A Survey on Stanley Depth

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Introduction

At the MONICA conference "MONomial Ideals, Computations and Applications" at the CIEM, Castro Urdiales (Cantabria, Spain) in July 2011, I gave three lectures covering different topics of Combinatorial Commutative Algebra: (1) A survey on Stanley decompositions. (2) Generalized Hibi rings and Hibi ideals. (3) Ideals generated by two-minors with applications to Algebraic Statistics. In this article I will restrict myself to give an extended presentation of the first lecture. The CoCoA tutorials following this survey will deal also with topics related to the other two lectures. Complementing the tutorials, the reader finds in [165] a CoCoA routine to compute the Stanley depth for modules of the form I/J, where $J \subset I$ are monomial ideals.

In his famous article "Linear Diophantine equations and local cohomology", Richard Stanley [181] made a striking conjecture predicting an upper bound for the depth of a multigraded module. This conjectured upper bound is nowadays called the Stanley depth of a module. The Stanley depth is of a rather combinatorial nature while the depth is a homological invariant. The definition of Stanley depth is given in Sect. 1. The conjecture is on so far striking as it compares two invariants of modules of very different nature. At a first glance it seems to be no relation among these two invariants. The conjecture was made in 1982, and to best of my knowledge it is Apel who first studied this conjecture intensively and proved it in some special cases in his papers [12, 13]. His papers appeared in 2003. Stanley decompositions were then considered again in 2006 in my joint paper [95] with Popescu and since then the topic has become very popular with numerous publications regarding different

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aspects of Stanley depth. Though there has been a lot of efforts to prove or disprove Stanley's conjecture, it is still widely open.

I tried to give a rather comprehensive list of references to papers dealing with Stanley depth. It is impossible to describe and to discuss the content of all these papers in this survey. In the very last section of this survey however I try to list some of the most remarkable results which are not discussed in these notes to give the reader an orientation on what is known, and I also list a few open questions which I think would be interesting to deal with.

This survey on Stanley depth naturally describes the theory from a viewpoint based on my own and my coauthors contributions, as presented in the papers [95, 98, 99, 102]. In Sect. 1 we recall Stanley's conjecture in its original form and fix the terminology and notation used throughout this notes. In Sect. 2 we introduce the concept of prime filtrations and show how they induce Stanley decompositions, thereby proving that Stanley decompositions actually exist. The discussions there lead to an invariant which is denoted by fdepth (filter depth). Then in the following Sect. 3 we show that the invariant fdepth gives a lower bound for the Stanley depth and we also present an upper bound that was proved by Apel. This upper bound implies in particular that the Stanley depth is bounded above by the Krull dimension.

Section 4 is devoted to the theory initiated by Dress [50] who showed that a simplicial complex is shellable (in the non-pure sense) if and only if its Stanley–Reisner ring is clean, meaning that it has a prime filtration with all its prime factors being minimal prime ideals. Here we present the proof of the fact that the Stanley–Reisner ring of a shellable simplicial complex is clean. In the context of Stanley decompositions this result is remarkable as it implies that the Stanley–Reisner ring of a shellable simplicial complex satisfies Stanley's conjecture. To extend this result to K-algebras with monomial relations, not necessarily squarefree, pretty clean filtrations were introduced in [95]. Modules with pretty clean filtrations also satisfy Stanley's conjecture. A pretty clean module which has no embedded prime ideals is actually clean. At the end of Sect. 4 we show that a polynomial ring modulo a Borel type ideal (ideal of nested type) is always pretty clean.

Long before Stanley made his conjecture, Janet [115] in 1924 gave an explicit algorithm to produce a Stanley decomposition of a monomial ideal. In Sect. 5 we describe his algorithm. Janet decompositions from the viewpoint of Stanley depth are not optimal. They rarely give Stanley decompositions providing the Stanley depth of a monomial ideal. However one obtains the result that the Stanley depth of a monomial ideal is at least 1.

A well-known result in Commutative Algebra is the fact that the depth of a module after reduction by a non-zero divisor drops exactly by one. In Sect. 6 we present the result of Rauf which says that the Stanley depth drops at least by one after reduction by a variable which is a non-zero divisor on the module. This result has two nice consequences which we cite from [32]. The first is that Stanley depth zero implies depth zero, the second is that the kth syzygy module of a $\mathbb{Z}^n$-graded module has Stanley depth at least k. The second result (with a more difficult argument) was first shown in [62].

To compute the Stanley depth of a module one has in principle to consider all possible Stanley decompositions. These are infinitely many. So the question arises whether the Stanley depth of a module can actually be computed. Indeed, up to date, no algorithm for computing the Stanley depth of a multigraded module is known. Only the Stanley depth of I/J for monomial ideals $J \subset I$ can be computed. This is the content of Sect. 7. Here we reproduce the theory as given in [101]. A slightly revised version of the paper appeared later in Journal of Algebra where Miller's functors as defined in [140] are used. The method of computing the Stanley depth of I/J consists in attaching to I/J a finite poset $P^g_{I/J}$, called the characteristic poset of I/J, with the property that each of its interval partitions yields a Stanley decomposition of I/J. The remarkable fact is that among the Stanley decompositions obtained in this way there is one which gives the Stanley depth. So in principle, the Stanley depth of I/J can be a computed in a finite number of steps. From a computational point of view however the method is not very efficient. Rinaldo in [165] uses this method, eliminating unnecessary interval partitions, for a routine to compute the Stanley depth [165]. Apart from the computational aspects, this approach via interval partitions has also some interesting theoretical aspects. For example it can be shown, as outlined in Sect. 7, that Stanley's conjecture on Stanley decompositions implies another one of Stanley's conjectures which asserts that each Cohen–Macaulay simplicial complex is partionable.

The characteristic poset of I/J can be used to define the skeletons of I/J, which when specialized to the Stanley–Reisner ideal of a simplicial complex gives the Stanley–Reisner ideals of its skeletons. This concept is discussed in Sect. 8 following the article [99]. It is used to determine the depth of S/I for a monomial ideal $I \subset S$, generalizing a corresponding result of Hibi in [104]. Moreover, this concept of skeletons is used to show that Stanley's conjecture holds for all K-algebras with monomial relations if and only if it holds for all Cohen–Macaulay K-algebras with monomial relations.

The method described in Sect. 7 to compute the Stanley depth leads to interesting combinatorial considerations in even seemingly simple cases as for example in the case of the graded maximal ideal $\mathfrak{m}$ of a polynomial ring in n variables. While the depth of $\mathfrak{m}$ is easily seen to be 1, Biró et al. showed in [18] that the Stanley depth of $\mathfrak{m}$ is $\lceil n/2 \rceil$. We present in Sect. 9 the main ideas of their nice combinatorial arguments. The result shows that the Stanley depth of module may exceed its depth by any amount. The combinatorial techniques developed in [18] have been extended and refined later by many authors. Here we use these extensions to show, following the arguments of [62], that the lower bound for the Stanley depth of a squarefree monomial ideal in a polynomial ring in n variables is approximately of size $\sqrt{n}$.

In Sect. 10 we describe the relationship between squarefree Stanley decompositions of the Stanley–Reisner ring of a simplicial complex and its Alexander dual, as it is done in [179]. This considerations naturally suggest to define a new invariant for $\mathbb{Z}^n$-graded modules denoted sreg (Stanley regularity). The Stanley regularity of a Stanley–Reisner ideal of a simplicial complex on the vertex set $[n]$ and the Stanley depth of the Stanley–Reisner ring of its Alexander dual always add up to n. This is a formula analogue to that discovered by Terai relating depth and regularity.

The conjecture of Soleyman Jahan, in some sense dual to that of Stanley, says that the Stanley regularity of a $\mathbb{Z}^n$-graded S-module is always less than or equal to its regularity.

In [129], Lyubeznik gave a nice lower bound for the depth of S/I where I is a monomial ideal, called the size of I. Provided the primary decomposition of I is known, the size of I is an easy to compute invariant of S/I which does not depend on the characteristic of the base field. The main purpose of Sect. 11 is to show that the same lower bound holds for the Stanley depth which, assuming that Stanley's conjecture holds true, is not surprising. Considering Alexander duality one is guided to define the so-called cosize of an ideal, and obtains that both the regularity as well as the Stanley regularity are bounded above by the cosize. This section reflects the main results of the paper [97].

A $\mathbb{Z}^n$-graded S-module, S a polynomial ring, may naturally be considered a $\mathbb{Z}$-graded module over S, where S is given the standard grading. Then the Hilbert series of M viewed as a $\mathbb{Z}$-graded module can be computed from each of its Stanley decompositions. This leads to an upper bound for the Stanley depth which only depends on the Hilbert series of the module. This upper bound is called the Hilbert depth, introduced by Bruns and his coauthors in [31]. In Sect. 12 we outline the idea of this concept. Amazingly, the Hilbert depth always exceeds the depth of a module, as shown in [31]. A different proof of this fact is given here for S/I where I is a monomial ideal. One would expect that it is easy to compute the Hilbert depth of a module, once its Hilbert function is known. But it turns out that even for the powers of the maximal ideal, the computation of the Hilbert depth leads to difficult numerical computations. Some of the results by Bruns and his group on the Hilbert depth is quoted in the last section, where also a few more nice results, not discussed in this survey, are mentioned.

Of course the most challenging open problem is to prove or to disprove Stanley's conjecture. To find a counter example would be quite hard. By what has been shown in this survey one would find a possible counter example among the Stanley–Reisner rings of a Cohen–Macaulay simplicial complex which is not shellable. On the other hand, there are many other attractive problems related to Stanley depth, not directly related to Stanley's conjecture. Some of them are listed in the last section.

1 Basic Definitions and Concepts

Richard Stanley [181] in his article "Linear Diophantine equations and local cohomology", made the following striking conjecture concerning the depth of multigraded modules.

Conjecture 1 (Stanley). Let R be a finitely-generated $\mathbb{N}^n$-graded K-algebra (where $R_0 = K$ as usual), and let M be a finitely generated $\mathbb{Z}^n$-graded R-module. Then there exist finitely many subalgebras $S_1, \dots, S_t$ of R, each generated by algebraically independent $\mathbb{N}^n$-homogeneous elements of R, and there exist $\mathbb{Z}^n$-homogeneous elements $\eta_1, \ldots, \eta_t$ of M, such that

$$M = \bigoplus_{i=1}^{t} \eta_i S_i, \qquad \text{(vector space direct sum)}$$

where $\dim(S_i) \geq \operatorname{depth}(M)$ for all i, and where $\eta_i S_i$ is a free S_i-module (of rank one). Moreover, if K is infinite and under a given specialization to an $\mathbb{N}$-grading R is generated by R_1, then we can choose the ($\mathbb{N}^n$-homogeneous) generators of each S_i to lie in R_1.

Stanley calls the above direct sum decomposition of M a combinatorial decomposition of M. Nowadays such a decomposition is called a Stanley decomposition.

In the same article Stanley proved this conjecture for decompositions of monoid rings, see [182, Theorem 5.2].

In this survey we concentrate our attention to the case that M is a finitely generated $\mathbb{Z}^n$-graded S-module, where $S = K[x_1, \ldots, x_n]$ is the polynomial ring over a field K.

Definition 2. Let M be a $\mathbb{Z}^n$-graded S-module. Let $m \in M$ be homogeneous, and $Z \subset X = \{x_1, \ldots, x_n\}$. Then the $K[Z]$-submodule $mK[Z]$ of M is called a *Stanley space* of M, if $mK[Z]$ is a free $K[Z]$-submodule of M, and $|Z|$ is called the *dimension* of $mK[Z]$.

A *Stanley decomposition* $\mathscr{D}$ of M is a decomposition of M as a direct sum of $\mathbb{Z}^n$-graded K-vector space

$$\mathscr{D} : M = \bigoplus_{j=1}^{r} m_j K[Z_j],$$

where each $m_j K[Z_j]$ is a Stanley space of M.

We set

$$\operatorname{sdepth}(\mathscr{D}) = \min\{|Z_j| : \ j = 1, \ldots, r\},$$

and

$$\operatorname{sdepth}(M) = \max\{\operatorname{sdepth}(\mathscr{D}) \colon \mathscr{D} \text{ is a Stanley decomposition of } M\}.$$

The number $\operatorname{sdepth}(M)$ is called the *Stanley depth* of M.

With the notation introduced, Stanley's conjecture can be phrased as follows:

Conjecture 3 (Stanley). $\operatorname{sdepth}(M) \geq \operatorname{depth}(M)$.

The conjecture is widely open. Nevertheless there have been interesting developments in recent years in the context of this conjecture.

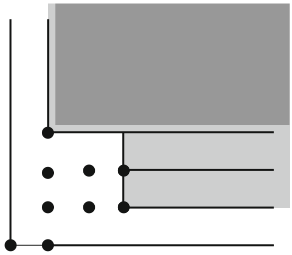


Fig. 1 A Stanley decomposition of I and S/I

Of particular interest is the case when M is isomorphic to a monomial ideal $I \subset S$ or isomorphic to the residue class ring S/I of a monomial ideal I. The monomials $u \in I$ form a homogeneous K-basis of I, while the residue classes modulo I of the monomials $u \in S \setminus I$ form a homogeneous basis of S/I. Therefore, we often identify S/I with the $\mathbb{Z}^n$-graded K-subvector space I^c of S which is generated by all monomials $u \in S \setminus I$.

Figure 1 displays Stanley decompositions of $I = (x_1^3x_2, x_1x_2^3)$ and S/I, namely

$$\mathscr{D}_1 : I = x_1x_2^3K[x_1, x_2] \oplus x_1^3x_2^2K[x_1] \oplus x_1^3x_2K[x_1],$$

and

$$\mathscr{D}_2 : S/I = K[x_2] \oplus x_1K[x_1] \oplus x_1x_2K \oplus x_1x_2^2K \oplus x_1^2x_2K \oplus x_1^2x_2^2K.$$

We have $\operatorname{sdepth}(I) \geq \operatorname{sdepth}(\mathscr{D}_1) = 1 = \operatorname{depth}(I)$, and $\operatorname{sdepth}(S/I) \geq \operatorname{sdepth}(\mathscr{D}_2) = 0 = \operatorname{depth}(S/I)$. Thus Stanley's conjecture holds in this particular case.

The natural question arises whether there exists always a Stanley decomposition (otherwise Definition 2 wouldn't make any sense), and whether we can compute the Stanley depth.

Note that whenever M contains a Stanley space $mK[Z]$ of dimension > 0, then M admits infinitely many Stanley decompositions. Indeed, suppose that a Stanley decomposition $\mathscr{D}$ contains the Stanley space $mK[Z]$ with $|Z| > 0$ as a summand. Then, among many other possibilities, we may replace the summand $mK[Z]$ for any r and any k with $x_k \in Z$ by the direct sum

$$\bigoplus_{i=0}^{r-1} x_k^i mK[Z'] \oplus x_k^r mK[Z],$$

where $Z' = Z \setminus \{x_k\}$.

Thus, since there are in general infinitely many different Stanley decompositions of a module, it is not at all clear how to compute the Stanley depth. We will deal with this problem in Sect. 7. In the next section we will show that any $\mathbb{Z}^n$-graded module admits indeed a Stanley decomposition.

2 Prime Filtrations and Stanley Decompositions

Let $S = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field K and M a finitely generated $\mathbb{Z}^n$-graded S-module. It is known that the associated prime ideals of M are monomial ideals, and that any monomial prime ideal is of the form $P_F = (x_i : i \notin F)$ for some $F \subset [n]$.

Definition 4. A chain

$$\mathscr{F} : 0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

of $\mathbb{Z}^n$-graded submodules of M such that $M_i/M_{i-1} \cong (S/P_i)(-a_i)$ with P_i a monomial prime ideal and $a_i \in \mathbb{Z}^n$ is called a *prime filtration* of M. The *support* of the prime filtration $\mathscr{F}$ is the set $\{P_1, \dots, P_r\}$, denoted $\mathrm{Supp}(\mathscr{F})$.

