

Chapter 2

Densities in Hermitian Matrix Models

Orthogonal polynomials are traditionally studied as special functions in mathematical theories such as in the Hilbert space theory, differential equations and asymptotics. In this chapter, a new purpose of the generalized Hermite polynomials will be discussed in detail. The Lax pair obtained from the generalized Hermite polynomials can be applied to formulate the eigenvalue densities in the Hermitian matrix models with a general potential. The Lax pair method then solves the eigenvalue density problems on multiple disjoint intervals, which are associated with scalar Riemann-Hilbert problems for multi-cuts. The string equation can be applied to derive the nonlinear algebraic relations between the parameters in the density models by reformulating the potential function in terms of the trace function of the coefficient matrix obtained from the Lax pair and using the Cayley-Hamilton theorem in linear algebra. The Lax pair method improves the traditional methods for solving the eigenvalue densities by reducing the complexities in finding the nonlinear relations, and the parameters are then well organized for further analyzing the free energy function to discuss the phase transition problems.

2.1 Generalized Hermite Polynomials

Let us consider the Hermitian matrix model with a general potential

$$V(z) = \sum_{j=0}^{2m} t_j z^j, \quad (2.1)$$

where z is a real or complex variable, t_j are real, and $t_{2m} > 0$ such that the partition function

$$Z_n = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\sum_{i=1}^n V(z_i)} \prod_{j < k} (z_j - z_k)^2 dz_1 \cdots dz_n \quad (2.2)$$

is well defined. The free energy function is defined as [1] $E = -\lim_{n \rightarrow \infty} \frac{1}{n} \ln Z_n$. By the scaling transformation $z = n^{\frac{1}{2m}} \eta$ and $t_j = n^{1 - \frac{j}{2m}} g_j$, the potential $V(z)$ be-

comes a new potential $W(\eta) = \sum_{j=0}^{2m} g_j \eta^j$. The eigenvalue density $\rho(\eta)$ on l_1 interval(s) $\Omega = \bigcup_{j=1}^{l_1} [\eta_-^{(j)}, \eta_+^{(j)}]$ is defined to minimize the free energy function

$$E = \int_{\Omega} W(\eta) \rho(\eta) d\eta - \int_{\Omega} \int_{\Omega} \ln |\lambda - \eta| \rho(\lambda) \rho(\eta) d\lambda d\eta. \quad (2.3)$$

The density $\rho(\eta)$ is required to satisfy the following conditions [1, 7]:

(i) ρ is non-negative when $\eta \in \Omega$,

$$\rho(\eta) \geq 0; \quad (2.4)$$

(ii) ρ is normalized,

$$\int_{\Omega} \rho(\eta) d\eta = 1; \quad (2.5)$$

(iii) ρ satisfies the following variational equation for an inner point η of Ω ,

$$(P) \int_{\Omega} \frac{\rho(\lambda)}{\eta - \lambda} d\lambda = \frac{1}{2} W'(\eta), \quad (2.6)$$

where (P) stands for the principal value of the integral, for example,

$$(P) \int_{\eta_-}^{\eta_+} \frac{\rho(\lambda)}{\eta - \lambda} d\lambda \doteq \lim_{\varepsilon \rightarrow 0} \left(\int_{\eta_-}^{\eta_- + \varepsilon} \frac{\rho(\lambda)}{\eta - \lambda} d\lambda + \int_{\eta_+ - \varepsilon}^{\eta_+} \frac{\rho(\lambda)}{\eta - \lambda} d\lambda \right).$$

The generalized Hermite polynomials will be applied to find the eigenvalue density $\rho(\eta)$. The last two conditions will be satisfied based on the asymptotic of an analytic function $\omega(\eta)$ as $\eta \rightarrow \infty$, and the first condition needs separate discussions. The density generally takes a form as the product of a polynomial in η and the square root of another polynomial in η of degree $l \geq l_1$, and the parameters in the density are generally restricted by complicated nonlinear relations that can be obtained by using the string equation associated with the Hermitian matrix models.

Now, let us use the planar diagram model [1] to briefly explain how the above basic concepts are connected each other. For the planar diagram eigenvalue density shown in Sect. 1.2 for $W(\eta) = \frac{1}{2}\eta^2 + g\eta^4$, if we define

$$\omega(\eta) = \left(\frac{1}{2} + 4gb^2 + 2g\eta^2 \right) \sqrt{\eta^2 - 4b^2}, \quad (2.7)$$

for $\eta \in \mathbb{C} \setminus \Omega$ where $\Omega = [-2b, 2b]$ and $\mathbb{C}$ stands for the complex plane, then there is the following asymptotics

$$\omega(\eta) = \frac{1}{2}(\eta + 4g\eta^3) - (b^2 + 12gb^4) \frac{1}{\eta} + O\left(\frac{1}{\eta^2}\right), \quad (2.8)$$

as $\eta \rightarrow \infty$. Let $\Omega^* = \Omega^- \cup \Omega^+$ be a closed counterclockwise contour, where Ω^- and Ω^+ are the lower and upper edges of Ω respectively. By Cauchy theorem, there is $\int_{\Omega^*} \omega(\eta) d\eta = -(b^2 + 12gb^4) \int_{|\eta|=R} \frac{d\eta}{\eta} = -2\pi i (b^2 + 12gb^4)$. The analytic function $\omega(\eta)$ has opposite signs on Ω^- and Ω^+ . If we define $\rho(\eta) = \frac{1}{\pi i} \omega(\eta)|_{\Omega^+}$, then $\rho(\eta)$ satisfies the normalized condition above if the parameters satisfy

$$b^2 + 12gb^4 = 1. \quad (2.9)$$

Further, if Ω^* is changed to Ω_ε^* by just changing the straight lines of the Ω^* at the neighborhood of a inner point η of Ω to the small semicircles of ε radius, then we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon^* \setminus \gamma_\varepsilon} \frac{\omega(\lambda)}{\lambda - \eta} d\lambda = \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon^*} \frac{\omega(\lambda)}{\lambda - \eta} d\lambda - \lim_{\varepsilon \rightarrow 0} \int_{\gamma_\varepsilon} \frac{\omega(\lambda)}{\lambda - \eta} d\lambda, \quad (2.10)$$

where γ_ε is the circle of ε radius with center η . Since $\omega(\eta)$ has opposite signs on the upper and lower edges, the second limit on the right hand side above is zero. By the asymptotics above, there is

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon^*} \frac{\omega(\lambda)}{\lambda - \eta} d\lambda &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon^*} \frac{\omega(\lambda) - \frac{1}{2} W'(\lambda)}{\lambda - \eta} d\lambda + \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon^*} \frac{\frac{1}{2} W'(\lambda)}{\lambda - \eta} d\lambda \\ &= \pi i W'(\eta). \end{aligned} \quad (2.11)$$

Therefore, $\rho(\eta)$ satisfies the variational equation given above for the eigenvalue density by noting that $\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon^* \setminus \gamma_\varepsilon} \frac{\omega(\lambda)}{\lambda - \eta} d\lambda = 2\pi i(P) \int_{\Omega} \frac{\rho(\eta)}{\eta - \lambda} d\eta$. We have shown the idea to use the $\omega(\eta)$ with the asymptotics to satisfy the conditions for the $\rho(\eta)$. The question then becomes how to get the $\omega(\eta)$ defined in the complex plane except the cut(s) with the corresponding asymptotics. The orthogonal polynomials and string equation can just provide such formulations to construct the $\omega(\eta)$.

In 1991, Fokas, Its and Kitaev [4] obtained that for the potential $V(z) = t_2 z^2 + t_4 z^4$, the coefficients v_n 's in the recursion formula $z p_n = p_{n+1} + v_n p_{n-1}$ satisfy the string equation

$$(2t_2 + 4t_4(v_{n-1} + v_n + v_{n+1}))v_n = n. \quad (2.12)$$

If all the v_n 's are replaced by $n^{1/2}b^2$ and t_j 's are replaced by $n^{1-\frac{j}{4}}g_j$, and if we choose $g_2 = 1/2$ and denote $g_4 = g$, then the string equation (2.12) is reduced to the condition (2.9), which is corresponding to (2.5). The relation between (2.12) and (2.9) indicates that the eigenvalue density problem and the string equation have some connections. We will see that these relations are so close not only for their algebraic formulas but also the roles they play in the corresponding models. The condition (2.9) for the eigenvalue density is a property for the exponent of the partition function. The string equation is a property for the exponent n of the orthogonal polynomials $p_n(z) = z^n + \dots$, that is about the wave function. We will experience more about the connections between these equations in the later discussions.

The main question is how to find the formula of the density ρ . It is discussed in [6] by McLeod and Wang in 2009 that the eigenvalue density can be formulated based on the square root of the determinant of matrix $A_n(z)$, $\sqrt{\det A_n(z)}$, where $A_n(z)$ is the coefficient matrix in the Lax pair

$$\Phi_{n+1} = L_n \Phi_n, \quad (2.13)$$

$$\frac{\partial}{\partial z} \Phi_n = A_n(z) \Phi_n. \quad (2.14)$$

Here $\Phi_n(z) = e^{-\frac{1}{2}V(z)}(p_n(z), p_{n-1}(z))^T$, the orthogonal polynomials $p_n = z^n + \dots$ are defined on the real line with the weight $\exp(-V(z)) : \langle p_n, p_{n'} \rangle = h_n \delta_{n,n'}$, and

L_n can be obtained from the recursion formula [8] $zp_n(z) = p_{n+1}(z) + u_n p_n(z) + v_n p_{n-1}(z)$. The consistency condition for the Lax pair is the string equation which is a set of two discrete equations for u_n and v_n : $\langle p_n(z), V'(z)p_{n-1}(z) \rangle = nh_{n-1}$ and $\langle p_n(z), V'(z)p_n(z) \rangle = 0$, where $h_n/h_{n-1} = v_n$. These two relations will be used to derive the conditions (2.5) and (2.6). For the planar diagram model, there is no the equation (2.6) since the potential is an even function.

