $u_N = \xi_N/\xi_i$ is part of a local parameter system at every point of X_N at which $u_N = 0$, and we saw in Section 1.2, Chapter 4 that none of the local parameters can be a zerodivisor. A fortiori u_N is not a zerodivisor in a neighbourhood of points where $u_N \neq 0$. Therefore $f = 0$ on every component of X, that is, $F \in \mathfrak{a}_X$.

Let us determine the module C. In what follows we use the notation introduced after the definition of projective scheme in Section 3.3, Chapter 5. By definition $C = S/(\xi_N, \mathfrak{a}_X)$. The ideal $(\xi_N, \mathfrak{a}_X)$ consists of forms that vanish on X', the section of X with the hyperplane $\xi_N = 0$. We prove that $(\xi_N, \mathfrak{a}_X) = \mathfrak{a}_{X'}$. For this it is enough to check on each affine open set U_i given by $\xi_i \neq 0$ that $(x_N, \mathfrak{a}_i) = \mathfrak{a}'_i$, where $x_N = \xi_N/\xi_i$ and $\mathfrak{a}'_i$ is the ideal of functions that vanish on the intersection of $X \cap U_i$ with $x_N = 0$. It is enough to prove that $(x_N, \mathfrak{a}_i)/\mathfrak{a}_i = \mathfrak{a}'_i/\mathfrak{a}_i$ in $k[X \cap U_i]$. For this it is enough to prove that if $\varphi \in k[X \cap U_i]$ and $\varphi^\rho \in (x_N)$ then $\varphi \in (x_N)$. This property holds locally in the neighbourhood of any point $\alpha \in X \cap U_i$. Indeed, as usual, it is enough to check this in the local ring $\mathcal{O}_\alpha$ of a point α. We need to prove that if $\varphi^\rho \in (x_N)$ then $\varphi \in (x_N)$ for $\varphi \in \mathcal{O}_\alpha$. But this follows at once because $\mathcal{O}_\alpha$ is a UFD (Theorem 2.10 of Section 3.1, Chapter 2), together with the fact that x_N is prime, as an element of a local system of parameters. Passing to the global situation, we can cover $X \cap U_i$ by open sets of the form $D(f_\lambda)$ and assume that $\varphi \in (x_N, \mathfrak{a}'_i)k[D(f_\lambda)]$ for every λ. Now it is enough to find for any arbitrarily large m functions $g_\lambda \in k[X \cap U_i]$ such that $\sum f_\lambda^m g_\lambda = 1$. Then $\varphi = \varphi \sum f_\lambda^m g_\lambda$ and we can assume that $\varphi f_\lambda^m \in (x_N, \mathfrak{a}'_i)$ by the choice of m.

Thus in the sequence (6.49) we now have $K = 0$ and $C = S'/\mathfrak{a}_{X'}$ where $S' = k[\xi_0, \dots, \xi_{N-1}]$ and X' is nonsingular, $(n-1)$-dimensional and of degree d. Using induction we can assume that the theorem holds for X'. We have an exact sequence

$$0 \to S^{(r)}/\mathfrak{a}_X^{(r)} \xrightarrow{\xi_N} S^{(r+1)}/\mathfrak{a}_X^{(r+1)} \to S'^{(r+1)}/\mathfrak{a}_{X'}^{(r+1)} \to 0,$$

and hence for sufficiently large r we have

$$P_X(r+1) - P_X(r) = P_{X'}(r+1),$$

that is,

$$P_X(T+1) - P_X(T) = P_{X'}(T+1). \tag{6.50}$$

By induction, we can assume that $P_{X'}(T)$ has leading term $(d/(n-1)!)T^{n-1}$. Writing the leading term of $P_X(T)$ as aT^m we see from (6.50) that $m = n$ and $a = d/n!$, as asserted in the theorem. □

The Hilbert polynomial provides the most natural answer to the question discussed at the beginning of this section of dividing up all projective subschemes $X \subset \mathbb{P}^N$ into natural classes, the classes of X with given Hilbert polynomial.

4.3 Flat Families

We proceed to consider families of closed subschemes $X \subset \mathbb{P}^N$ with a given Hilbert polynomial. First of all, we have to determine when all the schemes of a family with irreducible base have the same Hilbert polynomial. The fact that this does not always happen is shown by the following examples.