Prime filtrations always exist: Let $P_1 \in \mathrm{Ass}(M)$. Since M is $\mathbb{Z}^n$-graded, P_1 is a monomial prime ideal and $P_1 = \mathrm{Ann}(m_1)$ where m_1 is a homogeneous element of M. Set $M_1 = Sm_1$. Then $M_1 \cong (S/P_1)(-a_1)$ where $a_1 = \deg m_1$. If $M_1 = M$, the assertion follows. Otherwise $\mathrm{Ass}(M/M_1) \neq \emptyset$. Let $P_2 \in \mathrm{Ass}(M/M_1)$. Then there exists a $\mathbb{Z}^n$-graded submodule $M_2 \subset M$ with $M_1 \subset M_2$ such that $M_2/M_1 \cong (S/P_2)(-a_2)$ for some $a_2 \in \mathbb{Z}^n$. If $M_2 = M$, then $0 = M_0 \subset M_1 \subset M_2 = M$ is the desired prime filtration. If $M_2 \neq M$, we find in the same way as described before $M_3 \subset M$ with $M_3/M_2 \cong (S/P_3)(-a_3)$ where P_3 is a monomial prime ideal and $a_3 \in \mathbb{Z}^n$. Since M is Noetherian, this process must stop and hence yields the desired prime filtration.

This elementary fact from commutative algebra yield the desired existence theorem.

Theorem 5. *Let M be a finitely generated $\mathbb{Z}^n$-graded S-module. Then M admits a Stanley decomposition. More precisely, let*

$$\mathscr{F} : 0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

be a prime filtration of M with $M_i/M_{i-1} \cong (S/P_i)(-a_i)$ and $M_i = M_{i-1} + Sm_i$ with m_i homogeneous of degree a_i. Then one obtains the Stanley decomposition

$$\mathscr{D}(\mathscr{F}) : M = \bigoplus_{i=1}^{r} m_i K[Z_i],$$

where $Z_i = \{x_j : x_j \notin P_i\}$.

The proof of the theorem is immediate: observe that $M \cong \bigoplus_{i=1}^r M_i/M_{i-1}$ as a direct sum of $\mathbb{Z}^n$-graded K-vector spaces, and that $S/P_i \cong K[Z_i]$ for all i.

Not each Stanley decomposition of M is of the form $\mathscr{D}(\mathscr{F})$ for a suitable prime filtration $\mathscr{F}$ of M. Indeed, a prime filtration $\mathscr{F}$ of M is essentially given by a sequence of homogeneous elements $m_1, \dots, m_r$ of M with the property that

$M = Sm_1 + \cdots + Sm_r$ and that each of the colon ideals $(m_1, \ldots, m_{i-1}) : m_i$ is generated by a subset of the variables (in which case we say that the sequence $m_1, \ldots, m_r$ has linear quotients). Thus we see that a Stanley decomposition $\mathscr{D} : M = \bigoplus_{i=1}^r m_i K[Z_i]$ is induced by a prime filtration of M if and only if, after a suitable renumbering of the direct summands, we have that

$$(m_1, \ldots, m_{i-1}) : m_i = (x_j \colon x_j \notin Z_i) \quad \text{for all} \quad i.$$

Example 6. Consider the following Stanley decomposition

$$\mathscr{D} : (x_1, x_2, x_3) = x_1x_2x_3K[x_1, x_2, x_3] \oplus x_1K[x_1, x_2] \oplus x_2K[x_2, x_3] \oplus x_3K[x_1, x_3].$$

of the maximal ideal $(x_1, x_2, x_3) \subset K[x_1, x_2x_3]$.

It is easily seen that no order of the elements $x_1x_2x_3, x_1, x_2, x_3$ forms a sequence with linear quotients. Thus this Stanley decomposition does not arise from a prime filtration.

The following characterization of Stanley decompositions arising from prime filtrations is taken from [101].

Proposition 7. *Let M be a finitely generated $\mathbb{Z}^n$-graded S-module with* $\dim_K M_a \leq 1$ *for all $a \in \mathbb{Z}^n$, and let $\mathscr{D} : M = \bigoplus_{i=1}^r u_i K[Z_i]$ be a Stanley decomposition of M. Then the following conditions are equivalent:*

(a) *$\mathscr{D}$ is induced by a prime filtration.*
(b) *After a suitable relabeling of the summands in $\mathscr{D}$, for $j = 1, \ldots, r$ the direct sum $M_j = \bigoplus_{i=1}^j m_i K[Z_i]$ is a $\mathbb{Z}^n$-graded submodule of M.*

Proof. (a) $\Rightarrow$ (b) follows immediately from the construction of a Stanley decomposition which is induced by a prime filtration.

(b) $\Rightarrow$ (a): We claim that $\mathscr{F} \colon 0 \subset M_1 \subset M_2 \subset \ldots \subset M_r = M$ is a prime filtration of M. First notice that for each j, the module M_j/M_{j-1} is a cyclic module generated by the residue class $\bar{m}_j = m_j + M_{j-1}$. Indeed, each element $m \in M_j$ can be written as $m = \sum_{k=1}^j m_k f_k$ with $f_k \in K[Z_k]$ for $k = 1, \ldots, j$. Therefore $\bar{m} = \bar{m}_j f_j$.

Next we claim that the annihilator of $\bar{m}_j$ is equal to the monomial prime ideal P generated by the variables $x_k \notin Z_j$. Since $\dim_K M_a \leq 1$ for all a, it follows that if $m \in M$ is homogeneous, then there is a unique Stanley space $m_i K[Z_i]$ such that $m \in m_i K[Z_i]$. Thus, if $x_k \notin Z_j$, then $x_k m_j \notin m_j K[Z_j]$ since $\deg x_k m_j \neq \deg m_j v$ for all monomials $v \in K[Z_j]$. Therefore, $x_k m_j \in m_i K[Z_i]$ for some $i < j$. This implies that $x_k \bar{m}_j = 0$ and shows that P is contained in the annihilator of $\bar{m}_j$. On the other hand, if v is a monomial in $S \setminus P$, then $v \in K[Z_j]$, and so $m_j v$ is a nonzero element in $m_j K[Z_j]$. This implies that v does not belong to the annihilator of $\bar{m}_j$ and shows that P is precisely the annihilator of $\bar{m}_j$. From all this we conclude that $\mathscr{D}$ is induced by $\mathscr{F}$.

Definition 8. Let M be a finitely generated $\mathbb{Z}^n$-graded S-module. The number

$$\operatorname{fdepth}(M) = \min\{\operatorname{sdepth}(\mathscr{D}(\mathscr{F})) : \mathscr{F} \text{ is a prime filtration of } M\}$$

is called the *filtration depth* of M.

It is clear that $\operatorname{fdepth}(M) \leq \operatorname{sdepth}(M)$. Example 6 shows that this inequality may be strict. Indeed in that example we have $\operatorname{fdepth}(x_1, x_2, x_3) = 1$ and $\operatorname{sdepth}(x_1, x_2, x_3) = 2$.

We conclude this section by observing that for any prime filtration $\mathscr{F}$ of M the following inclusions

$$\operatorname{Ass}(M) \subset \operatorname{Supp}(\mathscr{F}) \subset \operatorname{Supp}(M) \tag{1}$$

of sets hold.

Of course, $\operatorname{Supp}(\mathscr{F}) \subset \operatorname{Supp}(M)$. In order to show that $\operatorname{Ass}(M) \subset \operatorname{Supp}(\mathscr{F})$, let $P \in \operatorname{Ass}(M)$. Then $PS_P \in \operatorname{Ass}(M_P)$, so that $\operatorname{depth}(M_P) = 0$. Hence, if $0 = M_0 \subset M_1 \subset \ldots \subset M_r = M$ with $M_i/M_{i-1} \cong (S/P_{F_i})(-a_i)$ is the prime filtration $\mathscr{F}$, then by applying the Depth Lemma [30, Proposition 1.2.9] we get

$$0 = \operatorname{depth}(M_P) \geq \min\{\operatorname{depth}((M_i/M_{i-1})_P) : (M_i/M_{i-1})_P \neq 0\}.$$

Hence there exists an index i such that $\operatorname{depth}(M_i/M_{i-1})_P = 0$. Therefore, $\operatorname{depth}(S_P/P_i S_P) = 0$, and this is only possible if $P_i = P$.

3 Upper and Lower Bounds for Stanley Depth

The main purpose of this section is to prove the following inequalities

Theorem 9. *Let M be a finitely generated $\mathbb{Z}^n$-graded S-module, and let $\mathscr{F}$ be a prime filtration of M. Then*

$$\min\{\dim(S/P) : P \in \operatorname{Supp}(\mathscr{F})\} \leq \operatorname{fdepth}(M) \leq \min\{\operatorname{depth}(M), \operatorname{sdepth}(M)\},$$

and

$$\max\{\operatorname{depth}(M), \operatorname{sdepth}(M)\} \leq \min\{\dim(S/P) : P \in \operatorname{Ass}(M)\} \leq \dim(M),$$

where for the validity of the upper bound for the Stanley depth we require that $\dim_K M_a \leq 1$ *for all* $a \in \mathbb{Z}^n$.

Proof. The lower bounds for the filtration depth follows from its definition. Let $\mathscr{F}_0 : 0 = M_0 \subset M_1 \subset \ldots \subset M_r = M$ be a prime filtration of M with

$\mathrm{supp}(\mathscr{F}_0) = \{P_1, \ldots, P_r\}$ and $\mathrm{fdepth}(M) = \min\{\dim(S/P_i)\colon\ i = 1, \ldots, r\}$. By applying the depth lemma we obtain

$$\mathrm{fdepth}(M) = \min_i\{\dim(S/P_i)\} = \min_i\{\mathrm{depth}(M_i/M_{i-1})\} \leq \mathrm{depth}(M).$$

The inequality $\mathrm{fdepth}(M) \leq \mathrm{sdepth}(M)$ is trivial.

The inequality $\mathrm{depth}(M) \leq \min\{\dim(S/P)\colon\ P \in \mathrm{Ass}(M)\}$ can be found in [30, Proposition 1.2.13], and the inequality $\min\{\dim(S/P)\colon\ P \in \mathrm{Ass}(M)\} \leq \dim(M)$ is trivial.

Thus it remains to show that $\mathrm{sdepth}(M) \leq \min\{\dim(S/P)\colon\ P \in \mathrm{Ass}(M)\}$. This inequality has first been shown by Apel in [13] in the case that $M = S/I$, where I is a monomial ideal. We follow his line of arguments to prove the general case. Let $\mathscr{D}\ :\ M = \bigoplus_{i=1}^r m_i K[Z_i]$ be a Stanley decomposition of M, and let $P \in \mathrm{Ass}(M)$. There exists a homogeneous element $m \in M$ with $P = \mathrm{Ann}(m)$. Since we assume that $\dim_K M_a \leq 1$ for all $a \in \mathbb{Z}^n$, it follows that $m \in m_i K[Z_i]$ for some i. Therefore, $x_j \notin P$ for all $x_j \in Z_i$. Hence

$$\mathrm{sdepth}(M) \leq |Z_i| \leq \dim(S/P).$$

This yields the desired inequality.

Question 10.
Is the inequality $\mathrm{sdepth}(M) \leq \min\{\dim(S/P)\colon\ P \in \mathrm{Ass}(M)\} \leq \dim(M)$ true without the extra assumption that $\dim_K M_a \leq 1$ for all $a \in \mathbb{Z}^n$?

Generally speaking, Stanley decompositions of $\mathbb{Z}^n$-graded S-modules which admit graded components of dimension > 1 are not well understood. We will come back to this problem later.

Recall that M is said to be Cohen–Macaulay, if $\mathrm{depth}(M) = \dim(M)$. We call a module *pseudo Cohen–Macaulay*, if $\mathrm{sdepth}(M) = \dim(M)$. It follows from Theorem 9 that a pseudo Cohen–Macaulay module M with $\dim_K M_a \leq 1$ for all $a \in \mathbb{Z}^n$ has no embedded prime ideals, and assuming Stanley's conjecture holds, M is pseudo Cohen–Macaulay if it is Cohen–Macaulay.

Problem 11. Find a pseudo Cohen–Macaulay module which is not Cohen–Macaulay.

The inequality $\mathrm{fdepth}(M) \leq \mathrm{depth}(M)$ may be strict. To give such an example, we recall a few concepts regarding simplicial complexes and Stanley–Reisner rings. Let $[n] = \{1, \ldots, n\}$. A collection Δ of subsets of $[n]$ is called a *simplicial complex on the vertex set* $[n]$ if it satisfies the condition that whenever $F \in \Delta$ and $G \subset F$, then $G \in \Delta$. The elements of Δ are called *faces*. The set of facets (maximal faces under inclusion) of Δ is denoted $\mathscr{F}(\Delta)$.

We fix a field K. The *Stanley–Reisner ideal* $I_\Delta \subset S = K[x_1, \ldots, x_n]$ of Δ is the ideal generated be the squarefree monomials $\mathbf{x}_F$ with $F \notin \Delta$. Here $\mathbf{x}_F = x_{i_1}x_{i_2}\cdots x_{i_k}$ for $F = \{i_1, i_2, \ldots, i_k\}$. The factor ring $K[\Delta] = S/I_\Delta$ is called the

Stanley–Reisner ring of Δ. The dimension, $\dim(F)$, of a face F of Δ is the number $|F|-1$, and the dimension of Δ is defined to be $\dim(\Delta) = \max\{\dim(F)\colon\ F \in \Delta\}$. One has $\dim(K[\Delta]) = \dim(\Delta) + 1$. We refer the reader to [30] and [91] regarding basic properties of Stanley–Reisner ideals.

Now we come to our example. Let Δ be the simplicial complex on the vertex set $\{1,\ldots,6\}$, associated to a triangulation of the real projective plane $\mathbb{P}^2$, whose facets are

$$\mathscr{F}(\Delta) = \{125, 126, 134, 136, 145, 234, 235, 246, 356, 456\}.$$

Here we write for simplicity $i_1 i_2 \ldots i_k$ for the set $\{i_1, i_2, \ldots, i_k\}$. The Stanley–Reisner ideal of Δ is

$$\begin{aligned} I_\Delta \;=\; & (x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_1x_4x_6, x_1x_5x_6, \\ & x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, x_3x_4x_6). \end{aligned}$$

It is known that $\operatorname{depth}(I_\Delta) = 4$ if $\operatorname{char} K \neq 2$ and $\operatorname{depth}(I_\Delta) = 3$ if $\operatorname{char} K = 2$. Since the inequality $\operatorname{fdepth}(I_\Delta) \leq \operatorname{sdepth}(I_\Delta)$ holds independent of the characteristic of the base field, we obtain that $\operatorname{fdepth}(I_\Delta) \leq 3$. Therefore, $\operatorname{fdepth}(I_\Delta) < \operatorname{depth}(I_\Delta)$ for any field K with $\operatorname{char} K \neq 2$.

Problem 12. Find a $\mathbb{Z}^n$-graded S module such that, independent of the characteristic of K, $\operatorname{fdepth}(M) < \operatorname{depth}(M)$.

For some of the results described in the next section we recall a few more concepts related to simplicial complexes. For a subset of faces $F_1, \ldots, F_r$ of Δ we denote by $\langle F_1, \ldots, F_r \rangle$ the smallest subcomplex of Δ containing the faces $F_1, \ldots, F_r$. The simplicial complex Δ is called *shellable*, if the facets of Δ can be ordered $F_1, \ldots, F_r$ such that for $i = 2, \ldots, r$ the facets of $\langle F_1, \ldots, F_i \rangle \cap \langle F_i \rangle$ are maximal proper faces of $\langle F_i \rangle$. For $i \geq 2$ we denote by a_i the number of facets of $\langle F_1, \ldots, F_{i-1} \rangle \cap \langle F_i \rangle$, and set $a_1 = 0$. We call the $a_1, \ldots, a_r$ the sequence of *shelling numbers* of the given shelling of Δ.