The coefficient matrix $A_n(z)$ above is generally a complicated 2×2 matrix. If we replace all the u_{n-k-1} and v_{n-k} in the Lax pair by x_n and y_n respectively, then $A_n(z)$ can be reduced to

$$\hat{A}_n(z) = D_n \hat{F}_n(z) D_n^{-1} - \frac{1}{2} V'(z) I, \quad (2.15)$$

where the matrix $\hat{F}_n(z)$ is a linear combination of positive powers of a matrix $\hat{J}_n$ derived from L_n ,

$$\hat{J}_n = \begin{pmatrix} 0 & 1 \\ -y_n & z - x_n \end{pmatrix}.$$

Here $D_n = \text{diag}(h_n, h_{n-1})$, and I is the identity matrix. By the Cayley-Hamilton theorem for $\hat{J}_n$ as known in the textbooks of linear algebra, we have

$$(z - x_n)I = \hat{J}_n + y_n \hat{J}_n^{-1}. \quad (2.16)$$

Applying this relation to $V'(z)I$ in (2.15), it is found that $D_n^{-1} \hat{A}_n(z) D_n$ can be factorized as a product of a polynomial and a simple matrix

$$D_n^{-1} \hat{A}_n(z) D_n = f_{2m-2}(z) (\hat{J}_n(z) - y_n \hat{J}_n^{-1}(z)), \quad (2.17)$$

where the polynomial $f_{2m-2}(z)$ will be given in the following sections. There is an important asymptotics

$$\sqrt{-\det \hat{A}_n(z)} = \frac{1}{2} V'(z) - \frac{n}{z} + O\left(\frac{1}{z^2}\right), \quad (2.18)$$

as $z \rightarrow \infty$ in the complex plane, obtained by using the string equation. This property will be applied to satisfy the conditions (2.5) and (2.6). Replacing the variable z and the parameters t_j , x_n and y_n by $n^{\frac{1}{2m}} \eta$, $n^{1-\frac{j}{m}} g_j$, $n^{\frac{1}{2m}} a$ and $n^{\frac{1}{m}} b^2$ respectively, the formula of the eigenvalue density $\rho(\eta)$ on the interval $[\eta_-, \eta_+] = [a - 2b, a + 2b]$ can be obtained by

$$\frac{1}{n\pi} \sqrt{\det \hat{A}_n(z)} dz = \rho(\eta) d\eta, \quad (2.19)$$

which follows the unified model discussed in Sect. 1.1. The eigenvalue density problem is then solved when condition (2.4) is satisfied.

The $\hat{A}_n(z)$ can be generalized to a new matrix $\hat{A}_n^{(l)}(z)$ to find the eigenvalue densities on multiple disjoint intervals. It should be noted that in some literatures the Lax pair is applied to study the asymptotics of u_n or v_n or the related functions, while the method here is to derive the density formula by referring the matrix $A_n(z)$ because $\sqrt{-\det A_n(z)}$ itself also has the same asymptotics (2.18) as $z \rightarrow \infty$. The details of the formulations and the relations to $A_n(z)$ will be discussed in the following sections.

2.2 Integrable System and String Equation

For the Hermitian matrix model with potential $W(\eta) = \sum_{j=1}^m g_j \eta^{2j}$, we have discussed in last section that the eigenvalue density $\rho(\eta)$ needs to satisfy the conditions (2.5) and (2.6). In this section, we discuss how to get an analytic function $\omega(\eta)$ with the asymptotics $\frac{1}{2}W'(\eta) - \frac{1}{\eta}$ as $\eta \rightarrow \infty$ in the complex plane, where $' = \partial/\partial\eta$.

Consider the orthogonal polynomials $p_n(z) = z^n + \dots$ on $(-\infty, \infty)$ defined by

$$\langle p_n, p_{n'} \rangle \equiv \int_{-\infty}^{\infty} p_n(z) p_{n'}(z) e^{-V(z)} dz = h_n \delta_{n,n'}, \quad (2.20)$$

where $V(z) = \sum_{j=0}^{2m} t_j z^j$, $t_{2m} > 0$. We have the following asymptotics $e^{-V(z)/2} \times p_n(z) \sim e^{-\frac{1}{2}V(z) + n \ln z}$ as $z \rightarrow \infty$. This asymptotics gives a hint that the differential equation for the orthogonal polynomials may help us to find the $\omega(\eta)$. In the following, we introduce the basic Lax pair theory to construct the coefficient matrix $A_n(z)$.

The orthogonal polynomials satisfy a recursion formula [8],

$$p_{n+1}(z) + u_n p_n(z) + v_n p_{n-1}(z) = z p_n(z). \quad (2.21)$$

By multiplying $p_{n-1}(z) e^{-V(z)}$ on both sides of this recursion formula and taking integral, we get $v_n = h_n/h_{n-1}$. This recursion formula will give the first equation in the Lax pair.

For the second equation in the pair, let us consider the differential equation in the z direction. When $n \geq 2m - 1$, express the derivative of p_n with respect to z as a linear combination of p_j 's,

$$\frac{\partial}{\partial z} p_n = a_{n,n-1} p_{n-1} + a_{n,n-2} p_{n-2} + \dots + a_{n,0} p_0, \quad (2.22)$$

where $a_{n,j}$ are independent of z . By integration by parts, there are

$$a_{n,j} h_j = \int_{-\infty}^{\infty} V'(z) p_j(z) p_n(z) e^{-V(z)} dz, \quad (' = \partial/\partial z)$$

for $j = 0, 1, \dots, n-1$, and $a_{n,j} = 0$ when $j < n - 2m + 1$ by the orthogonality. Then, by using the recursion formula, $\frac{\partial}{\partial z} p_n$ can be changed to a linear combination of p_n and p_{n-1} , but the new coefficients are dependent on z .

Denote $\Phi_n(z) = e^{-\frac{1}{2}V(z)} (p_n(z), p_{n-1}(z))^T$. By the discussions above, there are

$$\Phi_{n+1} = L_n \Phi_n, \quad (2.23)$$

where

$$L_n = \begin{pmatrix} z - u_n & -v_n \\ 1 & 0 \end{pmatrix};$$

and

$$\frac{\partial}{\partial z} \Phi_n = A_n(z) \Phi_n, \quad (2.24)$$

for a matrix $A_n(z)$. Equations (2.23) and (2.24) are called the Lax pair for the string equation. This Lax pair structure was given in [4], as well as in [3] (Part 2, Chap. 1).

The Lax pair method for the eigenvalue density starts from the construction of the matrix A_n . For $m \geq 1$ and $n \geq 2m$, consider

$$\begin{aligned} \frac{\partial}{\partial z} p_n &= a_{n,n-1} p_{n-1} + a_{n,n-2} p_{n-2} + \cdots + a_{n,n-2m+1} p_{n-2m+1}, \\ \frac{\partial}{\partial z} p_{n-1} &= a_{n-1,n-2} p_{n-2} + a_{n-1,n-3} p_{n-3} + \cdots + a_{n-1,n-2m} p_{n-2m}, \end{aligned}$$

where

$$a_{n',n'-k} h_{n'-k} = \int_{-\infty}^{\infty} V'(z) p_{n'-k} p_{n'} e^{-V(z)} dz, \quad (2.25)$$

for $n' = n$ or $n-1$, and $k = 1, 2, \dots, 2m-1$, with $V'(z) = \sum_{j=1}^{2m} j t_j z^{j-1}$. It follows that

$$\frac{\partial}{\partial z} \begin{pmatrix} p_n \\ p_{n-1} \end{pmatrix} = \sum_{k=1}^{2m-1} C_{n-k} \begin{pmatrix} p_{n-k} \\ p_{n-k-1} \end{pmatrix}, \quad (2.26)$$

where

$$C_{n-k} = \begin{pmatrix} a_{n,n-k} h_{n-k} & 0 \\ 0 & a_{n-1,n-k+1} h_{n-k+1} \end{pmatrix}, \quad (2.27)$$

for $k = 1, \dots, 2m-1$. And $P_j = p_j / h_j$ satisfy

$$\begin{pmatrix} P_j \\ P_{j-1} \end{pmatrix} = J_{j+1} \begin{pmatrix} P_{j+1} \\ P_j \end{pmatrix}, \quad J_{j+1} = \begin{pmatrix} 0 & 1 \\ -v_{j+1} & z - u_j \end{pmatrix}, \quad (2.28)$$

by using (2.23) and $v_{j+1} = h_{j+1} / h_j$. Denote $D_n = \text{diag}(h_n, h_{n-1})$. The above discussion gives

$$\frac{\partial}{\partial z} \begin{pmatrix} p_n \\ p_{n-1} \end{pmatrix} = D_n F_n D_n^{-1} \begin{pmatrix} p_n \\ p_{n-1} \end{pmatrix}, \quad (2.29)$$

where the matrix F_n is defined by

$$D_n F_n = C_{n-1} J_n + C_{n-2} J_{n-1} J_n + \cdots + C_{n-2m+1} J_{n-2m+2} J_{n-2m+3} \cdots J_n. \quad (2.30)$$

