Facets and Rank of Integer Polyhedra

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Dedicated to Martin Grötschel, my first and best former doctoral student and a personal friend for almost forty years now. Ad multos annos, Martin!

Abstract We discuss several methods of determining all facets of "small" polytopes and polyhedra and give several criteria for classifying the facets into different facet types, so as to bring order into the multitude of facets as, e.g., produced by the application of the double description algorithm (DDA). Among the forms that we consider are the normal, irreducible and minimum support representations of facets. We study symmetries of vertex and edge figures under permissible permutations that leave the underlying polyhedron unchanged with the aim of reducing the numerical effort to find all facets efficiently. Then we introduce a new notion of the rank of facets and integer polyhedra. In the last section, we present old and new results on the facets of the symmetric traveling salesman polytope $\mathcal{Q}_T^n$ with up to $n = 10$ cities based on our computer calculations and state a conjecture that, in the notion of rank $\rho(P)$ introduced here, asserts $\rho(\mathcal{Q}_T^n) = n - 5$ for all $n \geq 5$. This conjecture is supported by our calculations up to $n = 9$ and, possibly, $n = 10$.

1 Introduction

Let $P \subseteq \mathbb{R}^n$ be any *pointed* rational polyhedron of $\mathbb{R}^n$ and $\mathbf{X} = (\mathbf{x}^1 \ \cdots \ \mathbf{x}^p)$, $\mathbf{Y} = (\mathbf{y}^1 \ \cdots \ \mathbf{y}^r)$ be a minimal generator of P. The columns $\mathbf{x}^1, \ldots, \mathbf{x}^p$ of $\mathbf{X}$ are a list of the extreme points of P and, likewise, the columns $\mathbf{y}^1, \ldots, \mathbf{y}^r$ of $\mathbf{Y}$ are a list of the direction vectors of the extreme rays of P and thus,

$$\mathbf{x} \in P \iff \mathbf{x} = \sum_{i=1}^{p} \mu_i \mathbf{x}^i + \sum_{j=1}^{r} \lambda_j \mathbf{y}^j \quad \text{for some } \mu_i \geq 0 \text{ with } \sum_{i=1}^{p} \mu_i = 1 \text{ and } \lambda_j \geq 0,$$

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i.e., $P = \mathrm{conv}(\mathbf{x}^1, \dots, \mathbf{x}^p) + \mathrm{cone}(\mathbf{y}^1, \dots, \mathbf{y}^r)$ where $\mathrm{conv}(\cdot)$ means the convex hull and $\mathrm{cone}(\cdot)$ the conical hull of the respective point sets. For short, we denote by $\mathrm{vert}\, P$ the set of all extreme points of P and by $\mathrm{exray}\, P$ the set of the direction vectors of all extreme rays of P. Rationality of the polyhedron P means that we assume that both $\mathbf{X}$ and $\mathbf{Y}$ consist of rational numbers only and "pointed" means that we assume that $p \geq 1$. If $\mathbf{Y}$ is void, P is a *polytope* in $\mathbb{R}^n$ rather than a polyhedron. Given the *pointwise* description $\mathbf{X}$, $\mathbf{Y}$ of P we can determine for "small" n, p and r an ideal linear description of P by running the double description algorithm (DDA), see, e.g., [4] or [38], to find a basis $(\mathbf{v}^1, v_0^1), \dots, (\mathbf{v}^s, v_0^s)$ of the lineality space L_C and a minimal generator $(\mathbf{v}^{s+1}, v_0^{s+1}), \dots, (\mathbf{v}^{s+t}, v_0^{s+t})$ of the conical part of the polyhedral cone

$$C = \left\{(\mathbf{v}, v_0) \in \mathbb{R}^{n+1} : \mathbf{vX} - v_0\mathbf{e} \leq \mathbf{0}, \mathbf{vY} \leq \mathbf{0}\right\}, \tag{1}$$

where $\mathbf{e} = (1, \dots, 1) \in \mathbb{R}^p$ and $\mathbf{0}$ the null vector. Defining $\mathbf{a}^i = \mathbf{v}^i$, $b_i = v_0^i$ for $1 \leq i \leq s$, $\mathbf{b}^T = (b_1, \dots, b_s)$, $\mathbf{h}^i = \mathbf{v}^{s+i}$, $h_i = v_0^{s+i}$ for $1 \leq i \leq t$ and $\mathbf{h}^T = (h_1, \dots, h_t)$ then

$$P = \left\{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} = \mathbf{b}, \mathbf{Hx} \leq \mathbf{h}\right\} \tag{2}$$

is an *ideal*, i.e., a minimal and complete, linear description of P where $\mathbf{A}$ is the $s \times n$ matrix of rank s with rows $\mathbf{a}^1, \dots, \mathbf{a}^s$ and $\mathbf{H}$ is the $t \times n$ matrix with rows $\mathbf{h}^1, \dots, \mathbf{h}^t$. By the rationality assumption about $\mathbf{X}$ and $\mathbf{Y}$ both $(\mathbf{A}, \mathbf{b})$ and $(\mathbf{H}, \mathbf{h})$ are matrices of rational numbers and thus, after appropriate scaling, integer numbers. Moreover, the dimension of P satisfies $\dim P = n - s$. P is full dimensional if and only if $L_C = \emptyset$, i.e., if and only if the matrix $\mathbf{A}$ is void. If $\mathbf{A}$ is nonvoid, then P is a *flat* in $\mathbb{R}^n$, otherwise P is a *solid*. The system of equations $\mathbf{Ax} = \mathbf{b}$ is an ideal description of the affine hull $\mathrm{aff}(P)$ of P. Every row $\mathbf{hx} \leq h_0$, say, of $\mathbf{Hx} \leq \mathbf{h}$ defines a *facet* of P, i.e., a face of dimension $\dim P - 1$ of P, and vice versa, for every facet F of P there exists some row $\mathbf{hx} \leq h_0$ of $\mathbf{Hx} \leq \mathbf{h}$ such that $F = \{\mathbf{x} \in P : \mathbf{hx} = h_0\}$. The linear description (2) of P is *quasi-unique*: if $\mathbf{hx} \leq h_0$ and $\mathbf{gx} \leq g_0$ are two different representations of some facet F of P then $\mathbf{g} = \lambda\mathbf{h} + \boldsymbol{\mu}\mathbf{A}$ and $g_0 = \lambda h_0 + \boldsymbol{\mu}\mathbf{b}$ for some $\lambda > 0$ and $\boldsymbol{\mu} \in \mathbb{R}^s$, while the system of equations $\mathbf{Ax} = \mathbf{b}$ is unique *modulo* nonsingular transformations of $\mathbb{R}^s$. $\lambda > 0$ follows because $\mathbf{gx} - g_0 = \lambda(\mathbf{hx} - h_0) < 0$ for all $\mathbf{x} \in P - F$. The preceding is well known and we refer to [38] for a detailed treatment of polyhedra in $\mathbb{R}^n$ including the double description algorithm and any unexplained notation used in this paper. There are other methods to pass from the pointwise description of polyhedra to their ideal linear description (2) and vice versa, like the Fourier-Motzkin elimination algorithm, see, e.g., [50]. Since I have written most of this paper and carried out the computational work reported here in Sect. 7—which was around 1996/7—some progress has occurred in the algorithms for passing from the point-wise description of rational polyhedra to an ideal linear one and vice versa. I am indebted to Gerd Reinelt [46] for bringing his work with Thomas Christof to my attention, see [5] and [6–8] as well as the many references contained therein.

If P is a solid, then the ideal description of P is unique up to multiplication of $(\mathbf{H}, \mathbf{h})$ by positive scalars. Thus by scaling the data to be relatively prime integer numbers a *unique* ideal linear description of P is obtained. In this case the only issue that we need to address is how to compute a possible enormous number of the facets of P. Much of the material in this paper addresses this question and the question of how to obtain "reasonable" representations by way of linear inequalities of the facets of flats in $\mathbb{R}^n$.

The polyhedra of interest to us are subsets of $\mathbb{R}^n_+$ in the polyhedral case or subsets of the unit cube of $\mathbb{R}^n$ in the polytopal case, but we do not use that here. In either case one is frequently interested in obtaining an ideal description $\mathbf{Hx} \leq \mathbf{h}$ of the facets of P such that either $\mathbf{H} \geq \mathbf{O}$ or $\mathbf{H} \leq \mathbf{O}$ provided that such descriptions exist for P. No matter what *sign* convention is used in the linear description, one is frequently interested in finding *minimal support* representations of the facets of P. If P is flat, running the DDA to find an ideal linear description (2) of P does not automatically provide facets of this form. Rather we get a representation of any facet of P *modulo* some linear combination of the equations $\mathbf{Ax} = \mathbf{b}$. Thus the output of the DDA for a simple nonnegativity constraint $x_j \geq 0$ (if it defines a facet of P) can have many nonzeros and even a nonzero right-hand side. Intersecting the cone (1) with the nonnegativity constraints $\mathbf{v} \geq \mathbf{0}$ does not work: the result of running the DDA on the smaller cone gives all facets of P (*if* P has a description $\mathbf{Hx} \leq \mathbf{h}$ with $\mathbf{H} \geq \mathbf{O}$) plus typically considerably more inequalities that are valid, but not facet defining.

When we wish to analyze 1,000,000 or more "extreme rays" of the cone (1), we need to classify the 1,000,000 or more representations of facets produced by DDA *automatically* into equivalence classes. Here an "equivalence class" is understood to be a set of facets of P that are identical in some linear representation of them *modulo* linear combinations of the equations $\mathbf{Ax} = \mathbf{b}$ defining the affine hull of P and *modulo* permutations of the indices $1, \ldots, n$ of the components of $\mathbf{x} \in P$ which leave the polyhedron unchanged. Since every permutation of $1, \ldots, n$ can be described by some $n \times n$ permutation matrix $\boldsymbol{\Pi}$, a permutation $\boldsymbol{\Pi}$ leaves P unchanged if $P = \{\boldsymbol{\Pi}\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \in P\}$. We call such index permutations *permissible* for P. Denote by $\Pi(P)$ the set of all permissible index permutations for P. $\Pi(P) \neq \emptyset$ since $\mathbf{I}_n \in \Pi(P)$, where $\mathbf{I}_n$ is the $n \times n$ identity matrix. Let $\boldsymbol{\Pi} \in \Pi(P)$ and $\boldsymbol{\Pi}^T$ be the transpose of $\boldsymbol{\Pi}$. Since $\boldsymbol{\Pi}^T\boldsymbol{\Pi} = \boldsymbol{\Pi}\boldsymbol{\Pi}^T = \mathbf{I}_n$, $\boldsymbol{\Pi}^T \in \Pi(P)$ for every $\boldsymbol{\Pi} \in \Pi(P)$. Moreover, $\boldsymbol{\Pi}_a, \boldsymbol{\Pi}_b \in \Pi(P)$ implies that $\boldsymbol{\Pi}_a\boldsymbol{\Pi}_b \in \Pi(P)$ and $\boldsymbol{\Pi}_a, \boldsymbol{\Pi}_b, \boldsymbol{\Pi}_c \in \Pi(P)$ implies that $(\boldsymbol{\Pi}_a\boldsymbol{\Pi}_b)\boldsymbol{\Pi}_c = \boldsymbol{\Pi}_a(\boldsymbol{\Pi}_b\boldsymbol{\Pi}_c) = \boldsymbol{\Pi}_a\boldsymbol{\Pi}_b\boldsymbol{\Pi}_c \in \Pi(P)$. But $\boldsymbol{\Pi}_a\boldsymbol{\Pi}_b \neq \boldsymbol{\Pi}_b\boldsymbol{\Pi}_a$ is possible for $\boldsymbol{\Pi}_a, \boldsymbol{\Pi}_b \in \Pi(P)$ and thus $\Pi(P)$ is a (nonabelian) group of order at most $n!$ in general.

Definition 1 Let $F \neq F'$ be any two distinct *facets* of P. If $F' = \{\mathbf{x} \in P : \boldsymbol{\Pi}^T\mathbf{x} \in F\}$ for some $\boldsymbol{\Pi} \in \Pi(P)$, then F and F' are *equivalent* under $\Pi(P)$. $\kappa(P)$ is the *class number* of distinct facets of P that are *pairwise not equivalent* under $\Pi(P)$.

Every permissible index permutation corresponds to a linear transformation of the polyhedron P. More general, affine transformations exist that leave a polyhedron unchanged. This is the case, e.g., for the *Boolean quadric polytope* QP^n, see

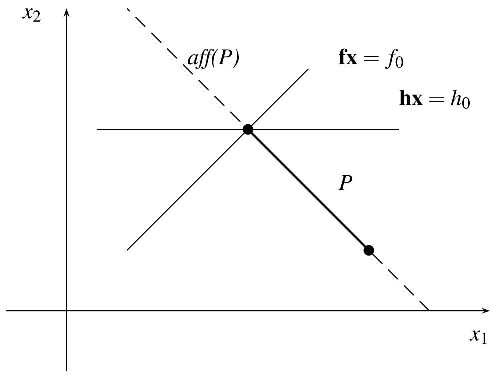


Fig. 1 Normal form representation of facets of a flat polyhedron $P \subseteq \mathbb{R}^2$

Theorem 6 of [37]. Here we study permissible index permutations only, even though several of the properties that we establish remain true mutatis mutandis for more general transformations of P.

2 Normal Form and Classification of Facets

The first task of the analysis is to find a "normal form" for the facets of flat polyhedra $P \subseteq \mathbb{R}^n$ that takes care of the multiplicity of the facet representations due to the equations defining the affine hull of P and that permits us to determine $\kappa(P)$ and some unique member of each equivalence class under $\Pi(P)$.

An inequality $\mathbf{fx} \le f_0$ is *valid* for P if $P \subseteq \{\mathbf{x} \in \mathbb{R}^n : \mathbf{fx} \le f_0\}$. Let $F \subseteq P$ be any facet of the polyhedron P. Every valid inequality $\mathbf{fx} \le f_0$ for P with

$$F = P \cap \left\{\mathbf{x} \in \mathbb{R}^n : \mathbf{fx} = f_0\right\}$$

is a *representation* of F.

Definition 2 ([28]) A representation $\mathbf{fx} \le f_0$ of a facet $F = \{\mathbf{x} \in P : \mathbf{fx} = f_0\}$ of P is in *normal form* if $\mathbf{Af}^T = \mathbf{0}$ where $\mathrm{aff}(P) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} = \mathbf{b}\}$.

Given any representation $\mathbf{hx} \le h_0$ of a facet F of P we can calculate its normal form representation by projecting $\mathbf{h}$ onto the subspace $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} = \mathbf{0}\}$ and adjusting the right-hand side accordingly; see Fig. 1.

Claim 1 *Let* $\mathbf{hx} \le h_0$ *be any representation of a facet* F *of* P. *Then* $\mathbf{fx} \le f_0$ *with*

$$\mathbf{f} = \mathbf{h}\big(\mathbf{I}_n - \mathbf{A}^T\left(\mathbf{AA}^T\right)^{-1}\mathbf{A}\big), \qquad f_0 = h_0 - \mathbf{hA}^T\left(\mathbf{AA}^T\right)^{-1}\mathbf{b} \tag{3}$$

is a representation of F *in normal form.*

Claim 2 *The normal form representation of a facet F of P is unique up to scaling.*

Proof Let $\mathbf{gx} \leq g_0$ be any representation of F and $\mathbf{fx} \leq f_0$ be a normal form representation of F. Then we know from the quasi-uniqueness that $\mathbf{g} = \lambda\mathbf{f} + \boldsymbol{\mu}\mathbf{A}$, $g_0 = \lambda f_0 + \boldsymbol{\mu}\mathbf{b}$ where $\lambda > 0$ is a positive scalar and $\boldsymbol{\mu} \in \mathbb{R}^s$ is arbitrary. Suppose that $\mathbf{gx} \leq g_0$ is also in normal form. Multiplying the equation for $\mathbf{g}$ by $\mathbf{A}^T$ we get $\mathbf{0} = \mathbf{gA}^T = \lambda\mathbf{fA}^T + \boldsymbol{\mu}\mathbf{AA}^T$ and thus $\boldsymbol{\mu} = \mathbf{0}$ since $\mathbf{fA}^T = \mathbf{0}$ and $r(\mathbf{AA}^T) = s$. Thus $(\mathbf{g}, g_0) = \lambda(\mathbf{f}, f_0)$ for some $\lambda > 0$. □

The normal form of the representation of some facet of the polyhedron $P \subseteq \mathbb{R}^n$ takes care—in essence—of the multiplicity of the representations of that facet that results from the flatness of the polyhedron. For rational polyhedra the remaining ambiguity can be eliminated by bringing the normal form representations of the facets of P into integer coefficient form with relatively prime integers.

Definition 3 A normal form representation $(\mathbf{f}, f_0)$ of a facet of P is in *primitive normal form* if the components of $(\mathbf{f}, f_0)$ are relatively prime integers.