Example 6.27 Let $\sigma\colon X \to Y$ be a blowup of a point $y_0 \in Y$ with $\dim X = \dim Y > 1$, and let $Z = \sigma^{-1}(y_0)$. Then for $y \in Y$, we have $\dim \sigma^{-1}(y) = 0$ if $y \neq y_0$ and $\dim \sigma^{-1}(y_0) > 0$. By Theorem 6.6, even the degree of the Hilbert polynomial changes.

Example 6.28 Let X be a curve with an ordinary double point x_0 and let X^ν be the normalisation of X. We consider the family $\nu\colon X^\nu \to X$ as a family of 0-dimensional schemes over the base X. Then for $x \neq x_0$ the fibre $\nu^{-1}(x)$ is a single point, and $\nu^{-1}(x_0)$ is two points, that is, $\nu^{-1}(x) \cong \operatorname{Spec} k$ and $\nu^{-1}(x_0) \cong \operatorname{Spec}(k \oplus k)$. By Example 6.26, we have $P_{\nu^{-1}(x)}(r) = \text{const.} = 1$ for $x \neq x_0$ but $P_{\nu^{-1}(x_0)}(r) = \text{const.} = 2$.

Example 6.29 Suppose that $\operatorname{char} k \neq 2$; let g be the automorphism of $X = \mathbb{A}^2$ of order 2 given by $g(x, y) = (-x, -y)$ and $S = X/G$ the quotient of X by the group $G = \{1, g\}$ (see Example 1.21 of Section 2.3, Chapter 1 and Section 2.1, Chapter 2).

Then $S \subset \mathbb{A}^3$ is given by $uv = w^2$, and the morphism $X \to S$ by $u = x^2$, $v = y^2$ and $w = xy$. We view $X \to S$ as a family of 0-dimensional subschemes of $\mathbb{A}^2$ with base S. For $s = (a, b, c) \in S$, the fibre $f^{-1}(s) = \operatorname{Spec} k[x, y]/I$ where I is the ideal $I = (x^2 - a, y^2 - b, xy - c)$. Multiplying $xy - c$ by x and by y, we see that $I \ni ay - cx$ and $bx - cy$. Thus if, say, $a \neq 0$, we have $I = (x^2 - a, y - (c/a)x)$, and $k[x, y]/I \cong k[x]/(x^2 - a)$, so that $\dim k[x, y]/I = 2$. By Example 6.27, this means that $P_{f^{-1}(s)}(r) = \text{const.} = 2$. The same holds if $b \neq 0$. However, if $s = (0, 0, 0)$ then $I = (x^2, xy, y^2)$ and $\dim k[x, y]/I = 3$, that is, $P_{f^{-1}(s)}(r) = \text{const.} = 3$.

Thus we can only expect the Hilbert polynomial of fibres to remain constant in a family under some condition of "continuity" or "fluidity" of the fibres of the family. There does indeed exist such a condition, reflecting perfectly the idea of "no jumping" of the fibres; it is the condition that the family is flat. The definition of flat may seem somewhat strange at first sight, since it is purely algebraic in nature. It is hard to lead the reader to this notion by pure logic; it is easier to define it first, then to show just how useful it is.

Definition A module M over a ring A is *flat* if for any ideal, $\mathfrak{a} \subset A$ the surjective map $\mathfrak{a} \otimes M \to \mathfrak{a}M$ defined by $\alpha \otimes m \mapsto \alpha m$ is an isomorphism. A family $f\colon X \to S$, where X and S are schemes, is *flat* if $\mathcal{O}_x$ is flat as a module over $\mathcal{O}_{f(x)}$ for every $x \in X$. We then also say that f is a *flat morphism*, or that X is *flat over* S.

To check that M is a flat A-module, it is enough to check that the homomorphism $\mathfrak{a} \otimes M \to M$ defined by $\alpha \otimes m \mapsto \alpha m$ has no kernel. In particular, if $\mathfrak{a} = (a)$ is a principal ideal and a is a non-zerodivisor, the condition reduces to saying that the only element of M killed by a is 0. Thus a flat module over an integral principal ideal domain is just a torsion-free module.

We note that an individual scheme over a field k (that is, $S = \operatorname{Spec} k$) is automatically flat; thus flatness is a dynamic property, reflecting the change of the schemes in a family over a base S.

We now enumerate a number of properties of flat morphisms that we neither prove nor make use of, and which characterise flat families as "families with no jumping". They are all geometric restatements of the corresponding properties of rings, and are proved in this form in Bourbaki [17].[2]

Proposition A *If X and S are irreducible schemes of finite type over a field k and $f\colon X \to S$ is a flat morphism, then all the fibres of f have the same dimension.* (*Compare Example* 6.27.)