Let I be the 2×2 identity matrix. Then, there is

$$A_n = D_n F_n D_n^{-1} - \frac{1}{2} V'(z) I, \quad n \geq 2m. \quad (2.31)$$

Let Δ be the operator for the index change acting only on the polynomials $\Delta^k p_n = p_{n+k}$, where k is an integer. This is the basic idea for the index folding technique for constructing the eigenvalue density on one interval. The recursion formula (2.21) becomes $(z - u_n) p_n = (\Delta + v_n \Delta^{-1}) p_n$. In the reduced model, we will consider x_n and y_n , and $(z - x_n)^q p_{n'-k}$ will be associated to

$$(\Delta + y_n \Delta^{-1})^q p_{n'-k} = \sum_{r=0}^q \binom{q}{r} y_n^r \Delta^{q-2r} p_{n'-k} = \sum_{r=0}^q \binom{q}{r} y_n^r p_{n'-k+q-2r},$$

for $n' = n$ or $n - 1$, $k = 1, 2, \dots, 2m - 1$, and $q = 0, 1, \dots, 2m - 1$. Then by orthogonality and

$$V'(z) = \sum_{j=1}^{2m} j t_j (x_n + (z - x_n))^{j-1} = \sum_{j=1}^{2m} j t_j \sum_{q=0}^{j-1} \binom{j-1}{q} x_n^{j-q-1} (z - x_n)^q,$$

we have that $a_{n', n'-k} h_{n'-k}$ is reduced to the following according to (2.25),

$$\begin{aligned} & \sum_{j=1}^{2m} j t_j \sum_{q=0}^{j-1} \binom{j-1}{q} x_n^{j-q-1} \sum_{r=0}^{2[q/2]-\mu_q+1} \binom{q}{r} y_n^r \int_{-\infty}^{\infty} p_{n'-k+q-2r} p_{n'} e^{-V(z)} dz \\ &= \sum_{j=1}^{2m} j t_j \sum_{q=0}^{j-1} \binom{j-1}{q} x_n^{j-q-1} \sum_{r=0}^{[q/2]-\mu_q} \binom{q}{r} y_n^r h_{n'} \delta_{q-k-2r,0}, \end{aligned} \quad (2.32)$$

for $n' = n$ or $n - 1$, where $[\cdot]$ denotes the integer part,

$$\mu_q = \frac{1 + (-1)^q}{2} = \begin{cases} 1, & q \text{ is even,} \\ 0, & q \text{ is odd,} \end{cases} \quad (2.33)$$

$$q = 2[q/2] - \mu_q + 1, \quad (2.34)$$

and for $k > 0$,

$$q - k - 2r = 2([q/2] - \mu_q - r) + 1 + \mu_q - k < 0, \quad \text{if } r > [q/2] - \mu_q,$$

which implies that $\delta_{q-k-2r,0} = 0$ when $r > [q/2] - \mu_q$, that is why the upper bound of the index r for the last summation above is changed to $[q/2] - \mu_q$. Consequently, the $D_n F_n$, defined by (2.30) is reduced to

$$\begin{aligned} & \sum_{k=1}^{2m-1} \sum_{j=1}^{2m} j t_j \sum_{q=0}^{j-1} \binom{j-1}{q} x_n^{j-q-1} \sum_{r=0}^{[q/2]-\mu_q} \binom{q}{r} y_n^r \delta_{q-k-2r,0} D_n \hat{J}_n^k \\ &= \sum_{j=1}^{2m} j t_j \sum_{q=0}^{j-1} \binom{j-1}{q} x_n^{j-q-1} \sum_{r=0}^{[q/2]-\mu_q} \binom{q}{r} y_n^r D_n \hat{J}_n^{q-2r}, \end{aligned}$$

and then F_n is reduced to

$$\hat{F}_n = \sum_{j=1}^{2m} j t_j \sum_{q=0}^{j-1} \binom{j-1}{q} x_n^{j-q-1} \sum_{r=0}^{[q/2]-\mu_q} \binom{q}{r} y_n^r \hat{J}_n^{q-2r}, \quad (2.35)$$

where

$$\hat{J}_n = \begin{pmatrix} 0 & 1 \\ -y_n & z - x_n \end{pmatrix}. \quad (2.36)$$

Let

$$\hat{A}_n(z) = D_n \hat{F}_n D_n^{-1} - \frac{1}{2} V'(z) I, \quad (2.37)$$

which is the matrix we need for the eigenvalue density on one interval. We use the hat symbol such as $\hat{J}_n$ and $\hat{A}_n$ for the reduced models, to distinguish from the J_n and A_n in the integrable system. Based on the factorization and asymptotics for this matrix, we will discuss how to find the formula for the eigenvalue density in the later sections. In the following, we discuss the condition for the parameters that are reduced from the string equation.

By the orthogonality of the polynomials $p_n(z) = z^n + \dots$ and integration by parts, we have the following string equation,

$$\langle p_n(z), V'(z)p_{n-1}(z) \rangle = nh_{n-1}, \quad (2.38)$$

$$\langle p_n(z), V'(z)p_n(z) \rangle = 0, \quad (2.39)$$

including two recursion formulas for the u_n 's and v_n 's. The set of (2.38) and (2.39) is called string equation. The string equation is the consistency condition for the Lax pair (2.23) and (2.24). The consistency can be discussed, for example, by referring the methods in [4, 5]. For the density problems, we only need the equations for restricting the parameters.

If the differential equation is written in the form

$$\frac{\partial}{\partial z} p_n = a_{n,n} p_n + a_{n,n-1} p_{n-1} + \dots + a_{n,n-2m+1} p_{n-2m+1},$$

where $a_{n,n} = 0$, then the formula (2.25) is still true for $k = 0$. Let us write (2.38) and (2.39) as

$$a_{n,n-1} h_{n-1} = nh_{n-1}, \quad (2.40)$$

$$a_{n,n} h_n = 0. \quad (2.41)$$

Based on the reduction (2.32) with $n' = n$ for $k = 1$ and $k = 0$ respectively, in our method for the eigenvalue density on one interval, we need x_n and y_n to satisfy the following equations

$$\sum_{j=1}^{2m} j t_j \sum_{q=0}^{j-1} \binom{j-1}{q} x_n^{j-q-1} \sum_{r=0}^{[q/2]-\mu_q} \binom{q}{r} y_n^{r+1} \delta_{q,2r+1} = n, \quad (2.42)$$

$$\sum_{j=1}^{2m} j t_j \sum_{q=0}^{j-1} \binom{j-1}{q} x_n^{j-q-1} \sum_{r=0}^{[q/2]-\mu_q} \binom{q}{r} y_n^r \delta_{q,2r} = 0. \quad (2.43)$$

Note that $\delta_{q,2r+1} = 0$ when q is even, and $\delta_{q,2r} = 0$ when q is odd. Take $q = 2p + 1$, $r = p$ in (2.42), and $q = 2p$, $r = p$ in (2.43), we get

$$\sum_{j=2}^{2m} j t_j \sum_{p=0}^{[\frac{j}{2}]-1} \binom{j-1}{2p+1} \binom{2p+1}{p} x_n^{j-2p-2} y_n^{p+1} = n, \quad (2.44)$$

$$\sum_{j=1}^{2m} j t_j \sum_{p=0}^{[\frac{j-1}{2}]} \binom{j-1}{2p} \binom{2p}{p} x_n^{j-2p-1} y_n^p = 0. \quad (2.45)$$

These two equations will be rescaled to satisfy (2.5) and (2.6).

Specially, when $V(z)$ is even, $V(-z) = V(z)$, or $t_1 = t_3 = \dots = t_{2m-1} = 0$, there is $p_n(-z) = p_n(z)$, which implies $u_n = 0$, and it follows that $x_n = 0$. Then (2.45) becomes $0 = 0$, because the terms on the left hand side of (2.45) has either a factor x_n or t_j with odd j . And (2.44) becomes

$$\sum_{j=1}^m 2j t_{2j} \binom{2j-1}{j} y_n^j = n, \quad (2.46)$$

by replacing j by $2j$ and taking $p = j - 1$ on the left hand side of (2.44).

For the density on multiple disjoint intervals, consider

$$\hat{J}_n^{(l)} = \begin{pmatrix} 0 & 1 \\ -y_n^{(1)} & z - x_n^{(1)} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -y_n^{(l)} & z - x_n^{(l)} \end{pmatrix}. \quad (2.47)$$

According to the Cayley-Hamilton theorem for $\hat{J}_n^{(l)}$, there is the following trace formula

$$(\text{tr } \hat{J}_n^{(l)})I = \hat{J}_n^{(l)} + (\det \hat{J}_n^{(l)})\hat{J}_n^{(l)-1}. \quad (2.48)$$

We can transform t_j ($j = 1, \dots, 2m$) into a new set of parameters t'_j ($j = 1, \dots, 2m$) by a linear transformation, such that

$$V'(z) = \sum_{s=0}^{l-1} z^s \sum_{q=0}^{m_s} t'_{lq+s+1} (\text{tr } \hat{J}_n^{(l)})^q, \quad (2.49)$$

where each m_s ($s = 0, \dots, l-1$) is the largest integer such that $s + lm_s \leq 2m-1$. In fact, by expanding the above expression in terms of z and comparing the coefficients with $V'(z) = \sum_{j=1}^{2m} j t_j z^{j-1}$, we can get a upper triangle matrix T_{2m} so that $T_{2m} \mathbf{t}' = \mathbf{t}$ where $\mathbf{t} = (t_1, 2t_2, \dots, 2mt_{2m})^T$ and $\mathbf{t}' = (t'_1, t'_2, \dots, t'_{2m})^T$. The derivative $\partial p_n / \partial z$ is now expanded as

$$\frac{\partial p_n}{\partial z} = \sum_{s=0}^{l-1} \sum_{q'=1}^{N_0} a_{n,n-lq'+s}^{(l)} z^s p_{n-lq'}(z) + \sum_{k=lN_0+1}^n a_{n,n-k}^{(l)} p_{n-k}(z), \quad (2.50)$$

where $n - lN_0 < l$ and the choice of N_0 is dependent on the value of m . This is the idea of the index folding technique for constructing the eigenvalue density on multiple disjoint intervals, so called because of the folding term lq' in the index above, which is one of the folding techniques in this subject area. As discussed in Sect. 1.3, the index change is referred as a role of the pressure. The index folding or periodic reduction reflects an even pressure property or multiple even pressure layers in the system that is often assumed in studying the application problems.

By the index change operator Δ , the coefficient $a_{n,n-lq'+s}^{(l)}$ is reduced to

$$\begin{aligned} & \sum_{q=1}^{m_s} t'_{lq+s} \int_{-\infty}^{\infty} p_{n-lq'+s} z^s (\Delta^l + (\det \hat{J}_n^{(l)}) \Delta^{-l})^q p_n e^{-V(z)} dz \\ &= \sum_{q=1}^{m_s} t'_{lq+s} \sum_{r=0}^{[q/2]-\mu_q} \binom{q}{r} (\det \hat{J}_n^{(l)})^{q-r} \delta_{q-q'-2r,0}, \quad q' \leq m_s. \end{aligned} \quad (2.51)$$

Also, $a_{n,n-lq'+s}^{(l)}$ for $q' > m_s$ and $a_{n,n-k}^{(l)}$ for $k = lN_0 + 1, \dots, n$ are reduced to 0 according to the term $\delta_{q-q'-2r,0}$ above. Then we get another reduced matrix

$$\hat{A}_n^{(l)}(z) = D_n \hat{F}_n^{(l)} D_n^{-1} - \frac{1}{2} V'(z) I, \quad (2.52)$$

where

$$\hat{F}_n^{(l)} = \sum_{s=0}^{l-1} z^s \sum_{q=1}^{m_s} t'_{lq+s+1} \sum_{r=0}^{[q/2]-\mu_q} \binom{q}{r} (\det \hat{J}_n^{(l)})^r (\hat{J}_n^{(l)})^{q-2r}, \quad (2.53)$$

by referring that $(p_{n-lq'}, p_{n-lq'-1})^T$ is connected to

$$D_n (\det \hat{J}_n^{(l)})^{-q'} (\hat{J}_n^{(l)})^{q'} D_n^{-1} (p_n, p_{n-1})^T.$$

The matrix $\hat{A}_n^{(l)}(z)$ will be applied to derive the formula of the density on multiple disjoint intervals to be discussed in Sect. 2.4. The restriction conditions for the parameters are similar to the one-interval case and will be given in Sect. 2.4.