Classically one requires from a shellable simplicial complex Δ that it is also pure. In that case $K[\Delta]$ is Cohen–Macaulay, see [30, Theorem 5.1.13] and [91, Theorem 8.2.6]. The definition of shellability given here which does not require that Δ is pure was first introduced by Björner and Wachs [22] and is sometimes called *nonpure shellability*. A shellable simplical complex as defined here has the nice property that $K[\Delta]$ is sequentially Cohen–Macaulay, see [91, Corollary 8.2.19].

Finally we recall that the *Alexander dual* $\Delta^\vee$ of Δ is defined to be the simplicial complex whose faces are $\{[n] \setminus F\colon\ F \notin \Delta\}$. The Stanley–Reisner ideal of $\Delta^\vee$ is minimally generated by all monomials $x_{i_1} \cdots x_{i_k}$ where $(x_{i_1}, \ldots, x_{i_k})$ is a minimal prime ideal of I_Δ, see [91, Corollary 1.5.5].

4 Clean and Pretty Clean Modules

Let M be a finitely generated $\mathbb{Z}^n$-graded S-module. According to Dress [50] a prime filtration $\mathscr{F}$ of M is called *clean* if $\operatorname{Supp}(\mathscr{F}) = \operatorname{Min}(M)$, and M itself is called *clean* if M admits a clean filtration.

Since $\operatorname{Ass}(M) \subset \operatorname{Supp}(\mathscr{F})$, it follows that $\operatorname{Ass}(M) = \operatorname{Min}(M)$, if M is clean. In other words, a clean module has no embedded prime ideals. Moreover, if $\dim_K M_{\mathbf{a}} \leq 1$ for all $\mathbf{a} \in \mathbb{Z}^n$, then Theorem 9 implies that

$$\operatorname{fdepth}(M) = \operatorname{sdepth}(M) = \operatorname{depth}(M), \quad \text{if } M \text{ is clean.} \tag{2}$$

Hence clean modules satisfy Stanley's conjecture.

The question is whether clean modules appear in nature. At least for modules of the form S/I, where I is a squarefree monomial ideal, there is nice interpretation of cleanness. Since I is a squarefree monomial ideal, there exists a (unique) simplicial complex Δ on the vertex set $[n]$ such that $I = I_\Delta$, where I_Δ is the Stanley Reisner ideal of Δ.

Theorem 13 (Dress). *The simplicial complex Δ is shellable if and only if $K[\Delta]$ is a clean ring.*

Here we only show that when Δ is shellable, then $K[\Delta]$ is clean. More precisely we present the following result taken from [95]:

Proposition 14. *Let Δ be a shellable simplicial complex with shelling $F_1, \ldots, F_r$ and shelling numbers $a_1, \ldots, a_r$.*

Then the filtration $(0) = M_0 \subset M_1 \subset \cdots \subset M_{r-1} \subset M_r = K[\Delta]$ of $K[\Delta]$ with

$$M_i = \Big(\bigcap_{j=1}^{r-i} P_{F_j}\Big)/I_\Delta \quad \textit{and} \quad M_i/M_{i-1} \cong S/P_{F_{r-i+1}}(-a_{r-i+1})$$

is a clean filtration of $K[\Delta]$.

Proof. Recall that $P_{F_i} = (\{x_j\}_{j \notin F_i})$, and $I_\Delta = \bigcap_{i=1}^r P_{F_i}$. Therefore, if $F_1, \ldots, F_r$ is a shelling of Δ, then for $i = 2, \ldots, r$ we have

$$\bigcap_{j=1}^{i-1} P_{F_j} + P_{F_i} = P_{F_i} + (f_i).$$

Here $f_i = \prod_k x_k$, where the product is taken over those $k \in F_i$ such that $F_i \setminus \{k\}$ is a facet of $\langle F_1, \ldots, F_{i-1}\rangle \cap \langle F_i \rangle$. In particular it follows that $\deg f_i$ equals the ith shelling number a_i.

We obtain the following isomorphisms of graded S-modules

$$\begin{aligned}\Big(\bigcap_{j=1}^{i-1} P_{F_j}\Big)/\Big(\bigcap_{j=1}^{i} P_{F_j}\Big) &\cong \Big(\bigcap_{j=1}^{i-1} P_{F_j} + P_{F_i}\Big)/P_{F_i} = (P_{F_i} + (f_i))/P_{F_i} \\ &\cong (f_i)/(f_i)P_{F_i} \cong S/P_{F_i}(-a_i).\end{aligned}$$

The isomorphism $(P_{F_i} + (f_i))/P_{F_i} \cong (f_i)/(f_i)P_{F_i}$ results from the fact that $(f_i) \cap P_{F_i} = (f_i)P_{F_i}$ since the set of variables dividing f_i and the set of variables generating P_{F_i} have no element in common.

The theorem of Dress combined with the discussion preceding it shows that

$$\operatorname{sdepth}(K[\Delta]) = \operatorname{depth}(K[\Delta])$$

for any shellable simplicial complex.

The following results demonstrates the use of the theorem of Dress.

Proposition 15. *Let $I \subset S$ a squarefree monomial ideal which is a complete intersection. Then S/I is clean, and hence* $\operatorname{fdepth}(S/I) = \operatorname{sdepth}(S/I) = \operatorname{depth}(S/I)$.

Proof. Let Δ be the simplicial complex whose Stanley–Reisner ideal I_Δ is equal to I.

Let $G(I) = \{v_1, \ldots, v_m\}$. Then $\operatorname{supp}(v_i) \cap \operatorname{supp}(v_j) = \emptyset$ for all $i \neq j$, since I is a complete intersection. Hence by what we said at the end of Sect. 3 it follows that $I_{\Delta^\vee}$ is minimally generated by the monomials of the form $x_{i_1} \ldots x_{i_m}$ where $x_{i_j} \in \operatorname{supp}(v_j)$ for $j = 1, \ldots, m$. Thus we see that $I_{\Delta^\vee}$ is the matroidal ideal of the transversal matroid attached to the sets $\operatorname{supp}(v_1), \ldots, \operatorname{supp}(v_m)$, see [41, Sect. 5]. In [100, Lemma 1.3] and [41, Sect. 5] it is shown that any polymatroidal ideal has linear quotients, and this implies that Δ is a shellable simplicial complex, see [91, Proposition 8.2.5]. Hence by the theorem of Dress, S/I_Δ is clean.

Problem 16. Let Δ be a simplicial complex. Show that $\operatorname{fdepth}(K(\Delta]) = \dim(K[\Delta])$ if and only if Δ is shellable.

In order to obtain a similar result as that of Dress for modules of the form S/I where I is a monomial ideal, but not necessarily squarefree, the definition of cleanness has been modified as follows [95].

Definition 17. Let M be a finitely generated $\mathbb{Z}^n$-graded S-module. A prime filtration $\mathscr{F}\colon 0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_r = M$ of M with $\operatorname{Ass}(M_i/M_{i-1}) \cong S/P_i(-\mathbf{a}_i)$ for $i = 1, \ldots, r$ is called *pretty clean*, if for all $i < j$ for which $P_i \subset P_j$ it follows that $P_i = P_j$. In other words, a proper inclusion $P_i \subset P_j$ is only possible if $i > j$.

The module M is called *pretty clean*, if it has a pretty clean filtration. A ring is called pretty clean if it is a pretty clean module, viewed as a module over itself.

It is clear that any clean module is also pretty clean. It has been shown in [95, Theorem 10.5] that S/I, I a monomial ideal, is pretty clean if and only if the associated multi-complex is shellable. Hence the concept of pretty clean modules is the natural extension of clean modules. As for clean modules we have

Proposition 18. *Let M be a pretty clean module. Then*

$$\operatorname{fdepth}(M) = \operatorname{sdepth}(M) = \operatorname{depth}(M).$$

Proof. Let $\mathscr{F}$ be a pretty clean filtration of M. We show that $\mathrm{Ass}(M) = \mathrm{Supp}(\mathscr{F})$. Then Theorem 9 yields the desired result.

Indeed, let $\mathscr{F}: 0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_r = M$ be a pretty clean filtration of M with $\mathrm{Ass}(M_i/M_{i-1}) \cong S/P_i(-a_i)$ for $i = 1, \ldots, r$. Now let $P_j \in \mathrm{Supp}(\mathscr{F})$. We may assume that $P_i \neq P_j$ for $i < j$. By localization we obtain the filtration

$$0 = (M_0)_{P_j} \subset (M_1)_{P_j} \subset (M_2)_{P_j} \subset \cdots \subset (M_r)_{P_j} = (M)_{P_j}$$

of M_{P_j} with $(M_i)_{P_j}/(M_{i-1})_{P_j} = (M_i/M_{i-1})_{P_j} = S_{P_j}/P_i S_{P_j} = 0$ for $i < j$. The last equation follows since $\mathscr{F}$ be a pretty clean filtration. It follows that $(M_j)_{P_j} = S_{P_j}/P_j S_{P_j}$. Therefore, $P_j \in \mathrm{Ass}(M_j)$, and hence $P_j \in \mathrm{Ass}(M)$ since $M_j \subset M$. This shows that $\mathrm{Supp}(\mathscr{F}) \subset \mathrm{Ass}(M)$. Since the other inclusion always holds, the claim follows.

There is a large and interesting class of ideals for which S/I is pretty clean. Let I and J be monomial ideals. We denote by $I: J^\infty$ the monomial ideal $\bigcup_{k\geq 1} I: J^k$. This ideal is called the *J-saturation* of I. A monomial ideal $I \subset S$ is called of *Borel type*, if

$$I : (x_j)^\infty = I : (x_1, \ldots, x_j)^\infty$$

for all $j = 1, \ldots, n$. Some authors call these ideals also ideals of nested type [19]. An important class of ideals of Borel type are the so-called Borel fixed ideals which play an important role in the theory of graded ideals, because they are just the generic initial ideals of graded ideals in a polynomial ring. Among the Borel fixed ideals are the strongly stable ideals.

Monomial ideals of Borel type can be characterized as follows [96, Proposition 2.2]:

Proposition 19. *Let I be a monomial ideal. The following conditions are equivalent:*

(a) *I is of Borel type.*
(b) *Let $u \in I$ be a monomial and suppose that $x_i^q | u$ for some $q > 0$. Then for all $j < i$ there exists an integer t such that $x_j^t(u/x_i^q) \in I$.*
(c) *Let $u \in I$ be a monomial; then for all integers i, j with $1 \leq j < i \leq n$ there exists an integer t such that $x_j^t(u/x_i^{\nu_i(u)}) \in I$, where $\nu_i(u)$ is the largest number for which $x_i^{\nu_i(u)}$ divides u.*

Proof. (a) $\Rightarrow$ (b): Let $u \in I$ be a monomial such that $x_i^q | u$ for some $q > 0$, and let $j < i$. Then $u = x_i^q v$ with $v \in I : (x_i)^\infty$. Condition (a) implies that $I : (x_i)^\infty \subset I : (x_j)^\infty$. Therefore, there exists t such that $x_j^t(u/x_i^q) = x_j^t v \in I$.

The implication (b) $\Rightarrow$ (c) is trivial. For the converse, let $u \in I$ be a monomial such that $x_i^q | u$ for some $q > 0$, and let $j < i$. By (c) there exists t such that $x_j^t(u/x_i^{\nu_i(u)}) \in I$. Therefore, $x_j^t(u/x_i^q) = x_i^{\nu_i(u)-q} x_j^t(u/x_i^{\nu_i(u)}) \in I$.

(b) $\Rightarrow$ (a): We will show that $I : x_i^\infty \subset I : (x_j)^\infty$ for $j < i$. This will imply (a). Let $u \in I : (x_i)^\infty$ be a monomial. Then $x_i^q u \in I$ for some $q > 0$, and so (b) implies that $x_j^t u \in I$ for some t, that is, $u \in I : (x_j)^\infty$.

For a monomial u we set $m(u) = \max\{i : \nu_i(u) \neq 0\}$, and for a monomial ideal $I \neq 0$ we set $m(I) = \max\{m(u) : u \in G(I)\}$.

Let $I \neq 0$ be a monomial ideal. Recursively we define an ascending chain of monomial ideals $I = I_0 \subset I_1 \subset \cdots$ as follows: We let $I_0 = I$. Suppose I_ℓ is already defined. If $I_\ell = S$, then the chain ends. Otherwise, we let $n_\ell = m(I_\ell)$, and set $I_{\ell+1} = I_\ell : (x_{n_\ell})^\infty$. Notice that $n \geq n_0 > n_1 > \cdots \geq 1$, so that this chain has length at most n. We call this chain of ideals, the *sequential chain of* I.

Lemma 20. *Let $I \neq 0$ be a monomial ideal of Borel type. Then*

$$I_{\ell+1} = I_\ell : (x_1, \ldots, x_{n_\ell})^\infty \quad \textit{for all} \quad \ell.$$

Proof. We will show that $I_\ell : (x_{n_\ell})^\infty = I_\ell : (x_1, \ldots, x_{n_\ell})^\infty$ for all ℓ. Let $u \in I_\ell : (x_{n_\ell})^\infty$ be a monomial. Then there exists an integer q_ℓ such that $x_{n_\ell}^{q_\ell} u \in I_\ell$. Since $I_\ell = I_{\ell-1} : (x_{n_{\ell-1}})^\infty$, there exists an integer $q_{\ell-1}$ such that $x_{n_\ell}^{q_\ell} x_{n_{\ell-1}}^{q_{\ell-1}} u \in I_{\ell-1}$. Proceeding in this way we find integers $q_0, \ldots, q_\ell$ such that $x_{n_\ell}^{q_\ell} \cdots x_{n_0}^{q_0} u \in I$. By Proposition 19(c) there exists for all $j < n_\ell$ an integer t_j such that $x_j^{t_j}(x_{n_{\ell-1}}^{q_{\ell-1}} \cdots x_{n_0}^{q_0}) u \in I$. This implies that $x_j^{t_j} u \in I_\ell$ for all $j < n_\ell$. In other words, $u \in I_\ell : (x_1, \ldots, x_{n_\ell})^\infty$.

Proposition 21. *Let I be a monomial ideal of Borel type. Then $R = S/I$ is pretty clean.*

Proof. We may assume that $I \neq 0$. Let $I = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_r = S$ be the sequential chain of I. Fix an integer $\ell < r$. Let $n_j = m(I_j)$ for all j, then the elements of $G(I_j)$ belong to $K[x_1, \ldots, x_{n_\ell}]$ for all $j \geq \ell$. Let J_l be the ideal generated by $G(I_\ell)$ in $K[x_1, \ldots, x_{n_\ell}]$. Then the saturation $J_\ell^{sat} = J_\ell : (x_1, \ldots, x_{n_\ell})^\infty$ is generated by the elements of $G(I_{\ell+1})$. It follows that

$$I_{\ell+1}/I_\ell = (J_\ell^{sat}/J_\ell)[x_{n_\ell+1}, \ldots, x_n]. \tag{3}$$

From this description it follows that $I_{\ell+1}/I_\ell$ has a prime filtration with the property that all of its factors are isomorphic to S/P with $P = (x_1, \ldots, x_{n_\ell})$. Composing the filtration of all the factors $I_{\ell+1}/I_\ell$ to a filtration of S/I, we see that S/I has a pretty clean filtration.

5 Janet Decompositions

In this section we describe an algorithm which goes back to Maurice Janet [115], a French mathematician. Janet's algorithm applied to monomial ideals gives an explicit Stanley decomposition of these ideals. We call the Stanley decomposition obtained in this way the *Janet decomposition* of I.