Proposition B *A finite morphism $f\colon X \to S$ of Noetherian schemes is flat if and only if $\ell_{k(s)}(f^{-1}(s))$ is a locally constant function of $s \in S$. Here $\ell_{k(s)}(f^{-1}(s)) =$* $\dim A_s$, *where the fibre is* $f^{-1}(s) = \operatorname{Spec}(A_s)$. (*Compare Examples* 6.28–6.29.)

Proposition C *If X and S are nonsingular varieties and $f\colon X \to S$ a morphism such that $\mathrm{d}f\colon \Theta_{X,x} \to \Theta_{S,f(x)}$ is surjective for every $x \in X$ then f is flat.*

Proposition D *If $f\colon X \to S$ is a flat morphism and $S' \to S$ an arbitrary morphism then $f'\colon X \times_S S' \to S'$ is again flat.*

Proposition E *For rings A and B and a homomorphism $f\colon A \to B$, the morphism* $\varphi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ *is flat if and only if the ring B is flat over A.*

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[2]Compare also Hartshorne [37, especially A: Proposition 9.5, B: Theorem 9.9, C: Proposition 10.4, D–E: Proposition 9.2, Chapter III].

In what follows, we need one very particular case of this final property.

Lemma *Let A be a principal ideal domain, and B an A-algebra; if* $\operatorname{Spec} B$ *is flat over* $\operatorname{Spec} A$ *then B is a flat A-algebra.*

Proof We need to show that a nonzero element $a \in A$ is a non-zerodivisor in B. The given information is that the localisation B_P of B at any prime ideal $P \subset B$ is flat over $A_{\mathfrak{p}}$, where $\mathfrak{p} = P \cap A$. Hence if $ab = 0$ for some $b \in B$ then $\varphi_P(b) = 0$, where $\varphi_P \colon B \to B_P$ is the localisation map (Section 1.1, Chapter 2). We prove that this implies $b = 0$; moreover, the conditions $\varphi_{\mathfrak{m}}(b) = 0$ for all maximal ideal of B is already sufficient. Indeed, it follows from this that for any maximal ideal $\mathfrak{m}$ there exists an element $c_{\mathfrak{m}} \in B$ such that $c_{\mathfrak{m}} \notin \mathfrak{m}$ and $bc_{\mathfrak{m}} = 0$. Then $bI = 0$, where I is the ideal generated by all the $c_{\mathfrak{m}}$. But I is not contained in any maximal ideal $\mathfrak{m}$, since it contains $c_{\mathfrak{m}} \notin \mathfrak{m}$. Hence $I = B$ and $b = 0$. The lemma is proved. □

For the questions we are interested in, the flat condition on a family is also related to "uniformity": the Hilbert polynomial is constant in a flat family of closed subschemes of $\mathbb{P}^n$ with a connected base S. Straightforward arguments reduce this assertion to the case that S is Spec of a 1-dimensional regular local ring. Namely, it is enough to prove the theorem for a 1-dimensional base S, since in the general case we need only join any two points of S by a chain of curves. Moreover, we can assume that S is irreducible and normal, since otherwise we need to pass to the normalisation S^ν and pullback our family to S^ν, that is, replace $X \to S$ by $X \times_S S^\nu \to S^\nu$. Finally, to prove that the Hilbert polynomial of the fibres over all points $s \in S$ coincide, it is enough to prove this for any closed point $s \in S$ and the generic point $\eta \in S$. We set $A = \mathcal{O}_s$ and pass to the family $X \times_S \operatorname{Spec} A$, thus reducing the assertion to the following: to prove that the Hilbert polynomial of the fibres over the closed and generic points of $\operatorname{Spec} A$ are equal. We now consider this case.

We will understand a family of closed subschemes of $\mathbb{P}^N$ over the base $S = \operatorname{Spec} A$ to mean a closed subscheme of $\mathbb{P}^N_A$. Since there is a canonical morphism $\mathbb{P}^N_A \to \operatorname{Spec} A$, a morphism $X \to \operatorname{Spec} A$ is defined for any closed subscheme $X \subset \mathbb{P}^N_A$, which allows us to view X as a family over the base $\operatorname{Spec} A$.