The matrix $\hat{A}_n^{(l)}(z)$ is obtained from A_n by replacing the $u_{n-lq+s-1}$ and v_{n-lq+s} by $x_n^{(s)}$ and $y_n^{(s)}$ respectively. One may ask whether the u_N and v_N functions must have such periodic behaviors. The explanation is that the string equations are applied to reorganize the wave functions, not the particles. In the momentum aspect, the parameters and the corresponding functions such as u_N and v_N control the wave functions of the random variables, so that the asymptotics of these functions are not directly related to the behaviors of the particles. If there is an asymptotic relation, it should be a “relative” asymptotics. These functions are closely connected to the moments of the eigenvalues. Each reduction from the integrable system is not a necessity, but a case of the probability. And the occurrence of each possible case is not based on certainty principle, but the uncertainty principle. The accumulations of the sequences and the distribution of the possibilities would give better explanations for the reduction method.

2.3 Factorization and Asymptotics

We have obtained in last section that $D_n^{-1} \hat{A}_n D_n = \hat{F}_n - \frac{1}{2} V'(z) I$, where $D_n = \text{diag}(h_n, h_{n-1})$. In this section, we are going to show that the matrix $\hat{F}_n - \frac{1}{2} V'(z) I$ can be factorized as a product of a polynomial in z of degree $2m - 2$ and the matrix $\hat{J}_n - y_n \hat{J}_n^{-1}$. The equation $\det(\hat{J}_n - y_n \hat{J}_n^{-1}) = 0$ has only two simply zeros which will be used as the bounds of the eigenvalue domain. In the following, we will denote $(\hat{J}_n^{-1})^k$ by $\hat{J}_n^{-k}$ where k is an integer.

Lemma 2.1 *If x_n , y_n and t_j ($j = 1, \dots, 2m$) satisfy (2.45), then for $\hat{A}_n(z)$ defined by (2.37) and $\mu_q = (1 + (-1)^q)/2$, there is*

$$\begin{aligned}
D_n^{-1} \hat{A}_n D_n &= \frac{1}{2} \sum_{j=1}^{2m} j t_j \sum_{q=0}^{j-1} \binom{j-1}{q} x_n^{j-q-1} \\
&\quad \times \sum_{r=0}^{[q/2]-\mu_q} \binom{q}{r} y_n^r (\hat{J}_n^{q-2r} - (y_n \hat{J}_n^{-1})^{q-2r}). \quad (2.54)
\end{aligned}$$

Proof Recall the matrix $\hat{J}_n$ defined by (2.36),

$$\hat{J}_n = \begin{pmatrix} 0 & 1 \\ -y_n & z - x_n \end{pmatrix}.$$

Applying the Cayley-Hamilton theorem for $\hat{J}_n$, there is $\hat{J}_n^2 - (z - x_n)\hat{J}_n + y_n I = 0$, which implies

$$(z - x_n)I = \hat{J}_n + y_n \hat{J}_n^{-1}. \quad (2.55)$$

Then by binomial expansion and $q = 2[q/2] - \mu_q + 1$, we have

$$\begin{aligned}
&(z - x_n)^q I \\
&= \left(\sum_{r=0}^{[q/2]-\mu_q} + \mu_q \sum_{r=[q/2]}^{[q/2]} + \sum_{r=[q/2]+1}^{2[q/2]-\mu_q+1} \right) \binom{q}{r} y_n^r \hat{J}_n^{q-2r} \\
&= \sum_{r=0}^{[q/2]-\mu_q} \binom{q}{r} y_n^r \hat{J}_n^{q-2r} + \mu_q \binom{q}{[q/2]} y_n^{[q/2]} \hat{J}_n^{q-2[q/2]} \\
&\quad + \sum_{s=0}^{[q/2]-\mu_q} \binom{q}{s} y_n^{q-s} \hat{J}_n^{-q+2s} \\
&= \sum_{r=0}^{[q/2]-\mu_q} \binom{q}{r} y_n^r (\hat{J}_n^{q-2r} + (y_n \hat{J}_n^{-1})^{q-2r}) + \mu_q \binom{q}{[q/2]} y_n^{[q/2]} \hat{J}_n^{q-2[q/2]},
\end{aligned}$$

where s comes out by the substitution $s = q - r$, and then replaced by r in the last step. Since

$$V'(z) = \sum_{j=1}^{2m} j t_j \sum_{q=0}^{j-1} \binom{j-1}{q} x_n^{j-q-1} (z - x_n)^q,$$

$V'(z)I$ can be expressed as a linear combination of the positive and negative powers of $\hat{J}_n$.

By $D_n^{-1} \hat{A}_n D_n = \hat{F}_n - \frac{1}{2} V'(z)I$ given by (2.37) and (2.35), we then have

$$D_n^{-1} \hat{A}_n D_n = \frac{1}{2} \sum_{j=1}^{2m} j t_j \sum_{q=0}^{j-1} \binom{j-1}{q} x_n^{j-q-1}$$

$$\begin{aligned}
& \times \sum_{r=0}^{\lfloor q/2 \rfloor - \mu_q} \binom{q}{r} y_n^r (\hat{J}_n^{q-2r} - (y_n \hat{J}_n^{-1})^{q-2r}) \\
& = -\frac{1}{2} \sum_{j=1}^{2m} j t_j \sum_{q=0}^{j-1} \binom{j-1}{q} x_n^{j-q-1} \mu_q \binom{q}{\lfloor q/2 \rfloor} y_n^{\lfloor q/2 \rfloor}.
\end{aligned}$$

Since $\mu_q = 1$ when q is even, and $\mu_q = 0$ when q is odd, the last part in the above vanishes by taking $q = 2p$ and applying (2.45). So we get the result in this lemma. $\square$

Let

$$\alpha_n = \frac{z - x_n + \sqrt{(z - x_n)^2 - 4y_n}}{2}, \quad (2.56)$$

which satisfies

$$\alpha_n + y_n \alpha_n^{-1} = z - x_n, \quad (2.57)$$

$$\alpha_n - y_n \alpha_n^{-1} = \sqrt{(z - x_n)^2 - 4y_n}. \quad (2.58)$$

And it is easy to check that

$$y_n \hat{J}_n^{-1} = \begin{pmatrix} z - x_n & -1 \\ y_n & 0 \end{pmatrix}.$$

We will need

$$\hat{J}_n - y_n \hat{J}_n^{-1} = \begin{pmatrix} -z + x_n & 2 \\ -2y_n & z - x_n \end{pmatrix},$$

which satisfies

$$\sqrt{-\det(\hat{J}_n - y_n \hat{J}_n^{-1})} = \sqrt{(z - x_n)^2 - 4y_n} = \alpha_n - y_n \alpha_n^{-1}. \quad (2.59)$$

Lemma 2.2 For the $\hat{J}_n$ defined by (2.36) and $k = 1, 2, \dots$, there are

$$\hat{J}_n^k + y_n^k \hat{J}_n^{-k} = (\alpha_n^k + y_n^k \alpha_n^{-k}) I, \quad (2.60)$$

and

$$\hat{J}_n^k - y_n^k \hat{J}_n^{-k} = \frac{\alpha_n^k - y_n^k \alpha_n^{-k}}{\alpha_n - y_n \alpha_n^{-1}} (\hat{J}_n - y_n \hat{J}_n^{-1}). \quad (2.61)$$

Proof By the relations $\hat{J}_n + y_n \hat{J}_n^{-1} = (z - x_n) I$, and $\alpha_n + y_n \alpha_n^{-1} = z - x_n$, we have

$$\hat{J}_n + y_n \hat{J}_n^{-1} = (\alpha_n + y_n \alpha_n^{-1}) I, \quad (2.62)$$

which is (2.60) for $k = 1$. Taking square on both sides of (2.62), we get

$$\hat{J}_n^2 + y_n^2 \hat{J}_n^{-2} = (\alpha_n^2 + y_n^2 \alpha_n^{-2}) I,$$

which is (2.60) for $k = 2$.

Now, by mathematical induction, suppose (2.60) is true for $k - 1$ and k , let us show it is also true for $k + 1$. Multiplying (2.60) with (2.62), we get

$$\begin{aligned} & \hat{J}_n^{k+1} + y_n^{k+1} \hat{J}_n^{-k-1} + y_n (\hat{J}_n^{k-1} + y_n^{k-1} \hat{J}_n^{-k+1}) \\ &= (\alpha_n^{k+1} + y_n^{k+1} \alpha_n^{-k-1}) I + y_n (\alpha_n^{k-1} + y_n^{k-1} \alpha_n^{-k+1}) I. \end{aligned}$$

By the assumption, we see that (2.60) is true for $k + 1$.