Let $\mathbf{fx} \leq f_0$ and $\mathbf{gx} \leq g_0$ be any two facet-defining inequalities for P in primitive normal form. Then the facets F and F' of P defined by $\mathbf{fx} \leq f_0$ and $\mathbf{gx} \leq g_0$ are equivalent under $\Pi(P)$ if and only if $f_0 = g_0$ and $\mathbf{f} = \mathbf{g}\boldsymbol{\Pi}$ for some $\boldsymbol{\Pi} \in \Pi(P)$. To find the class number $\kappa(P)$ of distinct facet "types" of P we must check all primitive normal form representations pairwise for equivalence under $\Pi(P)$. Testing each pair of inequalities for isomorphism is computationally expensive. This effort can be reduced by a "preclassification" using criteria that are necessary for the equivalence under $\Pi(P)$ of distinct facets of P.

Claim 3 *If $\mathbf{hx} \leq h_0$ and $\mathbf{gx} \leq g_0$ define two distinct facets of P that belong to the same equivalence class with respect to some $\boldsymbol{\Pi} \in \Pi(P)$, then*

(i) *$n^v_{\mathbf{h}} = n^v_{\mathbf{g}}$ and $n^{ex}_{\mathbf{h}} = n^{ex}_{\mathbf{g}}$, where*

$$n^v_{\mathbf{h}} = \left|\{\mathbf{x} \in \operatorname{vert} P : \mathbf{hx} = h_0\}\right| \quad \textit{and} \quad n^{ex}_{\mathbf{h}} = \left|\{\mathbf{y} \in \operatorname{exray} P : \mathbf{hy} = 0\}\right| \tag{4}$$

and $n^v_{\mathbf{g}}$ and $n^{ex}_{\mathbf{g}}$ are defined correspondingly for $(\mathbf{g}, g_0)$.

(ii) *$d_h = d_g$, where d_h are d_g are the distances of the center of gravity $\mathbf{x}^C$ of P, i.e.,*

$$\mathbf{x}^C = \frac{1}{p}\sum_{i=1}^{p}\mathbf{x}^i, \tag{5}$$

from $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} = \mathbf{b}, \mathbf{hx} = h_0\}$ and $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} = \mathbf{b}, \mathbf{gx} = g_0\}$, respectively.

(iii) *$a_0 = b_0$ and $\mathbf{a}^{ord} = \mathbf{b}^{ord}$, where $\mathbf{ax} \leq a_0$, $\mathbf{bx} \leq b_0$ are the primitive normal forms of $\mathbf{hx} \leq h_0$, $\mathbf{gx} \leq g_0$ and $\mathbf{a}^{ord}$, $\mathbf{b}^{ord}$ are the vectors obtained from $\mathbf{a}$, $\mathbf{b}$ by ordering their components by increasing value.*

Proof

(i) Extreme points of P are mapped into extreme points of P for $\boldsymbol{\Pi} \in \Pi(P)$. For let $\mathbf{x} \in \operatorname{vert} P$, $\boldsymbol{\Pi} \in \Pi(P)$ and suppose that $\mathbf{z} = \boldsymbol{\Pi}\mathbf{x} \notin \operatorname{vert} P$. Then there exist $\mathbf{z}^1 \neq \mathbf{z}^2 \in P$ and $0 < \mu < 1$ such that $\mathbf{z} = \mu\mathbf{z}^1 + (1-\mu)\mathbf{z}^2$. But $\boldsymbol{\Pi}^T \in \Pi(P)$ and $\mathbf{x} \in \operatorname{vert} P$. Hence $\mathbf{x} = \boldsymbol{\Pi}^T\mathbf{z} = \mu\boldsymbol{\Pi}^T\mathbf{z}^1 + (1-\mu)\boldsymbol{\Pi}^T\mathbf{z}^2$ implies $\boldsymbol{\Pi}^T\mathbf{z}^1 = \boldsymbol{\Pi}^T\mathbf{z}^2$ and thus $\mathbf{z}^1 = \mathbf{z}^2$, which is a contradiction. We prove likewise that extreme rays of P are mapped into extreme rays of P for $\boldsymbol{\Pi} \in \Pi(P)$. Thus, since the facets defined by $\mathbf{hx} \le h_0$ and $\mathbf{gx} \le g_0$ are equivalent under $\Pi(P)$, the respective counts are equal.

(ii) To calculate d_h^2 we have to solve

$$\min\left\{\left\|\mathbf{x} - \mathbf{x}^C\right\|^2 : \mathbf{x} \in \mathcal{A}\right\}, \tag{6}$$

where $\mathcal{A} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} = \mathbf{b}, \mathbf{hx} = h_0\}$. It follows that $\mathcal{A} \subseteq \{\mathbf{x} \in \mathbb{R}^n : \mathbf{f}^1\mathbf{x} = f_0^1\}$ where $(\mathbf{f}^1, f_0^1)$ is the normal form of $(\mathbf{h}, h_0)$. Solving $\min\{\|\mathbf{x} - \mathbf{x}^C\|^2 : \mathbf{f}^1\mathbf{x} = f_0^1\}$ by the Lagrangean multiplier technique we calculate $\mathbf{x}^* = \mathbf{x}^C - \frac{(\mathbf{f}^1\mathbf{x}^C - f_0^1)}{\|\mathbf{f}^1\|^2}(\mathbf{f}^1)^T$. Since $\mathbf{hx} - h_0 = \mathbf{f}^1\mathbf{x} - f_0^1$ for all $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{Ax} = \mathbf{b}$ we calculate $\mathbf{x}^* \in \mathcal{A}$, because

$$\mathbf{f}^1\mathbf{h}^T = \|\mathbf{h}\|^2 - \mathbf{hA}^T\left(\mathbf{AA}^T\right)^{-1}\mathbf{Ah}^T = \left\|\mathbf{f}^1\right\|^2. \tag{7}$$

Since $\mathcal{A} \subseteq \{\mathbf{x} \in \mathbb{R}^n : \mathbf{f}^1\mathbf{x} = f_0^1\}$ and $\mathbf{x}^* \in \mathcal{A}$ it follows that

$$d_h = \frac{|\mathbf{hx}^C - h_0|}{\|\mathbf{f}^1\|} \quad \text{and likewise} \quad d_g = \frac{|\mathbf{gx}^C - g_0|}{\|\mathbf{f}^2\|}, \tag{8}$$

where $(\mathbf{f}^2, f_0^2)$ is the normal form of $(\mathbf{g}, g_0)$. Since $\mathbf{hx} \le h_0$ and $\mathbf{gx} \le g_0$ belong to the same equivalence class and the normal form is unique up to scaling, there exists $\lambda > 0$ such that $\mathbf{f}^1 = \lambda\mathbf{f}^2\boldsymbol{\Pi}$ and $f_0^1 = \lambda f_0^2$ for some $\boldsymbol{\Pi} \in \Pi(P)$. Consequently,

$$\begin{aligned} p\left|\mathbf{hx}^C - h_0\right| &= \left|\sum_{i=1}^{p}\left(\mathbf{f}^1\mathbf{x}^i - f_0^1\right)\right| = \lambda\left|\sum_{i=1}^{p}\left(\mathbf{f}^2\boldsymbol{\Pi}\mathbf{x}^i - f_0^2\right)\right| \\ &= \lambda\left|\sum_{i=1}^{p}\left(\mathbf{f}^2\mathbf{x}^i - f_0^2\right)\right| = p\lambda\left|\mathbf{gx}^C - g_0\right|. \end{aligned}$$

Since $\|\mathbf{f}^1\| = \lambda\|\mathbf{f}^2\|$ it follows that $d_h = d_g$.

(iii) By assumption $\mathbf{hx} \le h_0$ and $\mathbf{gx} \le g_0$ belong to the same equivalence class with respect to $\boldsymbol{\Pi}$ and their primitive normal forms are unique. Thus $a_0 = b_0$ and $\mathbf{a} = \mathbf{b}\boldsymbol{\Pi}$. Hence if $a_0 \neq b_0$ or $\mathbf{a}^{ord} \neq \mathbf{b}^{ord}$ then $\boldsymbol{\Pi} \in \Pi(P)$ cannot exist, contradiction. □

Applying Claim 3 to the 1,000,000 or more inequalities $(\mathbf{H}, \mathbf{h})$ produced by the DDA we can partition the system $(\mathbf{H}, \mathbf{h})$ into q, say, disjoint subsystems

$$\mathbf{H}_1\mathbf{x} \le \mathbf{h}_1, \qquad \mathbf{H}_2\mathbf{x} \le \mathbf{h}_2, \qquad \ldots, \qquad \mathbf{H}_q\mathbf{x} \le \mathbf{h}_q, \tag{9}$$

such that for each row of $(\mathbf{H}_i, \mathbf{h}_i)$ the criteria of Claim 3 are met. This breaks the 1,000,000 or more rows of $(\mathbf{H}, \mathbf{h})$ typically down into "chunks" of 5,000, 10,000, etc. rows that we need to analyze further.

Example 1 The distance and normal form calculation can be effectively shortened in most cases when a particular structure is present. To illustrate this for the symmetric traveling salesman problem, we can ignore the basis of the lineality space as calculated by, e.g., the DDA and let $\mathbf{A}$ be the node versus edge incidence matrix of the complete graph having m nodes, say, so that $n = m(m-1)/2$ is the number of variables in the problem. Every *permutation* of the m nodes of the graph induces a permissible permutation of the indices $1, \ldots, n$ because re-indexing the nodes of the graph leaves the associated symmetric traveling salesman (STS) polytope unchanged. We wish to find the equivalence classes of its facets for such permutations. We know, see [19], that the affine hull for the symmetric traveling salesman polytope is given by the system of equations $\mathbf{A}\mathbf{x} = \mathbf{2}$ where $\mathbf{2}$ is a vector of m entries equal to 2. Thus

$$\mathbf{A}\mathbf{A}^T = (m-2)\mathbf{I}_m + \mathbf{e}_m\mathbf{e}_m^T, \qquad \left(\mathbf{A}\mathbf{A}^T\right)^{-1} = \frac{1}{m-2}\left(\mathbf{I}_m - \frac{1}{2(m-1)}\mathbf{e}_m\mathbf{e}_m^T\right),$$

where $\mathbf{e}_m$ is a column vector of m entries equal to 1. It follows that (3) and the calculation of $\|\mathbf{f}\|^2$ simplify to

$$\begin{aligned} \mathbf{f} &= \mathbf{h} - \frac{1}{m-2}\mathbf{h}^*\mathbf{A} + \frac{2(\mathbf{h}\mathbf{e}_n)}{(m-1)(m-2)}\mathbf{e}_n^T, \\ f_0 &= h_0 - \frac{2}{m-1}(\mathbf{h}\mathbf{e}_n), \end{aligned} \tag{10}$$

$$\|\mathbf{f}\|^2 = \|\mathbf{h}\|^2 - \frac{1}{m-2}\left\|\mathbf{h}^*\right\|^2 + \frac{2}{(m-1)(m-2)}(\mathbf{h}\mathbf{e}_n)^2, \tag{11}$$

where $\mathbf{h}^* = \mathbf{h}\mathbf{A}^T$ is the vector with components $h_v^* = \sum_{e\in\delta(v)} h_e$ for all nodes $1 \le v \le m$ of the graph, $\delta(v)$ is the set of edges e of the graph meeting the node v and h_e for $1 \le e \le n$ are the components of $\mathbf{h}$. Thus the numerical inversion of $\mathbf{A}\mathbf{A}^T$ can be avoided. Moreover, the gravity center $\mathbf{x}^C$ of the STS polytope is given by $x_e^C = \frac{2}{m-1}$ for $1 \le e \le n$ and so the formula (8) for the squared distance calculation simplifies to

$$d_h^2 = \frac{(m-2)(2\mathbf{h}\mathbf{e}_n - (m-1)h_0)^2}{(m-1)((m-1)(m-2)\|\mathbf{h}\|^2 - (m-1)\|\mathbf{h}^*\|^2 + 2(\mathbf{h}\mathbf{e}_n)^2)}. \tag{12}$$

If the components of $\mathbf{h}$ are integer, d_h^2 can thus be computed exactly in rational form. Multiplying $(\mathbf{f}, f_0)$ as defined by (10) by $(m-1)(m-2)$ and clearing the greatest common divisor we get for integer $(\mathbf{h}, h_0)$ the *unique* representation in primitive normal form of the facet of the STS polytope defined by $\mathbf{hx} \le h_0$.

Having obtained the partitioning (9) we see at present no other way than to check the remaining members in each class of (9) pairwise for equivalence under $\Pi(P)$, unless there are other affine linear transformations of P that leave P unchanged. The remaining isomorphism test can be done, e.g., *enumeratively* on the primitive normal form representation of the facets of P and is computationally expensive. But for relatively "small" polyhedra it can be done in reasonable computing times. This way we can find the class number $\kappa(P)$ of the distinct types of facets of flat or solid polyhedra $P \subseteq \mathbb{R}^n$ as well as a particular representation for each facet class.

3 Irreducible Representations of Facets

Denote f_j for $j = 1, \ldots, n$ the n first components of $(\mathbf{f}, f_0) \in \mathbb{R}^{n+1}$.

Definition 4 A representation $(\mathbf{f}, f_0)$ of a facet of P is in *irreducible form* if either

(i) $\mathbf{f} \ge \mathbf{0}$ and $|\{j \in \{1, \ldots, n\} : f_j > 0\}|$ is as small as possible or
(ii) $\mathbf{f} \le \mathbf{0}$ and $|\{j \in \{1, \ldots, n\} : f_j < 0\}|$ is as small as possible or
(iii) $|\{j \in \{1, \ldots, n\} : f_j \neq 0\}|$ is as small as possible.

In case (i) of the definition $(\mathbf{f}, f_0)$ has minimum positive, in case (ii) it has minimum negative and in case (iii) minimum support. This concept of irreducibility of facet representations does not imply uniqueness, but it reduces substantially the number of ways in which we can represent the facets of *flat* polyhedra which is, a priori, a continuum. See Fig. 5 of Sect. 7 where we show 20 different facet types of the 194,187 facets of the traveling salesman polytope on eight cities in irreducible nonnegative form (except the four remaining types given by the nonnegativity and subtour elimination constraints), i.e., $\kappa(P) = 24$ in this case. The question is how to find them (if they exist at all). To do so we have to use the DDA again.

We will discuss only the polytopal case, i.e., when $\mathbf{Y}$ is void, but the following applies mutatis mutandis to the polyhedral case as well. Let $(\mathbf{h}, h_0)$ be a representation of some facet F of P and partition $\mathbf{X}$ into two matrices $\mathbf{X}_1$, $\mathbf{X}_2$ such that

$$\mathbf{hX}_1 - h_0\mathbf{e}_1 = \mathbf{0}, \qquad \mathbf{hX}_2 - h_0\mathbf{e}_2 < \mathbf{0}, \tag{13}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$ are compatible vectors of ones. Consider the polyhedral cone

$$C_h^+ = \left\{(\mathbf{w}, w_0) \in \mathbb{R}^{n+1} : \mathbf{wX}_1 - w_0\mathbf{e}_1 = \mathbf{0}, \mathbf{wX}_2 - w_0\mathbf{e}_2 \le \mathbf{0}, \mathbf{w} \ge \mathbf{0}\right\} \tag{14}$$

and denote by $F_\mathbf{h}$ the facet of P defined by $\mathbf{hx} \le h_0$, i.e.,

$$F_\mathbf{h} = P \cap \left\{\mathbf{x} \in \mathbb{R}^n : \mathbf{hx} = h_0\right\}. \tag{15}$$

If $F_{\mathbf{h}}$ has a representation $\mathbf{fx} \le f_0$ with $\mathbf{f} \ge \mathbf{0}$, then $(\mathbf{f}, f_0) \in C_h^+$. Thus any extreme ray of the conical part of C_h^+ with minimum positive support is an irreducible representation of $F_{\mathbf{h}}$ and can be found by running the DDA with (14) as input. To find a minimum negative support representation of the facet $F_{\mathbf{h}}$ of P we use the polyhedral cone C_h^- which is (14) with the constraints $\mathbf{wX}_2 - w_0\mathbf{e}_2 \le \mathbf{0}$ replaced by $\mathbf{wX}_2 - w_0\mathbf{e}_2 \ge \mathbf{0}$ and run the DDA.

Since the polyhedron $P \subseteq \mathbb{R}^n$ may or may not admit representations of its facets satisfying either sign restriction, we are lead to consider the cone $C_h^\pm$ which is (14) but without the constraints $\mathbf{w} \ge \mathbf{0}$. It is not difficult to show that a minimal generator for $C_h^\pm$ consists of a basis of its lineality space, i.e., of the rows of the matrix $(\mathbf{A}, \mathbf{b})$ defining the affine hull of the polyhedron P, plus the direction vector given by the (unique) normal form of the facet of P defined by $\mathbf{hx} \le h_0$. So this construction does not help us to find "minimal support" representations of the facets of flat polyhedra.