Theorem 6.7 *Let A be the local ring of a nonsingular point of a curve over an algebraically closed field, and $X \subset \mathbb{P}^N_A$ a closed subscheme such that the morphism $X \to$* $\operatorname{Spec} A$ *is flat. Then the fibres of X over the closed and generic points of* $\operatorname{Spec} A$ *have the same Hilbert polynomial.*

Proof Let

$$\mathfrak{a}_X = \bigoplus_{r \geq 0} \mathfrak{a}_X^{(r)} \subset \Gamma = A[T_0, \dots, T_N]$$

be the homogeneous ideal corresponding to the closed subscheme X. Set $B = \Gamma/\mathfrak{a}_X = \bigoplus_{r\geq 0} B^{(r)}$. Then each $B^{(r)}$ is a finite A-module. Let K be the field of fractions of A and $(\tau) \subset A$ the maximal ideal. The fibre $X \otimes_A K$ of X over the generic point of $\operatorname{Spec} A$ is defined by the ideal $\mathfrak{a}_X \otimes K \subset K[T_0, \dots, T_N]$; and the fibre $X \otimes_A k$ over the closed point is defined by the ideal $\mathfrak{a}_X/\tau\mathfrak{a}_X \subset k[T_0, \dots, T_N]$. Hence the Hilbert polynomial of the fibre over the generic point is defined by the dimensions of the K-vector spaces $B^{(r)} \otimes_A K$ and that of the fibre over the closed point by the dimensions of the k-vector spaces $B^{(r)}/\tau B^{(r)}$. Since $B^{(r)}$ is a finite A-module, the equality

$$\dim_K\bigl(B^{(r)} \otimes_A K\bigr) = \dim_k\bigl(B^{(r)}/\tau B^{(r)}\bigr)$$

just means that $B^{(r)}$ is torsion-free for all sufficiently large r, and, for this, it is enough to check that $\tau b = 0$ with $b \in B^{(r)}$ is only possible for $b = 0$.

The ring B defines an affine scheme $Z = \operatorname{Spec} B$. This is called the *affine cone* over X, and X is the *base* of the cone Z; compare Exercise 8 of Section 4.5. The intersection $\mathfrak{a}_X \cap A$ is an ideal of A. If this ideal were nonzero it would be of the form (τ^k) for some $k \ge 0$, and thus $\tau^k B = 0$; one sees easily that this would imply $\tau^k \mathcal{O}_X = 0$, so that $\mathcal{O}_X$ would not be flat over A. Hence $\mathfrak{a}_X \cap A = 0$, that is, $B^{(0)} = A$.

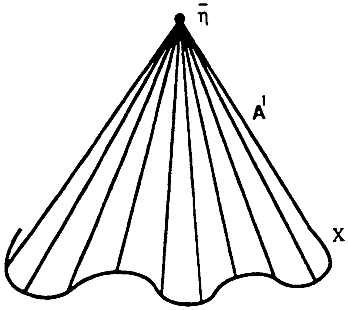
Write η_i for the images of the T_i in B, and I for the ideal $(\eta_0, \dots, \eta_N)$. By what we just said $B/I \cong A$, so that I is a prime ideal. Write ζ for the point of Z corresponding to this prime ideal. We call it the *vertex* of Z. Obviously the subscheme defined by $\eta_0, \dots, \eta_N$ is the closure of ζ, that is, $\bigcap_0^N V(\eta_i) = \overline{\zeta}$, and hence $Z \setminus \overline{\zeta} = \bigcup_0^N D(\eta_i)$.

Consider a set $D(\eta_i)$. By definition $D(\eta_i) = \operatorname{Spec}(B_{\eta_i})$, where B_{η_i} is the ring of fractions u/η_i^ν with $u \in B$ and $\nu \ge 0$. If $u = \sum u^{(r)}$ then u/η_i^ν can be written uniquely in the form $\sum \eta_i^{\nu_r}(u^{(r)}/\eta_i^r)$. Here $\nu_r = r - \nu$ is an integer, possibly negative, so that u/η_i^ν can be written as a polynomial in η_i and η_i^{-1} with coefficients of the form $u^{(r)}/\eta_i^r$. We have seen (Section 3.3, Chapter 5) that elements $u^{(r)}/\eta_i^r$ form a ring $C_i = A_i/\mathfrak{a}_i$ with $\operatorname{Spec} C_i = V_i \subset X$. Hence $B_{\eta_i} = C_i[\eta_i, \eta_i^{-1}]$. Since $\operatorname{Spec}(\mathbb{Z}[T, T^{-1}]) \cong \mathbb{A}^1 \setminus 0$, $D(\eta_i) \cong V_i \times (\mathbb{A}^1 \setminus 0)$. It is easy to see (we do not require this) that the projections $D(\eta_i) \to V_i$ glue together to a global morphism $Z \setminus \overline{\zeta} \to X$. That is, removing the origin, the cone has a projection to its base with fibre $\mathbb{A}^1 \setminus 0$ (because we removed the origin). Note that we have proved more: this is a locally trivial fibration—over each V_i it turns into a direct product. (See Figure 28.)