In order to describe his construction, let $I \subset S = K[x_1, \dots, x_n]$ be a monomial ideal. For each integer $k \geq 0$ we define the monomial ideals $I_k \subset S' = K[x_1, \dots, x_{n-1}]$ by the equation

$$I \cap x_n^k K[x_1, \dots, x_{n-1}] = I_k x_n^k.$$

It is clear that

$$I_0 \subset I_1 \subset I_2 \subset \dots.$$

Since S' is Noetherian, we have that $I_k = I_{k+1}$ for $k \gg 0$. We define the integers

$$a = \min\{k\colon\ I_k \neq 0\},$$

and

$$b = \min\{d\colon\ I_k = I_{k+1} \text{ for all } k \geq d\}.$$

Then

$$I = \bigoplus_{a \leq k < b} I_k x_n^k \oplus I_b x_n^b K[x_n]. \tag{4}$$

Now Janet's decomposition is defined by induction on n. If $n = 1$, then $I = (x_1)^k$ and $I = x_1^k K[x_1]$ is the Janet decomposition of I. Suppose the Janet decomposition for any ideal in the polynomial ring S' is known. Let $I \subset S$ be a monomial ideal and consider its decomposition (4). By induction hypothesis we have for each I_k a Janet decomposition $I_k = \bigoplus_{j=1}^{r_k} u_{kj} K[Z_{kj}]$. By using (4) we obtain the Stanley decomposition

$$I = \bigoplus_{a \leq k < b} \bigoplus_{j=1}^{r_k} u_{kj} x_n^k K[Z_{kj}] \oplus \bigoplus_{j=1}^{r_b} u_{bj} x_n^b K[Z_{bj}, x_n].$$

The Stanley decomposition so obtained is called the *Janet decomposition* of I.

Example 22. Let $I = (x_1, x_2)^2 \subset K[x_1, x_2]$. Then with the notation introduced, $I_0 = x_1^2 K[x_1]$, $I_1 = x_1 K[x_1]$ and $I_k = K[x_1]$ for $k \geq 2$. Hence by (4), $I = x_1^2 K[x_1] \oplus (x_1 K[x_1]) x_2 \oplus (K[x_1]) x_2^2 K[x_2]$, which gives the Janet decomposition

$$I = x_1^2 K[x_1] \oplus x_1 x_2 K[x_1] \oplus x_2^2 K[x_1, x_2].$$

Problem 23. For any integer $k \geq 1$ compute the Janet decomposition of $(x_1, \dots, x_n)^k$.

The following result is an immediate consequence of the Janet construction.

Corollary 24. *Let I be a monomial ideal in the polynomial in n variables $n \geq 1$. Then* $\operatorname{sdepth}(I) \geq 1$.

A Janet decomposition can also be constructed for S/I where I is a monomial ideal. For this purpose we identify S/I as $\mathbb{Z}^n$-graded K-vector space with I^c whose monomial K-basis $\mathscr{B}$ consists of all monomials $u \in S \setminus I$. The monomial basis $\mathscr{B}$ of I^c forms an *order ideal* in the poset of all monomials ordered by divisibility, that is, if $u \in \mathscr{B}$ and $v|u$, then $v \in \mathscr{B}$. On the other hand, if $\mathscr{B}'$ is any order ideal, then the vector space spanned the monomials $u \in S \setminus \mathscr{B}'$ is a monomial ideal in S.

We observe that for all k there exist (uniquely determined) ideals $I_k \subset S'$ such that

$$I^c \cap x_n^k K[x_1, \ldots, x_{n-1}] = I_k^c x_n^k.$$

Since $I_0^c \supset I_1^c \supset \cdots$, it follows that $I_0 \subset I_1 \subset \cdots$ is an ascending chain of ideals in S' which stabilizes, because S' is Noetherian. Thus we may define the integers

$$a = \min\{k\colon\ I_k^c \neq 0\},$$

and

$$b = \min\{d\colon\ I_k^c = I_{k+1}^c \text{ for all } k \geq d\},$$

and obtain

$$I^c = \bigoplus_{a \leq k < b} I_k^c x_n^k \oplus I_b^c x_n^b K[x_n].$$

Equivalently, we obtain a $\mathbb{Z}^n$-graded direct sum decomposition of S/I, as an S'-module

$$S/I = \bigoplus_{a \leq k < b} (S'/I_k)x_n^k \oplus (S'/I_b)x_n^b \otimes_K K[x_n]. \tag{5}$$

As before, this is the first step in the inductive definition of a Janet decomposition of S/I. We leave it to the reader to complete the construction of the Janet decomposition for S/I. As an example the reader may work out the following

Problem 25. Compute the Janet decomposition of

$$S/(x_1^n, x_1^{n-1}x_2, x_1^{n-2}x_3, \cdots, x_1x_n).$$

6 Stanley Depth and Regular Sequences

In this section we present a theorem of Rauf [164, Proposition 1.10] which describes how the Stanley depth of a $\mathbb{Z}^n$-graded module behaves under reduction modulo a variable which is regular on M.

Theorem 26. *Let M be a finitely generated $\mathbb{Z}^n$-graded S-module, and let x_k be regular on M. If $M/x_kM = \bigoplus_{i=1}^r n_i K[Z_i]$ is a Stanley decomposition of M/x_kM, where $n_i = m_i + x_kM$ and m_i is homogeneous, then $M = \bigoplus_{i=1}^r m_i K[Z_i, x_k]$ is a Stanley decomposition of M. In particular,* $\operatorname{sdepth}(M) \geq \operatorname{sdepth}(M/x_kM) + 1$.

Proof. Let $N = \sum_{i=1}^r m_i K[Z_i, x_k]$. Then $N \subset M$ and $M = N + x_kM$, by the choice of N. By induction on d one shows that $M = N + x_k^d M$. Indeed, suppose this is known for $d-1$. Then $M = N + x_k^{d-1}(N + x_kM) = N + x_k^{d-1}N + x_k^d M = N + x_k^d M$. From the fact that $M = N + x_k^d M$ for all d, one easily deduces $M = N$.

Suppose the sum $\sum_{i=1}^r m_i K[Z_i, x_k]$ is not direct. Then for $i=1,\dots,r$ there exist monomials $q_i \in K[Z_i, x_k]$, not all zero, such that $\sum_{i=1}^r m_i q_i = 0$. There exists a largest integer s such that x_k^s divides all q_i. Then $x_k^s(\sum_{i=1}^r m_i q_i') = 0$, where $q_i' = q_i/x_k^s$. Since x_k is a non-zero divisor on M it follows that $\sum_{i=1}^r m_i q_i' = 0$. This implies that $\sum_{i=1}^r n_i q_i' = 0$, which, since not all $q_i' = 0$, contradicts the assumption that $\bigoplus_{i=1}^r n_i K[Z_i]$ is a Stanley decomposition of M/x_kM.

It remains to be shown that each $m_i K[Z_i, x_k]$ is a Stanley space. Suppose that $m_i f = 0$ for some non-zero polynomial $f \in K[Z_i, x_k]$. We may assume that x_k does not divide f, since x_k is a non-zero divisor on M. Then $f = f_0 + f_1 x_k$ with $f_0 \in K[Z_i]$ and $f_0 \neq 0$. It follows that $n_i f_0 = 0$, and this is a contradiction since $n_i K[Z_i]$ is a Stanley space.

As a first application we have the following result which was stated in [36]. We present its proof following [31, Proposition 2.13].

Theorem 27. *Let M be a $\mathbb{Z}^n$-graded S-module. Then*

(a) $\operatorname{depth}(M) = 0$ *if* $\operatorname{sdepth}(M) = 0$*;*
(b) $\operatorname{sdepth}(M) = 0$ *if* $\operatorname{depth}(M) = 0$ *and* $\dim_K M_{\mathbf{a}} \leq 1$ *for all* $\mathbf{a}$.

Proof. (a) We show that if $\operatorname{depth}(M) > 0$, then $\operatorname{sdepth}(M) > 0$. We set $U_{n+1} = M$, $U_0 = 0$, and $U_i = \{m \in U_{i+1}\colon\ x_i^k m = 0 \text{ for some } k > 0\}$ for $i = 1, \dots, n$. Then we obtain the following chain of $\mathbb{Z}^n$-graded submodules

$$0 = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_{n+1} = M.$$

This implies that $\operatorname{sdepth}(M) \geq \min\{\operatorname{sdepth}(U_{i+1}/U_i)\colon\ i = 0, \dots, n\}$.

Since $U_1/U_0 = U_1 = \{m \in M\colon\ (x_1, \dots, x_n)^k m = 0 \text{ for some } k\}$, and since $\operatorname{depth}(M) > 0$ it follows that $U_1/U_0 = 0$. By the definition of U_{i+1}/U_i we have that x_i is regular on U_{i+1}/U_i for $i > 0$. Thus Theorem 26 implies $\operatorname{sdepth}(U_{i+1}/U_i) > 0$ for $i > 0$. This yields the desired conclusion.

(b) Assuming that $\operatorname{depth}(M) = 0$, there exists a homogeneous element $m \neq 0$ such that $x_j m = 0$ for all j. Let $\mathscr{D}$: $M = \bigoplus_{i=1}^r m_i K[Z_i]$ be a Stanley decomposition with $\operatorname{sdepth}(\mathscr{D}) = \operatorname{sdepth}(M)$. Since $\dim_K M_{\mathbf{a}} \leq 1$ for all $\mathbf{a}$, it follows that $m \in m_i K[Z_i]$ for some i. However since $m_i K[Z_i]$ is a free $K[Z_i]$-submodule of M and since $x_j m = 0$ for all j, this is only possible if $K[Z_i] = K$. Thus $\operatorname{sdepth}(M) = 0$.

As a second application we obtain the following result of [62, Theorem 2.2], see also [31, Corollary 2.12].

Theorem 28. *Let M be a $\mathbb{Z}^n$-graded S-module, and*

$$\mathbb{F}\colon \cdots \to F_p \to F_{p-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

a $\mathbb{Z}^n$-graded free resolution of M. Let Z_p be the pth syzygy module of M with respect to this resolution, i.e., the image of $F_p \to F_{p-1}$. Then $\operatorname{sdepth}(Z_p) \geq \min\{p, n\}$.

Proof. One easily proves by induction on p that $x_1, \ldots, x_p$ is a regular sequence on Z_p. Thus the assertion follows immediately from Theorem 26.

7 How to Compute the Stanley Depth

In this section we present an algorithm, introduced in [101], in order to compute the Stanley depth of a module of the form I/J where $J \subset I \subset S$ are monomial ideals. The algorithm amounts to attach to each partition of a certain posets associated with I/J a Stanley decompositions of I/J, and to show that among the finitely many Stanley decompositions so obtained there is one which gives the Stanley depth of I/J. Here in this presentation, the proofs given are more straightforward than those in [102] where Miller's functors as defined in [140] are used.

We define a natural partial order on $\mathbb{N}^n$ as follows: $a \leq b$ if and only if $a(i) \leq b(i)$ for $i = 1, \ldots, n$. Note that $x^a | x^b$ if and only if $a \leq b$. Here, for any $c \in \mathbb{N}^n$ we denote as usual by x^c the monomial $x_1^{c(1)} x_2^{c(2)} \cdots x_n^{c(n)}$. Observe that $\mathbb{N}^n$ with the partial order introduced is a distributive lattice with meet $a \wedge b$ and join $a \vee b$ defined as follows: $(a \wedge b)(i) = \min\{a(i), b(i)\}$ and $(a \vee b)(i) = \max\{a(i), b(i)\}$. We also denote by ϵ_j the jth canonical unit vector in $\mathbb{Z}^n$.

Suppose I is generated by the monomials $x^{a_1}, \ldots, x^{a_r}$ and J by the monomials $x^{b_1}, \ldots, x^{b_s}$. We choose $g \in \mathbb{N}^n$ such that $a_i \leq g$ and $b_j \leq g$ for all i and j, and let $P_{I/J}^g$ be the set of all $c \in \mathbb{N}^n$ with $c \leq g$ and such that $a_i \leq c$ for some i and $c \not\geq b_j$ for all j. The set $P_{I/J}^g$ viewed as a subposet of $\mathbb{N}^n$ is a finite poset. We call it the *characteristic poset* of I/J with respect to g. There is a natural choice for g, namely the join of all the a_i and b_j. For this g, the poset $P_{I/J}^g$ has the least number of elements, and we denote it simply by $P_{I/J}$. Note that if Δ is a simplicial complex on the vertex set $[n]$, then P_{S/I_Δ} is just the face poset of Δ.

Figure 2 shows the characteristic poset for the maximal ideal $\mathfrak{m} = (x_1, x_2, x_3) \subset K[x_1, x_2, x_3]$. The elements of this poset correspond to the squarefree monomials x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3 and $x_1x_2x_3$. Thus the corresponding labels in Fig. 2 should be $(1,0,0), (0,1,0), \ldots, (1,1,1)$. In the squarefree case, like in this example, it is however more convenient and shorter to replace the $(0,1)$-vectors (which label the vertices in the characteristic poset) by their support. In other words, each $(0,1)$-vector with support $\{i_1 < i_2 < \cdots < i_k\}$ is replaced by $i_1 i_2 \cdots i_k$, as done in Fig. 2.

Given any poset P and $a, b \in P$ we set $[a,b] = \{c \in P\colon\ a \leq c \leq b\}$ and call $[a,b]$ an *interval*. Of course, $[a,b] \neq \emptyset$ if and only if $a \leq b$. Suppose P is a finite poset. A *partition* of P is a disjoint union

Fig. 2 A poset P

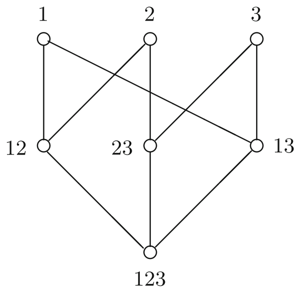
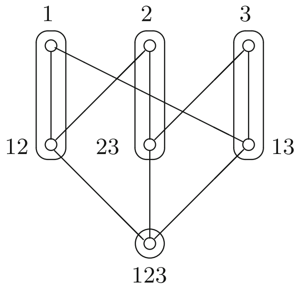


Fig. 3 A partition of the poset P



$$\mathscr{P}\colon\ P = \bigcup_{i=1}^{r} [a_i, b_i]$$

of intervals.

Figure 3 displays a partition of the poset given in Fig. 2. The framed regions in Fig. 3 indicate that $P_{\mathfrak{m}} = [1, 12] \cup [2, 23] \cup [3, 13] \cup [123, 123]$.

We will show that each partition of $P^g_{I/J}$ gives rise to a Stanley decomposition of I/J.

In order to describe the Stanley decomposition of I/J coming from a partition of $P^g_{I/J}$ we shall need the following notation: for each $b \in P^g_{I/J}$, we set $Z_b = \{x_j\colon\ b(j) = g(j)\}$. We also introduce the function

$$\rho\colon\ P^g_{I/J} \to \mathbb{Z}_{\geq 0}, \quad c \mapsto \rho(c),$$

where $\rho(c) = |\{j\colon\ c(j) = g(j)\}| (= |Z_c|)$. We then have

Theorem 29. (a) *Let $\mathscr{P}\colon P^g_{I/J} = \bigcup_{i=1}^r [c_i, d_i]$ be a partition of $P^g_{I/J}$. Then*

$$\mathscr{D}(\mathscr{P})\colon I/J = \bigoplus_{i=1}^{r} (\bigoplus_{c} x^c K[Z_{d_i}]) \tag{6}$$

is a Stanley decomposition of I/J, where the inner direct sum is taken over all $c \in [c_i, d_i]$ for which $c(j) = c_i(j)$ for all j with $x_j \in Z_{d_i}$. Moreover,

$$\operatorname{sdepth}(\mathscr{D}(\mathscr{P})) = \min\{\rho(d_i)\colon\ i = 1, \ldots, r\}.$$

(b) *One has*

$$\operatorname{sdepth}(I/J) = \max\{\operatorname{sdepth}(\mathscr{D}(\mathscr{P})) : \mathscr{P} \textit{ is a partition of } P^g_{I/J}\}.$$

In particular, there exists a partition $\mathscr{P}$: $P^g_{I/J} = \bigcup_{i=1}^r [c_i, d_i]$ *of* $P^g_{I/J}$ *such that*

$$\operatorname{sdepth}(I/J) = \min\{\rho(d_i) : i = 1, \ldots, r\}.$$

Proof. (a) We first show that the sum of the K-vector spaces in (6) is equal to the K-vector space spanned by all monomials $u \in I \setminus J$ (which of course is isomorphic to the K-vector space I/J).