Consider instead

$$C_h = \big\{(\mathbf{u}, \mathbf{v}, w_0) \in \mathbb{R}^{2n+1} : \\ \mathbf{uX}_1 - \mathbf{vX}_1 - w_0\mathbf{e}_1 = \mathbf{0}, \mathbf{uX}_2 - \mathbf{vX}_2 - w_0\mathbf{e}_2 \le \mathbf{0}, \mathbf{u} \ge \mathbf{0}, \mathbf{v} \ge \mathbf{0}\big\}. \qquad (16)$$

Since the extreme rays of C_h are defined by submatrices of the constraint set of (16) of rank $2n$, it follows from a rank consideration that every extreme ray $(\mathbf{u}, \mathbf{v}, w_0)$ of C_h with $u_j \ne v_j$ satisfies either $u_j = 0$ or $v_j = 0$ for all $1 \le j \le n$. Moreover, it follows also from a rank consideration that every extreme ray of C_h^+ and every extreme ray of C_h^- defines an extreme ray of C_h. Let $\mathbf{fx} \le f_0$ be any representation of the facet $F_{\mathbf{h}}$ of *minimal support*. Setting $\mathbf{u} = \max\{\mathbf{0}, \mathbf{f}\}$, $\mathbf{v} = \max\{\mathbf{0}, -\mathbf{f}\}$ and $w_0 = f_0$ it follows that $(\mathbf{u}, \mathbf{v}, w_0) \in C_h$. Consequently, every minimum support representation of $F_{\mathbf{h}}$ defines an extreme ray of the conical part of C_h. The cone C_h exhibits a lot of "symmetry" and has many extreme rays.

We can thus find the *irreducible representations* of any facet $F_{\mathbf{h}}$ of any flat P by running the DDA with the respective cones C_h^+, C_h^-, and C_h as input.

In a computer implementation we translate the polyhedron P so that $F_{\mathbf{h}}$ contains the origin of $\mathbb{R}^n$. Consequently, the "homogenizing" variable w_0 in (14) and (16) can be dropped. Moreover, we need to generate only a single row for the changed $\mathbf{X}_2$ part of the constraint set of, e.g., (16). For the changed $\mathbf{X}_1$ part of it we need only a submatrix of maximal rank. This reduces the size of the input to the DDA substantially. More precisely, in all three cases exactly $\dim P$ constraints plus the nonnegativity conditions suffice to find an irreducible representation of $F_{\mathbf{h}}$.

4 Symmetry of Vertex Figures

Given a pointwise description of a polyhedron $P \subseteq \mathbb{R}^n$ the numerical effort to find all facets of P consists of running the DDA, or some similar algorithm, with the cone (1) as input. E.g., for the symmetric traveling salesman problem with m cities on a complete graph the number of inequalities in (1) equals $p = \frac{1}{2}(m-1)!$. For

$m = 8$ we have $p = 2{,}520$, for $m = 9$ we have $p = 20{,}160$, for $m = 10$ we have $p = 181{,}440$ and so forth. Given the current state of computing machinery it is out of the question to attack this problem directly by analyzing the cone (1) for $m \geq 9$. In this section and the next we discuss ways of reducing the computational effort for general polyhedra and polytopes for which $\Pi(P) \neq \{\mathbf{I}_n\}$.

Definition 5 For $\mathbf{x}^0 \in \operatorname{vert} P$ let $\mathbf{x}^1, \ldots, \mathbf{x}^a$ be all of its *adjacent vertices* and $\mathbf{y}^1, \ldots, \mathbf{y}^b$ represent all the extreme rays P such that $\mathbf{x}^0 + \lambda \mathbf{y}^i$ for $\lambda \geq 0$ is a 1-dimensional face of P. The displaced cone with apex at $\mathbf{x}^0$

$$OC\big(\mathbf{x}^0, P\big) = \left\{ \mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{x}^0 + \sum_{i=1}^{a} \lambda_i \big(\mathbf{x}^i - \mathbf{x}^0\big) + \sum_{i=1}^{b} \mu_i \mathbf{y}^i, \lambda_i \geq 0, \mu_i \geq 0 \right\} \quad (17)$$

is the *vertex figure* (or the *outer cone*) for P at $\mathbf{x}^0$.

Note that our definition of a vertex figure is different from the one frequently found in the literature on polytopes, see, e.g., [22] or [50]. By construction, $OC(\mathbf{x}^0, P) \supset P$, $\dim OC(\mathbf{x}^0, P) = \dim P$ and every facet of $OC(\mathbf{x}^0, P)$ is a facet of P, but not vice versa since all facets of $OC(\mathbf{x}^0, P)$ contain $\mathbf{x}^0$. Let $\boldsymbol{\Pi} \in \Pi(P)$ and $\mathbf{x}^* = \boldsymbol{\Pi}\mathbf{x}^0$. Then $\mathbf{x}^* \in \operatorname{vert} P$ and we claim that the vertex figure $OC(\mathbf{x}^*, P)$ for P at $\mathbf{x}^*$ is given by

$$OC\big(\mathbf{x}^*, P\big) = \left\{ \mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{x}^* + \sum_{i=1}^{a} \lambda_i \big(\boldsymbol{\Pi}\mathbf{x}^i - \mathbf{x}^*\big) + \sum_{i=1}^{b} \mu_i \boldsymbol{\Pi}\mathbf{y}^i, \lambda_i \geq 0, \mu_i \geq 0 \right\}.$$

Claim 4 *For any $\mathbf{x}^0 \neq \mathbf{x}^1 \in P$ let $F(\mathbf{x}^0, \mathbf{x}^1)$ be the face of* minimal dimension *of P containing both $\mathbf{x}^0$ and $\mathbf{x}^1$. Then $F(\boldsymbol{\Pi}\mathbf{x}^0, \boldsymbol{\Pi}\mathbf{x}^1) = \{\mathbf{x} \in \mathbb{R}^n : \boldsymbol{\Pi}^T\mathbf{x} \in F(\mathbf{x}^0, \mathbf{x}^1)\}$ and* $\dim F(\mathbf{x}^0, \mathbf{x}^1) = \dim F(\boldsymbol{\Pi}\mathbf{x}^0, \boldsymbol{\Pi}\mathbf{x}^1)$ *for all $\boldsymbol{\Pi} \in \Pi(P)$.*

Proof Let $F = F(\mathbf{x}^0, \mathbf{x}^1)$ and likewise $\Pi F = F(\boldsymbol{\Pi}\mathbf{x}^0, \boldsymbol{\Pi}\mathbf{x}^1)$. Since F is a face of P, there exists $(\mathbf{f}, f_0) \in \mathbb{R}^{n+1}$ such that $\mathbf{fx} = f_0$ for all $\mathbf{x} \in F$ and $\mathbf{fx} < f_0$ for all $\mathbf{x} \in P$, $\mathbf{x} \notin F$. Since $\mathbf{x}^0, \mathbf{x}^1 \in F$ and $\boldsymbol{\Pi}\mathbf{x}^0, \boldsymbol{\Pi}\mathbf{x}^1 \in \Pi F$ it follows from the minimality of ΠF that $\Pi F \subseteq \{\mathbf{x} \in \mathbb{R}^n : (\mathbf{f}\boldsymbol{\Pi}^T)\mathbf{x} = f_0\}$. Consequently, we have $\Pi F \subseteq \{\mathbf{x} \in \mathbb{R}^n : \boldsymbol{\Pi}^T\mathbf{x} \in F\}$ since $\boldsymbol{\Pi}^T\mathbf{x} \in P$ for all $\mathbf{x} \in \Pi F$. We conclude likewise that $F \subseteq \{\mathbf{x} \in \mathbb{R}^n : \boldsymbol{\Pi}\mathbf{x} \in \Pi F\}$. Consequently, if $\boldsymbol{\Pi}^T\mathbf{x} \in F$ for some $\mathbf{x} \in \mathbb{R}^n$ then $\boldsymbol{\Pi}(\boldsymbol{\Pi}^T\mathbf{x}) = \mathbf{x} \in \Pi F$. Thus $\Pi F = \{\mathbf{x} \in \mathbb{R}^n : \boldsymbol{\Pi}^T\mathbf{x} \in F\}$ and the first part follows. Let $P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{Hx} \leq \mathbf{h}\}$ be any linear description of P. Since P is pointed we have $r(\mathbf{H}) = n$. Denote by $(\mathbf{H}_F, \mathbf{h}_F)$ the *largest* submatrix of $(\mathbf{H}, \mathbf{h})$ such that $\mathbf{H}_F\mathbf{x} = \mathbf{h}_F$ for all $\mathbf{x} \in F$. Thus $\dim F = n - r(\mathbf{H}_F)$. Likewise, let $(\mathbf{H}_{\Pi F}, \mathbf{h}_{\Pi F})$ be the largest submatrix $(\mathbf{H}, \mathbf{h})$ such that $\mathbf{H}_{\Pi F}\mathbf{x} = \mathbf{h}_{\Pi F}$ for all $\mathbf{x} \in \Pi F$. By the preceding argument it follows that—up to row permutations—$\mathbf{H}_{\Pi F} = \mathbf{H}_F\boldsymbol{\Pi}^T$ and thus $\dim \Pi F = \dim F$ since $\boldsymbol{\Pi}$ is nonsingular. □

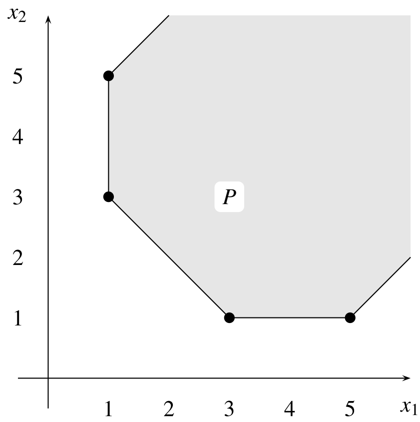


Fig. 2 Vertex classes of polyhedra

Thus the face of minimal dimension of P containing $\boldsymbol{\Pi}\mathbf{x}^0$ and $\boldsymbol{\Pi}\mathbf{x}^1$ is the image under $\boldsymbol{\Pi}$ of the face of minimal dimension of P that contains $\mathbf{x}^0$ and $\mathbf{x}^1$. Thus permissible index permutations for P preserve the adjacency of extreme points of P and hence $OC(\mathbf{x}^*, P)$ is the vertex figure for P at $\mathbf{x}^* = \boldsymbol{\Pi}\mathbf{x}^0$ for any $\boldsymbol{\Pi} \in \Pi(P)$ as claimed.

Claim 5 *Let* $\mathbf{x}^0 \neq \mathbf{x}^* \in \operatorname{vert} P$ *be such that* $\mathbf{x}^* = \boldsymbol{\Pi}\mathbf{x}^0$ *for some* $\boldsymbol{\Pi} \in \Pi(P)$. *Then* $\mathbf{fx} \le f_0$ *defines a facet of* $OC(\mathbf{x}^0, P)$ *if and only if* $(\mathbf{f}\boldsymbol{\Pi}^T)\mathbf{x} \le f_0$ *defines a facet of* $OC(\mathbf{x}^*, P)$.

Proof Let $\mathbf{fx} \le f_0$ define a facet of $OC(\mathbf{x}^0, P)$. Consequently, $\mathbf{fx}^0 = f_0$ and $\mathbf{fx} \le f_0$ for all $\mathbf{x} \in P$. Hence $f_0 = \mathbf{fx}^0 = (\mathbf{f}\boldsymbol{\Pi}^T)\boldsymbol{\Pi}\mathbf{x}^0 = (\mathbf{f}\boldsymbol{\Pi}^T)\mathbf{x}^*$ and $(\mathbf{f}\boldsymbol{\Pi}^T)\mathbf{y} = \mathbf{f}\boldsymbol{\Pi}^T\boldsymbol{\Pi}\mathbf{x} = \mathbf{fx} \le f_0$ for all $\mathbf{y} \in P$ because $\mathbf{y} \in P$ implies $\mathbf{y} = \boldsymbol{\Pi}\mathbf{x}$ for some $\mathbf{x} \in P$. Let $\widehat{\mathbf{x}} \in P$ be such that $\mathbf{f}\widehat{\mathbf{x}} = f_0$ and $\dim F(\mathbf{x}^0, \widehat{\mathbf{x}}) = \dim P - 1$. Since $\mathbf{fx} \le f_0$ defines a facet of P such an $\widehat{\mathbf{x}} \in P$ exists. Then $\boldsymbol{\Pi}\widehat{\mathbf{x}} \in P$ and $(\mathbf{f}\boldsymbol{\Pi}^T)\boldsymbol{\Pi}\widehat{\mathbf{x}} = \mathbf{f}\widehat{\mathbf{x}} = f_0$. From Claim 4 it follows that $\dim F(\mathbf{x}^*, \boldsymbol{\Pi}\widehat{\mathbf{x}}) = \dim P - 1$. Thus the inequality $(\mathbf{f}\boldsymbol{\Pi}^T)\mathbf{x} \le f_0$ defines a facet of $OC(\mathbf{x}^*, P)$. The rest follows by symmetry. □

It follows from Claim 5 that we know all facets of $OC(\mathbf{x}^*, P)$ if we know all facets of $OC(\mathbf{x}^0, P)$ where $\mathbf{x}^* = \boldsymbol{\Pi}\mathbf{x}^0$ for some $\boldsymbol{\Pi} \in \Pi(P)$. Consequently, if for *every pair* $\mathbf{x}^0 \neq \mathbf{x}^* \in \operatorname{vert} P$ there exists some $\boldsymbol{\Pi} \in \Pi(P)$ such that $\mathbf{x}^* = \boldsymbol{\Pi}\mathbf{x}^0$, then all vertex figures of P are identical *modulo* $\Pi(P)$. The task of finding all facets of P is thus reduced to finding all facets of the vertex figure $OC(\mathbf{x}^0, P)$, where $\mathbf{x}^0$ is some extreme point of P. This observation makes the task of finding an ideal linear description of P easier from a computational point of view: the number of the facets of $OC(\mathbf{x}^0, P)$ is typically considerably smaller than the number of the facets of P.

In general, we cannot expect that for *every* pair $\mathbf{x}^0 \neq \mathbf{x}^* \in \operatorname{vert} P$ there exists some $\boldsymbol{\Pi} \in \Pi(P)$ such that $\mathbf{x}^* = \boldsymbol{\Pi}\mathbf{x}^0$. In Fig. 2 we show a "symmetric" polyhedron where this is not the case. The index permutation

$$\boldsymbol{\Pi} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is permissible for the polyhedron P of Fig. 2 and the extreme points of P fall into the *two* vertex classes $\{(1,5),(5,1)\}$ and $\{(1,3),(3,1)\}$, of which it suffices to analyze one representative in each class. More generally, $\Pi(P)$ partitions the set of all extreme points into equivalence classes of which it suffices to analyze one representative in each class in order to find all the facets of P. If the number of such equivalence classes is relatively small—as compared, e.g., to the total number of extreme points of P—then substantial savings in the computational effort for the problem of finding all the facets of P result. In the case of the Boolean quadric polytope QP^n $\Pi(\mathrm{QP}^n)$ induces precisely n vertex classes. However, in this case there is also a "symmetry theorem", see [37], that permits one to reduce the study of the facial structure of QP^n to a single vertex figure as well.

For simplicity of exposition we will assume that all extreme points of P fall into a single equivalence class with respect to $\Pi(P)$ and that P is a polytope rather than a polyhedron. The following applies, however, mutatis mutandis to the general case of pointed polyhedra having several vertex classes as well.

We can thus replace the problem of finding all facets of P by the problem of finding all the facets of the vertex figure $OC(\mathbf{x}^0, P)$ at *any* extreme point $\mathbf{x}^0$ of P. To do so numerically we translate the displaced cone $OC(\mathbf{x}^0, P)$ with apex at $\mathbf{x}^0 \in P$ to the origin of $\mathbb{R}^n$ and consider instead of $OC(\mathbf{x}^0, P)$ the cone

$$CC(\mathbf{x}^0, P) = \left\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \sum_{i=1}^{a} \lambda_i (\mathbf{x}^i - \mathbf{x}^0), \lambda_i \geq 0\right\}. \tag{18}$$

Now we use, e.g., the double description algorithm DDA to find a linear description of $CC(\mathbf{x}^0, P)$. To do so we find a minimal generator of the "polar" cone

$$PCC(\mathbf{x}^0, P) = \left\{\mathbf{f} \in \mathbb{R}^n : \mathbf{f}(\mathbf{x}^i - \mathbf{x}^0) \leq 0 \text{ for } 1 \leq i \leq a\right\}, \tag{19}$$

where $\mathbf{f} \in \mathbb{R}^n$ is a row vector and a is the number of extreme points adjacent to $\mathbf{x}^0$. We write $PCC_0 = PCC(\mathbf{x}^0, P)$, for short. The following two facts are well known. If $\mathbf{f} \in PCC_0$ belongs to the lineality space of PCC_0, then $\mathbf{fx} = \mathbf{fx}^0$ belongs to the system of equations describing the affine hull of the polytope P. If $\mathbf{f} \in PCC_0$ belongs to the conical part of the minimal generator of PCC_0, then $\mathbf{fx} \leq \mathbf{fx}^0$ defines a facet of P that is tight at $\mathbf{x}^0$.

These considerations bring about a substantial reduction in the number of rows of the constraint matrix for the cone PCC_0 (19) that is the input for DDA which reduces the computations to find all facets of P. We illustrate the reduction for the symmetric traveling salesman problem on m cities: for $m = 8$ the input cone (1) has $p = 2{,}520$ rows while (19) has $a = 730$ rows, for $m = 9$ we have $p = 20{,}160$ while $a = 3{,}555$, and for $m = 10$ we have $p = 181{,}440$ and $a = 19{,}391$. Of course, we must *find* all extreme points of the polyhedron that are adjacent to $\mathbf{x}^0 \in P$ and this is difficult in general. However, it can be done, e.g., enumeratively for "small" polyhedra and all of the previous numbers were computed by a computer program that we have written for this purpose.