Thus $Z \setminus \overline{\zeta}$ is covered by $N + 1$ open sets each of which is isomorphic to $V_i \times (\mathbb{A}^1 \setminus 0)$ where $V_i \subset X$ are open sets. Since X is flat over A so are the schemes V_i. It follows that the $V_i \times (\mathbb{A}^1 \setminus 0)$ are also flat over A; since in our case flat is equivalent to torsion-free, this follows from the fact that $V_i = \operatorname{Spec} C_i$ and $V_i \times (\mathbb{A}^1 \setminus 0) = \operatorname{Spec} C_i[T, T^{-1}]$. Finally, since flat is a local condition, we conclude that $Z \setminus \overline{\zeta}$ is flat over A.

What does this mean from the point of view of B? If we recall the definition of the ring $\mathcal{O}(U)$ for an open set $U \subset \operatorname{Spec} B$ (Section 2.2, Chapter 5), the answer is as follows: suppose that $b \in B$ and $\tau b = 0$; then for any $f \in (\eta_0, \dots, \eta_N)$ the element b is zero on the open set $D(f)$, that is, $f^s b = 0$ for some $s > 0$. In particular $\eta_i^{s_i} b = 0$ for some s_i, and hence $I^t b = 0$ for $t \ge s_0 + \cdots + s_N$.

Figure 28 The affine cone



All elements $b \in B$ with $\tau b = 0$ form an ideal J, which, since B is Noetherian, has a finite basis, say, $J = a_1 B + \cdots + a_m B$. From the fact that $I^{t_i} a_i = 0$ for some $t_i > 0$, it follows that all components of $a_i B$ of sufficiently high degree are zero, and therefore the same holds for J. This means that $J \cap B^{(r)} = 0$ for all sufficiently large r; that is, $B^{(r)}$ is a torsion-free module, and this is what we had to prove. $\square$

4.4 The Hilbert Scheme

We can now state the fundamental existence theorem. Let S be a scheme over a field k. A *family of closed subschemes* of $\mathbb{P}^N$ with *base* S is a closed subscheme $X \subset \mathbb{P}^N \times_k S$ with the natural projection morphism $X \to S$. Let $P \in \mathbb{Q}[T]$ be a polynomial. Consider the functor Ψ^P that sends a scheme S to the set $\Psi^P(S)$ of all flat families of closed subschemes of $\mathbb{P}^N$ with base S and Hilbert polynomial P. For a morphism $f\colon S' \to S$, we define $\Psi^P(f)$ to be the map $\Psi^P(S) \to \Psi^P(S')$ which sends a family $X \to S$ into the pullback family $X' = X \times_S S' \to S'$.

Theorem F *There exists a universal scheme* $\operatorname{Hilb}^P_{\mathbb{P}^N}$ *for the functor* Ψ^P; *it is a projective scheme over* k, *called the* Hilbert scheme *of* $\mathbb{P}^N$.

The proof of this theorem is not difficult, but we cannot give it here because it uses cohomological methods. Roughly speaking, one proves that for sufficiently large r, the homogeneous ideal $\mathfrak{a}_X$ of any flat family $X \to S$ with Hilbert polynomial P has every homogeneous component $\mathfrak{a}_X^{(t)}$ with $t \geq r$ generated by forms of degree r, that is, $\mathfrak{a}_X^{(t)} = \Gamma^{(t-r)} \cdot \mathfrak{a}_X^{(r)}$. For r sufficiently large, the codimension of $\mathfrak{a}_X^{(r)} \subset \Gamma^{(r)}$ equals $P(r)$, and it determines a point of the Grassmannian $\operatorname{Grass}(\binom{N+r}{r}, P(r))$. Conversely, this point determines $\mathfrak{a}_X^{(r)}$. One checks furthermore that, for sufficiently large r, the points of $\operatorname{Grass}(\binom{N+r}{r}, P(r))$ for which the corresponding space of forms $\mathfrak{a}^{(r)}$ generates a homogeneous ideal $\mathfrak{a}$ defining a closed subscheme with Hilbert polynomial P is itself a closed subscheme of $\operatorname{Grass}(\binom{N+r}{r}, P(r))$. This is the universal scheme $\operatorname{Hilb}^P_{\mathbb{P}^N}$.