Equation (2.61) can also be proved by using mathematical induction. It is easy to check that

$$\hat{J}_n^2 - y_n^2 \hat{J}_n^{-2} = (\hat{J}_n + y_n \hat{J}_n^{-1})(\hat{J}_n - y_n \hat{J}_n^{-1}) = (z - x_n)(\hat{J}_n - y_n \hat{J}_n^{-1}),$$

and then (2.61) is true for $k = 1$ and 2 . Suppose (2.61) is true for $k - 1$ and k . We show it is true for $k + 1$. Multiplying (2.61) with (2.62), we have

$$\begin{aligned} & \hat{J}_n^{k+1} - y_n^{k+1} \hat{J}_n^{-k-1} + y_n (\hat{J}_n^{k-1} - y_n^{k-1} \hat{J}_n^{-k+1}) \\ &= \frac{\alpha_n^{k+1} - y_n^{k+1} \alpha_n^{-k-1}}{\alpha_n - y_n \alpha_n^{-1}} (\hat{J}_n - y_n \hat{J}_n^{-1}) + y_n \frac{\alpha_n^{k-1} - y_n^{k-1} \alpha_n^{-k+1}}{\alpha_n - y_n \alpha_n^{-1}} (\hat{J}_n - y_n \hat{J}_n^{-1}). \end{aligned}$$

By the assumption, we have that (2.61) is true for $k + 1$. $\square$

Lemma 2.3 For the α_n defined by (2.56) and $k = 1, 2, \dots$, there are

$$\alpha_n^k + y_n^k \alpha_n^{-k} = \frac{1}{2^{k-1}} \sum_{s=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2s} (z - x_n)^{k-2s} ((z - x_n)^2 - 4y_n)^s, \quad (2.63)$$

and

$$\frac{\alpha_n^k - y_n^k \alpha_n^{-k}}{\alpha_n - y_n \alpha_n^{-1}} = \frac{1}{2^{k-1}} \sum_{s=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{2s+1} (z - x_n)^{k-2s-1} ((z - x_n)^2 - 4y_n)^s. \quad (2.64)$$

Proof By (2.57) and (2.58), we have

$$\begin{aligned} \alpha_n &= \frac{1}{2} (z - x_n + ((z - x_n)^2 - 4y_n)^{1/2}), \\ y_n \alpha_n^{-1} &= \frac{1}{2} (z - x_n - ((z - x_n)^2 - 4y_n)^{1/2}). \end{aligned}$$

Then the binomial formula implies

$$\begin{aligned} & \alpha_n^k + y_n^k \alpha_n^{-k} \\ &= \frac{1}{2^k} \sum_{j=0}^k \binom{k}{j} (z - x_n)^{k-j} (((z - x_n)^2 - 4y_n)^{\frac{j}{2}} + (-1)^j ((z - x_n)^2 - 4y_n)^{\frac{j}{2}}) \\ &= \frac{1}{2^{k-1}} \sum_{s=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2s} (z - x_n)^{k-2s} ((z - x_n)^2 - 4y_n)^s, \end{aligned}$$

where the terms with odd j are canceled, and the terms with even j are combined by taking $j = 2s$. So (2.63) is obtained.

Similarly, there is

$$\begin{aligned} & \alpha_n^k - y_n^k \alpha_n^{-k} \\ &= \frac{1}{2^k} \sum_{j=0}^k \binom{k}{j} (z - x_n)^{k-j} \left(((z - x_n)^2 - 4y_n)^{\frac{j}{2}} - (-1)^j ((z - x_n)^2 - 4y_n)^{\frac{j}{2}} \right) \\ &= \frac{1}{2^{k-1}} \sum_{s=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{2s+1} (z - x_n)^{k-2s-1} ((z - x_n)^2 - 4y_n)^{s+\frac{1}{2}}, \end{aligned}$$

where the terms with even j are canceled, and the terms with odd j are combined by taking $j = 2s + 1$. So the lemma is proved. $\square$

Now, let

$$f_{2m-2}(z) = \frac{1}{2} \sum_{j=1}^{2m} j t_j \sum_{q=0}^{j-1} \binom{j-1}{q} x_n^{j-q-1} \sum_{r=0}^{\lfloor q/2 \rfloor - \mu_q} \binom{q}{r} \frac{y_n^r}{2^{q-2r-1}} f^{(q,r)}(z), \quad (2.65)$$

where

$$f^{(q,r)}(z) = \sum_{s=0}^{\lfloor \frac{q-2r-1}{2} \rfloor} \binom{q-2r}{2s+1} (z - x_n)^{q-2r-2s-1} ((z - x_n)^2 - 4y_n)^s. \quad (2.66)$$

Theorem 2.1 *If the x_n , y_n and t_j ($j = 1, \dots, 2m$) satisfy (2.45), then for any $z \in \mathbb{C}$, there is*

$$D_n^{-1} \hat{A}_n(z) D_n = f_{2m-2}(z) (\hat{J}_n(z) - y_n \hat{J}_n^{-1}(z)), \quad (2.67)$$

where $\hat{A}_n(z)$ is defined by (2.37), $f_{2m-2}(z)$ is a polynomial of degree $2m - 2$ given by (2.65), and $\hat{J}_n(z)$ is given by (2.36).

Proof By Lemma 2.1 and (2.61) in Lemma 2.2, $D_n^{-1} \hat{A}_n D_n$ is equal to

$$\begin{aligned} & \frac{1}{2} \sum_{j=1}^{2m} j t_j \sum_{q=0}^{j-1} \binom{j-1}{q} x_n^{j-q-1} \\ & \times \sum_{r=0}^{\lfloor q/2 \rfloor - \mu_q} \binom{q}{r} y_n^r \frac{\alpha_n^{q-2r} - (y_n \alpha_n^{-1})^{q-2r}}{\alpha_n - y_n \alpha_n^{-1}} (\hat{J}_n - y_n \hat{J}_n^{-1}). \end{aligned}$$

Applying (2.54) for $k = q - 2r$ in Lemma 2.3 to the above, we then have the result. $\square$

The next goal is to study the asymptotics of $(-\det(\hat{A}_n))^{1/2}$ as $z \rightarrow \infty$ in the complex plane. The asymptotics comes out based on (2.44) and (2.45) which are reduced from the string equation.

Theorem 2.2 *If the x_n , y_n and $t_j (j = 1, \dots, 2m)$ satisfy (2.44) and (2.45), then as $z \rightarrow \infty$ in the complex plane, there is the asymptotics*

$$\sqrt{-\det \hat{A}_n(z)} = \frac{1}{2} V'(z) - \frac{n}{z} + O\left(\frac{1}{z^2}\right), \quad (2.68)$$

where $V(z) = \sum_{j=0}^{2m} t_j z^j$, $t_{2m} > 0$ and $' = \partial/\partial z$.

Proof By (2.35), (2.36) and (2.37), there is $D_n^{-1} \hat{A}_n(z) D_n \sim m t_{2m} \text{diag}(-z^{2m-1}, z^{2m-1})$ as $z \rightarrow \infty$. Since $t_{2m} > 0$, the branch of the square root is determined by $(-\det \hat{A}_n(z))^{1/2} \sim m t_{2m} z^{2m-1}$ as $z \rightarrow +\infty$ on the real line.

If we take $k = q - 2r$ in (2.61), then

$$\hat{J}_n^{q-2r} - (y_n \hat{J}_n^{-1})^{q-2r} = \frac{\alpha_n^{q-2r} - (y_n \alpha_n^{-1})^{q-2r}}{\alpha_n - y_n \alpha_n^{-1}} (\hat{J}_n - y_n \hat{J}_n^{-1}).$$

Since $\sqrt{-\det(\hat{J}_n - y_n \hat{J}_n^{-1})} = \alpha_n - y_n \alpha_n^{-1}$, the formula (2.61) implies

$$\sqrt{-\det(\hat{J}_n^{q-2r} - (y_n \hat{J}_n^{-1})^{q-2r})} = \alpha_n^{q-2r} - (y_n \alpha_n^{-1})^{q-2r}.$$

By (2.54), it follows that

$$\begin{aligned} \sqrt{-\det \hat{A}_n} &= \frac{1}{2} \sum_{j=1}^{2m} j t_j \sum_{q=0}^{j-1} \binom{j-1}{q} x_n^{j-q-1} \\ &\quad \times \sum_{r=0}^{\lfloor q/2 \rfloor - \mu_q} \binom{q}{r} y_n^r (\alpha_n^{q-2r} - (y_n \alpha_n^{-1})^{q-2r}). \end{aligned} \quad (2.69)$$

Here, when $q = 0$, we denote $\sum_{r=0}^{-1} \cdot = 0$ for convenience in the discussions. Let $s = q - r = (\lfloor q/2 \rfloor - \mu_q - r) + \lfloor q/2 \rfloor + 1$ for the terms $(y_n \alpha_n^{-1})^{q-2r}$ above. We arrive

$$\begin{aligned} \sqrt{-\det \hat{A}_n} &= \frac{1}{2} \sum_{j=1}^{2m} j t_j \sum_{q=0}^{j-1} \binom{j-1}{q} x_n^{j-q-1} \left[\sum_{r=0}^{\lfloor q/2 \rfloor - \mu_q} \binom{q}{r} \alpha_n^{q-r} (y_n \alpha_n^{-1})^r \right. \\ &\quad \left. - \sum_{s=\lfloor q/2 \rfloor + 1}^q \binom{q}{s} \alpha_n^{q-s} (y_n \alpha_n^{-1})^s \right]. \end{aligned}$$

Furthermore, by the binomial formula we have

$$\begin{aligned} \sqrt{-\det \hat{A}_n} &= \frac{1}{2} \sum_{j=1}^{2m} j t_j \sum_{q=0}^{j-1} \binom{j-1}{q} x_n^{j-q-1} \left[(\alpha_n + y_n \alpha_n^{-1})^q \right. \\ &\quad \left. - \mu_q \binom{q}{\lfloor q/2 \rfloor} \alpha_n^{q-\lfloor q/2 \rfloor} (y_n \alpha_n^{-1})^{\lfloor q/2 \rfloor} \right. \\ &\quad \left. - 2 \sum_{s=\lfloor q/2 \rfloor + 1}^q \binom{q}{s} y_n^s \alpha_n^{-(2s-q)} \right]. \end{aligned}$$

For the first part in the bracket, since $\alpha_n + y_n \alpha_n^{-1} = z - x_n$, it is easy to check that

$$\frac{1}{2} \sum_{j=1}^{2m} j t_j \sum_{q=0}^{j-1} \binom{j-1}{q} x_n^{j-q-1} (\alpha_n + y_n \alpha_n^{-1})^q = \frac{1}{2} \sum_{j=1}^{2m} j t_j z^{j-1} = \frac{1}{2} V'(z).$$