From the numbers that we give for the symmetric traveling salesman polytope it is clear that with our current computing machinery we can find the linear description of $CC(\mathbf{x}^0, P)$ for $m = 8$ at best. We have indeed succeeded to compute the corresponding linear description on a SunSparc 4 computer this way. However, for $m = 9$ this is again out of the question—at least at present.

5 Symmetry of Edge Figures

To generalize the previous device that we have used to reduce the computational effort let us state it as follows: we replace the problem of finding all facets of P by the problem of finding all facets of P that contain some 0-dimensional face of P, namely $\mathbf{x}^0 \in \operatorname{vert} P$. If P has several vertex classes then we do likewise by choosing some representative in each class. This brings about a substantial reduction in the size of the problem that we need to solve using the DDA. A further reduction can be expected if we *increase* the dimension of the face of P that we require the facets to contain. So we want to find all facets of P that contain a k-dimensional face of P that contains $\mathbf{x}^0$ where $k \geq 0$ is relatively small. For $k = 0$ we retrieve the previous trick, for $k = 1$ we ask for all facets of P that contain an edge $\mu\mathbf{x}^0 + (1-\mu)\mathbf{x}^i$ of P, where $0 \leq \mu \leq 1$ and $\mathbf{x}^i$ is some extreme point of P that is adjacent to $\mathbf{x}^0$. We can do likewise for any $k \geq 2$. Let us consider the case $k = 1$ explicitly, since for $k \geq 2$ the notation becomes a bit messy.

For $k = 1$ we replace (17) in the polytopal case by the a displaced cones

$$EC_P\left(\mathbf{x}^0, \mathbf{x}^\ell\right) = \left\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{x}^0 + \mu\left(\mathbf{x}^\ell - \mathbf{x}^0\right) + \sum_{\ell \neq i = 1}^{a} \lambda_i \left(\mathbf{x}^i - \mathbf{x}^0\right), \lambda_i \geq 0\right\}, \quad (20)$$

where $1 \leq \ell \leq a$ and $\mu \in \mathbb{R}$ is arbitrary because we want all facets of P containing the edge $\mu\mathbf{x}^0 + (1-\mu)\mathbf{x}^\ell$ for $0 \leq \mu \leq 1$ of P and thereby, necessarily, the entire line $\mu\mathbf{x}^0 + (1-\mu)\mathbf{x}^\ell$ for all $\mu \in \mathbb{R}$. By construction, $\dim EC_P(\mathbf{x}^0, \mathbf{x}^\ell) = \dim P$ and every facet of $EC_P(\mathbf{x}^0, \mathbf{x}^\ell)$ defines a facet of P, but not vice versa since the facets of $EC_P(\mathbf{x}^0, \mathbf{x}^\ell)$ all contain both $\mathbf{x}^0$ and $\mathbf{x}^\ell$. Of course, to find all the facets of P we must a priori analyze the displaced cones $EC_P(\mathbf{x}^0, \mathbf{x}^\ell)$ for all values of $\ell = 1, \ldots, a$, but as we shall see this number can be reduced substantially if $\Pi(P) \neq \{\mathbf{I}_n\}$.

Let us replace $\sum_{\ell \neq i = 1}^{a} \lambda_i (\mathbf{x}^i - \mathbf{x}^0)$ in the definition of $EC_P(\mathbf{x}^0, \mathbf{x}^\ell)$ by a list N_ℓ^0 of *all* extreme points of P other than $\mathbf{x}^0$ and $\mathbf{x}^\ell$. The $\mathbf{x}^i - \mathbf{x}^0$ with $\mathbf{x}^i$ *not adjacent* to $\mathbf{x}^0$ are not extremal in $OC(\mathbf{x}^0, P)$ nor in $EC(\mathbf{x}^0, \mathbf{x}^\ell)$. This does not change the displaced cone (20) because $EC_P(\mathbf{x}^0, \mathbf{x}^\ell) \supseteq OC(\mathbf{x}^0, P) \supseteq P$. We calculate

$$EC_P\left(\mathbf{x}^0, \mathbf{x}^\ell\right)$$
$$= \left\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{x}^0 + \mu\left(\mathbf{x}^\ell - \mathbf{x}^0\right) + \sum_{i \in N_\ell^0} \lambda_i \left(\mathbf{x}^i - \mathbf{x}^0\right), \lambda_i \geq 0\right\}$$

$$= \left\{ \mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{x}^\ell + \left(1 - \mu - \sum_{i \in N_\ell^0} \lambda_i\right)\left(\mathbf{x}^0 - \mathbf{x}^\ell\right) + \sum_{i \in N_\ell^0} \lambda_i \left(\mathbf{x}^i - \mathbf{x}^\ell\right), \lambda_i \geq 0 \right\}$$

$$= EC_P\left(\mathbf{x}^\ell, \mathbf{x}^0\right),$$

because $\nu = 1 - \mu - \sum_{i \in N_\ell^0} \lambda_i$ runs through all reals when $\mu \in \mathbb{R}$ is arbitrary. This calculation reflects the fact that the direction in which we traverse the edge of P defined by $\mathbf{x}^0$ and $\mathbf{x}^\ell$ is immaterial for the "shape" of $EC_P(\mathbf{x}^0, \mathbf{x}^\ell)$. So the displaced cone (20) is well defined by the edge given by the pair of adjacent extreme points $\mathbf{x}^0$ and $\mathbf{x}^\ell$ of P and we call it the *edge figure* of P relative to $\mathbf{x}^0$ and $\mathbf{x}^\ell$.

Shifting the displaced cone (20) to contain the origin we get the cone

$$ECC_P\left(\mathbf{x}^0, \mathbf{x}^\ell\right)$$
$$= \left\{ \mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mu\left(\mathbf{x}^\ell - \mathbf{x}^0\right) + \sum_{\ell \neq i = 1}^{a} \lambda_i \left(\mathbf{x}^i - \mathbf{x}^0\right), \lambda_i \geq 0, \mu \in \mathbb{R} \right\}, \qquad (21)$$

which is an infinite "wedge" in $\mathbb{R}^n$, i.e., its lineality space has a dimension of 1. Moreover, while (21) is a valid pointwise description of $ECC_\ell = ECC_P(\mathbf{x}^0, \mathbf{x}^\ell)$, it is typically far from minimal. By testing each direction vector $\mathbf{x}^i - \mathbf{x}^0$ of ECC_ℓ with $1 \leq i \neq \ell \leq a$ for extremality in the cone ECC_ℓ we can reduce the pointwise description of ECC_ℓ to a minimal one. To keep the notation simple, let us assume that—after reindexing if necessary—the direction vectors $\mathbf{x}^i - \mathbf{x}^0$ for $1 \leq i \leq a(\ell)$ remain. Forming the polar cone like in (19) we thus get

$$PEC_P\left(\mathbf{x}^0, \mathbf{x}^\ell\right) = \left\{ \mathbf{f} \in \mathbb{R}^n : \mathbf{f}\left(\mathbf{x}^\ell - \mathbf{x}^0\right) = 0, \mathbf{f}\left(\mathbf{x}^i - \mathbf{x}^0\right) \leq 0 \text{ for } 1 \leq i \leq a(\ell) \right\} \qquad (22)$$

as the input cone for the double description algorithm. Its number of rows $a(\ell) + 1$ is typically considerably smaller than the corresponding number for the cone (19), though $a(\ell)$ can vary significantly with ℓ. As mentioned above, all of this can be extended to k-dimensional faces of P that contain $\mathbf{x}^0$, where $k \geq 0$ is arbitrary. The cost of the "input preparation" for the corresponding cones to the analyzed by the DDA increases, but the mechanics of carrying out the analysis are clear.

Using this new trick we can thus replace the single problem (19) by a total of a typically smaller problems (22) to be analyzed by the DDA and this is the way it was done by Christof, Jünger and Reinelt [9] who used this methodology for $k = 1$ to determine all facets for the symmetric traveling salesman polytope on a complete graph with $m = 8$ nodes. But rather than solving the $a = 730$ problems that result from the number of extreme points that are adjacent to any given one of this polytope, they reduced the number of edge figures to be analyzed to a total of 59 and this is how.

Claim 6 *Let $\mathbf{x}^a$, $\mathbf{x}^b$ and $\mathbf{x}^c$, $\mathbf{x}^d$ be any two pairs of adjacent extreme points of P and suppose that $\mathbf{x}^c = \boldsymbol{\Pi}\mathbf{x}^a$, $\mathbf{x}^d = \boldsymbol{\Pi}\mathbf{x}^b$ for some $\boldsymbol{\Pi} \in \Pi(P)$. Then $\mathbf{f}\mathbf{x} \leq f_0$ defines a facet of $EC_P(\mathbf{x}^a, \mathbf{x}^b)$ if and only if $(\mathbf{f}\boldsymbol{\Pi}^T)\mathbf{x} \leq f_0$ defines a facet of $EC_P(\mathbf{x}^c, \mathbf{x}^d)$.*

Proof Let $\mathbf{fx} \le f_0$ define a facet of $EC_P(\mathbf{x}^a, \mathbf{x}^b)$. Then $\mathbf{fx}^a = \mathbf{fx}^b = f_0$ and $\mathbf{fx} \le f_0$ for all $\mathbf{x} \in P$. Consequently, $(\mathbf{f}\boldsymbol{\Pi}^T)\mathbf{x}^c = \mathbf{fx}^a = f_0$, $(\mathbf{f}\boldsymbol{\Pi}^T)\mathbf{x}^d = \mathbf{fx}^b = f_0$ and for any $\mathbf{y} \in P$ we get $(\mathbf{f}\boldsymbol{\Pi}^T)\mathbf{y} = \mathbf{fx} \le f_0$ since $\mathbf{y} = \boldsymbol{\Pi}\mathbf{x}$ for some $\mathbf{x} \in P$. Let $\widehat{\mathbf{x}} \in P$ be such that $\mathbf{f}\widehat{\mathbf{x}} = f_0$ and $\dim F(\mathbf{x}^a, \widehat{\mathbf{x}}) = \dim P - 1$, where $F(\mathbf{x}^a, \widehat{\mathbf{x}})$ is the face of smallest dimension of P containing both $\mathbf{x}^a$ and $\widehat{\mathbf{x}}$. Since $\mathbf{fx} \le f_0$ defines a facet of P such an $\widehat{\mathbf{x}} \in P$ exists. Then $\boldsymbol{\Pi}\widehat{\mathbf{x}} \in P$, $(\mathbf{f}\boldsymbol{\Pi}^T)\boldsymbol{\Pi}\widehat{\mathbf{x}} = f_0$ and by Claim 4, $\dim F(\boldsymbol{\Pi}\widehat{\mathbf{x}}, \mathbf{x}^c) = \dim P - 1$. Thus $(\mathbf{f}\boldsymbol{\Pi}^T)\mathbf{x} \le f_0$ defines a facet of $EC_P(\mathbf{x}^c, \mathbf{x}^d)$ and the rest follows by symmetry. □

We do not require distinctness of the extreme points $\mathbf{x}^a$, $\mathbf{x}^b$, $\mathbf{x}^c$ and $\mathbf{x}^d$ in Claim 6. So it may be that $\mathbf{x}^a = \mathbf{x}^c = \mathbf{x}^0$, say, and let

$$\Pi\left(\mathbf{x}^0\right) = \left\{\boldsymbol{\Pi} \in \Pi(P) : \mathbf{x}^0 = \boldsymbol{\Pi}\mathbf{x}^0\right\}, \tag{23}$$

be the permissible permutations for P that leave the extreme point $\mathbf{x}^0 \in P$ invariant. Since $\mathbf{I}_n \in \Pi(\mathbf{x}^0)$ this set is always nonempty and it forms evidently a subgroup of $\Pi(P)$. E.g., for the symmetric traveling salesman polytope every subgroup $\Pi(\mathbf{x}^0)$ has precisely $2m$ elements where m is the number of nodes of the graph. The $2m$ elements in $\Pi(\mathbf{x}^0)$ come about by reindexing the nodes of the graph—in both a forward and backward sense—while maintaining the same order of the nodes as in the tour $\mathbf{x}^0$.

The reduction in the number of cones (22) to be analyzed by the DDA that results from an application of Claim 6 with the special permutations in $\Pi(\mathbf{x}^0)$ can be enormous: for the symmetric traveling salesman problem with $m = 8$ nodes 59 cones (edge figures) suffice rather than the 730 original ones (which is exactly the number analyzed in [9]), for $m = 9$ we get 216 instead of 3,555 and for $m = 10$ we get 1,032 cones to analyze instead of the 19,391 original ones. By a complete application of Claim 6 these numbers can be reduced even further; see Table 5 of Sect. 7.

6 Rank of Facets and Integer Polyhedra

To bring order into the facial structure of polyhedra related to combinatorial optimization problems we introduce next the notions of the "rank" of a facet and of a polyhedron.

Let $\mathcal{F}$ be the family of row vectors $(\mathbf{h}, h_0) \in \mathbb{R}^{n+1}$ of the matrix $(\mathbf{H}, \mathbf{h})$ of an ideal description (2) of a pointed rational polyhedron $P \subseteq \mathbb{R}^n$. So every $(\mathbf{h}, h_0) \in \mathcal{F}$ defines a facet $F_{\mathbf{h}}$ of P via (15) and vice versa, for every facet of P there is some row of $(\mathbf{H}, \mathbf{h})$ that defines it. For simplicity of notation we say that $\mathcal{F}$ is ideal for P. The set

$$\operatorname{relint} P = \left\{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} = \mathbf{b}, \mathbf{fx} < f_0 \ \forall (\mathbf{f}, f_0) \in \mathcal{F}\right\}$$

is the *relative interior* of P. As before we denote by vert P the set of extreme points of P and by exray P the set of the direction vectors of all extreme rays of P.

To make the notion of rank precise we need some knowledge about the facets of the facets of a polyhedron, i.e., the *ridges* of the polyhedron P. For $(\mathbf{f}, f_0) \in \mathcal{F}$ the facet

$$F_{\mathbf{f}} = \{\mathbf{x} \in P : \mathbf{fx} = f_0\}$$

of P defined by $\mathbf{fx} \leq f_0$ is itself a *pointed* polyhedron of dimension $\dim P - 1$ in $\mathbb{R}^n$ satisfying vert $F_{\mathbf{f}} \subseteq$ vert P and exray $F_{\mathbf{f}} \subseteq$ exray P. Let

$$\mathcal{H}^f = \big\{(\mathbf{h}, h_0) \in \mathcal{F} : \dim F_{\mathbf{f}} \cap F_{\mathbf{h}} = \dim P - 2\big\}, \tag{24}$$

i.e., $(\mathbf{h}, h_0) \in \mathcal{H}^f$ if and only if the facet-defining inequality $\mathbf{hx} \leq h_0$ of P defines a facet of the polyhedron $F_{\mathbf{f}}$, and $(\mathbf{H}^f, \mathbf{h}^f)$ be the matrix of all $(\mathbf{h}, h_0) \in \mathcal{H}^f$. If $\dim P < 2$ then $\mathcal{H}^f = \emptyset$ and $P = \emptyset$, or $P = \{\mathbf{x}\}$ is a singleton, or P is a line segment or a halfline in $\mathbb{R}^n$. We define the rank of P to equal -1 in these cases and assume throughout that $\dim P \geq 2$ and relint $P \neq \emptyset$.

Lemma 1 *For every* $(\mathbf{f}, f_0) \in \mathcal{F}$ $\mathcal{H}^f$ *is ideal for* $F_{\mathbf{f}}$ *and*

$$F_{\mathbf{f}} = \big\{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} = \mathbf{b}, \mathbf{fx} = f_0, \mathbf{H}^f \mathbf{x} \leq \mathbf{h}^f\big\}$$

is an ideal linear description of $F_{\mathbf{f}}$.

Proof Suppose $\dim P \geq 2$ and that $\mathcal{H}^f$ is not complete. Then there exists some facet H of $F_{\mathbf{f}}$ such that $H \neq \{\mathbf{x} \in F_{\mathbf{f}} : \mathbf{hx} = h_0\}$ for all $(\mathbf{h}, h_0) \in \mathcal{H}^f$. Since H is a facet of $F_{\mathbf{f}}$ there exists some $(\mathbf{g}, g_0) \in \mathbb{R}^{n+1}$ such that $\mathbf{gx} \leq g_0$ for all $\mathbf{x} \in F_{\mathbf{f}}$ and $H = \{\mathbf{x} \in F_{\mathbf{f}} : \mathbf{gx} = g_0\}$. Let

$$\nu_{\mathbf{x}} = \max\left\{\frac{\mathbf{gx} - g_0}{f_0 - \mathbf{fx}} : \mathbf{x} \in \text{vert}\, P \text{ with } \mathbf{fx} < f_0\right\} \tag{25}$$

and $\nu_{\mathbf{x}} = -\infty$ if (25) does not exist.