Let $u = x^e$ be a monomial in $I \setminus J$ and let $c' = e \wedge g$. Then, $c' \in P^g_{I/J}$ and consequently, there exists $i \in \{1, \ldots, r\}$ such that $c' \in [c_i, d_i]$. Let c be the vector with

$$c(j) = \begin{cases} c_i(j), \text{ if } d_i(j) = g(j), \\ c'(j), \text{ otherwise.} \end{cases}$$

It follows from the definition of c that $x^c K[Z_{d_i}]$ is one of the Stanley spaces appearing in (6). We claim that $u \in x^c K[Z_{d_i}]$, equivalently, that $x^{e-c} \in K[Z_{d_i}]$. Indeed, if $x_j \in Z_{d_i}$, then $d_i(j) = g(j)$, and hence $e(j) \geq c'(j) \geq c_i(j) = c(j)$. On the other hand, if $x_j \notin Z_{d_i}$, then $g(j) > d_i(j) \geq c'(j) = c(j)$. Since $c'(j) = \min\{e(j), g(j)\}$, it therefore follows that $e(j) = c(j)$, as desired.

In order to prove that the sum (6) is direct, it suffices to show that any two different Stanley spaces in (6) have no monomial in common. Suppose to the contrary that $x^b \in x^p K[Z_{d_i}] \cap x^q K[Z_{d_j}]$ and that $x^p K[Z_{d_i}] \neq x^q K[Z_{d_j}]$ are both summands in (6). Since each of the inner sums in (6) is direct, we have that $i \neq j$.

We claim that $x^b \in x^p K[Z_{d_i}]$ yields $b \wedge g \in [c_i, d_i]$. Indeed, since $c_i \leq b \wedge g$, the claim follows once it is shown that $b \wedge g \leq d_i$. If $d_i(j) = g(j)$, then

$$(b \wedge g)(j) = \min\{b(j), g(j)\} \leq g(j) = d_i(j).$$

If $d_i(j) < g(j)$, then $x_j \notin Z_{d_i}$ and hence $b(j) = p(j)$. Together with the inequality $p(j) \leq d_i(j) < g(j)$, we obtain that $(b \wedge g)(j) = p(j) \leq d_i(j)$. In both cases the claim follows.

Similarly, since $x^b \in x^q K[Z_{d_j}]$ we see that $b \wedge g \in [c_j, d_j]$. This is a contradiction, since $[c_i, d_i] \cap [c_j, d_j] = \emptyset$.

The statement about the Stanley depth of $\mathscr{D}(\mathscr{P})$ follows immediately from the definitions.

(b) Let $\mathscr{D}$ be an arbitrary Stanley decomposition of I/J. We are going to construct a partition $\mathscr{P}$ of $P^g_{I/J}$ such that $\operatorname{sdepth}(\mathscr{D}(\mathscr{P})) \geq \operatorname{sdepth}(\mathscr{D})$. This will then yield the desired conclusion. First, to each $b \in P^g_{I/J}$ we assign an interval

$[c,d] \subset P^g_{I/J}$: since $x^b \in I \setminus J$, there exists a Stanley space $x^c K[Z]$ in the decomposition $\mathscr{D}$ of I/J with $x^b \in x^c K[Z]$. It follows that $c \in P^g_{I/J}$ and $b(j) = c(j)$ for all j with $x_j \notin Z$. Now, we define $d \in \mathbb{N}^n$ by setting

$$d(j) = \begin{cases} g(j), \text{ if } x_j \in Z, \\ c(j), \text{ if } x_j \notin Z. \end{cases}$$

Observe that $[c,d] \subset P^g_{I/J}$. We noticed already that $c \in P^g_{I/J}$. It remains to be shown that $d \in P^g_{I/J}$. Since $x^c K[Z] \in I \setminus J$, it follows that $x^{c+\sum_j n_j \epsilon_j} \in I \setminus J$, where the sum is taken over all j with $x_j \in Z$ and where for all j we have $n_j \in \mathbb{Z}_{\geq 0}$. Therefore $d = c + \sum_j (g(j) - c(j))\epsilon_j \in P^g_{I/J}$.

Next we show that $b \in [c,d]$. For this we need to show that $b \leq d$. Indeed, if $x_j \in Z$, then $b(j) \leq g(j) = d(j)$. Otherwise $d(j) = c(j) = b(j)$ and consequently the inequality holds. Since $b \in [c,d]$, we obtain that $x^b \in x^c K[Z_d]$, and $Z \subseteq Z_d$, according to the definition of d.

In order to complete the proof we now show that the intervals constructed above provide a partition $\mathscr{P}$ of $P^g_{I/J}$ and that $\operatorname{sdepth}(\mathscr{D}(\mathscr{P})) \geq \operatorname{sdepth}(\mathscr{D})$.

It is clear that these intervals cover $P^g_{I/J}$. Therefore it is enough to check that for any $b_1, b_2 \in P^g_{I/J}$ with $b_1 \neq b_2$, the corresponding intervals obtained from our construction, say $[c_1, d_1]$ and $[c_2, d_2]$, satisfy either $[c_1, d_1] = [c_2, d_2]$ or $[c_1, d_1] \cap [c_2, d_2] = \emptyset$.

To each c_i corresponds a Stanley space $x^{c_i} K[Z_i]$ in the given Stanley decomposition $\mathscr{D}$. We consider two cases. In the first case, we assume that $c_1 = c_2$. Then $Z_1 = Z_2$, and consequently $d_1 = d_2$. Hence $[c_1, d_1] = [c_2, d_2]$. In the second case, we assume $c_1 \neq c_2$. In this case we prove that $[c_1, d_1] \cap [c_2, d_2] = \emptyset$. Assume, by contradiction, that there exists $e \in P^g_{I/J}$ such that $e \in [c_1, d_1] \cap [c_2, d_2]$. It follows from the construction of the interval $[c_1, d_1]$ that $c_1(j) = d_1(j)$ if $x_j \notin Z_1$. Therefore, $e \in [c_1, d_1]$ implies that $e(j) = c_1(j)$, for all j with $x_j \notin Z_1$, and hence we obtain that $x^e \in x^{c_1} K[Z_1]$. Analogously, one obtains that $x^e \in x^{c_2} K[Z_2]$, a contradiction since $x^{c_1} K[Z_1] \cap x^{c_2} K[Z_2] = 0$.

To establish the inequality $\operatorname{sdepth}(\mathscr{D}(\mathscr{P})) \geq \operatorname{sdepth}(\mathscr{D})$, we observe that $\operatorname{sdepth}(\mathscr{D}(\mathscr{P}))$ is equal to the minimum of all integers $|Z_d|$ where $[c,d]$ belongs to $\mathscr{P}$. On the other hand, we have already shown that for each Stanley space $x^c K[Z]$ in $\mathscr{D}$ such that $c \in P^g_{I/J}$ we have $|Z_d| \geq |Z|$. This yields the desired inequality.

Problem 30. Let $J \subset I$ be monomial ideals of $S = K[x_1, \ldots, x_n]$, and let $T = S[x_{n+1}]$ be the polynomial ring over S in the variable x_{n+1}. Then $\operatorname{sdepth}(IT/JT) = \operatorname{sdepth}(I/J) + 1$.

Problem 31. Use Theorem 29 to show that if I is a squarefree monomial ideal, then $\operatorname{sdepth}(I) \geq \min\{\deg(u)\colon u \in G(I)\}$.

As a first application we will show that Stanley's conjecture on the Stanley depth implies another conjecture by Stanley regarding partitions of simplicial complexes.

Let Δ be a simplicial complex of dimension $d-1$ on the vertex set $[n]$. A subset $\mathscr{I} \subset \Delta$ is called an *interval*, if there exist faces $F, G \in \Delta$ such that $\mathscr{I} = \{H \in \Delta\colon F \subseteq H \subseteq G\}$. We denote this interval given by F and G also by $[F, G]$. A *partition* $\mathscr{P}$ of Δ is a presentation of Δ as a disjoint union of intervals.

Let $I_\Delta \subset S$ be the Stanley–Reisner ideal of Δ and $K[\Delta] = S/I_\Delta$ its Stanley–Reisner ring. Obviously partitions of $P^g_{K[\Delta]}$ with $g = (1, 1, \ldots, 1)$ correspond to partitions of Δ. To simplify notation we write $P_{K[\Delta]}$ for $P^g_{K[\Delta]}$ when $g = (1, \ldots, 1)$, and similarly P_{I_Δ} for $P^g_{I_\Delta}$.

A Stanley space $uK[Z]$ is called a *squarefree Stanley space*, if u is a squarefree monomial and $\mathrm{supp}(u) \subseteq Z$. We shall use the following notation: for $F \subseteq [n]$ we set $x_F = \prod_{i\in F} x_i$ and $Z_F = \{x_i\colon i \in F\}$. Then a Stanley space is squarefree if and only if it is of the form $x_F K[Z_G]$ with $F \subseteq G \subseteq [n]$.

A Stanley decomposition of S/I is called a *squarefree Stanley decomposition* of S/I if all Stanley spaces in the decomposition are squarefree.

Having in mind the relationship between partitions of simplicial complexes and partitions of the characteristic poset of the corresponding Stanley–Reisner ideal, we obtain as an immediate consequence of Theorem 29 the following

Corollary 32. *Let Δ be a simplicial complex on the vertex set $[n]$, and let $\mathscr{P}\colon\ \Delta = \bigcup_{i=1}^r [F_i, G_i]$ be a partition of Δ. Then*

(a) *$\mathscr{D}(\mathscr{P})\colon\ K[\Delta] = \bigoplus_{i=1}^r x_{F_i} K[Z_{G_i}]$ is a squarefree Stanley decomposition of $K[\Delta]$.*
(b) *The map $\mathscr{P} \mapsto \mathscr{D}(\mathscr{P})$ establishes a bijection between partitions of Δ and squarefree Stanley decompositions of $K[\Delta]$.*
(c) *There exists a partition $\mathscr{P}$ of Δ such that* $\mathrm{sdepth}(K[\Delta]) = \mathrm{sdepth}(\mathscr{D}(\mathscr{P}))$.

It follows from statement (c) of the preceding corollary that Stanley's conjecture holds for the Stanley–Reisner ring $K[\Delta]$ if and only if there exists a partition $\Delta = \bigcup_{i=1}^r [F_i, G_i]$ of Δ with $|G_i| \geq \mathrm{depth}(K[\Delta])$ for all i.

Let $\mathscr{F}(\Delta)$ denote the set of facets of Δ. Stanley calls a simplicial complex Δ *partitionable* if there exists a partition $\Delta = \bigcup_{i=1}^r [F_i, G_i]$ with $\mathscr{F}(\Delta) = \{G_1, \ldots, G_r\}$. Stanley conjectures [182, Conjecture 2.7] (see also [183, Problem 6]) that each Cohen–Macaulay simplicial complex is partitionable. In view of Corollary 32(c) and the comments following it, we see that the conjecture on Stanley depth implies the conjecture on partitionable simplicial complexes.

In the next section we will see that Stanley's conjecture on the Stanley depth, restricted to Stanley–Reisner rings, is indeed equivalent to his conjecture on partionable simplicial complexes.

8 Characteristic Posets and Skeletons

The main purpose of this section is the proof of the fact, shown in [99], that Stanley's conjecture holds for all S/I, I a monomial ideal, if and only if it holds for all such algebras S/I which are Cohen–Macaulay.

Let $J \subset I \subset S$ be monomial ideals. We choose $g \in \mathbb{N}^n$ for which the characteristic poset $P^g_{I/J}$ of I/J is defined. In the previous section we introduced the ρ-function, whose definition we now extend as follows: for any $b \in \mathbb{N}^n$ we define subsets $Y_b = \{x_j : b(j) \neq g(j)\}$ and $Z_b = \{x_j : b(j) = g(j)\}$ of $X = \{x_1, \ldots, x_n\}$ and set $\rho(b) = |\{j : b(j) = g(j)\}| = |Z_b|$.

Lemma 33. *With the notation introduced we have*

(a) $\dim(I/J) = \max\{\rho(b) : b \in P^g_{I/J}\}$.
(b) $\rho(b) \leq \dim(I/J)$ *for all* $b \in \mathbf{N}^n$ *with* $x^b \in I \setminus J$.

Proof. (a) We choose a partition $\mathscr{P}: P^g_{I/J} = \bigcup_{i=1}^r [c_i, d_i]$ of $P^g_{I/J}$. Then Theorem 29 yields the Stanley decomposition

$$\mathscr{D}(\mathscr{P}): I/J = \bigoplus_{i=1}^r \Big(\bigoplus_c x^c K[Z_{d_i}]\Big) \tag{7}$$

of I/J, where the inner direct sum is taken over all $c \in [c_i, d_i]$ for which $c(j) = c_i(j)$ for all j with $x_j \in Z_{d_i}$.

We use this Stanley decomposition to compute the Hilbert series of I/J and obtain

$$\mathrm{Hilb}(I/J) = \sum_{i=1}^r \sum_c \frac{t^{|c|}}{(1-t)^{\rho(d_i)}}$$

with the summation on the c as above. Here $|c| = \sum_{i=1}^n c_i$.

We may assume that $\rho(d_r) \geq \rho(d_i)$ for all i. Then

$$\mathrm{Hilb}(I/J) = \frac{\sum_{i=1}^r \sum_c t^{|c|}(1-t)^{\rho(d_r)-\rho(d_i)}}{(1-t)^{\rho(d_r)}} = \frac{Q(t)}{(1-t)^{\rho(d_r)}}$$

with $Q(1) \neq 0$. Thus it follows from [91, Theorem 6.1.3] that $\rho(d_r) = \dim(I/J)$. Since $\rho(d_r) = \max\{\rho(b) : b \in P^g_{J/I}\}$, the assertion follows.

(b) Let $g' = g \vee b$. Then $b \in P^{g'}_{I/J}$, and hence $|\{j : b(j) = g'(j)\}| \leq \dim(I/J)$, by (a). Since $\rho(b) = |\{j : b(j) = g(j)\}| \leq |\{j : b(j) \geq g(j)\}| = |\{j : b(j) = g'(j)\}|$, the assertion follows.

Motivated by Lemma 33(a) we consider for each $j \leq d$, the monomial ideal I_j generated by I together with all monomials x^b such that $\rho(b) > j$. We then obtain a chain of monomial ideals

$$I = I_d \subset I_{d-1} \subset \cdots \subset I_0 \subset S.$$

Of course this chain of ideals depends not only on I, but also on the choice of g.

Consider the special case, where $I = I_\Delta$ is the Stanley–Reisner ideal of a simplicial complex Δ on the vertex set $\{1, \ldots, n\}$. Then for $g = (1, \ldots, 1)$ we have $I_j = I_{\Delta^{(j)}}$ where $\Delta^{(j)}$ is the jth skeleton of Δ, that is the subcomplex of Δ with faces $F \in \Delta$ with $\dim(F) \le j$. This observation justifies to call I_j the *jth skeleton ideal* of I (with respect to g).

Theorem 34. *For each $0 \le j \le d$, the factor module I_{j-1}/I_j is a direct sum of cyclic Cohen–Macaulay modules of dimension j. In particular, I_{j-1}/I_j is a j-dimensional Cohen–Macaulay module.*

Proof. Replacing I by I_j it suffices to consider the case $j = d$. Let

$$J = (I, \{x^b : b \in A\}), \quad \text{where} \quad A = \{b \in P^g_{S/I} : \rho(b) = d\},$$

then $I_{d-1}/I_d = J/I$.