The second part in the bracket can be dropped off by considering the outside summations and using (2.45). For $s = [q/2] + 1$ in the third part in the bracket, we have the following by separating the odd q and even q terms and noticing that q starts from $q = 1$,

$$\begin{aligned} & \sum_{j=1}^{2m} j t_j \sum_{q=1}^{j-1} \binom{j-1}{q} x_n^{j-q-1} \binom{q}{[q/2]+1} y_n^{[q/2]+1} \alpha_n^{q-2[q/2]-2} \\ &= \alpha_n^{-1} \sum_{j=1}^{2m} j t_j \sum_{p=0}^{[\frac{j}{2}]-1} \binom{j-1}{2p+1} \binom{2p+1}{p} x_n^{j-2p-2} y_n^{p+1} \\ &+ \alpha_n^{-2} \sum_{j=1}^{2m} j t_j \sum_{p=1}^{[\frac{j-1}{2}]} \binom{j-1}{2p} \binom{2p}{p+1} x_n^{j-2p-1} y_n^{p+1}, \end{aligned}$$

where $q = 2p + 1$ when q is odd, and $q = 2p$ when q is even. As $z \rightarrow \infty$, by (2.57) and (2.58), we have

$$\alpha_n^{-1} = \frac{z - x_n - (z - x_n)(1 - \frac{4y_n}{(z-x_n)^2})^{1/2}}{2y_n} = \frac{1}{z - x_n} + O\left(\frac{1}{(z - x_n)^2}\right).$$

Then, combining the discussions above, we obtain

$$\sqrt{-\det(\hat{A}_n)} = \frac{1}{2} \sum_{j=1}^{2m} j t_j z^{j-1} - \frac{n}{z} + O\left(\frac{1}{z^2}\right),$$

by using (2.44), and the theorem is proved. $\square$

In the following, we show that $(-\det A_n(z))^{1/2}$ has similar asymptotics as discussed for $(-\det \hat{A}_n(z))^{1/2}$ as $z \rightarrow \infty$ [6]. Since the restriction conditions for A_n and $\hat{A}_n$ are different in the asymptotics, separate proofs are needed. The Cauchy kernel discussed in [4] is applied in the following proof.

Theorem 2.3 *For A_n defined by (2.31) with $n \geq 2m$, as $z \rightarrow \infty$, there is*

$$\sqrt{-\det A_n(z)} = \frac{1}{2} V'(z) - \frac{n}{z} + O\left(\frac{1}{z^2}\right), \quad (2.70)$$

when the parameters satisfy (2.40).

Proof Denote

$$\hat{p}_n(z) = \int_{-\infty}^{\infty} p_n(\zeta) \frac{e^{-V(\zeta)}}{\zeta - z} d\zeta, \quad \text{and} \quad \Psi_n = \begin{pmatrix} p_n & \hat{p}_n \\ p_{n-1} & \hat{p}_{n-1} \end{pmatrix} e^{-\frac{1}{2}\sigma_3 V(z)}, \quad (2.71)$$

where $\sigma_3 = \text{diag}(1, -1)$. It is not hard to see that $V'(z)$ and $F_n(z)$ are both of degree $2m - 1$ in z . Since $n \geq 2m$, by the orthogonality there is

$$\int_{-\infty}^{\infty} [D_n(F(\zeta) - F_n(z))D_n^{-1} - (V'(\zeta) - V'(z))] \begin{pmatrix} p_n(\zeta) \\ p_{n-1}(\zeta) \end{pmatrix} \frac{e^{-V(\zeta)}}{\zeta - z} d\zeta = 0.$$

Then it can be verified that

$$\frac{\partial}{\partial z} \Psi_n = D_n F_n D_n^{-1} \Psi_n - \frac{1}{2} V'(z) \Psi_n, \quad (2.72)$$

that means Ψ_n is a matrix solution for $\frac{\partial}{\partial z} \Psi_n = A_n \Psi_n$ when $n \geq 2m$. The orthogonality of the polynomials also implies

$$\det \Psi_n = \int_{-\infty}^{\infty} (p_n(z)p_{n-1}(\zeta) - p_n(\zeta)p_{n-1}(z)) \frac{e^{-V(\zeta)}}{\zeta - z} d\zeta = -h_{n-1}.$$

Then there is $\text{tr}(\Psi_n' \Psi_n^{-1}) = (\ln \det \Psi_n)' = 0$ by using the Liouville's formula [2] and $\det \Psi_n = -h_{n-1}$, where $' = \partial/\partial z$. Multiplying Ψ_n^{-1} on both sides of the equation (2.72) and taking trace, we get the following,

$$\text{tr} F_n(z) = V'(z), \quad (2.73)$$

that implies $-\det A_n(z) = \frac{1}{4}(V'(z))^2 - \det F_n(z)$.

According to (2.30), there is

$$\begin{aligned} D_n F_n &= [C_{n-1} J_{n-1}^{-1} \cdots J_{n-m+1}^{-1} + \cdots \\ &\quad + C_{n-2m-1} J_{n-2m+2} \cdots J_{n-m}] J_{n-m+1} \cdots J_{n-1} J_n. \end{aligned}$$

Considering the leading terms as $z \rightarrow \infty$, we have

$$\begin{aligned} D_n F_n &= [\det(J_{n-1} \cdots J_{n-m+1})^{-1} z^{m-1} \text{diag}(a_{n,n-1} h_{n-1}, 0) + \cdots \\ &\quad + z^{m-1} \text{diag}(0, a_{n-1,n-2m} h_{n-2m})] J_{n-m+1} \cdots J_{n-1} J_n. \end{aligned}$$

It can be calculated by using (2.25) that $a_{n-1,n-2m} h_{n-2m} = 2mt_{2m} h_{n-1}$. Since $\det D_n = h_n h_{n-1}$ and $v_n = h_n/h_{n-1}$, there is $\det F_n = 2mt_{2m} a_{n,n-1} z^{2m-2} (1 + O(z^{-1}))$. By (2.40), we have

$$\det F_n(z) = 2mnt_{2m} z^{2m-2} (1 + O(z^{-1})). \quad (2.74)$$

Then (2.70) is proved. $\square$

2.4 Density Models

For the density on one interval, denote $z/n^{\frac{1}{2m}}$, $t_j/n^{1-\frac{j}{2m}}$, $x_n/n^{\frac{1}{2m}}$, and $y_n/n^{\frac{1}{m}}$ by η , g_j , a , and b^2 respectively, where $b > 0$. The a and b will be called center and radius parameters respectively in the later discussions. Let $\alpha_n = n^{\frac{1}{2m}} \alpha$, and then $y_n \alpha_n^{-1} = n^{\frac{1}{2m}} (b^2 \alpha^{-1})$, where $\alpha = (\eta - a + \sqrt{(\eta - a)^2 - 4b^2})/2$, and

$b^2\alpha^{-1} = (\eta - a - \sqrt{(\eta - a)^2 - 4b^2})/2$. By Theorem 2.1, it follows that for $z \in \mathbb{C} \setminus [x_n - 2\sqrt{y_n}, x_n + 2\sqrt{y_n}]$,

$$\sqrt{-\det \hat{A}_n(z)} = n^{1-\frac{1}{2m}} k_{2m-2}(\eta) \sqrt{(\eta - a)^2 - 4b^2}, \quad \eta \in \mathbb{C} \setminus [a - 2b, a + 2b], \quad (2.75)$$

where

$$k_{2m-2}(\eta) = \sum_{j=1}^{2m} j g_j \sum_{q=0}^{j-1} \binom{j-1}{q} a^{j-q-1} \sum_{r=0}^{\lfloor \frac{q}{2} \rfloor - \mu_q} \binom{q}{r} \frac{b^{2r}}{2^{q-2r}} k^{(q,r)}(\eta), \quad (2.76)$$

and

$$k^{(q,r)}(\eta) = \sum_{s=0}^{\lfloor \frac{q-2r-1}{2} \rfloor} \binom{q-2r}{2s+1} (\eta - a)^{q-2r-2s-1} ((\eta - a)^2 - 4b^2)^s, \quad (2.77)$$

where $\binom{m}{n} = m!/(n!(m-n)!)$, $\mu_q = (1 + (-1)^q)/2$, and $[\cdot]$ stands for the integer part.

Define an analytic function [6]

$$\omega(\eta) = k_{2m-2}(\eta) \sqrt{(\eta - a)^2 - 4b^2}, \quad \eta \in \mathbb{C} \setminus [a - 2b, a + 2b]. \quad (2.78)$$

The parameters a , b and g_j ($j = 1, \dots, 2m$) are required to satisfy the conditions:

(i) When $\eta \in [\eta_-, \eta_+]$,

$$k_{2m-2}(\eta) \geq 0; \quad (2.79)$$

$$(ii) \quad \sum_{j=2}^{2m} j g_j \sum_{p=0}^{\lfloor \frac{j}{2} \rfloor - 1} \binom{j-1}{2p+1} \binom{2p+1}{p} a^{j-2p-2} b^{2p+2} = 1; \quad (2.80)$$

$$(iii) \quad \sum_{j=1}^{2m} j g_j \sum_{p=0}^{\lfloor \frac{j-1}{2} \rfloor} \binom{j-1}{2p} \binom{2p}{p} a^{j-2p-1} b^{2p} = 0. \quad (2.81)$$

By Theorem 2.2, if a , b , and g_j ($j = 1, \dots, 2m$) satisfy (2.80) and (2.81), then for $\eta \in \mathbb{C} \setminus [a - 2b, a + 2b]$ there is

$$\omega(\eta) = \frac{1}{2} \sum_{j=1}^{2m} j g_j \sum_{q=0}^{j-1} \binom{j-1}{q} a^{j-q-1} \sum_{r=0}^{\lfloor q/2 \rfloor - \mu_q} \binom{q}{r} b^{2r} (\alpha^{q-2r} - (b^2 \alpha^{-1})^{q-2r}). \quad (2.82)$$

As $\eta \rightarrow \infty$, there is

$$\omega(\eta) = \frac{1}{2} W'(\eta) - \frac{1}{\eta} + O\left(\frac{1}{\eta^2}\right). \quad (2.83)$$

In (2.82), the index j actually starts from $j = 2$, and index q starts from 1. We keep this form just for convenience in the later discussion for free energy when we use (2.81) where j is from $j = 1$ and p is from $p = 0$. Let

$$\rho(\eta) = \frac{1}{\pi} k_{2m-2}(\eta) \sqrt{(\eta_+ - \eta)(\eta - \eta_-)}, \quad \eta \in [\eta_-, \eta_+], \quad (2.84)$$

where $\eta_- = a - 2b$, $\eta_+ = a + 2b$, $b > 0$, and $k_{2m-2}(\eta)$ is given by (2.76). By (2.78) and (2.84), there is

$$\omega(\eta)|_{[\eta_-, \eta_+]^\pm} = \pm \pi i \rho(\eta)|_{[\eta_-, \eta_+]}, \quad (2.85)$$

where $[\eta_-, \eta_+]^+$ and $[\eta_-, \eta_+]^-$ stand for the upper and lower edges of the interval $[\eta_-, \eta_+]$ respectively. Since $\rho(\eta)$ is non-negative, we also need

$$k_{2m-2}(\eta) \geq 0, \quad (2.86)$$

for $\eta \in [\eta_-, \eta_+]$.