Let vert $P = \{\mathbf{x}^1, \ldots, \mathbf{x}^p\}$ and exray $P = \{\mathbf{y}^1, \ldots, \mathbf{y}^r\}$. Since $\mathbf{fx} \leq f_0$ defines a facet of P we have $\mathbf{fy}^i \leq 0$ for all $1 \leq i \leq r$. Moreover, $\mathbf{fy}^i = 0$ implies $\mathbf{gy}^i \leq 0$ for any $i \in \{1, \ldots, r\}$ since $\mathbf{gx} \leq g_0$ defines the facet H of $F_{\mathbf{f}}$. If $\nu_{\mathbf{x}} > -\infty$ and $\mathbf{gy}^i + \nu_{\mathbf{x}}\mathbf{fy}^i \leq 0$ for $1 \leq i \leq r$ then we claim that $\mathbf{hx} \leq h_0$ where $\mathbf{h} = \mathbf{g} + \nu_{\mathbf{x}}\mathbf{f}$ and $h_0 = g_0 + \nu_{\mathbf{x}} f_0$ defines a facet F' of P which is different from $F_{\mathbf{f}}$. By (25)

$$\big(\mathbf{gx}^i - g_0\big) \leq \nu_{\mathbf{x}}\big(f_0 - \mathbf{fx}^i\big) \quad \forall \mathbf{x}^i \in \text{vert}\, P \text{ with } \mathbf{fx}^i < f_0$$

and thus $\mathbf{hx}^i = (\mathbf{g} + \nu_{\mathbf{x}}\mathbf{f})\mathbf{x}^i \leq g_0 + \nu_{\mathbf{x}} f_0 = h_0$ for $\mathbf{x}^i \in$ vert P with $\mathbf{fx}^i < f_0$. If $\mathbf{x}^i \in$ vert P with $\mathbf{fx}^i = f_0$ then $\mathbf{gx}^i \leq g_0$ and thus $\mathbf{hx}^i \leq h_0$ as well. Let $\mathbf{x} \in P$. Then $\mathbf{x} = \sum_{i=1}^p \mu_i \mathbf{x}^i + \sum_{j=1}^r \lambda_j \mathbf{y}^j$ for some $\mu_i \geq 0$ with $\sum_{i=1}^p \mu_i = 1$ and $\lambda_j \geq 0$. Thus $\mathbf{hx} \leq h_0$ for all $\mathbf{x} \in P$. Since H is a facet of $F_{\mathbf{f}}$ there exist $\ell = \dim P - 1$ affinely independent $\mathbf{x}^1, \ldots, \mathbf{x}^\ell$ with $\mathbf{x}^i \in F_{\mathbf{f}}$ and $\mathbf{hx}^i = h_0$ for $1 \leq i \leq \ell$. Let $\mathbf{x}^0$ be a vertex of P for which the maximum in (25) is attained. Then $\mathbf{hx}^0 = h_0$ and $\mathbf{x}^0, \mathbf{x}^1, \ldots, \mathbf{x}^\ell$ are affinely independent because $\mathbf{fx}^0 < f_0$ and $\mathbf{fx}^i = f_0$ for $i = 1, \ldots, \ell$. Thus $\mathbf{hx} \leq h_0$

defines a facet of P which is different from $F_{\mathbf{f}}$ and $\dim F_{\mathbf{f}} \cap F_{\mathbf{h}} = \dim P - 2$. Since $\mathcal{F}$ is ideal for P, there exists some $(\mathbf{f}', f_0') \in \mathcal{F}$ such that $F' = \{\mathbf{x} \in P : \mathbf{f}'\mathbf{x} = f_0'\} = \{\mathbf{x} \in P : \mathbf{h}\mathbf{x} = h_0\}$. But

$$
\begin{aligned}
F_{\mathbf{f}} \cap F' &= \left\{\mathbf{x} \in P : \mathbf{f}\mathbf{x} = f_0, \mathbf{f}'\mathbf{x} = f_0'\right\} = \{\mathbf{x} \in P : \mathbf{f}\mathbf{x} = f_0, \mathbf{h}\mathbf{x} = h_0\} \\
&= \{\mathbf{x} \in P : \mathbf{f}\mathbf{x} = f_0, \mathbf{g}\mathbf{x} = g_0\} = H,
\end{aligned}
$$

contradicting $H \neq \{\mathbf{x} \in F_{\mathbf{f}} : \mathbf{h}\mathbf{x} = h_0\}$ for all $(\mathbf{h}, h_0) \in \mathcal{H}^f$. Thus $\mathcal{H}^f$ is complete for $F_{\mathbf{f}}$. Suppose that $\mathbf{g}\mathbf{y}^i + \nu_{\mathbf{x}}\mathbf{f}\mathbf{y}^i > 0$ for some $1 \leq i \leq r$ or that $\nu_{\mathbf{x}} = -\infty$. If $\nu_{\mathbf{x}} = -\infty$ then $\mathbf{f}\mathbf{x} = f_0$ for all $\mathbf{x} \in \operatorname{vert} P$. If $\mathbf{f}\mathbf{y} = 0$ for all $\mathbf{y} \in \operatorname{exray} P$, then $F_{\mathbf{f}} = P$ which contradicts $\dim F_{\mathbf{f}} < \dim P$ and thus $\mathbf{f}\mathbf{y}^i < 0$ for some $i \in \{1, \ldots, r\}$. So suppose $\nu_{\mathbf{x}} > -\infty$ and $\mathbf{g}\mathbf{y}^i + \nu_{\mathbf{x}}\mathbf{f}\mathbf{y}^i > 0$ for some $i \in \{1, \ldots, r\}$. If $\mathbf{f}\mathbf{y}^i = 0$ for all i then $0 < \mathbf{g}\mathbf{y}^i + \nu_{\mathbf{x}}\mathbf{f}\mathbf{y}^i \leq 0$ for some $i \in \{1, \ldots, r\}$ is a contradiction because $\mathbf{g}\mathbf{y}^i \leq 0$ for all i in this case as well. Thus $\mathbf{f}\mathbf{y}^i < 0$ for at least one $i \in \{1, \ldots, r\}$ in both subcases and the scalar

$$
\nu_{\mathbf{y}} = \max\left\{-\frac{\mathbf{g}\mathbf{y}^i}{\mathbf{f}\mathbf{y}^i} : \mathbf{f}\mathbf{y}^i < 0, 1 \leq i \leq r\right\}
$$

is well defined in the second case. We set $\nu = \max\{\nu_{\mathbf{x}}, \nu_{\mathbf{y}}\}$ and claim $\mathbf{h}\mathbf{x} \leq h_0$ with $\mathbf{h} = \mathbf{g} + \nu\mathbf{f}$ and $h_0 = g_0 + \nu f_0$ defines a facet F' of P with $F' \neq F_{\mathbf{f}}$. By the definition of ν

$$
\mathbf{h}\mathbf{y}^i = \mathbf{g}\mathbf{y}^i + \nu\mathbf{f}\mathbf{y}^i \leq 0 \quad \forall i \in \{1, \ldots, r\} \text{ with } \mathbf{f}\mathbf{y}^i < 0
$$

and $\mathbf{h}\mathbf{y}^i \leq 0$ for all $i \in \{1, \ldots, r\}$ with $\mathbf{f}\mathbf{y}^i = 0$, since $\mathbf{g}\mathbf{y}^i \leq 0$ in this case. Thus $\mathbf{h}\mathbf{y} \leq 0$ for all $\mathbf{y} \in \operatorname{exray} P$. Like above, using $\nu \geq \nu_{\mathbf{x}}$, we show that $\mathbf{h}\mathbf{x} \leq h_0$ for all $\mathbf{x} \in \operatorname{vert} P$ and thus $\mathbf{h}\mathbf{x} \leq h_0$ for all $\mathbf{x} \in P$. If $\nu = \nu_{\mathbf{x}}$ then the claim follows like in the first case and we are done. If $\nu = \nu_{\mathbf{y}}$ let $\mathbf{y} \in \operatorname{exray} P$ be such that equality in the definition of $\nu_{\mathbf{y}}$ is attained. Since H is a facet of $F_{\mathbf{f}}$ there exist $\ell = \dim P - 1$ affinely independent $\mathbf{x}^1, \ldots, \mathbf{x}^\ell$ with $\mathbf{x}^i \in F_{\mathbf{f}}$ and $\mathbf{h}\mathbf{x}^i = h_0$ for $1 \leq i \leq \ell$. Moreover, by construction $\mathbf{h}(\mathbf{x}^i + \alpha\mathbf{y}) = h_0$ for $1 \leq i \leq \ell$ and $\alpha \geq 0$. Thus $\mathbf{x}^1, \ldots, \mathbf{x}^\ell, \mathbf{x}^1 + \mathbf{y}$ are all in P and affinely independent. For suppose not. Then $\lambda_0(\mathbf{x}^1 + \mathbf{y}) + \sum_{i=1}^{\ell} \lambda_i \mathbf{x}^i = \mathbf{0}$ for some nonnull $\lambda_0, \lambda_1, \ldots, \lambda_\ell$ with $\lambda_0 + \sum_{i=1}^{\ell} \lambda_i = 0$ and thus $\lambda_0 \neq 0$ since $\mathbf{x}^1, \ldots, \mathbf{x}^\ell$ are affinely independent. Thus $\mathbf{x}^1 + \mathbf{y} = \sum_{i=1}^{\ell} \lambda_i' \mathbf{x}^i$ for some $\lambda_1', \ldots, \lambda_\ell'$ with $\sum_{i=1}^{\ell} \lambda_i' = 1$ and $\mathbf{f}\mathbf{y} = 0$ because $\mathbf{f}\mathbf{x}^i = f_0$ for $1 \leq i \leq \ell$. This contradicts $\mathbf{f}\mathbf{y} < 0$ and we conclude like in the first case that $\mathcal{H}^f$ is complete for $F_{\mathbf{f}}$.

Suppose that $\mathcal{H}^f$ is not minimal. Since $\mathcal{H}^f$ is complete for $F_{\mathbf{f}}$ there exists $(\mathbf{h}, h_0) \in \mathcal{H}^f$ and a facet $H_{\mathbf{h}}^{\mathbf{f}}$ of $F_{\mathbf{f}}$ such that

$$
H_{\mathbf{h}}^{\mathbf{f}} = \left\{\mathbf{x} \in P : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{f}\mathbf{x} = f_0, \mathbf{h}\mathbf{x} = h_0, \mathbf{g}^1\mathbf{x} = g_0^1, \ldots, \mathbf{g}^k\mathbf{x} = g_0^k\right\}
$$

for $(\mathbf{g}^1, g_0^1), \ldots, (\mathbf{g}^k, g_0^k) \in \mathcal{H}^f - (\mathbf{h}, h_0)$ and some $k \geq 1$. Since $\dim H_{\mathbf{h}}^{\mathbf{f}} = \dim P - 2 = n - r(\mathbf{A}) - 2$ it follows that $\mathbf{g}^i = \boldsymbol{\lambda}^i\mathbf{A} + \alpha^i\mathbf{f} + \beta^i\mathbf{h}$ and $g_0^i = \boldsymbol{\lambda}^i\mathbf{b} + \alpha^i f_0 + \beta^i h_0$ for some vectors $\boldsymbol{\lambda}^i$ and scalars α^i and β^i, i.e., the $(\mathbf{g}^i, g_0^i)$ are linearly dependent on $(\mathbf{A}, \mathbf{b})$, $(\mathbf{f}, f_0)$ and $(\mathbf{h}, h_0)$. Let $\mathbf{x}^{\mathbf{f}}$ and $\mathbf{x}^{\mathbf{h}}$ be any points in $\operatorname{relint} F_{\mathbf{f}}$ and $\operatorname{relint} F_{\mathbf{h}}$, respectively. It follows from $F_{\mathbf{h}} \neq F_{\mathbf{f}}$ that $\mathbf{h}\mathbf{x}^{\mathbf{h}} = h_0$, $\mathbf{f}\mathbf{x}^{\mathbf{h}} < f_0$, $\mathbf{g}^i\mathbf{x}^{\mathbf{h}} < g_0^i$ and $\mathbf{f}\mathbf{x}^{\mathbf{f}} = f_0$, $\mathbf{h}\mathbf{x}^{\mathbf{f}} < h_0$, $\mathbf{g}^i\mathbf{x}^{\mathbf{f}} < g_0^i$, for all $i = 1, \ldots, k$. Thus, e.g.,

$$\mathbf{g}^i\mathbf{x}^{\mathbf{h}} = \boldsymbol{\lambda}^i\mathbf{A}\mathbf{x}^{\mathbf{h}} + \alpha^i\mathbf{f}\mathbf{x}^{\mathbf{h}} + \beta^i\mathbf{h}\mathbf{x}^{\mathbf{h}} = \boldsymbol{\lambda}^i\mathbf{b} + \alpha^i\mathbf{f}\mathbf{x}^{\mathbf{h}} + \beta^i h_0 < g_0^i = \boldsymbol{\lambda}^i\mathbf{b} + \alpha^i f_0 + \beta^i h_0.$$

Consequently, $\alpha^i(f_0 - \mathbf{f}\mathbf{x}^{\mathbf{h}}) > 0$ and $\alpha^i > 0$. Multiplying by $\mathbf{x}^{\mathbf{f}}$ we find likewise that $\beta^i > 0$. Consequently, $\mathbf{g}^i\mathbf{x} = g_0^i$ if and only if $\mathbf{f}\mathbf{x} = f_0$ and $\mathbf{h}\mathbf{x} = h_0$ for all $\mathbf{x} \in P$. Thus none of the $\mathbf{g}^i\mathbf{x} \le g_0^i$ defines a facet of P. This contradicts the assumption that $\mathcal{F}$ is ideal for P. □

The last part of the proof of the lemma shows also that every ridge of P is the intersection of two *unique* facets of P. A shorter, nonconstructive proof of this fact is possible using the methods of Chap. 7.2.2 of [38]. The proof given here is useful: given a pointwise generator of P it describes a *lifting procedure* to compute facets of P from the facets of a facet of P.

Definition 6 For every $(\mathbf{f}, f_0) \in \mathcal{F}$ we call the polyhedron

$$P(\mathcal{H}^f) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{f}\mathbf{x} \le f_0, \mathbf{H}^f\mathbf{x} \le \mathbf{h}^f\} \tag{26}$$

the *facet figure* of P at the facet $F_{\mathbf{f}}$.

Clearly, $P(\mathcal{H}^f) \neq \emptyset$, $P(\mathcal{H}^f)$ is pointed, the linear description of $P(\mathcal{H}^f)$ is ideal,

$$F_{\mathbf{f}} \subsetneq P \subsetneq P(\mathcal{H}^f) \quad \text{and} \quad P = \bigcap_{(\mathbf{f}, f_0) \in \mathcal{F}} P(\mathcal{H}^f).$$

Let $P \subseteq \mathbb{R}^n$ be an *integer polyhedron*, i.e., every $\mathbf{x} \in \operatorname{vert} P$ satisfies $\mathbf{x} \in \mathbb{Z}^n$, and moreover, $\operatorname{relint} P \neq \emptyset$. For any such polyhedron let $\mathcal{F}_{\min} \subseteq \mathcal{F}$ be such that

$$P_{\min} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{f}\mathbf{x} \le f_0 \ \forall (\mathbf{f}, f_0) \in \mathcal{F}_{\min}\} \tag{27}$$

meets the following two requirements

(i) $P = \operatorname{conv}(P_{\min} \cap \mathbb{Z}^n)$
(ii) $P \cap \mathbb{Z}^n \subsetneq P_{\min}^{\mathbf{h}} \cap \mathbb{Z}^n \ \forall (\mathbf{h}, h_0) \in \mathcal{F}_{\min}$, where

$$P_{\min}^{\mathbf{h}} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{f}\mathbf{x} \le f_0 \ \forall (\mathbf{f}, f_0) \in \mathcal{F}_{\min} - (\mathbf{h}, h_0)\}$$

and the containment in (ii) is proper. Given $\mathcal{F}$ we can construct $\mathcal{F}_{\min}$ e.g. as follows: Initially we set $\mathcal{F}_{\min} = \mathcal{F}$. If dropping $(\mathbf{h}, h_0)$ does not change the convex hull (i), we drop it from $\mathcal{F}_{\min}$ and continue to do so until every remaining element in $\mathcal{F}_{\min}$ satisfies (ii). Since $|\mathcal{F}| < \infty$ this procedure is finite. By construction we have $\dim P_{\min} = \dim P$, $P_{\min}$ is pointed, $\mathcal{F}_{\min} \neq \emptyset$ since $\operatorname{relint} P \neq \emptyset$.

We call any subset $\mathcal{F}_{\min} \subseteq \mathcal{F}$ satisfying (i) and (ii) a *minimal formulation* for the integer polyhedron. Minimal formulations of integer polyhedra need not be unique.

An example of nonuniqueness is given by the *set packing* or *vertex packing polytope*

$$VP(G) = \operatorname{conv}\left(\left\{\mathbf{x} \in \{0,1\}^{|V|} : x_u + x_v \le 1 \ \forall e = (u,v) \in E\right\}\right),$$

where $G = (V, E)$ is a finite undirected graph on $n = |V|$ nodes; see [32, 33, 35–37]. In this case every minimal "covering" of the edge set E of G by some of its "cliques" (= maximal complete vertex-induced subgraphs) together with the nonnegativity conditions provides a minimal formulation for the integer polytope and a *unique* minimal formulation simply does not exist in the general case. On the other hand, in many cases of interest to us, such as, e.g., in the case of the symmetric and asymmetric traveling salesman polytopes, the corresponding minimal formulations appear to be unique (see Sect. 7).