Let $\{Z_1, \ldots, Z_r\}$ be the collection of those subsets of X with the property that for each $i = 1, \ldots, r$ there exists $b \in A$ such that $Z_i = Z_b$. Let $A_i = \{b \in A : Z_b = Z_i\}$, and let $b, b' \in A_i$. Then $b \wedge b' \in A_i$. Thus the meet of all the elements in A_i is the unique smallest element in A_i. We denote this element by b_i. Then $Z_i = Z_{b_i}$. Obviously the elements $f_i = x^{b_i} + I$, $i = 1, \ldots, r$ generate I_{d-1}/I_d. We claim that

$$I_{d-1}/I_d = \bigoplus_{i=1}^{r} S f_i.$$

The cyclic module Sf_i is $\mathbb{Z}^n$-graded with a K-basis $x^a + I$ with $a \ge b_i$ and $x^a \notin I$. Given $c \in \mathbf{N}^n$ with $c \ge b_i$ and $c \ge b_j$ for some $1 \le i < j \le r$, then $\rho(c) > d$, and so $x^c \in I$, by Lemma 33. This shows that the sum of the cyclic modules Sf_i is indeed direct.

Next we notice that if $x^c = x^{c_1} x^{c_2}$ with $x^{c_1} \in K[Z_{b_i}]$ and $x^{c_2} \in K[Y_{b_i}]$ belong to $\mathrm{Ann}(Sf_i)$, then $x^{c_2} \in \mathrm{Ann}(Sf_i)$. Indeed, $x^c = x^{c_1} x^{c_2} \in \mathrm{Ann}(Sf_i)$ if and only if $a_j \le b_i + c_1 + c_2$ for some j. Since $c_1(k) = 0$ for all k with $x_k \in Y_{b_i}$, it follows that $a_j(k) \le (b_i + c_2)(k)$ for all $k \in Y_{b_i}$, while for k with $x_k \in Z_{b_i}$ we have $a_j(k) \le g(k) = b_i(k) = (b_i + c_2)(k)$. Hence $a_j \le b_i + c_2$, which implies that $x^{c_2} \in \mathrm{Ann}(Sf_i)$.

It follows that $\mathrm{Ann}(Sf_i)$ is generated by monomials in $K[Y_{b_i}]$. In other words, there exists a monomial ideal $M_i \subset K[Y_{b_i}]$ such that $\mathrm{Ann}(Sf_i) = M_i S$.

For each k with $x_k \in Y_{b_i}$ we have $b_i(k) < g(k)$ and $\rho(b_i + (g(k) - b_i(k))\epsilon_k) = d + 1$. Therefore Lemma 33 implies that $x^{b_i} x_k^{g(k)-b_i(k)} \in I$. It follows that $x_k^{g(k)-b_i(k)} \in M_i$ for all k with $x_k \in Y_{b_i}$. Hence we see that $\dim(K[Y_{b_i}]/M_i) = 0$. This implies that $Sf_i = S/M_i S$ is Cohen–Macaulay of dimension d.

As a first application of Theorem 34 we obtain the following characterization of the depth of S/I which generalizes a classical result of Hibi [104, Corollary 2.6].

Corollary 35. *Let $I \subset S$ be a monomial ideal. Then*

$$\text{depth}(S/I) = \max\{j : S/I_j \text{ is Cohen–Macaulay}\},$$

and S/I_j is Cohen–Macaulay for all $j \leq \text{depth}(S/I)$.

Proof. Let $d = \dim(S/I)$ and $t = \text{depth}(S/I)$. Since $I_j = (I_{d-1})_j$ for $j \leq d-1$, both assertions follow by induction on d once we can show the following:

(i) If $t < d$, then $\text{depth}(S/I_{d-1}) = t$.
(ii) If S/I is Cohen–Macaulay, then S/I_{d-1} is Cohen–Macaulay.

Proof of (i): The exact sequence

$$0 \longrightarrow I_{d-1}/I \longrightarrow S/I \longrightarrow S/I_{d-1} \longrightarrow 0$$

implies that

$$\text{depth}(S/I_{d-1}) \geq \min\{\text{depth}(I_{d-1}/I) - 1, \text{depth}(S/I)\}, \tag{8}$$

with equality if $t < d - 1$, see [30, Proposition 1.2.9]. By Theorem 34, $\text{depth}(I_{d-1}/I) - 1 = d - 1$. It follows that $\text{depth}(S/I_{d-1}) = t$, if $t < d - 1$. On the other hand, if $t = d - 1$, then $\text{depth}(S/I_{d-1}) \geq d - 1$. However, since $\dim(S/I_{d-1}) = d - 1$, we again get $\text{depth}(S/I_{d-1}) = d - 1 = t$.

Proof of (ii): If S/I is Cohen–Macaulay, then $\text{depth}(S/I) = d$. Hence Theorem 34 and inequality (8) imply that $\text{depth}(S/I_{d-1}) \geq d - 1$. Since $\dim(S/I_{d-1}) = d - 1$, the assertion follows.

Now we come to the application to Stanley depth.

Proposition 36. *For all $0 \leq j \leq d = \dim(S/I)$ we have*

$$\text{sdepth}(S/I) \geq \text{sdepth}(S/I_j).$$

Proof. Observe that $P^g_{S/I_j} = \{a \in P^g_{S/I} : \rho(a) \leq j\}$. Let t be the Stanley depth of S/I_j. By Theorem 29 there exists a partition $\mathscr{P}: P^g_{S/I_j} = \bigcup_{i=1}^r [c_i, d_i]$ with $\rho(\mathscr{P}) = t$. We complete the partition of $\mathscr{P}$ to a partition $\mathscr{P}'$ of S/I by adding the intervals $[a, a]$ with $a \in P^g_{S/I} \setminus P^g_{S/I_j}$. Since $\rho(a) > j$ for all $a \in P^g_{S/I} \setminus P^g_{S/I_j}$ it follows that $\rho(\mathscr{P}') = t$. Hence, again by Theorem 29, $\text{sdepth}(S/I) \geq t$, as desired.

We denote by $\mathscr{A}_n$ the set of all K-algebras of the form S/I where $S = K[x_1, \ldots, x_n]$ and I is a monomial ideal.

Corollary 37. *Suppose Stanley's conjecture holds for all Cohen–Macaulay K-algebras in $\mathscr{A}_n$. Then Stanley's conjecture holds for all K-algebras in $\mathscr{A}_n$.*

Proof. Let I be a monomial ideal and suppose that $t = \text{depth}(S/I)$. Then S/I_t is Cohen–Macaulay of dimension t, see Corollary 35. Our assumption implies that $\text{sdepth}(S/I_t) = t$. Thus the assertion follows from Proposition 36.

Remark 38. Combining the above Corollary 37 with the equations of (2) and with Theorem 13 we see that if there exists a simplicial complex Δ for which $K[\Delta]$ Stanley's conjecture does not hold, then there exists such a such simplicial complex Δ with the additional property that $K[\Delta]$ is Cohen–Macaulay, and this simplicial complex Δ cannot be shellable. Thus possible counterexamples to Stanley's conjecture should be searched among the non-shellable Cohen–Macaulay simplicial complexes

9 Special Partitions of Posets

In [102] it was conjectured that $\operatorname{sdepth}(\mathfrak{m}) = \lceil n/2 \rceil$ where $\mathfrak{m} = (x_1, \dots, x_n)$ is the graded maximal ideal of $S = K[x_1, \dots, x_n]$ and where $\lceil n/2 \rceil$ denotes smallest integer $\geq n/2$. This conjecture has been proved by Biró et al. [18]. We will present their beautiful solution here and refer to their paper for the details of the proof.

According to Theorem 29 one has to find a partition of $\mathscr{P}$: $P_{\mathfrak{m}} = \bigcup_{i=1}^{r} [c_i, d_i]$ with $\lceil n/2 \rceil = \min\{\rho(d_i) : i = 1, \dots, r\}$ in order to show that $\operatorname{sdepth}(\mathfrak{m}) \geq \lceil n/2 \rceil$. Finding such a partition is what Biró et al. succeeded to do.

Let us first show why $\operatorname{sdepth}(\mathfrak{m}) \leq \lceil n/2 \rceil$. For the proof of both inequalities, the easy and the difficult one, we may choose $g = (1, \dots, 1)$ Then the poset $P_{\mathfrak{m}}^{g}$ can be identified with poset P_n of all non-empty subsets $A \subset [n]$ ordered by inclusion, and the function ρ applied to $A \in P_n$ is just the cardinality $|A|$ of A.

We first assume that n is odd, say $n = 2k+1$, and consider an interval partition of $\mathfrak{m}$. Any such partition has to contain intervals of the form $[\{i\}, D_i]$. Since the number of 2-sets in P_n is $(2k+1)(k+1)$, it follows that $\sum_{i=1}^{2k+1} |D_i| = (2k+1)(k+1)$. If the given partition would have Stanley depth $> k+1$ it would follow that $|D_i| > k+1$ for all i, contracting the equation $\sum_{i=1}^{2k+1} |D_i| = (2k+1)(k+1)$. Next assume that n is even, say $n = 2k$. Then the number of 2-sets of P_n is $(2k+1)k$. Thus assuming that the given Stanley decomposition has Stanley depth $> k$ would contradict the equation $\sum_{i=1}^{2k+1} |D_i| = (2k+1)k$.

Now for the proof of the inequality $\operatorname{sdepth}(\mathfrak{m}) \geq \lceil n/2 \rceil$ we may restrict ourselves to the case that $n = 2k+1$. Indeed, if $\mathscr{P}$ is a partition of P_n with $|D| \geq n/2$ for each interval $[C, D] \in \mathscr{P}$, then $\mathscr{Q} = \mathscr{P} \cup \{[\{n+1\}, [n+1]]\}$ is a partition of P_{n+1} with $|D| \geq (n+1)/2$ for all $[C, D] \in P_{n+1}$.

Now we assume that $n = 2k+1$. Following [18] we arrange the elements of $[2k+1]$ around the circle C_n in clockwise star order: the integer i, going clockwise, is followed by the integer j where $j \cong i \bmod 2k+1$ and $1 \leq j \leq 2k+1$. For $k = 5$, Fig. 4 displays this order.

Any subset $S \subset [n]$ is the disjoint union of intervals on C_n, called the *blocks* of S, see Fig. 5 where the blocks of $S = \{3, 6, 8, 9, 11\}$ are $\{6, 11\}$ and $\{3, 8, 9\}$ while the gap blocks are $\{1, 2, 7\}$ and $\{4, 5, 10\}$.

A subset $B \subset [n]$ is called *balanced* if each block of $[n] \setminus B$ has even size. The set S in Fig. 5 displays a non-balanced set while the set $B = \{2, 3, 6, 8, 10\}$ is balanced, see Fig. 6. It is clear that the number of elements of each balanced subset of $[n]$ is odd

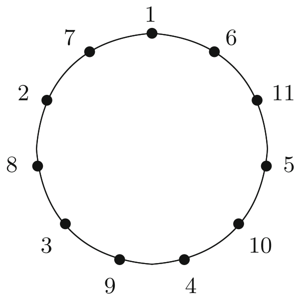


Fig. 4 Clockwise star order

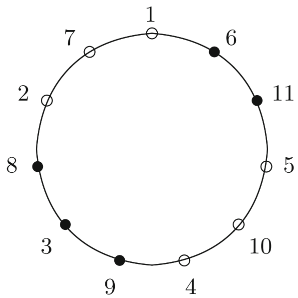


Fig. 5 The blocks for $S = \{3, 6, 8, 9, 11\}$

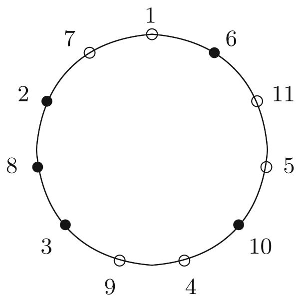


Fig. 6 The balanced set $B = \{2, 3, 6, 8, 10\}$

Let B be a balanced subset of $[n]$, say $|B| = 2s + 1$. Let $j \in [n] \setminus B$. Then $|B \cap \{j + 1, \ldots, j + k\}| = s$ or $|B \cap \{j, j + 1, \ldots, j + k\}| = s + 1$, see [18, Lemma 4.2]. The set of elements $j \in [n] \setminus B$ for which the previous intersection contains s elements is denoted L_B, the set of the remaining elements of $[n] \setminus B$ is denoted R_B. In [18, Lemma 4.3] it is shown that $|L_B| = |R_B| = k - s$. It follows that $|B \cup L_B| = k + s + 1$.

For the balanced set $B = \{2, 3, 6, 8, 10\}$ we have $L_B = \{4, 7, 11\}$ and $R_B = \{1, 5, 9\}$.

Now the crucial results is the following

Theorem 39 (Lemma 4.6 in [18]). *Let k be a non-negative integer and let $S \in P_n$ where $n = 2k+1$. Then there is a unique balanced set B such that $S \in [B, B \cup L_B]$.*

As a consequence we obtain

Corollary 40. *Let k be a non-negative integer and $n = 2k+1$. Then $\mathscr{P}$: $P_n = \bigcup_{B \text{ is balanced}}[B, B \cup L_B]$ is a partition of the poset $P_n = P_{\mathfrak{m}}^g$ with $|B \cup L_B| \geq k+1$ for all B. In particular,* $\operatorname{sdepth}(\mathfrak{m}) \geq \operatorname{sdepth}(\mathscr{D}(\mathscr{P})) \geq k+1$.

In the following example we apply Theorems 29 and 39 to obtain a Stanley decomposition $\mathscr{D}$ of $\mathfrak{m} = (x_1, x_2, \dots, x_5)$ with $\operatorname{sdepth}(\mathscr{D}) = \operatorname{sdepth}(\mathfrak{m}) = 3$. The method of Biró et al. gives us a partition of P_5 with the following intervals:

$$[1, 123], [2, 234], [3, 345], [4, 145], [5, 125],$$

$$[124, 1234], [134, 1345], [135, 1235], [235, 2345], [245, 1245], [12345, 12345].$$

Here $i_1 i_2 \cdots i_r$ stands for $\{i_1, i_2, \dots, i_r\}$.

The corresponding Stanley decomposition is the following:

$$\begin{aligned}(x_1, x_2, x_3, x_4, x_5) &= x_1 K[x_1, x_2, x_3] \oplus x_2 K[x_2, x_3, x_4] \oplus x_3 K[x_3, x_4, x_5] \\ &\oplus x_4 K[x_1, x_4, x_5] \oplus x_5 K[x_1, x_2, x_5] \\ &\oplus x_1 x_2 x_4 K[x_1, x_2, x_3, x_4] \oplus x_1 x_3 x_4 K[x_1, x_3, x_4, x_5] \\ &\oplus x_1 x_3 x_5 K[x_1, x_2, x_3, x_5] \oplus x_2 x_3 x_5 K[x_2, x_3, x_4, x_5] \\ &\oplus x_2 x_4 x_5 K[x_1, x_2, x_4, x_5] \\ &\oplus x_1 x_2 x_3 x_4 x_5 x K[x_1, x_2, x_3, x_4, x_5].\end{aligned}$$

The methods of interval partitions of posets have been extended and refined by Keller et al. [119] to obtain the following results on the Stanley depth of the squarefree Veronese ideal $I_{n,d}$ which is the ideal of all squarefree monomials in n variables generated in degree d.

Theorem 41 (Keller et al. [119]). *Let $I_{n,d}$ be the squarefree Veronese ideal in $S = K[x_1, \dots, x_n]$, generated by all squarefree monomials of degree d. Then*

(a) *If* $1 \leq d \leq n < 5d+4$, *then* $\operatorname{sdepth}(I_{n,d}) = \lfloor \frac{n-d}{d+1} \rfloor + d$.
(b) *If* $d \geq 1$ *and* $n \geq 5d+4$, *then* $d+3 \leq \operatorname{sdepth}(I_{n,d}) \leq \lfloor \frac{n-d}{d+1} \rfloor + d$.