For the density on multiple disjoint intervals, consider

$$J^{(l)} = \begin{pmatrix} 0 & 1 \\ -b_1^2 & \eta - a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -b_l^2 & \eta - a_l \end{pmatrix}, \quad (2.87)$$

where $l \geq 1$. According to the Cayley-Hamilton theorem for $J^{(l)}$, choose $\alpha^{(l)} = (\Lambda + \sqrt{\Lambda^2 - 4b^{(l)2}})/2$, where $\Lambda = \Lambda(\eta) = \text{tr } J^{(l)}$, $b^{(l)2} = \det J^{(l)}$ and $b^{(l)} > 0$. We can transform g_j ($j = 1, \dots, 2m$) into a new set of parameters g'_j ($j = 1, \dots, 2m$) by a linear transformation so that $W'(\eta) = \sum_{s=0}^{l-1} \eta^s \sum_{q=0}^{m_s} g'_{lq+s+1} \Lambda^q$, where each m_s ($s = 0, \dots, l-1$) is the largest integer such that $s + lm_s \leq 2m - 1$.

Define another analytic function [6]

$$\omega_l(\eta) = \frac{1}{2} \sum_{s=0}^{l-1} \eta^s \sum_{q=1}^{m_s} g'_{lq+s+1} \sum_{r=0}^{[q/2]-\mu_q} \binom{q}{r} b^{(l)2r} (\alpha^{(l)q-2r} - (b^{(l)2} \alpha^{(l)-1})^{q-2r}), \quad (2.88)$$

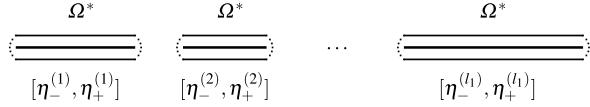
for η in the outside of the cuts to be discussed in the following. Then there is $\omega_l(\eta) = \frac{1}{2} W'(\eta) + y(\eta)$, where $-y(\eta)$ is equal to

$$\begin{aligned} & \sum_{s=0}^{l-1} \eta^s \sum_{q=0}^{m_s} g'_{lq+s+1} \left[\frac{\mu_q}{2} \binom{q}{[q/2]} b^{(l)2[q/2]} \alpha^{(l)q-2[q/2]} \right. \\ & \quad \left. + \sum_{r=[q/2]+1}^q \binom{q}{r} b^{(l)2r} \alpha^{(l)q-2r} \right]. \end{aligned} \quad (2.89)$$

It is the same argument as discussed for $\omega(\eta)$ that if the parameters satisfy the conditions

$$\sum_{p=0}^{[\frac{m_l-1}{2}]} g'_{2lp+2l} \binom{2p+1}{p} b^{(l)2p+2} = 1, \quad (2.90)$$

Fig. 2.1 Multiple cuts and their upper and lower edges



$$\sum_{p=0}^{\lfloor \frac{m_s}{2} \rfloor} g'_{2l_p+s+1} \binom{2p}{p} b^{(l)2p} = 0, \quad (2.91)$$

for $s = 0, 1, \dots, l-1$, then

$$\omega_l(\eta) = \frac{1}{2} W'(\eta) - \frac{1}{\eta} + O\left(\frac{1}{\eta^2}\right), \quad (2.92)$$

as $\eta \rightarrow \infty$.

Now, consider the cuts for $\omega_l(\eta)$, determined by $\alpha^{(l)} - b^{(l)2} \alpha^{(l)-1}$ which is equal to $\sqrt{\Lambda^2 - 4b^{(l)2}}$. Equation $\Lambda^2 - 4b^{(l)2} = 0$ has $2l$ roots, real or complex. If there is a complex root, its complex conjugate is also a root. If there is repeated root, the factor can be moved out from the inside of the square root in the expression of $\omega_l(\eta)$. Therefore, without loss of generality, we consider the equation $\Lambda^2 - 4b^{(l)2} = 0$ has $2l_1$ simple real roots $\eta_-^{(s)}, \eta_+^{(s)}$, $s = 1, \dots, l_1$, and $2l_2$ simple complex roots $\eta_s, \bar{\eta}_s$, $s = 1, \dots, l_2$, where $\bar{\eta}_s$ is the complex conjugate of η_s , $\text{Im } \eta_s > 0$, and $l = l_1 + l_2$. Suppose the real roots are so ordered that $[\eta_-^{(s)}, \eta_+^{(s)}]$, $s = 1, \dots, l_1$, form a set of disjoint intervals, $\Omega = \bigcup_{s=1}^{l_1} [\eta_-^{(s)}, \eta_+^{(s)}]$, see Fig. 2.1. Define

$$\rho_l(\eta) = \text{Re} \left. \frac{1}{\pi i} \omega_l(\eta) \right|_{\Omega^+}, \quad (2.93)$$

for $\eta \in \Omega$ as the general eigenvalue density on multiple disjoint intervals in the Hermitian matrix models. It can be seen that when $l = l_1 = 1$, $\omega_1 = \omega$, $\rho_1(\eta) = \rho(\eta)$, and the conditions (2.90) and (2.91) become (2.80) and (2.81) respectively.

Choose l_2 points $\eta_s^{(0)}$ on the real line outside Ω , such that the straight lines Γ_s 's, each one connecting η_s and $\eta_s^{(0)}$ for $s = 1, \dots, l_2$, do not intersect each other. Now, $\omega_l(\eta)$ is well defined and analytic in the outside of $\Omega \cup \bigcup_{s=1}^{l_2} (\Gamma_s \cup \bar{\Gamma}_s)$, where $\bar{\Gamma}_s$ is the straight line connecting $\bar{\eta}_s$ and $\eta_s^{(0)}$. Let Γ_s^* be the closed counterclockwise contour along the edges of $\Gamma_s \cup \bar{\Gamma}_s$, and define

$$I_s = \int_{\Gamma_s^*} \omega_l(\eta) d\eta, \quad \text{and} \quad \hat{I}_s(\eta) = \int_{\Gamma_s^*} \frac{\omega_l(\lambda)}{\lambda - \eta} d\lambda, \quad \eta \in \Omega,$$

for $s = 1, \dots, l_2$. According to the definition of Γ_s^* , I_s and $\hat{I}_s(\eta)$ are real.

Theorem 2.4 *If the parameters a_s, b_s ($s = 1, \dots, l$), and g_j ($j = 1, \dots, 2m$) satisfy the conditions (2.90) and (2.91), then $\rho_l(\eta)$ defined by (2.93) on Ω satisfies (2.5) and (2.6).*

Proof Let Γ be a large counterclockwise circle of radius R , and Ω^* be the union of closed counterclockwise contours around the upper and lower edges of all the intervals in Ω . Then by Cauchy theorem and (2.92),

$$\int_{\Omega^*} \left(\omega_l(\eta) - \frac{1}{2} W'(\eta) \right) d\eta + \sum_{s=1}^{l_2} I_s = \int_{\Gamma} \left(\omega_l(\eta) - \frac{1}{2} W'(\eta) \right) d\eta \rightarrow -2\pi i,$$

as $R \rightarrow \infty$, which implies $\int_{\Omega} \rho(\eta) d\eta = 1$ by (2.85), $\int_{\Omega^*} W'(\eta) d\eta = 0$, and I_s are real. So $\rho(\eta)$ satisfies the condition (2.5).

Change the Ω^- and Ω^+ discussed above just at $\eta \in \Omega$ as semicircles of ε radius. By (2.92) and $\int_{\Gamma_s^*} \frac{W'(\lambda)}{\lambda - \eta} d\lambda = 0$, there is

$$\frac{1}{2\pi i} \int_{\Omega^*} \frac{\omega_l(\lambda) - \frac{1}{2} W'(\lambda)}{\lambda - \eta} d\lambda + \frac{1}{2\pi i} \sum_{s=1}^{l_2} \hat{I}_s = \frac{1}{2\pi i} \int_{\Gamma} \frac{\omega_l(\lambda) - \frac{1}{2} W'(\lambda)}{\lambda - \eta} d\lambda \rightarrow 0,$$

as $R \rightarrow \infty$. Then taking the real parts of both sides, we get

$$\frac{1}{2} W'(\eta) = \frac{1}{2\pi} \int_{\Omega^*} \frac{\operatorname{Re} \frac{1}{i} \omega_l(\lambda)}{\lambda - \eta} d\lambda \rightarrow (\text{P}) \int_{\Omega} \frac{\rho(\lambda)}{\eta - \lambda} d\lambda,$$

as $\varepsilon \rightarrow 0$ by using (2.93). □

By the discussions above, it can be seen that when $l_2 = 0$, a_s, b_s ($s = 1, \dots, l$), and g_j ($j = 1, \dots, 2m$) satisfy the relations (2.90) and (2.91), then $y(\eta) = \omega_l(\eta) - \frac{1}{2} W'(\eta)$ satisfies the following relations: $y(\eta)$ is analytic when $\eta \in \mathbb{C} \setminus \Omega$; $y(\eta)|_{\Omega^+} + y(\eta)|_{\Omega^-} = -W'(\eta)$; $y(\eta) \rightarrow 0$, as $\eta \rightarrow \infty$. These relations are important in complex analysis, called scalar Riemann-Hilbert problem. If the parameters a_s and b_s can be chosen such that

$$(\operatorname{tr} J^{(l)})^2 - 4 \det J^{(l)} = \prod_{j=1}^l (\eta - \eta_-^{(j)}) (\eta - \eta_+^{(j)}), \quad (2.94)$$

then the density models and the corresponding scalar Riemann-Hilbert problems can be well solved. Note that the left hand side of (2.94) is also equal to $-\det(J^{(l)} - (\det J^{(l)}) J^{(l)-1})$ by considering $(J^{(l)} - \sqrt{\det J^{(l)}})(J^{(l)} + \sqrt{\det J^{(l)}})$ and calculating the determinants.