Given an ideal family $\mathcal{F}$ for an integer polyhedron $P \subseteq \mathbb{R}^n$ we call

$$\mathcal{F}_0 = \left\{(\mathbf{f}, f_0) \in \mathcal{F} : (\mathbf{f}, f_0) \text{ belongs to } \textit{some} \text{ minimal formulation } \mathcal{F}_{\min} \text{ of } P\right\}$$

the *rank zero facets* of P. It follows that $\mathcal{F}_0$ is a uniquely defined subset of $\mathcal{F}$. We call

$$P_0 = \left\{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{f}\mathbf{x} \le f_0 \ \forall (\mathbf{f}, f_0) \in \mathcal{F}_0\right\}$$

the *rank zero formulation* of P. In the case of the set packing or vertex packing polytope

$$\begin{aligned} VP(G) &= \operatorname{conv}\left(\left\{\mathbf{x} \in \mathbb{Z}^n : \mathbf{A}_C\mathbf{x} \le \mathbf{e}_C, \mathbf{x} \ge \mathbf{0}\right\}\right) \\ &= \left\{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}_C\mathbf{x} \le \mathbf{e}_C, \mathbf{A}_F\mathbf{x} \le \mathbf{f}^0, \mathbf{x} \ge \mathbf{0}\right\}, \end{aligned}$$

where $\mathbf{A}_C\mathbf{x} \le \mathbf{e}_C$ are all *clique* constraints, $\mathbf{x} \ge \mathbf{0}$ are the nonnegativity constraints and $\mathbf{A}_F\mathbf{x} \le \mathbf{f}^0$ all other facet-defining inequalities of $VP(G)$ in primitive normal form. Every clique constraint is of the form $\sum_{j \in K} x_j \le 1$ where $K \subseteq V$ is the node set of a clique in G. Every row $\mathbf{f}\mathbf{x} \le f_0$ of $\mathbf{A}_F\mathbf{x} \le \mathbf{f}^0$ consists of relatively prime integers $0 \le f_j \le f_0$ for all $j \in V$ and $f_0 \ge 2$. $\mathcal{F}_0$ consists of *all* clique inequalities of G plus *all* nonnegativity constraints. [The referee pointed out that a formal proof is needed that no facet-defining inequality with $f_0 \ge 2$ belongs to $\mathcal{F}_0$. I agree with the referee, but have at present no such proof. This makes the notion of rank proposed here, possibly, formulation-dependent. However, I continue to think that this is not the case.]

If $\mathcal{F} = \mathcal{F}_0$ then all facets of P have rank zero and we define the rank $\rho(P)$ of the polyhedron P to be zero. Examples of integer polyhedra of rank zero are the n-dimensional simplex S_n, the unit hypercube C_n

$$S_n = \left\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \ge \mathbf{0}, \sum_{j=1}^{n} x_j \le 1\right\}, \qquad C_n = \left\{\mathbf{x} \in \mathbb{R}^n : 0 \le x_j \le 1 \text{ for } 1 \le j \le n\right\}$$

and the vertex packing polytope $VP(G) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}_C\mathbf{x} \le \mathbf{e}_C, \mathbf{x} \ge \mathbf{0}\}$ of a perfect graph G where $\mathbf{A}_C$ is defined above, see [33, 34]. There are other examples, e.g., polyhedra that are defined with respect to totally unimodular, ideal 0–1 matrices, etc.

If $\mathcal{F}_0 \neq \mathcal{F}$ then we proceed inductively as follows. Given $\rho + 1$ nonempty, pairwise disjoint subsets $\mathcal{F}_0, \ldots, \mathcal{F}_\rho$ of $\mathcal{F}$ we let

$$P_\rho = \left\{ \mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{h}\mathbf{x} \leq h_0 \ \forall (\mathbf{h}, h_0) \in \bigcup_{\ell=0}^{\rho} \mathcal{F}_\ell \right\}. \tag{28}$$

A facet of P defined by $(\mathbf{f}, f_0) \in \mathcal{F}_\ell$ is a facet of rank ℓ and $\mathbf{f}\mathbf{x} \leq f_0$ is a rank ℓ inequality where $0 \leq \ell \leq \rho$. If $\mathcal{F} = \bigcup_{\ell=0}^{\rho} \mathcal{F}_\ell$ then $\rho(P) = \rho$, i.e., the integer polyhedron P has rank ρ. Otherwise, let

$$\begin{aligned} \mathcal{F}_{\rho+1} = \Bigg\{ & (\mathbf{f}, f_0) \in \mathcal{F} - \bigcup_{\ell=0}^{\rho} \mathcal{F}_\ell : \\ & \exists (\mathbf{h}, h_0) \in \bigcup_{\ell=0}^{\rho} \mathcal{F}_\ell \text{ s.t. } \dim F_{\mathbf{f}} \cap F_{\mathbf{h}} = \dim P - 2 \Bigg\} \end{aligned} \tag{29}$$

Theorem 1 *If $\mathcal{F} \neq \bigcup_{\ell=0}^{\rho} \mathcal{F}_\ell$, then $\mathcal{F}_{\rho+1} \neq \emptyset$ and $\mathbf{x}^* \notin P_{\rho+1}$ for all $\mathbf{x}^* \in \operatorname{vert} P_\rho - \mathbb{Z}^n$.*

Proof If $\mathcal{F} \neq \bigcup_{\ell=0}^{\rho} \mathcal{F}_\ell$, then $P \neq P_\rho$ and there exists $\mathbf{x}^* \in \operatorname{vert} P_\rho - \mathbb{Z}^n$. Since $\mathbf{x}^* \in \operatorname{vert} P_\rho$ there exists $(\mathbf{h}, h_0) \in \bigcup_{\ell=0}^{\rho} \mathcal{F}_\ell$ such that $\mathbf{h}\mathbf{x}^* = h_0$. If $\mathbf{x}^* \in F_{\mathbf{h}}$, then $\mathbf{x}^* \in P$ since $F_{\mathbf{h}} \subset P$, which contradicts $\mathbf{x}^* \notin P$. Consequently, $\mathbf{x}^* \notin F_{\mathbf{h}}$ and by the lemma there exists a $(\mathbf{f}, f_0) \in \mathcal{H}^h$ such that $\mathbf{f}\mathbf{x}^* > f_0$ and thus $(\mathbf{f}, f_0) \notin \bigcup_{\ell=0}^{\rho} \mathcal{F}_\ell$, i.e., $(\mathbf{f}, f_0) \in \mathcal{F}_{\rho+1}$. Since $\mathbf{x}^* \in \operatorname{vert} P_\rho - \mathbb{Z}^n$ is arbitrary, the theorem follows. □

Since $|\mathcal{F}| < \infty$ and since we augment the set $\bigcup_{\ell=0}^{\rho} \mathcal{F}_\ell$ at every step of the above inductive process by at least one new element of $\mathcal{F}$, the inductive process is finite. Hence the rank $\rho(P)$ of every integer polyhedron $P \subseteq \mathbb{R}^n$ is some well-defined finite number. Moreover, the process produces a finite sequence of polyhedra $P_0, P_1, \ldots$ satisfying

$$P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_{\rho(P)} = P$$

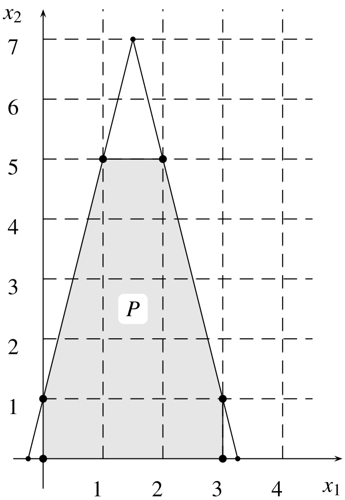
with the property that all *noninteger* extreme points $\mathbf{x}^* \in P_\ell$ are eliminated from $P_{\ell+1}$ where $0 \leq \ell < \rho(P)$ is arbitrary.

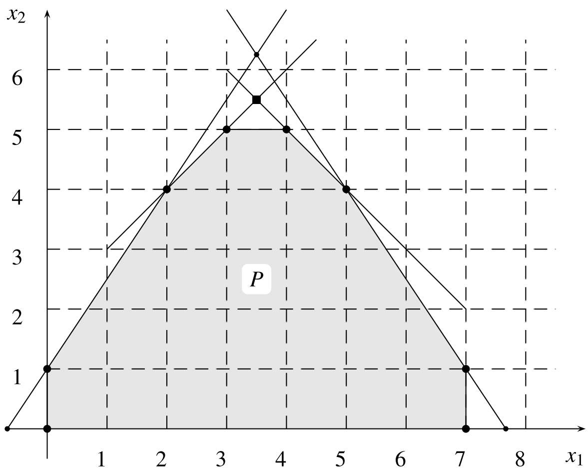
The notion of rank introduced here remains correct for *mixed-integer rational polyhedra* $P \subseteq \mathbb{R}^n$. In the case of mixed-integer polyhedra we have a partitioning of $\mathbf{x} \in \mathbb{R}^n$ into n_1, say, "integer" variables $\boldsymbol{\xi} \in \mathbb{R}^{n_1}$ and $n - n_1$ "flow" variables $\boldsymbol{\phi} \in \mathbb{R}^{n-n_1}$. The requirement $\mathbf{x} \in \mathbb{Z}^n$ is replaced by $\boldsymbol{\xi} \in \mathbb{Z}^{n_1}$. With the corresponding notational changes we define the notions of the rank of facets as well as of a mixed-integer rational polyhedron like above; see [40] and Chaps. 10.2–10.3 of [38] for details.

In Fig. 3 we show two families of polytopes in $\mathbb{R}^2$ of rank one and two, respectively. The integer polytope on the top is given by

$$P = \operatorname{conv}\left\{ \mathbf{x} \in \mathbb{Z}^2 : -Mx_1 + x_2 \leq 1, Mx_1 + x_2 \leq 3M + 1, x_2 \geq 0 \right\},$$

Fig. 3 Polytopes in $\mathbb{R}^2$ of rank one and two





where $M \geq 2$ is an arbitrary integer number, and the one on the bottom by

$$P = \operatorname{conv}\big\{\mathbf{x} \in \mathbb{Z}^2 : -Mx_1 + 2x_2 \leq 2,\, Mx_1 + 2x_2 \leq 7M + 2,\, x_2 \geq 0\big\},$$

where $M \geq 3$ is an arbitrary *odd* integer number. The corresponding minimal formulations are the polytopes P_0 given by the respective linear programming relaxations of the two constraint sets shown here and the corresponding subfamily $\mathcal{F}_0$ of $\mathcal{F}$ is unique in both cases for the permitted values of the parameter M. The polytope on the top has three facets of rank 0 and three facets of rank 1, namely $x_1 \geq 0$, $x_1 \leq 3$, and $x_2 \leq M + 1$. The polytope on the bottom has three facets of rank 0 as well. The inequalities $x_1 \geq 0$, $x_1 \leq 7$, $-\lfloor M/2 \rfloor x_1 + x_2 \leq 2$, and $\lfloor M/2 \rfloor x_1 + x_2 \leq 7\lfloor M/2 \rfloor + 2$ define the four facets of rank 1 of the corresponding polytope while the facet defined by $x_2 \leq 3\lfloor M/2 \rfloor + 2$ has a rank of two. The bottom figure also shows the polytopes

P_0, P_1 and $P_2 = P$ and their respective containment as an illustration of the second part of the theorem.

Note 1 Different notions of the rank of an integer polyhedron can be found, e.g., in the books by Nemhauser and Wolsey (Chap. II.1.2 in [30]) and Schrijver (Chap. 23.4 in [48]). Their concepts are based on the algorithm for integer programming due to Gomory [16]; see also [15] and [10]. If one uses the Nemhauser and Wolsey notion, the rank of a facet of a *flat* integer polyhedron may differ according to its different representations by linear inequalities which is absolutely undesirable. An example to this effect are the following three inequalities

$$
\begin{aligned}
\mathbf{a}^1\mathbf{x} &:= x_{12} + x_{13} + x_{14} + x_{23} + x_{25} + x_{36} \le 4 \\
\mathbf{a}^2\mathbf{x} &:= x_{12} + x_{13} + 2x_{14} + x_{23} + 2x_{25} + 2x_{36} + x_{45} + x_{46} + x_{56} \le 8 \\
\mathbf{a}^3\mathbf{x} &:= x_{12} - 4x_{13} + 6x_{14} - 4x_{14} + x_{16} + 6x_{23} - 4x_{24} - 4x_{25} \\
&\qquad + x_{26} + x_{34} + x_{35} - 4x_{36} + x_{45} - 4x_{46} + 6x_{56} \\
&\le 16
\end{aligned}
$$

which define the same facet of $\mathcal{Q}_T^6$, i.e., the symmetric traveling salesman polytope on 6 cities. Since $\max\{\mathbf{a}^1\mathbf{x} : \mathbf{x} \in \mathcal{Q}_S^6\} = 4.5$, where $\mathcal{Q}_S^6$ is the relaxation with all subtour elimination constraints, the inequality $\mathbf{a}^1\mathbf{x} \le 4$ gets a rank of 1 in this notion, since $\max\{\mathbf{a}^2\mathbf{x} : \mathbf{x} \in \mathcal{Q}_S^6\} = 9$, the inequality $\mathbf{a}^2\mathbf{x} \le 8$ has probably a rank of 2 and since $\max\{\mathbf{a}^3\mathbf{x} : \mathbf{x} \in \mathcal{Q}_S^6\} = 21$ the inequality $\mathbf{a}^3\mathbf{x} \le 16$ has a rank of at least two according to this notion. Yet they all define the same facet of $\mathcal{Q}_T^6$. Thus the rank of $\mathcal{Q}_T^6$ that we get from this concept depends entirely upon the *representation* of the facets of $\mathcal{Q}_T^6$ that we work with. Such an outcome is not possible with the notion of rank that we propose here because the inductive process does not work with any particular representation of the facets of P. Schrijver's notion of rank gets formally around the representation dependency by choosing a formulation of P and determining the rank of P relative to the given formulation. In the case of *flat* polyhedra many different formulations of an integer polyhedron exist and it is not known in what way the resulting "rank" depends on the formulation. Like the previous one his notion employs Gomory's 1958 algorithm which has been known from its beginnings to be a bad algorithm for integer programs. In either case, if one calculates, for instance, the rank of the top integer polytopes of Fig. 3 from the textbooks, one finds a rank of something like M which is absurd and easily explained by the poor convergence properties of Gomory's algorithm; see [33]. Except in rare instances, the textbook notions and the concept of the rank of integer P introduced here will lead to radically different conclusions about the rank and the complexity of the facial structure of integer polyhedra; see [40] for more on this.

The reason for the irrelevancy of the textbook notions for the rank of integer polyhedra and especially of those related to combinatorial problems can be explained as follows: Let $\mathbf{A}$ be any matrix of rationals of size $m \times n$ and of rank m, $\mathbf{b}$ a vector of

m rationals and

$$P = \operatorname{conv}\left\{\mathbf{x} \in \mathbb{Z}^n : \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\right\} \subseteq P_{LP} = \left\{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\right\}.$$

For any $\boldsymbol{\mu} \in \mathbb{R}^m$ it follows that $\lfloor \boldsymbol{\mu}\mathbf{A} \rfloor \leq \boldsymbol{\mu}\mathbf{A}$ where $\lfloor \alpha \rfloor$ is the largest integer less than or equal to α and $\lfloor \boldsymbol{\mu}\mathbf{A} \rfloor$ is the same componentwise. Since $\mathbf{x} \geq \mathbf{0}$ we get $\lfloor \boldsymbol{\mu}\mathbf{A} \rfloor \mathbf{x} \leq \boldsymbol{\mu}\mathbf{Ax}$, $\boldsymbol{\mu}\mathbf{Ax} = \boldsymbol{\mu}\mathbf{b}$ for all for all $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{Ax} = \mathbf{b}$ and thus $P \subseteq P_{LP} \subseteq \{\mathbf{x} \in \mathbb{R}^n : \lfloor \boldsymbol{\mu}\mathbf{A} \rfloor \mathbf{x} \leq \boldsymbol{\mu}\mathbf{b}\}$. But $\lfloor \boldsymbol{\mu}\mathbf{A} \rfloor \mathbf{x} \in \mathbb{Z}$ and $\lfloor \boldsymbol{\mu}\mathbf{A} \rfloor \mathbf{x} \leq \lfloor \boldsymbol{\mu}\mathbf{b} \rfloor$ for all $\mathbf{x} \in \mathbb{Z}^n$, and *any* $\mathbf{x} \in \mathbb{Z}^n$ with $\lfloor \boldsymbol{\mu}\mathbf{A} \rfloor \mathbf{x} \leq \boldsymbol{\mu}\mathbf{b}$ stops the rounddown of $\boldsymbol{\mu}\mathbf{b}$ to $\lfloor \boldsymbol{\mu}\mathbf{b} \rfloor$. Thus, coincidentally, $\lfloor \boldsymbol{\mu}\mathbf{A} \rfloor \mathbf{x} \leq \lfloor \boldsymbol{\mu}\mathbf{b} \rfloor$ for all $\mathbf{x} \in P$ as well. Hence no particularities of P or of a specific combinatorial problem are used in the derivation of these "cuts". To get Gomory's classical "fractional" cut, just choose $\boldsymbol{\mu} = \mathbf{u}^i\mathbf{B}^{-1}$ where $\mathbf{B}$ is a (feasible) basis for the associated linear program, $\mathbf{u}^i \in \mathbb{R}^m$ any i^{th} unit vector such that $\mathbf{u}^i\mathbf{B}^{-1}\mathbf{b} \notin \mathbb{Z}$ and do the algebra or consult [39] or do both. The fact that Gomory's 1958 cuts worked for Edmonds' (1965) matching polytope [13] is absolutely no reason to expect the same for other combinatorial problems like for vertex packing or traveling salesman polytopes and even less for general integer polyhedra.