In general it is conjectured by Cimpoeas [38] and Keller et al. [119] that $\operatorname{sdepth}(I_{n,d}) = \lfloor \frac{n-d}{d+1} \rfloor + d$ for all positive integers $1 \leq d \leq n$. Some more positive results in this direction have been obtained by Gi et al. [76].

For our next result we will use the following proposition which can be extracted from the work of Keller et al. [119] and Gi et al. [76] and as it is summarized in [62].

Proposition 42. *For each c-set $C \subset [n]$ with $c \leq \lfloor \frac{n+1}{2} \rfloor$, and each non-negative integer s such that $n+1 \geq (c+1)(s+1)$ there exists a superset $C(s)$ of C of cardinality $c+s$ such that the following condition is satisfied: suppose that A and B are subsets of $[n]$ of cardinalities $a \leq b \leq \lfloor \frac{n+1}{2} \rfloor$, and $s' \leq s$ are non-negative integers such that*

$$n+1 \geq (b+1)(s'+1) \geq (a+1)(s+1).$$

If B does not belong to $[A, A(s)]$, then this interval is disjoint from $[B, B(s')]$.

As an application of this proposition we have the following result [62, Theorem 3.4].

Theorem 43. *Let t be the largest integer such that $n+1 \geq (2t+1)(t+1)$. Then the Stanley depth of any squarefree monomial ideal in a polynomial ring in n variables is greater than or equal to $2t+1$. In particular, this lower bound is approximately of size $\sqrt{n}$.*

Proof. We consider the sequence $s_1, s_2, \ldots, s_{2t}$ with $s_i = 2t - i + 1$ for $i = 1, \ldots, t$ and $s_i = t$ for $i = t+1, \ldots, 2s$. Then

$$n + 1 \geq (i+1)(s_i + 1) \geq i(s_{i-1} + 1) \quad \text{for} \quad i = 2, \ldots, 2t.$$

Now let $I \subset S = K[x_1, \ldots, x_n]$ be a squarefree monomial ideal. Having chosen this sequence we construct a partition $\mathscr{P}$ of the poset P_I^g (with $g = (1, \ldots, 1)$) which we may consider as a subposet of the poset $P(n)$ of all non-empty subsets of $[n]$. For each $\{i\} \in P_I^g$ we take the interval $[\{i\}, \{i\}(s_1)]$. In the next step we take all intervals $[\{i_1, i_2\}, \{i_1, i_2\}(s_2)]$ with $\{i_1, i_2\}$ not contained in any of the intervals $[\{i\}, \{i\}(s_1)]$. In this way we proceed until we reach intervals of the form $[A, A(s_{2t})]$ with $2t$-sets A. By Proposition 42 these intervals are pairwise disjoint and cover the a-sets in P_I^g for all $a \leq 2t$. The remaining set $A \in P_I^g$ which are not covered by the above intervals are all of cardinality $\geq 2t+1$. Each such a set A we cover by the interval $[A, A]$ to finally obtain the partition $\mathscr{P}$ of the poset P_I^g. Since $A(s_i)$ is a set of cardinality $i + (2t - i + 1) = 2t + 1$ for all our intervals $[A, A(s_i)]$ of $\mathscr{P}$ it follows from Theorem 29 that the corresponding Stanley decomposition has Stanley depth $2t+1$, so that $\operatorname{sdepth}(I) \geq 2t+1$.

10 Stanley Decompositions and Alexander Duality

In this section we study the effect of Alexander duality to Stanley decompositions and describe in the special case of Stanley–Reisner rings and Stanley–Reisner ideals the behavior of Stanley decompositions under this duality following the paper [179] where more generally squarefree modules are considered.

Given a simplicial complex Δ on the vertex set $[n]$, we define $\Delta^\vee$ by

$$\Delta^\vee = \{F^c \colon\ F \notin \Delta\}.$$

Here for $F \subset [n]$ we denote by F^c the complement of F in $[n]$, that is, $F^c = [n] \setminus F$.

It is easily checked that $\Delta^\vee$ is again a simplicial complex on the vertex set $[n]$. The simplicial complex $\Delta^\vee$ is called the *Alexander dual* of Δ. Obviously, one has $\Delta = (\Delta^\vee)^\vee$. More about the Alexander duality can be found in the book [91, Sect. 1.5.3, Chaps. 5.1 and 8.1].

For any collection $\mathscr{A}$ of subsets of $[n]$ we denote by $\mathscr{A}^c$ the set of subsets $\{F^c\colon\ F \in \mathscr{A}\}$ of $[n]$.

As an immediate consequence of the definition of the Alexander dual we obtain the following result and its corollary.

Proposition 44. *Let Δ by a simplicial complex. Then $P_{K[\Delta^\vee]} = (P_{I_\Delta})^c$.*

Corollary 45. *Let Δ by a simplicial complex. Then*
$\mathscr{P}\colon\ P_{K[\Delta^\vee]} = \bigcup_{i=1}^r [F, G]$ is a partition of $P_{K[\Delta^\vee]}$ if and only if
$\mathscr{P}^c\colon\ P_{I_\Delta} = \bigcup_{i=1}^r [G^c, F^c]$ is a partition of P_{I_Δ}.

In terms of Stanley decompositions the preceding corollary can be interpreted as follows:

$$\mathscr{D}\colon\ K[\Delta^\vee] = \bigoplus_{i=1}^r u_i K[Z_i]$$

is a Stanley decomposition of $K[\Delta^\vee]$ if and only if

$$\mathscr{D}^\vee\colon\ I_\Delta = \bigoplus_{i=1}^r v_i K[W_i]$$

is a Stanley decomposition of I_Δ. Here $v_i = \prod_{x_j \in Z_i} x_j$ and $W_i = \{x_j\colon\ x_j | u_i\}$.

Let $J \subset I \subset S$ be monomial ideals. For a Stanley decomposition $\mathscr{D}\colon\ I/J = \bigoplus_{i=1}^r u_i K[Z_i]$ we define the number

$$\operatorname{sreg}(\mathscr{D}) = \max\{\rho(\deg(u_i))\colon\ i = 1, \ldots, r\},$$

and set $\operatorname{sreg}(I/J) = \min\{\operatorname{sreg}(\mathscr{D})\colon\ \mathscr{D}$ is a Stanley decomposition of $I/J\}$.

The relationship of this invariant to the regularity of a module will be explained in a moment.

We first observe the following fact which can be easily deduced from Corollary 45.

Corollary 46. *Let Δ be a simplicial complex. Then*

$$\operatorname{sreg}(I_\Delta) + \operatorname{sdepth}(K[\Delta^\vee]) = n \quad \textit{and} \quad \operatorname{sdepth}(I_\Delta) + \operatorname{sreg}(K[\Delta^\vee]) = n.$$

A remarkable generalization of this result due to Okazaki and Yanagawa can be found in [155, Theorem 3.13].

Example 47. Let $\Delta = \{\emptyset\}$. Then $I_\Delta = \mathfrak{m} = (x_1, \ldots, x_n)$ and $K[\Delta^\vee] = S/(x_1 \cdots x_n)$. It follows from Corollary 46 and Proposition 15 that

$$\operatorname{sreg}(\mathfrak{m}) = n - \operatorname{sdepth}(S/(x_1 \cdots x_n)) = n - (n-1) = 1,$$

and from Corollary 46 and the result of Biró et al. that

$$\operatorname{sreg}(S/(x_1,\ldots,x_n)) = n - \operatorname{sdepth}(\mathfrak{m}) = n - \lceil n/2 \rceil = \lfloor n/2 \rfloor.$$

Comparing in this example sreg with the Castelnuovo–Mumford regularity, we see that $\operatorname{sreg}(\mathfrak{m}) = \operatorname{reg}(\mathfrak{m})$ and $\operatorname{sreg}(S/(x_1\cdots x_n)) < \operatorname{reg}(S/(x_1\cdots x_n)) = n-1$. Is there any relationship between these invariants? To answer this question we recall the following result of Terai, see [91, Proposition 8.1.10]:

$$\operatorname{reg}(I_\Delta) + \operatorname{depth}(K[\Delta^\vee]) = n \text{ and } \operatorname{depth}(I_\Delta) + \operatorname{reg}(K[\Delta^\vee]) = n \tag{9}$$

Comparing this with the identities in Corollary 46 we see that

$$\operatorname{sreg}(I_\Delta) \le \operatorname{reg}(I_\Delta) \quad \text{and} \quad \operatorname{sreg}(K[\Delta]) \le \operatorname{reg}(K[\Delta]),$$

provided Stanley's conjecture is true. Thus it is natural to conjecture the following inequality.

Conjecture 48 ([179]). Let $J \subset I \subset S$ be monomial ideals. Then $\operatorname{sreg}(I/J) \le \operatorname{reg}(I/J)$.

Actually, Soleyman Jahan conjectured the same inequality for any squarefree $\mathbb{Z}^n$-graded module.

In the following we describe a case where $\operatorname{sreg}(M) \le \operatorname{reg}(M)$. A monomial ideal $I \subset S$ is said to have *linear quotients* if the monomial set of monomial generators of I can be ordered $u_1, u_2, \ldots, u_m$ such that for all i the colon ideal $(u_1,\ldots,u_{i-1}) : u_i$ is generated by variables. In [91, Proposition 8.2.5] it is shown that I_Δ has linear quotients if and only if $\Delta^\vee$ is shellable.

Proposition 49. *Let I be a squarefree monomial ideal with linear quotients. Then* $\operatorname{sreg}(I) = \operatorname{reg}(I)$*, and* $\operatorname{reg}(I)$ *is equal to the maximal degree of an element in the minimal set of monomial generators of I.*

Proof. Let Δ be the simplicial complex with $I = I_\Delta$. Then, by what we said before, the Alexander dual $\Delta^\vee$ is shellable. Therefore it follows from Theorem 13 and Proposition 18 that $\operatorname{sdepth}(S/I_{\Delta^\vee}) = \operatorname{depth}(S/I_{\Delta^\vee})$. Thus the formulas of Terai (9) together with Corollary 46 implies that $\operatorname{sreg}(I) = \operatorname{reg}(I)$. Since I has linear quotients, it follows that I is componentwise linear, see [91, Theorem 8.2.15]. Thus [91, Corollary 8.2.14] completes the proof.

11 The Size of an Ideal

In [129] Lyubeznik introduced the *size* of a monomial ideal and showed that $\operatorname{depth}(S/I) \ge \operatorname{size}(I)$ for any monomial ideal I. In this section we present the result of [97] where it is shown that $\operatorname{sdepth}(I) \ge \operatorname{size}(I) + 1$. Of course, assuming that Stanley's conjecture holds, one expects exactly this inequality.

Let $I \subset S$ be a monomial ideal and $I = \bigcap_{i=1}^{s} Q_i$ a primary decomposition of I, where the Q_i are monomial ideals. Let Q_i be P_i-primary. Then each P_i is a monomial prime ideal and $\mathrm{Ass}(S/I) = \{P_1, \ldots, P_s\}$.

According to [129, Proposition 2] the *size* of I, denoted size(I), is the number $v + (n - h) - 1$, where v is the minimum number t such that there exist $j_1 < \cdots < j_t$ with

$$\sqrt{\sum_{k=1}^{t} Q_{j_k}} = \sqrt{\sum_{j=1}^{s} Q_j},$$

and where $h = \mathrm{height}(\sum_{j=1}^{s} Q_j)$.

Notice that $\sqrt{\sum_{k=1}^{t} Q_{j_k}} = \sum_{k=1}^{t} P_{j_k}$ and $\sqrt{\sum_{j=1}^{s} Q_j} = \sum_{j=1}^{s} P_j$, so that the size of I depends only on the set of associated prime ideals of S/I.

We will indicate the proof of the inequality $\mathrm{sdepth}(I) \geq \mathrm{size}(I) + 1$. The details of the proof can be found in [97]. For the proof of this inequality the technique of splitting variables is used, as it was introduced in [159, 160]. Let $I \subset S$ be a monomial ideal. We will decompose I into a direct sum of $\mathbb{Z}^n$-graded subspaces. The decomposition depends on the choice of a subset Y of the set of variables $X = \{x_1, \ldots, x_n\}$, and is also determined by the unique irredundant presentation of I as an intersection $I = \bigcap_{j=1}^{s} Q_j$ of its minimal irreducible monomial ideals. As before each Q_j is a P_j-primary ideal with $P_j \in \mathrm{Ass}(S/I)$.

Without loss of generality we may assume that $Y = \{x_1, \ldots, x_r\}$ for some number r such that $0 \leq r \leq n$. Then the set of variables splits into the two sets $\{x_1, \ldots, x_r\}$ and $\{x_{r+1}, \ldots, x_n\}$.

Given a subset $\tau \subset [s]$, we let I_τ be the $\mathbb{Z}^n$-graded K-vector space spanned by the set of monomials of the form $w = uv$ where u and v are monomials with

$$u \in K[x_1, \ldots, x_r] \quad \text{and} \quad u \in \bigcap_{j \notin \tau} Q_j \setminus \sum_{j \in \tau} Q_j,$$

$$v \in K[x_{r+1}, \ldots, x_n] \quad \text{and} \quad v \in \bigcap_{j \in \tau} Q_j.$$

The following result (taken from [97]) extends the corresponding statement shown by Popescu [160] for squarefree monomial ideals.

Proposition 50. *With the notation introduced, the ideal I has a decomposition $\mathscr{D}_Y\colon I = \bigoplus_{\tau \subset [s]} I_\tau$ as a direct sum of $\mathbb{Z}^n$-graded K-subspaces of I.*

Proof. It is clear from the definition of I_τ that $I_\tau \subset I$, so that $\sum_{\tau \subset [s]} I_\tau \subset I$. Conversely, let $w = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ be a monomial in I. Then $w = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ can be written in a unique way as a product $w = uv$ of monomials with $u \in K[x_1, \ldots, x_r]$ and $v \in K[x_{r+1}, \ldots, x_n]$. Let $\tau = \{j \in [s]\colon\ u \notin Q_j\}$. Then $u \in \bigcap_{j \notin \tau} Q_j$ and $u \notin \sum_{j \in \tau} Q_j$.

Let $j \in \tau$. Since $uv \in I$ we have $uv \in Q_j$. Thus, if $Q_j = (x_{i_1}^{b_{i_1}}, \ldots, x_{i_k}^{b_{i_k}})$, then there exists an integer ℓ with $a_{i_\ell} \geq b_{i_\ell}$. On the other hand, since $u \notin Q_j$, it follows that $a_{i_t} < b_{i_t}$ for all $i_t \leq r$. This implies that $i_\ell \geq r+1$, and consequently $v \in Q_j$. Hence we see that $v \in \bigcap_{j \in \tau} Q_j$, and conclude that $w \in I_\tau$.

In order to see that the sum is direct assume that $w = uv \in I_\tau \cap I_\sigma$. Suppose that $\tau \neq \sigma$. Then we may assume that $\sigma \setminus \tau \neq \emptyset$. Let $j \in \sigma \setminus \tau$. Then $u \in Q_j$ by the definition of I_τ, and $u \notin Q_j$, by the definition of I_σ, a contradiction.

Now we are ready to indicate the proof of the following

Theorem 51. *Let I be a monomial ideal of S. Then*

$$\operatorname{sdepth}(I) \geq \operatorname{size}(I) + 1.$$

Proof. Let $I = \bigcap_{j=1}^{s} Q_j$ be the unique irredundant presentation of I as an intersection of its minimal irreducible monomial ideals. Each of the Q_j is a primary ideal whose associated monomial prime ideal we denote, as before, by P_j.