By Theorem 2.3, when $n \geq 2m$ and the parameters satisfy (2.40), the $\sigma_n(z)$ defined by

$$\sigma_n(z) = \frac{1}{\pi} \operatorname{Re} \sqrt{\det A_n(z)}, \quad -\infty < z < \infty, \quad (2.95)$$

satisfies $\int_{-\infty}^{\infty} \sigma_n(z) dz = n$ and $(\text{P}) \int_{-\infty}^{\infty} \frac{\sigma_n(z')}{z - z'} dz' = \frac{1}{2} V'(z)$, as the level density [7], that is consistent with the unified model discussed in Sect. 1.1. When the density involves the parameter n , the string equation and the initial conditions when n is less than $2m$ need to be considered to calculate the functions u_n and v_n .

2.5 Special Densities

When $m = 1$ and $W(\eta) = \eta^2$, there is

$$\rho(\eta) = \frac{1}{\pi} \sqrt{2 - \eta^2}, \quad (2.96)$$

for $\eta \in [-\sqrt{2}, \sqrt{2}]$, which is the well known Wigner semicircle.

When $m = 2$ and $W(\eta) = g_2\eta^2 + g_4\eta^4$, by the discussions before we have

$$\rho(\eta) = \frac{1}{\pi} (g_2 + 2g_4(\eta^2 + 2b^2)) \sqrt{4b^2 - \eta^2}, \quad (2.97)$$

for $\eta \in [-2b, 2b]$, with the restriction conditions

$$g_2 + 2g_4(\eta^2 + 2b^2) \geq 0, \quad \eta \in [-2b, 2b], \quad (2.98)$$

$$2g_2b^2 + 12g_4b^4 = 1. \quad (2.99)$$

The results are consistent with the case $W(\eta) = \frac{1}{2}\eta^2 + g\eta^4$ obtained by Brezin, Itzykson, Parisi and Zuber [1] that

$$\rho(\eta) = \frac{1}{\pi} \left(\frac{1}{2} + 4gb^2 + 2g\eta^2 \right) \sqrt{4b^2 - \eta^2}, \quad (2.100)$$

for $\eta \in [-2b, 2b]$, where

$$b^2 + 12gb^4 = 1. \quad (2.101)$$

When $W(\eta) = g_0 + g_1\eta + g_2\eta^2 + g_3\eta^3 + g_4\eta^4$, by Theorem 2.4, we have the following general density formula

$$\rho(\eta) = \frac{1}{2\pi} (2g_2 + 3g_3(\eta + a) + 4g_4(\eta^2 + a\eta + a^2 + 2b^2)) \sqrt{4b^2 - (\eta - a)^2}, \quad (2.102)$$

where the parameters satisfy the following conditions

$$2g_2 + 3g_3(\eta + a) + 4g_4(\eta^2 + a\eta + a^2 + 2b^2) \geq 0, \quad \eta \in [\eta_-, \eta_+], \quad (2.103)$$

$$2g_2b^2 + 6g_3ab^2 + 12g_4(a^2 + b^2)b^2 = 1, \quad (2.104)$$

$$g_1 + 2g_2a + 3g_3(a^2 + 2b^2) + 4g_4a(a^2 + 6b^2) = 0, \quad (2.105)$$

obtained from (2.4), (2.5) and (2.6), where $\eta_- = a - 2b$, $\eta_+ = a + 2b$. The density formula and the conditions coincide with the results (45) and (46) in [1] for the case $W(\eta) = \frac{1}{2}\eta^2 + g_3\eta^3$.

When $g_1 = g_2 = 0$, i.e., $W(\eta) = g_0 + g_3\eta^3 + g_4\eta^4$, the conditions become

$$3g_3(\eta + a) + 4g_4(\eta^2 + a\eta + a^2 + 2b^2) \geq 0, \quad \eta \in [\eta_-, \eta_+], \quad (2.106)$$

$$g_3 = -\frac{8a(a^2 + 6b^2)}{3b^2(5a^4 + 3(a^2 - 4b^2)^2)}, \quad (2.107)$$

$$g_4 = \frac{2(a^2 + 2b^2)}{b^2(5a^4 + 3(a^2 - 4b^2)^2)}. \quad (2.108)$$

The first condition (2.106) is satisfied if and only if $\tau = \frac{4b^2}{a^2}$ is restricted in the interval $0 < \tau \leq \tau_-$ or $\tau_+ \leq \tau$, where $\tau_+ = 1 + \sqrt{5}$, and τ_- is uniquely determined by the conditions: $0 < \tau_- < 1/2$ and $1 - 2\tau_-^{1/2} + \frac{3}{4}\tau_-^2 = 0$. Approximately we have $\tau_- \approx 0.28$ and $\tau_+ \approx 3.24$. The density function in this case can be further rescaled into the following forms. Let $\eta = ax$ and $\tau = c^2$ ($c > 0$). Then

$$\rho(\eta)d\eta = \frac{16}{\pi} \frac{(\frac{x}{c} - \frac{c}{2})^2 + \frac{x^2-1}{2}}{5 + 3(1-c^2)^2} \sqrt{c^2 - (x-1)^2} dx, \quad (2.109)$$

for $x \in [1-c, 1+c]$, where $c \in (0, c_-] \cup [c_+, \infty)$, $c_- = \sqrt{\tau_-}$ and $c_+ = \sqrt{\tau_+}$. On the other hand, if $\eta = -ax$ and $\tau = c^2$ ($c > 0$), then

$$\rho(\eta)d\eta = \frac{16}{\pi} \frac{(\frac{x}{c} + \frac{c}{2})^2 + \frac{x^2-1}{2}}{5 + 3(1-c^2)^2} \sqrt{c^2 - (x+1)^2} dx, \quad (2.110)$$

for $x \in [-1-c, -1+c]$, where $c \in (0, c_-] \cup [c_+, \infty)$, $c_- = \sqrt{\tau_-}$ and $c_+ = \sqrt{\tau_+}$.

The density on two disjoint intervals can also be obtained by using the method discussed before. Briefly, we have

$$\rho(\eta) = \frac{1}{2\pi} (3g_3 + 4g_4(a_1 + a_2 + \eta)) \operatorname{Re} \sqrt{4b_1^2 b_2^2 - ((\eta - a_1)(\eta - a_2) - b_1^2 - b_2^2)^2}, \quad (2.111)$$

where $-\infty < \eta < \infty$, and

$$4g_4 b_1^2 b_2^2 = 1, \quad (2.112)$$

$$2g_2 + (3g_3 + 4g_4(a_1 + a_2))(a_1 + a_2) - 4g_4(a_1 a_2 - b_1^2 - b_2^2) = 0, \quad (2.113)$$

$$g_1 - (3g_3 + 4g_4(a_1 + a_2))(a_1 a_2 - b_1^2 - b_2^2) = 0. \quad (2.114)$$

It can be checked that if we take $a_1 = a_2 = a$ and $b_1 = b_2 = b$ in the above, then a and b satisfy (2.104) and (2.105). In addition, the non-negative condition for this $\rho(\eta)$ will be discussed in next chapter.

When $m = 3$, by Theorem 2.4, there is

$$\rho(\eta) = \frac{1}{\pi} (g_2 + 2g_4(\eta^2 + 2b^2) + 3g_6(\eta^4 + 2b^2\eta^2 + 6b^4)) \sqrt{4b^2 - \eta^2}, \quad (2.115)$$

for $\eta \in [-2b, 2b]$, and (2.103) and (2.104) become

$$g_2 + 2g_4(\eta^2 + 2b^2) + 3g_6(\eta^4 + 2b^2\eta^2 + 6b^4) \geq 0, \quad \eta \in [-2b, 2b], \quad (2.116)$$

$$2g_2 b^2 + 12g_4 b^4 + 60g_6 b^6 = 1. \quad (2.117)$$

Generally, for the symmetric density with the potential $W(\eta) = \sum_{j=1}^m g_j \eta^{2j}$, we have the following by Theorem 2.4

$$\rho(\eta) = \frac{1}{\pi} k_{2m-2}(\eta) \sqrt{4b^2 - \eta^2}, \quad \eta \in [-2b, 2b], \quad (2.118)$$

where

$$k_{2m-2}(\eta) = \sum_{j=1}^m j g_{2j} \sum_{p=1}^j \binom{2j-1}{j-p} \frac{b^{2(j-p)}}{4^{p-1}} \sum_{s=0}^{p-1} \binom{2p-1}{2s+1} \eta^{2(p-s-1)} (\eta^2 - 4b^2)^s, \quad (2.119)$$

and

$$k_{2m-2}(\eta) \geq 0, \quad \eta \in [-2b, 2b], \quad (2.120)$$

$$\sum_{j=1}^m 2j g_{2j} \binom{2j-1}{j} b^{2j} = 1. \quad (2.121)$$

Here, the formula (2.119) is obtained from (2.76) and (2.77) by choosing $g_1 = g_3 = \dots = g_{2m-1} = 0$ and $a = 0$ first, then replacing j by $2j$ and taking the substitutions $q = 2j - 1$ and $r = j - p$. By the asymptotics (2.83), we also have that for large $R > 0$, there is

$$k_{2m-2}(\eta) = \frac{1}{2\pi i} \oint_{|\lambda|=R} \frac{\omega(\lambda)}{\sqrt{\lambda^2 - 4b^2}} \frac{d\lambda}{\lambda - \eta} = \frac{1}{2\pi i} \oint_{|\lambda|=R} \frac{\frac{1}{2}W'(\lambda)}{\sqrt{\lambda^2 - 4b^2}} \frac{d\lambda}{\lambda - \eta}. \quad (2.122)$$

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