7 The Facial Structure of "Small" STS Polytopes

We have applied the foregoing methodology to analyze the facial structure of symmetric traveling salesman (STS) polytopes where the underlying graph is complete and has up to 10 nodes. As it is usual we denote from now on by n the number of nodes of the graph and thus the dimension of the space that we work in equals $n(n-1)/2$.

The analysis of "small" STS polytopes has quite some history which begins apparently with work done in the 1950s—see [23, 26] and [31]. These early works were thoroughly forgotten, though, until Martin Grötschel uncovered them during an extensive literature search while writing his dissertation around 1975. In the case of the STS polytope complete descriptions of $\mathcal{Q}_T^n$ for $n \leq 6$ and a partial description for $n = 7$ were known in the fifties. The case $n = 7$ was completed by Boyd and Cunningham [3]. For $n = 8$ Naddef and Rinaldi [28] obtained among other results all but 60,480 of the 194,187 facets of this polytope, the complete set was computed and published by Christof, Jünger, and Reinelt [9]. Until very recently, I believed that my calculations for $n = 9$ and $n = 10$ were new. However, as Gerd Reinelt [46] pointed out to me, identical results for these two cases, see Table 1, were obtained independently by him and Christof using a similar methodology, see, e.g., [6, 8].

Similar work has been carried out for the asymmetric traveling salesman polytope, see [2, 14, 27], the multicut polytope, see [12], the linear ordering polytope, see [21], the minimum cut polyhedron, see [1], the quadratic assignment polytope, see [25] and [43], and possibly other polytopes or polyhedra as well. All of these works were concerned with finding explicit "lists" of all facet-defining inequalities for the respective problems. Here we study the STS polytope with up to 10 nodes.

Table 1 The facial structure of the polytopes Q_T^n for $3 \le n \le 10$

Nodes	3	4	5	6	7	8	9	10	n
Variables	3	6	10	15	21	28	36	45	$\frac{1}{2}n(n-1)$
dim Q_T^n	0	2	5	9	14	20	27	35	$\frac{1}{2}n(n-3)$
$\pi(n)$	1	3	12	60	360	2,520	20,160	181,440	$\frac{1}{2}(n-1)!$
$\pi_0(n)$	0	2	10	41	168	730	3,555	19,391	?
$\phi(n)$	0	3	20	100	3,437	194,187	42,104,442	$\ge$51,043,900,866	?
$\phi_0(n)$	0	2	10	27	196	2,600	88,911	$\ge$13,607,980	?
$\kappa(Q_T^n)$	0	1	2	4	6	24	192	$\ge$15,379	?
$\rho(Q_T^n)$	-1	0	0	1	2	3	4	?	?
diam Q_T^n	0	1	2	2	2	2	2	2	$\le$4

It remains to bring some order into the resulting information about this incredibly complex family of polytopes; see Table 1.

Let $\mathbf{A}$ be the node *versus* edge incidence matrix of the complete graph $G = (V, E)$, say, having n nodes and $n(n-1)/2$ edges. As usually, we denote by

$$Q_A^n = \left\{\mathbf{x} \in \mathbb{R}^{n(n-1)/2} : \mathbf{Ax} = \mathbf{2}, \mathbf{0} \le \mathbf{x} \le \mathbf{1}\right\}$$

the *assignment* or *2-matching relaxation* of the STS problem. Q_A^n has integer and noninteger extreme points. The incidence vector of every tour in G is an extreme point of Q_A^n, but Q_A^n also has integer extreme points that correspond to subtours. These are ruled out by the subtour elimination constraints (SECs) of Dantzig, Fulkerson, and Johnson [11]. They give rise to the *subtour polytope relaxation* of the STS problem

$$Q_S^n = \left\{\mathbf{x} \in \mathbb{R}^{n(n-1)/2} : \mathbf{Ax} = \mathbf{2}, \mathbf{x}(S) \le |S| - 1 \text{ for } S \subset V \text{ with } |S| \ge 2, \mathbf{x} \ge \mathbf{0}\right\},$$

where $\mathbf{x}(S)$ is the sum of all components x_e of $\mathbf{x}$ such that e is an edge of G with both endpoints in the set $S \subset V = \{1, \ldots, n\}$.

The polytope Q_S^n provides a formulation for the STS problem: every integer extreme point of Q_S^n corresponds to the incidence vector of some tour in G and vice versa. Thus the STS polytope, i.e., the convex hull of the incidence vectors of all tours in G, satisfies

$$Q_T^n = \operatorname{conv}\left(Q_S^n \cap \mathbb{Z}^{n(n-1)/2}\right).$$

The STS problem consists of minimizing some linear function over the polytope Q_T^n which has a dimension of $n(n-3)/2$ for all $n \ge 3$; see, e.g., [17].

There are $2^n - n - 2$ possible SECs in a complete graph and not all of them are needed because $\mathbf{x}(S) \le |S| - 1$ if and only if $\mathbf{x}(V - S) \le |V - S| - 1$ for all $\mathbf{x} \in Q_A^n$ and nonempty $S \subset V$. It was shown among other results by Grötschel and Padberg [17–20] that the SECs define precisely $2^{n-1} - n - 1$ distinct facets of Q_T^n for all $n \ge 5$. Moreover, the corresponding smaller set of SECs together with the equations $\mathbf{Ax} = \mathbf{2}$ and the nonnegativity $\mathbf{x} \ge \mathbf{0}$ suffice to formulate the STS problem correctly.

We also showed that the nonnegativity conditions define distinct facets of $\mathcal{Q}_T^n$ for all $n \geq 5$ that are distinct from those defined by the SECs.

For the purpose of a rank analysis of the facets of $\mathcal{Q}_T^n$, we take the reduced set of SECs plus the nonnegativity constraints as the rank zero facets of $\mathcal{Q}_T^n$. They constitute the family $\mathcal{F}_0$ and the subtour polytope $\mathcal{Q}_S^n$ is a minimal formulation $\mathcal{F}_{\min}^S$ for $\mathcal{Q}_T^n$. It is readily shown using [18, 19] that $\mathcal{F}_{\min}^S$ satisfies the requirements (27) of Sect. 6 and thus $\mathcal{F}_{\min}^S \subseteq \mathcal{F}_0$ as defined there. I state that $\mathcal{F}_{\min}^S = \mathcal{F}_0$ and thus that the STS problem has a *unique* minimal formulation. [The referee pointed out that $\mathcal{F}_{\min}^S = \mathcal{F}_0$ needs a formal proof and I agree. However, I will leave such a formal proof or (or disproof ?) as "food for thought" for the younger talents in our field.]

In Table 1 we summarize the key characteristics of the facial structure of the STS polytopes $\mathcal{Q}_T^n$ for $3 \leq n \leq 10$. We use the following notation:

$$\begin{aligned}
\pi(n) &= \text{the number of tours (extreme points) of } \mathcal{Q}_T^n, \\
\pi_0(n) &= \text{the number of tours } \textit{adjacent} \text{ to any given tour of } \mathcal{Q}_T^n, \\
\phi(n) &= \text{the number of facets of } \mathcal{Q}_T^n, \\
\phi_0(n) &= \text{the number of facets that are tight at any given tour of } \mathcal{Q}_T^n, \\
\kappa\left(\mathcal{Q}_T^n\right) &= \text{the class number of different facet types of } \mathcal{Q}_T^n, \\
\rho\left(\mathcal{Q}_T^n\right) &= \text{the rank of } \mathcal{Q}_T^n.
\end{aligned}$$

diam $\mathcal{Q}_T^n$ is the diameter of the STS polytope $\mathcal{Q}_T^n$, i.e., the maximum over all ordered pairs $(\mathbf{x}, \mathbf{y})$ of extreme points of $\mathcal{Q}_T^n$ of the minimum number of edges of $\mathcal{Q}_T^n$ that must be traversed to reach $\mathbf{x}$ from $\mathbf{y}$. For *asymmetric* traveling salesman polytopes we know that the diameter equals two for all $n \geq 6$, i.e., it is in particular independent upon the number n of nodes of the underlying directed graph; see [42]. In [20] we conjectured that diam $\mathcal{Q}_T^n = 2$ for $n \geq 5$. For $5 \leq n \leq 12$ this conjecture has been verified by way of a computer program; see [49]. Rispoli and Cosares [47] have shown that diam $\mathcal{Q}_T^n \leq 4$ for all $n \geq 5$, thus establishing a small upper bound on the diameter of this family of polytopes that does not depend on the number n of nodes of the graph.

Facet defining inequalities other than the SECs are known for the STS polytope $\mathcal{Q}_T^n$. See [24] for an excellent survey. From a computational point of view the following *r-comb inequalities* are the ones that are most frequently used. Let $H \subseteq V$ and $T_i \subseteq V$ for $1 \leq i \leq k$ be any subsets of nodes satisfying

$$\begin{aligned}
|H \cap T_i| \geq 1, \qquad |T_i - H| \geq 1 \quad \text{for } 1 \leq i \leq k, \\
T_i \cap T_j = \emptyset \quad \text{for } 1 \leq i < j \leq k,
\end{aligned}$$

where $k \geq 3$ is an odd integer. The configuration $C = (H, T_1, \ldots, T_k)$ in the graph G is called a *comb* in G with H being the "handle" and $T_1, \ldots, T_k$ the "teeth" of the comb. We partition each T_i into $r_i \geq 1$ nonempty sets T_i^j such that

$$\begin{aligned} &\left|H \cap T_i^j\right| \geq 1, \qquad \left|T_i^j - H\right| \geq 1 \quad \text{for } 1 \leq j \leq r_i, \\ &T_i^j \cap T_i^\ell = \emptyset \quad \text{for } 1 \leq j < \ell \leq r_i, \end{aligned}$$

for all $1 \leq i \leq k$. Then the r-comb inequality, see [38],

$$\begin{aligned} &\mathbf{x}(H) + \sum_{i=1}^{k}\left(\sum_{j=1}^{r_i}\mathbf{x}\big(T_i^j\big) + \sum_{1\leq \ell < j \leq r_i}\big(\mathbf{x}\big(T_i^\ell \cap H : T_i^j - H\big) + \mathbf{x}\big(T_i^\ell - H : T_i^j \cap H\big)\big)\right) \\ &\quad \leq |H| + \sum_{i=1}^{k}\big(|T_i| - r_i - 1\big) + \left\lfloor \frac{k}{2} \right\rfloor \end{aligned}$$

is a facet defining inequality for $\mathcal{Q}_T^n$. If $r_i = 1$ for $1 \leq i \leq k$, then we have the *comb inequalities* that were shown to define facets of $\mathcal{Q}_T^n$ by Grötschel and Padberg [19]. If $r_1 = 2$, $r_i = 1$ otherwise then we have the *chain inequalities* of Padberg and Hong [41]. The assertion that r-comb inequalities are facets defining for $\mathcal{Q}_T^n$ follows, e.g., from the work of Naddef and Rinaldi [29]. The prototype inequalities for r-combs can be seen in Fig. 5 and Fig. 6. They were known to us in early 1975 including the only chain inequality that exists for $\mathcal{Q}_T^8$. They were found by us empirically by trying to cut off "fractional" extreme points of LP relaxations that we encountered in small-scale numerical experimentation. When the comb and chain inequalities were tested by Martin Grötschel (September 1975) for their facet defining properties by way of a computer program at the University of Bonn, it turned out that the dimension of the face of $\mathcal{Q}_T^8$ defined by the chain inequality (facet type 7 in Fig. 5) was smaller than required, whereas the comb inequality (in particular the right-most one in Fig. 6) turned out to be facet defining for $\mathcal{Q}_T^8$. As a result we concentrated our effort on proving comb inequalities to be facet defining for $\mathcal{Q}_T^n$, but ignored the chain constraints. By the time it was found that the computer program used by Grötschel in the rank calculation had a "bug"—to fix the bug was one of Michael Jünger's and Gerd Reinelt's first tasks as research assistants at Bonn University—we had indeed more or less "forgotten" the chain constraints. They were shown to be facet defining by Sylvia Boyd and Mark Hartmann around 1990; see [3].

In Table 2 we summarize the knowledge about the number of facet defining inequalities for $\mathcal{Q}_T^n$ where $6 \leq n \leq 10$ up to about 1980. The exact number $\nu^C(n)$ of comb facets of $\mathcal{Q}_T^n$, see [19], is given by

$$\begin{aligned} \nu^C(n) = &\sum_{q=3}^{n-3}\sum_{j=3}^{n-q}\sum_{\substack{k=3\\ k \text{ odd}}}^{\min\{j,q\}}\sum_{p=k}^{q}\sum_{\ell=0}^{k}\sum_{h=0}^{k}\frac{(-1)^{\ell+h}}{2k!} \\ &\times \binom{n}{q}\binom{n-q}{j}\binom{q}{p}\binom{k}{\ell}\binom{k}{h}(k-\ell)^j(k-h)^p. \end{aligned}$$

Table 2 Known and unknown facets of Q^n_T for $6 \le n \le 10$ as of 1980

Nodes	6	7	8	9	10	n
Vars	15	21	28	36	45	$\frac{1}{2}n(n-1)$
SECs	25	56	119	246	501	$2^{n-1}-n-1$
Combs	60	2,100	42,840	667,800	8,843,940	$\nu^C(n)$
$\phi(n)$	100	3,437	194,187	42,104,442	$\ge$51,043,900,866	?
	100 %	63 %	22 %	1.6 %	$\le$0.017 %	?

The formula for $\nu^C(n)$ was computed—with the help of Rabe von Randow of Bonn University—around 1976. Given the enormous number of facet defining inequalities that were known to us we began to optimize larger scale instances of symmetric traveling salesman problems around that time. With hindsight, it is fair to say that our ignorance of the true numbers $\phi(n)$ for Q^n_T was beneficial in this endeavor because, otherwise, we might have been discouraged to try out numerical experimentation. On the other hand, today instances with 10,000 cities or more have been optimized using essentially only comb inequalities. It is therefore fair to conjecture that some facets of Q^n_T, such as, e.g., those defined by the r-comb inequalities, are relatively more important than other ones. Whence our desire to bring some order into the facial structure of Q^n_T by "ranking" its facets.

To arrive at the numbers displayed in all tables (except $\dim Q^n_T$, $\pi(n)$ and $\operatorname{diam} Q^n_T$), we have written several computer programs. To find the number $\pi_0(n)$ of adjacent extreme points to a given one, e.g., the one corresponding to the tour $1, \ldots, n$, we use the CPLEX subroutines in a straightforward enumerative way. Then each edge of Q^n_T is subjected to the symmetry tests of Sect. 4 and Sect. 5 to reduce the number of edge figures to be analyzed. The output of this part of the overall program is a "problem file" of edge figures to be analyzed by the double description algorithm DDA. The facets for each edge figure from the problem file are calculated in a second program and classified like discussed in Sect. 2. This gives the entries for $\phi(n)$, $\phi_0(n)$, and $\kappa(Q^n_T)$ of Table 1. Only one representative for each facet type is generated and stored, the counting of the respective totals $\phi(n)$ and $\phi_0(n)$ is done in a separate subroutine. This program also calculates the normal form representation and the minimum positive support representations of each facet type. The rank analysis is done in a third program that implements the mathematics of Sect. 6 in an enumerative way.

As always in computer-based calculation we must leave a margin for error due to possibly remaining "bugs" in the programs. The numbers for Q^n_T—except for the rank $\rho(Q^n_T)$ which is a new concept—agree, however, exactly with the previously published results for $3 \le n \le 8$ and the same programs were used unchanged for $7 \le n \le 9$ to produce all numbers of this paper. For $n \le 6$ all edges of Q^n_T (after the corresponding reductions) were used, while for $7 \le n \le 9$ all edges of Q^n_T that are edges of Q^n_A were also excluded from consideration, which has a mathematical justification.

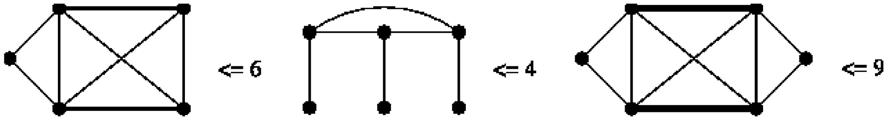


Fig. 4 Irreducible representations of facet types 4, 5, 6 of Q_T^7

For $n = 10$ we computed an initial set of facet types by this procedure as well. The corresponding edge figures, however, turned out to have more facets than we could calculate with our computing equipment at the time and so we changed the procedure as follows. For the initial set of about 2,000 facet types we compute all facets of each facet and classify them as known and unknown ones. For "unknown" facets we store a representative which is added to the list of facet types and later subjected to the same procedure as before. This procedure is then iterated until the program finds no new unknown facets. Clearly, this procedure need not find *all* facets of Q_T^{10} and we believe that Q_T^{10} has indeed more facets than we have found.