It follows easily from Theorem F (or it can be proved directly in the same way as Theorem F) that if $Y \subset \mathbb{P}^N$ is a closed subscheme then closed subschemes of Y with given Hilbert polynomial P also have a universal family Hilb_Y^P. The proof of these theorems are given in condensed form in Grothendieck's Bourbaki seminars [35]. For the case of 1-dimensional subschemes of a surface Y it is given in Mumford [62]. The general case is worked out in Altman and Kleiman [5, 6].

It is also proved that for a given polynomial $P(T)$ the Hilbert scheme $\mathrm{Hilb}_{\mathbb{P}^N}^P$ is connected; for a simple proof of this theorem of Hartshorne, see Cartier's Bourbaki seminar [21]. Thus the Hilbert polynomial is a complete set of discrete invariants of projective schemes.

Applying Theorem F, we now show how one can find the tangent space to a point of $\mathrm{Hilb}_{\mathbb{P}^N}^P$.

Theorem 6.8 *Let $X \subset \mathbb{P}^N$ be a closed subscheme. The tangent space to the Hilbert scheme $\mathrm{Hilb}_{\mathbb{P}^N}^{P_X}$ at the point corresponding to X is isomorphic to the space $\mathcal{N}_{\mathbb{P}^N/X}(X)$ of sections of the normal sheaf $\mathcal{N}_{\mathbb{P}^N/X}$ (Example 6.21).*

Proof Write $x \in \mathrm{Hilb}_{\mathbb{P}^N}^{P_X}$ for the point corresponding under the universal property of the Hilbert scheme Hilb to the scheme X. The tangent space to Hilb, as for any scheme, equals $\mathcal{M}_x(\operatorname{Spec} D, \mathrm{Hilb}_{\mathbb{P}^N}^{P_X})$, where $D = k[\varepsilon]/(\varepsilon^2)$ (by Proposition of Section 3.4, Chapter 5). If we now use the universal property of the Hilbert scheme, this set can be given another interpretation: it equals the set of flat families of closed subschemes $\widetilde{X} \subset \mathbb{P}_D^N$ with base $\operatorname{Spec} D$ whose fibre over the closed point of $\operatorname{Spec} D$ coincides with X. We now describe this set.

We start with the analogous problem for affine schemes. Let A and B be algebras over k with $B = A/I$, so that $\operatorname{Spec} B \subset \operatorname{Spec} A$ is a closed subscheme. Write $\widetilde{A} = A \otimes_k D = A \oplus \varepsilon A$. A closed subscheme of $\operatorname{Spec} \widetilde{A}$ is of the form $\operatorname{Spec} \widetilde{B}$, where $\widetilde{B} = \widetilde{A}/\widetilde{I}$, and $\widetilde{I} \subset \widetilde{A}$ is an ideal such that $(\widetilde{I} + \varepsilon A)/\varepsilon A = I$. Since D has a unique nonzero ideal (ε), flatness over D means the isomorphism $\varepsilon \otimes \widetilde{B} \cong \varepsilon \widetilde{B}$. In other words, this means that for $\widetilde{b} \in \widetilde{B}$, we have $\varepsilon \widetilde{b} = 0 \iff \widetilde{b} = \varepsilon \widetilde{c}$. Or in terms of the ideal $\widetilde{I}$, if $\varepsilon \widetilde{a} \in \widetilde{I}$ for $\widetilde{a} \in \widetilde{A}$ then $\widetilde{a} \equiv \varepsilon \widetilde{x} \bmod I$; then $\widetilde{a} = \varepsilon \widetilde{x} + i$ for some $i \in I$ and $\varepsilon \widetilde{a} = \varepsilon i$. That is, $\widetilde{B}$ flat over D is the condition that

$$\varepsilon A \cap \widetilde{I} = \varepsilon I. \tag{6.51}$$

By assumption, $(\widetilde{I} + \varepsilon A)/\varepsilon A = I$, that is, any element $j \in \widetilde{I}$ is of the form $j = i + \varepsilon a$ with $a \in A$, and conversely, for any $i \in I$ one can find $a \in A$ such that $i + \varepsilon a \in \widetilde{I}$. By (6.51), $\varepsilon I \subset \widetilde{I}$, and hence a is only defined modulo I. But for given $i \in I$, it follows from (6.51) that the residue class modulo I consisting of elements a such that $i + \varepsilon a \in \widetilde{I}$ is uniquely determined. Thus by condition (6.51), that is, by the flatness of $\widetilde{B}$ over D, the ideal $\widetilde{I}$ is determined by a homomorphism $\varphi \colon I \to A/I = B$, and consists of elements $i + \varepsilon a$ such that $a \in \varphi(i)$. We see that the set of closed subschemes of $\operatorname{Spec} \widetilde{A}$ flat over $\operatorname{Spec} D$ which intersect the closed fibre in the given scheme $\operatorname{Spec} B$ is the set $\operatorname{Hom}_A(I, B)$. Since $IB = 0$, any $\varphi \in \operatorname{Hom}_A(I, B)$ has $\varphi(I^2) = 0$, so that $\operatorname{Hom}_A(I, B) = \operatorname{Hom}_A(I/I^2, B)$.

In the case of any scheme $\mathcal{P}$ (for example $\mathbb{P}^N$), closed subschemes of $\mathcal{P} \times \operatorname{Spec} D$ that are flat over D are described in an entirely analogous way. We have to cover $\mathcal{P}$ by affine pieces $U_\alpha = \operatorname{Spec} A_\alpha$. The closed subscheme $X \subset \mathcal{P}$ defines in U_α a subscheme $U_\alpha \cap X = U_\alpha \times_{\mathcal{P}} X = \operatorname{Spec}(A_\alpha/I_\alpha)$. A family $\widetilde{X} \subset \mathcal{P} \times \operatorname{Spec} D$ with closed fibre equal to X determines by what we said above homomorphisms $\varphi_\alpha \in \operatorname{Hom}_{A_\alpha}(I_\alpha/I_\alpha^2, A_\alpha/I_\alpha)$. These homomorphisms must be compatible on $U_\alpha \cap U_\beta$, and hence they define a global homomorphism $\varphi\colon \mathcal{I}_X/\mathcal{I}_X^2 \to \mathcal{O}_X$ of coherent sheaves on X, where $\mathcal{I}_X$ is the sheaf of ideals defining the subscheme $X \subset \mathcal{P}$. Conversely, any homomorphism of coherent sheaves $\varphi\colon \mathcal{I}_X/\mathcal{I}_X^2 \to \mathcal{O}_X$ defines flat subschemes $\widetilde{X}_\alpha \subset U_\alpha \times \operatorname{Spec} D$ that are compatible, that is, a subscheme $\widetilde{X} \subset \mathcal{P} \times \operatorname{Spec} D$.

We see that all families of the type we are interested in are described by homomorphisms $\varphi\colon \mathcal{I}_X/\mathcal{I}_X^2 \to \mathcal{O}_X$ of sheaves of $\mathcal{O}_X$-modules. The homomorphism φ is a section over X of the sheaf $\mathcal{H}om(\mathcal{I}_X/\mathcal{I}_X^2, \mathcal{O}_X)$. Since $\mathcal{H}om(\mathcal{I}_X/\mathcal{I}_X^2, \mathcal{O}_X) = \mathcal{N}_{\mathcal{P}/X}$ (see Example 6.21), the families under consideration are in one-to-one correspondence with elements of the set $\mathcal{N}_{\mathcal{P}/X}(X)$. By what we said at the start of the proof we thus establish a one-to-one correspondence between the set $\mathcal{N}_{\mathcal{P}/X}(X)$ and the tangent space to the Hilbert scheme $\operatorname{Hilb}^{P_X}_{\mathbb{P}^N}$. A routine verification shows that this correspondence is an isomorphism of vector spaces; we need to use the interpretation of the algebraic operations in the tangent space indicated after Proposition of Section 3.4, Chapter 5. The theorem is proved. $\square$

Mumford [62, Lecture 22] gives an example (already known in different terminology to the ancient Italian geometers) of a nonsingular projective surface Y containing a curve C which does not move on Y, but for which the tangent space to the scheme $\operatorname{Hilb}^{P_C}_Y$ at the point ξ corresponding to C is 1-dimensional. That is, the reduced subscheme of $\operatorname{Hilb}^{P_C}_Y$ in a neighbourhood of ξ consists of the single point ξ, but the local ring of this point on $\operatorname{Hilb}^{P_C}_Y$ has nonzero nilpotent elements; in other words, this component of $\operatorname{Hilb}^{P_C}_Y$ is of the form $\operatorname{Spec} A$ where A is a finite dimensional k-algebra with radical $\mathfrak{m}$ and $A/\mathfrak{m} = k$. This result shows that the curve C on Y can be moved infinitesimally to first order, but not moved globally. It again demonstrates vividly that schemes with nilpotent elements appear naturally in entirely classical questions of algebraic geometry.