In Fig. 4 and Fig. 5 we depict the graphs of the minimal positive support representations of the facets of Q_T^n having rank 1 or higher for $n = 7$ and $n = 8$. The thickness of each arc corresponds to the numerical value of the respective inequality in less-than-or-equal-to form and the nonzero coefficients range from 1 to 4. Coefficients of 4 are drawn as heavy dotted lines. Thus Fig. 4 depicts 3,360 and Fig. 5 194,040 distinct facets of Q_T^n for $n = 7$ and $n = 8$. Figure 6 shows two "comb" forms of the respective facet types of Q_T^8. Note that the facet types 4, 5 and 6 of Q_T^7 are "inherited" by Q_T^8 (types 5, 6 and 12) and that, e.g., the facet type 15 of Q_T^8 is obtained by "lifting" facet type 5 of Q_T^7.

A more detailed breakdown of the facets of Q_T^n for $6 \le n \le 10$ is described below and the numerical experiments (see Table 1 and Table 8) suggest the following conjecture.

Conjecture 1 $\rho(Q_T^n) = n - 5$ *for all* $n \ge 5$.

Let $\mathbf{x}^0$ be an extreme point of Q_T^n and $\mathbf{x}^1 \in Q_T^n$ be an extreme point of Q_T^n that is adjacent to $\mathbf{x}^0$ on the polytope Q_T^n, i.e., $\mathbf{x}^1$ is a *neighbor* for $\mathbf{x}^0$. The face of minimal dimension of Q_A^n, of Q_S^n containing both $\mathbf{x}^0$ and $\mathbf{x}^1$ is denoted by $F_A(\mathbf{x}^0, \mathbf{x}^1)$ and $F_S(\mathbf{x}^0, \mathbf{x}^1)$, respectively. Table 3 and Table 4 show the breakdown of the neighbors of any extreme point of Q_T^n according to the dimensions $1, 2, 3, \ldots$ of $F_A(\mathbf{x}^0, \mathbf{x}^1)$ and $F_S(\mathbf{x}^0, \mathbf{x}^1)$ for $5 \le n \le 10$.

Common to the numbers of the two tables is a "linear" growth of the maximum dimension of $F_A(\mathbf{x}^0, \mathbf{x}^1)$ and $F_S(\mathbf{x}^0, \mathbf{x}^1)$, respectively.

Conjecture 2 *Let* $\mathbf{x}^0 \ne \mathbf{x}^1 \in Q_T^n$ *be any two extreme points and* $F_A(\mathbf{x}^0, \mathbf{x}^1)$, $F_S(\mathbf{x}^0, \mathbf{x}^1)$ *be the face of smallest dimension of* Q_A^n, *of* Q_S^n, *respectively, that contains both* $\mathbf{x}^0$ *and* $\mathbf{x}^1$. *If* $n \ge 5$ *and* $\dim F_A(\mathbf{x}^0, \mathbf{x}^1) \ge n - 3$ *or* $n \ge 7$ *and* $\dim F_S(\mathbf{x}^0, \mathbf{x}^1) \ge n - 5$, *then* $\mathbf{x}^0$ *and* $\mathbf{x}^1$ *are not adjacent on* Q_T^n.

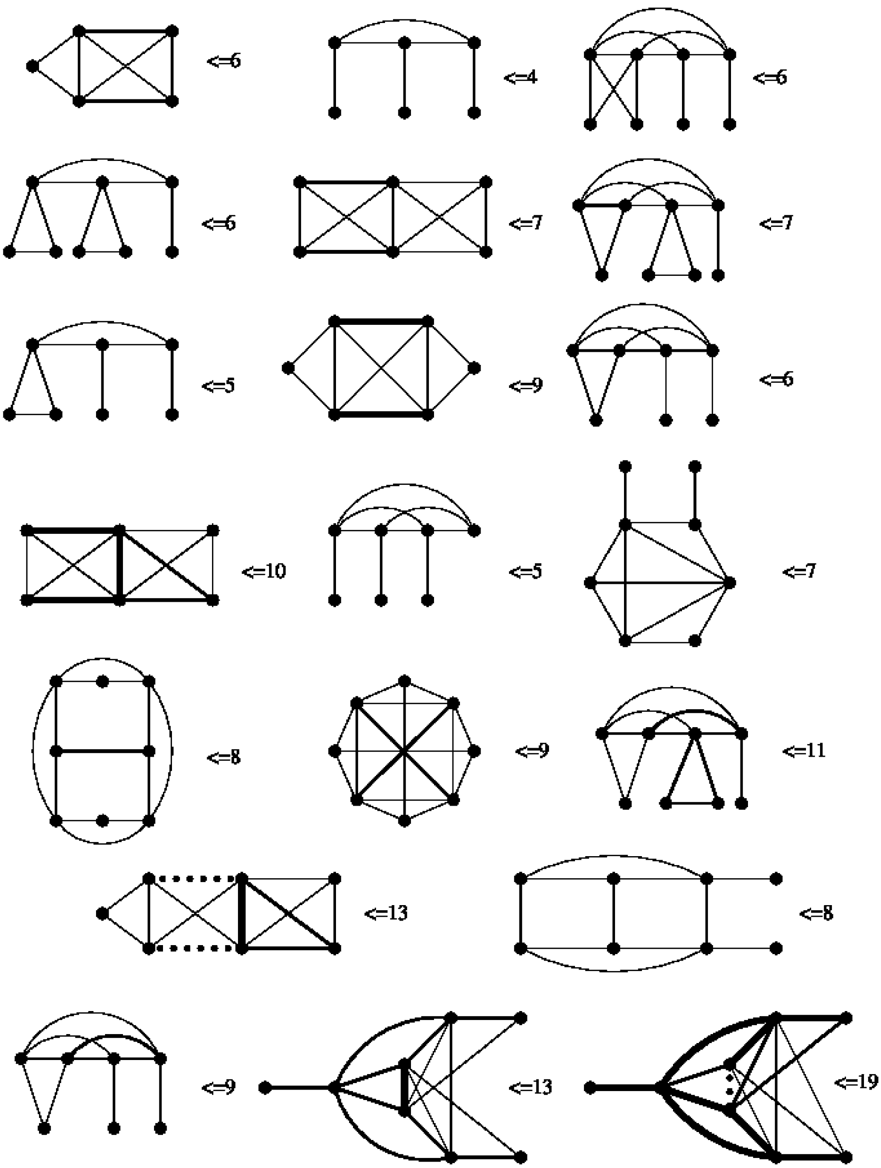


Fig. 5 Irreducible representations of facet types $5, \ldots, 24$ of $\mathcal{Q}_T^8$

Note 2 I have included Conjecture 2 for the simple reason that it is suggested by my calculations (see Table 3 and Table 4) and that Papadimitriou's 1978 result [45] on the NP-completeness of the problem of checking the adjacency of tours on $\mathcal{Q}_T^n$ just goes against my intuition. Ting-Yi Sung and I [44] showed a related negative statement about the non-polytime solvability of certain TSPs called "traps" to be

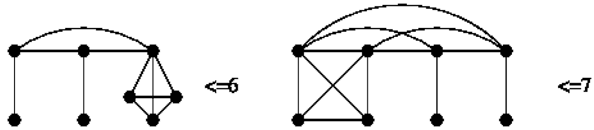


Fig. 6 Comb forms of facet types 5 and 9 of Q_T^8

Table 3 Adjacency on Q_T^n and Q_A^n

n	5	6	7	8	9	10
1	10	29	91	252	894	2,851
2		12	49	318	1,125	6,755
3			28	96	1,224	4,285
4				64	168	4,860
5					144	320
6						320
$\pi_0(n)$	10	41	168	730	3,555	19,391

Table 4 Adjacency on Q_T^n and Q_S^n

n	5	6	7	8	9	10
1	10	41	168	714	3,213	15,531
2				16	234	2,300
3					108	1,340
4						220
$\pi_0(n)$	10	41	168	730	3,555	19,391

Table 5 Reduction of the number of edge figures of Q_T^n using Claim 6

n	5	6	7	8	9	10
$\pi_0(n)$	10	41	168	730	3,555	19,391
$\sigma_0(n)$	2	7	16	59	216	1,032
$\iota_0(n)$	2	4	8	20	42	123

wrong. By refining the work on Conjecture 2 you may be able to prove $P = NP$ or to invalidate Papadimitriou's claim. [The referee pointed out that adjacency on Q_T^n is coNP-complete rather than NP-complete.]

Table 5 summarizes the effect of applying Claim 6 of Sect. 5 to reduce the number of edge figures of Q_T^n that have to be analyzed to determine the class number $\kappa(Q_T^n)$ of distinct facet types of Q_T^n. Like above $\pi_0(n)$ is the number of neighbors of any extreme point of Q_T^n and thus the number of edge figures to be analyzed a priori, $\sigma_0(n)$ is the number of edge figures that results when Claim 6 is applied for all permissible index permutations that leave $\mathbf{x}^0$ invariant, see (23), and $\iota_0(n)$ the number of "nonisomorphic" graphs that result from a full application of Claim 6.

Table 6 Classification of the facets of $\mathcal{Q}_T^n$ for $5 \le n \le 8$

n	κ	$\nu_\kappa(n)$	$\nu_0^\kappa(n)$	$d_\kappa^2(n)$	$d_\kappa(n)$	$n_\kappa^\pi(n)$	$\rho_\kappa(n)$
5	1	10	5	$\frac{1}{2}$	0.707	6	0
	2	10	5	$\frac{1}{2}$	0.707	6	0
6	1	15	9	$\frac{4}{15}$	0.516	36	0
	2	15	6	$\frac{3}{5}$	0.775	24	0
	3	10	3	$\frac{32}{45}$	0.843	18	0
	4	60	9	$\frac{128}{105}$	1.104	9	1
7	1	21	14	$\frac{1}{6}$	0.408	240	0
	2	21	7	$\frac{2}{3}$	0.816	120	0
	3	35	7	$\frac{5}{6}$	0.913	72	0
	4	1,260	70	$\frac{245}{156}$	1.253	20	1
	5	840	42	$\frac{5}{3}$	1.291	18	1
	6	1,260	56	$\frac{605}{372}$	1.275	16	2
8	1	28	20	$\frac{4}{35}$	0.338	1,800	0
	2	28	8	$\frac{5}{7}$	0.845	720	0
	3	56	8	$\frac{32}{35}$	0.956	360	0
	4	35	4	$\frac{27}{28}$	0.982	288	0
	5	3,360	88	$\frac{12}{7}$	1.309	66	1
	6	3,360	72	$\frac{256}{133}$	1.387	54	1
	7	2,520	52	$\frac{486}{259}$	1.370	52	1
	8	5,040	88	$\frac{1452}{749}$	1.392	44	1
	9	2,520	44	$\frac{27}{14}$	1.389	44	1
	10	5,040	88	$\frac{529}{280}$	1.375	44	1
	11	10,080	160	$\frac{361}{182}$	1.408	40	1
	12	5,040	80	$\frac{2883}{1498}$	1.387	40	1
	13	10,080	152	$\frac{600}{301}$	1.412	38	1
	14	10,080	152	$\frac{256}{133}$	1.387	38	1
	15	3,360	48	$\frac{289}{140}$	1.437	36	1
	16	20,160	264	$\frac{1587}{742}$	1.462	33	1
	17	10,080	124	$\frac{2}{1}$	1.414	31	1
	18	2,520	36	$\frac{729}{364}$	1.415	36	2
	19	20,160	272	$\frac{4107}{2114}$	1.394	34	2
	20	10,080	128	$\frac{1849}{952}$	1.394	32	2
	21	10,080	120	$\frac{900}{427}$	1.452	30	2
	22	20,160	240	$\frac{2883}{1400}$	1.435	30	2
	23	20,160	176	$\frac{4563}{1988}$	1.515	22	3
	24	20,160	176	$\frac{3025}{1344}$	1.500	22	3

Table 7 Facets-of-facets analysis of $\mathcal{Q}_T^n$ for $6 \le n \le 8$

κ	1	2	3	4	TT
1	14	14	6	24	58
2	14	6	4	12	36
3	9	6			15
4	6	3			9
$\rho_\kappa(6)$	0	0	0	1	

κ	1	2	3	4	5	6	TT
1	20	20	30	720	480	600	1,870
2	20	10	15	180	120	120	465
3	18	9	4	36			67
4	12	3	1			2	18
5	13	3				3	18
6	10	2		2	2		16
$\rho_\kappa(7)$	0	0	0	1	1	2	

κ	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	TT
5	19	5	1	1			3					3		3		12									47
6	21	4												3		12	6			6	6	6	6		70
7	20	6		1	4				4			4				16		4			4			8	71
8	18	3	2														4		4						31
9	22	4		1			4		4					4		16					4				59
10	19	3	2																4						28
11	18	3	1									1					4		2			2			31
12	19	3			2		2				2									2					30
13	19	3	1											1					2			2			28
14	18	3			1	1			1				1			4				2	2				33
15	19	3																		3		6			31
16	15	4		1	2	2	2		2					2		4		1			3		2	2	42
17	17	5	2			2		2			4								4			4			40
18	16	4					4									8							8	8	48
19	17	2	1					1		1	1		1				2								26
20	17	2				2						1		2	1						1	2			28
21	16	4				2	1		1					2		6				1			4	2	39
22	17	2				1					1		1		1		2			1					26
23	13	4				1										2		1			2			2	25
24	11	4					1									2		1			1		2		22
$\rho_\kappa(8)$	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	3	3	

Table 8 Breakdown of the facet types and the facets of $\mathcal{Q}_T^n$ by rank for $5 \le n \le 10$

ρ/n	5	6	7	8	9	10
0	2	3	3	4	4	5
1		1	2	13	64	604
2			1	5	83	4,506
3				2	39	6,176
4					2	?
5						?
$\kappa(\mathcal{Q}_T^n)$	2	4	6	24	192	≥15,379

ρ/n	5	6	7	8	9	10
0	20	40	77	147	282	546
1		60	2,100	90,720	6,163,920	?
2			1,260	63,000	22,150,800	?
3				40,320	13,063,680	?
4					725,760	?
5						?
$\phi(n)$	20	100	3,437	194,187	42,104,442	≥51,043,900,866

Table 6 provides more detail on the classification of the facets of $\mathcal{Q}_T^n$ into equivalence classes for $5 \le n \le 8$. The additional notation used is as follows:

$$\nu_\kappa(n) = \text{the number of facets of } \mathcal{Q}_T^n \text{ of type } \kappa,$$

$$\nu_0^\kappa(n) = \text{the number of facets of type } \kappa \text{ that are tight at any given tour},$$

$$\rho_\kappa(n) = \text{the rank of facet of type } \kappa,$$

$$d_\kappa(n) = \text{the Euclidean distance from the center of facet of type } \kappa,$$

$$n_\kappa^\pi(n) = \text{the number of tours on facet of type } \kappa,$$

where $1 \le \kappa \le \kappa(\mathcal{Q}_T^n)$. The $\lfloor n/2 \rfloor$ first types of the facets in each table correspond to the nonnegativity constraints (type 1) and the upper bounds (type 2) followed by the SECs with increasing $|S| \ge 3$.

Table 7 and Table 8 give the results of our rank analysis of the facets of $\mathcal{Q}_T^n$ for $5 \le n \le 9$. The top and middle parts of Table 7 give a complete picture for $6 \le n \le 7$, while for $n = 8$ we left out the facets-of-facets analysis for the rank zero facets of $\mathcal{Q}_T^8$. Thus the facets defined by the nonnegativity constraints $x_e \ge 0$ have a total of 58 facets for $n = 6$ (1,870 facets for $n = 7$). For $n = 6$ there are 14 facets of the same type, 14 are upper bounds, 6 are SECs with $|S| = 3$ and 24 are "matching constraints" (see the picture in the middle of Fig. 4). The remaining entries in these tables are interpreted accordingly. TT are the respective totals, i.e., they are equal to the sum of the entries in each row. In Table 8 the rows correspond to rank $0, \ldots, 5$ and the columns to the number of nodes $5, \ldots, 10$ of the graph.

Work on ideal linear descriptions of "small" problems related to combinatorial optimization problems has become much easier with the arrival of software for the double description algorithm or the Fourier-Motzkin elimination algorithm. In the early 1970s we had to "guess" some linear inequality by trial and error and then prove it to be facet-defining (if it was). Now we can use the computer to experiment with educated guesses from the linear description of small instances. As in the work of Christof and Reinelt [7] we can use the descriptions of small instances for the optimization of larger instances. We can form conjectures about the problem in question—like the conjectures above. The observed linearity of the rank $\rho(\mathcal{Q}_T^n)$ for $n = 5, \ldots, 9$ suggests to study the facial structure of $\mathcal{Q}_T^n$ *inductively*, i.e., to try to determine an ideal linear description of $\mathcal{Q}_T^{n+1}$ from the one of $\mathcal{Q}_T^n$, to prove or disprove Conjecture 1, to find an expression, e.g., for $\phi(n)$ of the form $\phi(n+1) = g(\phi(n))$ for $n \geq 5$ and some suitable function $g()$ and to continue the work that Martin Grötschel and I began so many years ago.

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