The Hilbert scheme plays a basic role not only in studying subschemes of $\mathbb{P}^N$, but also in the study of algebraic varieties in the "abstract" setting, that is, up to isomorphism. The reason, of course, is that one problem can be reduced to the other. Thus we saw in Section 7.1, Chapter 3 that for a nonsingular projective curve X of genus $g > 1$ the map φ_{3K} corresponding to the divisor class $3K$ is an isomorphic embedding $X \hookrightarrow \mathbb{P}^{5g-6}$. The images of curves of genus g under this embedding are curves of degree $6g-6$, and their Hilbert polynomial is easily seen to be $P(T) = (6g-6)T - g + 1$. They are thus parametrised by points of the scheme $\operatorname{Hilb}^{P}_{\mathbb{P}^{5g-6}}$: more precisely, by points of the locally closed subset H_g corresponding to nonsingular curves for which the hyperplane section is in the class $3K$. Points $x, y \in H_g$ correspond to isomorphic curves if and only if the curves in $\mathbb{P}^{5g-6}$

parametrised by x and y are taken into one another by a projective transformation. Thus H_g has an action of the group G of projective transformations of $\mathbb{P}^{5g-6}$, and all nonsingular projective curves of genus g (up to isomorphism) are parametrised by the points of the quotient space H_g/G. A treatment of this theory is contained in Mumford and Fogarty [64].

4.5 Exercises to Section 4

1 Prove that for a closed subscheme $X \subset \mathbb{P}_k^N$ the power series $\sum_{r\geq 0} a_r(X)T^r$ represents a rational function.

2 Find the numbers $a_r(X)$ and the Hilbert polynomial $P_X(T)$ for a projective curve $X \subset \mathbb{P}^2$ of degree d. From what value of r is it true that $a_r(X) = P_X(r)$?

3 Find the Hilbert polynomial of a hypersurface of degree d in $\mathbb{P}^N$.

4 Find and prove a relation analogous to (6.50) in the case that X' is the intersection of X with a hypersurface of degree d transversal to X.

5 Find the Hilbert polynomial for the variety that is the intersection of two nonsingular transversal hypersurfaces of degree d_1 and d_2 in $\mathbb{P}^N$.

6 Is the ring $B = k[T]$ flat over its subring consisting of polynomials $F(T)$ such that $F'(T) = 0$?

7 Prove that a localisation A_S of any ring A is flat over A.

8 Prove that if $X \subset \mathbb{P}^N$ is a closed variety then the cone Z over it (introduced in proof of Theorem 6.7) is contained in $\mathbb{A}^{N+1}$.

9 Prove that if $a, b \in k$ with $4a^3 + 27b^2 \neq 0$ and $c(t) \in k[t]$ then the family of elliptic curves $y^2 = x^3 + ac(t)^2x + bc(t)^3$ has all the fibres over t with $c(t) \neq 0$ isomorphic. Prove that if $c(t)$ is not a perfect square in $k[t]$ then the family is not isomorphic to a direct product over any open set $U \subset \mathbb{A}^1$. Deduce from this that for elliptic curves there does not exist a universal family.

10 Find the Hilbert polynomial for the two curves of degree 2 in $\mathbb{P}^3$: a plane irreducible conic and a pair of skew lines.

11 Let $\varphi\colon X \to \mathbb{A}^1 = \operatorname{Spec} k[t]$ be a family of curves of degree 2 in $\mathbb{P}^3$ whose fibres over $t \neq 0$ are pairs of skew lines, and over $t = 0$ a pair of intersecting lines. Describe the scheme $\varphi^{-1}(0)$.

12 Prove the converse of Theorem 6.8: if $X \to \operatorname{Spec} A$ is a projective scheme over a 1-dimensional regular local ring A, and the fibres of X over the closed and generic points of $\operatorname{Spec} A$ have the same Hilbert polynomial, then X is flat over $\operatorname{Spec} A$.

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Basic Algebraic Geometry 2
Schemes and Complex Manifolds
Shafarevich, I.R.
2013, XIV, 262 p. 12 illus., Hardcover
ISBN: 978-3-642-38009-9