We may assume that $\sum_{j=1}^{s} P_j = \mathfrak{m}$. Indeed, let $Z = \{x_i\colon\ x_i \notin \sum_{j=1}^{s} P_j\}$, $T = K[X \setminus Z]$ and $J = I \cap T$. Then the sum of the associated prime ideals of J is the graded maximal ideal of T, and

$$\operatorname{sdepth}(I) = \operatorname{sdepth}(J) + |Z|, \quad \text{and} \quad \operatorname{size}(I) = \operatorname{size}(J) + |Z|.$$

The first equation follows from [102, Lemma 3.6], while the second equation follows from the definition of size.

We choose the splitting set Y to be the set $\{x_i : x_i \in P_1\}$, and we may assume that $Y = \{x_1, \ldots, x_r\}$ for some number r such that $1 \leq r \leq n$. If $r = n$, then the desired inequality follows at once since in this case size $I = 0$, and since for every monomial ideal I we have that $\operatorname{sdepth}(I) \geq 1$. Therefore from now on we assume that $r < n$. We will prove the assertion of the theorem by induction on s. The case $s = 1$ follows immediately from [174, Theorem 2.4] and Problem 30 since $\operatorname{sdepth}(Q_1) = \lceil |Y|/2 \rceil + n - |Y|$ and size $Q_1 = n - |Y|$.

Assume now that the assertion is proved for all monomial ideals which are intersections of at most $s-1$ irreducible monomial ideals. Since $Y = \{x_1, \ldots, x_r\}$, it follows from the method described before Proposition 50 that $I = \bigoplus_{\tau \subset [s]} I_\tau$ with $I_{[s]} = 0$. We obtain from the decomposition of I that

$$\operatorname{sdepth}(I) \geq \min\{\operatorname{sdepth}(I_\tau)\colon\ \tau \subset [s] \text{ and } I_\tau \neq 0\}.$$

The proof is completed by showing that for any subset τ of $[s]$ such that $I_\tau \neq 0$ we have that $\operatorname{sdepth}(I_\tau) \geq \operatorname{size}(I) + 1$. Here the induction hypothesis will be used that the inequality is valid for monomial ideals with at most $s-1$ irreducible components.

Now we consider a concept which in some sense is dual to that of the size of an ideal. Let I be a squarefree monomial ideal minimally generated by the monomials

$u_1, \dots, u_m$. Let w be the smallest number t with the property that there exist integers $1 \le i_1 < i_2 < \cdots < i_t \le m$ such that

$$\operatorname{lcm}(u_{i_1}, u_{i_2}, \dots, u_{i_t}) = \operatorname{lcm}(u_1, u_2, \dots, u_m).$$

Then we call the number $\deg \operatorname{lcm}(u_1, u_2, \dots, u_m) - w$ the *cosize* of I, denoted $\operatorname{cosize} I$.

Now we have

Proposition 52. *Let $I \subset S$ be a squarefree monomial ideal. Then* $\operatorname{reg} S/I \le \operatorname{cosize} I$.

Proof. Let Δ be the simplicial complex with the property that $I = I_\Delta$. By using the result of Lyubeznik as well as Terai's result [91, Proposition 8.1.10], we obtain

$$n - \operatorname{reg} K[\Delta] = n - \operatorname{proj\,dim} I_{\Delta^\vee} = \operatorname{depth}(I_{\Delta^\vee}) \ge \operatorname{size} I_{\Delta^\vee} + 1,$$

so that $n - \operatorname{reg} K[\Delta] \ge v + (n - h)$. This implies that $\operatorname{reg} K[\Delta] \le h - v$.

Let $I_{\Delta^\vee} = P_{F_1} \cap \cdots \cap P_{F_m}$ where the P_{F_i} are the minimal prime ideals of $I_{\Delta^\vee}$. (Here $P_G = (\{x_i\}_{i \in G})$ for $G \subset [n]$.) Then $\{x_{F_1}, \dots, x_{F_m}\}$ is the minimal monomial set of generators of $I_{\Delta^\vee}$, see [91, Corollary 1.5.5]. (Here $x_G = \prod_{i \in G} x_i$ for $G \subset [n]$.) By using this fact we see that the number v for $I_{\Delta^\vee}$ is equal to the number w for I_Δ, and that the number h for $I_{\Delta^\vee}$ is equal to the number $\deg \operatorname{lcm}(u_1, u_2, \dots, u_m)$ for I_Δ. This yields the desired result.

Our main result Theorem 51 yields

Corollary 53. *Let $I \subset S$ be a squarefree monomial ideal. Then* $\operatorname{sreg} S/I \le \operatorname{cosize} I$.

Proof. Let Δ be the simplicial complex with $I = I_\Delta$. Then, by using Corollary 46 as well as Theorem 51, we obtain

$$\operatorname{sreg} S/I_\Delta = n - \operatorname{sdepth}(I_{\Delta^\vee}) \le n - (\operatorname{size} I_{\Delta^\vee} + 1) = \operatorname{cosize} I_\Delta.$$

12 The Hilbert Depth

Let M be a $\mathbb{Z}^n$-graded S-module and $\mathscr{D}\colon\ M = \bigoplus_{i=1}^r m_i K[Z_i]$ a Stanley decomposition. As in the proof of Proposition 33 this decomposition yields the following formula for the Hilbert series of M (viewed as a $\mathbb{Z}$-graded module over the standard graded polynomial ring S):

$$\mathscr{H}(\mathscr{D})\colon \operatorname{Hilb}(M) = \sum_{i=1}^{r} \frac{t^{a_i}}{(1-t)^{b_i}}, \tag{10}$$

where $a_i = \deg(m_i)$ and $b_i = |Z_i|$.

Let $\mathscr{H}$ be a sum presentation of $\mathrm{Hilb}(M)$ as in (10) (not necessarily induced by a Stanley decomposition). We call such a sum presentation a *Hilbert decomposition* of $\mathrm{Hilb}(M)$, and call the number

$$\mathrm{hdepth}(\mathscr{H}) = \min\{b_i\colon\ i = 1,\dots,r\}$$

the *Hilbert depth* of $\mathscr{H}$. Finally we define the *Hilbert depth* of M to be the number

$$\mathrm{hdepth}(M) = \max\{\mathrm{hdepth}(\mathscr{H})\colon\ \mathscr{H} \text{ is a Hilbert decomposition of } \mathrm{Hilb}(M)\}.$$

This invariant has been introduced in [31] where it is denoted $\mathrm{Hdepth}_1(M)$. In that paper one can find a multigraded version of this definition as well, which the authors denote by $\mathrm{Hdepth}_n(M)$.

It is obvious from the definition of the Hilbert depth that the following inequality holds

$$\mathrm{hdepth}(M) \geq \mathrm{sdepth}(M). \tag{11}$$

Thus one would expect that $\mathrm{hdepth}(M) \geq \mathrm{depth}(M)$. That this is indeed the case was shown in [31, Theorem 2.7]. Here we prove this result for the special case $M = S/I$, by using a different argument.

Theorem 54. *Let $I \subset S$ be a monomial ideal. Then*

$$\mathrm{hdepth}(S/I) \geq \mathrm{depth}(S/I).$$

Proof. Let $\mathrm{gin}(I)$ be the generic initial ideal of I with respect to the reverse lexicographic order. By Galligo, Bayer and Stillman (see [91, Theorem 4.2.1] and [91, Theorem 4.2.10]) it is known that $\mathrm{gin}(I)$ is of Borel type. Thus it follows from Proposition 21 that $S/\mathrm{gin}(I)$ is pretty clean which by Proposition 18 implies that $\mathrm{sdepth}(S/\mathrm{gin}(I)) = \mathrm{depth}(S/\mathrm{gin}(I)) = \mathrm{depth}(S/I)$, see [91, Corollary 4.3.18]. Thus there exists a Stanley decomposition of $S/\mathrm{gin}(I)$ such that Hilbert depth of $\mathscr{H}(\mathscr{D})$ is equal to $\mathrm{depth}(S/I)$. Hence, since $\mathrm{Hilb}(S/I) = \mathrm{Hilb}(S/\mathrm{in}(I))$, we see that $\mathrm{hdepth}(S/I) \geq \mathrm{depth}(S/I)$.

As an example for the computation of the Hilbert depth consider the maximal ideal $\mathfrak{m} = (x_1,\dots,x_n)$ in S. Let

$$\mathscr{H}\ :\ \mathrm{Hilb}(\mathfrak{m}) = \sum_{i=1}^{r} \frac{t}{(1-t)^{b_i}} + \sum_{i=1}^{s} \frac{t^2}{(1-t)^{c_i}} + \cdots = \sum_{i=1}^{r} t(1+b_it+\cdots) + st^2 + \cdots$$

be a Hilbert decomposition of $\mathrm{Hilb}(\mathfrak{m})$. Since $\mathrm{Hilb}(\mathfrak{m}) = nt + \binom{n+1}{2}t^2 + \cdots$, comparison of coefficients gives $r = n$ and $\sum_{i=1}^{n} b_i \leq \binom{n}{2}$. This implies that $\min\{b_i\colon\ i = 1,\dots,n\} \leq (n+1)/2$. It follows that $\mathrm{hdepth}(\mathscr{H}) \leq \lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil$ Therefore $\mathrm{hdepth}(\mathfrak{m}) \leq \lceil n/2 \rceil = \mathrm{sdepth}(\mathfrak{m})$, see Sect. 9.

Since $\operatorname{hdepth}(M) \geq \operatorname{sdepth}(M)$ for any $\mathbb{Z}^n$-graded S-module, we finally get that $\operatorname{hdepth}(\mathfrak{m}) = \lceil n/2 \rceil$.

Comparing Formula (10) with the definition of the Stanley regularity as defined in Sect. 10 it is natural to define the Hilbert regularity of a Hilbert decomposition $\mathscr{H}$: $\operatorname{Hilb}(M) = \sum_{i=1}^{r} \frac{t^{a_i}}{(1-t)^{b_i}}$ of a $\mathbb{Z}^n$-graded module M to be the number $\operatorname{hreg}(M) = \max\{a_i : i = 1, \ldots, r\}$, and to define the *Hilbert regularity* of M as the number

$$\operatorname{hreg}(M) = \min\{\operatorname{hreg}(\mathscr{H}) \colon \mathscr{H} \text{ is a Hilbert decomposition of } \operatorname{Hilb}(M)\}.$$

Obviously one has that $\operatorname{hreg}(M) \leq \operatorname{sreg}(M)$, but also

Theorem 55. *Let M be a $\mathbb{Z}^n$-graded S-module. Then* $\operatorname{hreg}(M) \leq \operatorname{reg}(M)$.

Proof. We prove the assertion by induction on $\dim(M)$. If $\dim(M) = 0$, then $\operatorname{Hilb}(M)$ is a polynomial of degree $\operatorname{reg}(M)$. Therefore, in this case, $\operatorname{hreg}(M) = \operatorname{reg}(M)$. Now assume that $\dim(M) > 0$, and let $N \subset M$ be the maximal submodule of M with finite length. Then we obtain the exact sequence

$$0 \to N \longrightarrow M \longrightarrow W \longrightarrow 0,$$

where W is a module of positive depth. By the definition of Hilbert regularity we get

$$\operatorname{hreg}(M) \leq \max\{\operatorname{hreg}(N), \operatorname{hreg}(W)\} = \max\{\operatorname{reg}(N), \operatorname{hreg}(W)\}. \tag{12}$$

Without restriction we may assume that the base field of S is infinite. Then there exists a nonzero divisor $y \in S$ on W of degree 1. Let $\sum_{i=1}^{r} t^{a_i}/(1-t)^{b_i}$ be a Hilbert decomposition of $H_{W/yW}(t)$, then $\sum_{i=1}^{r} t^{a_i}/(1-t)^{b_i+1}$ is a Hilbert decomposition of W. This implies that $\operatorname{hreg}(W) \leq \operatorname{hreg}(W/yW)$. Thus applying our induction hypothesis we see that

$$\operatorname{hreg}(W) \leq \operatorname{hreg}(W/yW) \leq \operatorname{reg}(W/yW) = \operatorname{reg}(W).$$

Therefore (12) implies that $\operatorname{hreg}(M) \leq \max\{\operatorname{reg}(N), \operatorname{reg}(W)\} = \operatorname{reg}(M)$, as desired. For the last equation we used [52, Corollary 20.19].

Problem 56. Find an example of a $\mathbb{Z}^n$-graded S-module M such that $\operatorname{hdepth}(M) > \operatorname{sdepth}(M)$.

Problem 57. Given a $\mathbb{Z}^n$-graded module M with Hilbert decomposition as in (10). Show that $b_i \leq d = \dim(M)$ for all i and that the number of elements i with $b_i = d$ is equal to the multiplicity of the module M.

13 Further Results and Open Problems

In this section we summarize what is known about Stanley decompositions and related topics which have not been discussed in the previous sections.

1. *What is know about Stanley's conjecture?*

Apel in his papers [12, 13] was the first to prove Stanley's conjecture in a several interesting special cases. In his first paper [12] he proved Stanley's conjecture for generic monomial ideals. Precisely his result is the following:
([12, Theorem 2]) Let $I \subset S$ be a monomial ideal. Furthermore, assume that for some variable x_k the ideal I has the following property: for any two distinct minimal generators m_i and m_j of I such that $\deg_{x_k}(m_i) = \deg_{x_k}(m_j) = d$ there exists is a third minimal generator m_r which divides $\operatorname{lcm}(m_i, m_j)$ and satisfies $\deg_{x_k} m_r < d$. Then $\operatorname{sdepth}(I) \geq \operatorname{depth}(I)$.

Here Apel used a refinement (see [142]) of the original definition [16] of generic monomial ideals. In the same paper he proved
([12, Theorem 1]) Stanley's conjecture holds true for any monomial ideal in a polynomial ring with at most three variables.

These results have been completed by Apel in the second paper [13] as follows: in Corollary 2 he shows (with a different argument as presented here in Sect. 4) that S/I satisfies Stanley's conjecture for any Borel-fixed ideal, and in Corollary 3 he shows that S/I satisfies Stanley's conjecture whenever $\dim(S/I) \leq 1$. Moreover, he showed
([13, Theorem 3]) Let I be a generic monomial ideal. Then

$$\operatorname{sdepth}(S/I) = \min\{\operatorname{depth}(S/P)\colon\ P \in \operatorname{Ass}(S/I)\}.$$

In particular, together with Theorem 9, it follows that $\operatorname{sdepth}(S/I) \geq \dim(S/I)$ if I is a generic monomial ideal. In [13, Theorem 5] Apel also showed that S/I satisfies Stanley's conjecture if S/I is Cohen–Macaulay and I is a cogeneric monomial ideal.

According to Sturmfels [184] a monomial ideal I with the irredundant irreducible decomposition $I = \bigcap_{i=1}^{m} \mathfrak{m}^{a_i}$ is said to be *cogeneric*, if any distinct $\mathfrak{m}^{a_i}$ and $\mathfrak{m}^{a_j}$ do not have the same minimal monomial generators. Here, for $a \in \mathbb{N}$, $\mathfrak{m}^{a}$ denotes the irreducible ideal $(x_i^{a_i}\colon\ a_i > 0)$. Okazaki and Yanagawa improved Apel's result as follows:
([155, Theorem 6.5]) If I is a cogeneric monomial ideal, then $\operatorname{sdepth}(S/I) \geq \operatorname{depth}(S/I)$. That is, Stanley's conjecture holds for the quotient by a cogeneric monomial ideal (no matter whether S/I is Cohen–Macaulay or not).
Finally I want to mention the following result of Apel:
([13, Theorem 4]) Let $I \subset S = K[x_1, x_2, x_3]$ be a monomial ideal. Then $\operatorname{sdepth}(S/I) = \min\{\operatorname{depth}(S/P)\colon\ P \in \operatorname{Ass}(S/I)\}$. In particular, Stanley's conjecture holds for S/I, when S is a polynomial ring in at most three variables.

The most far reaching result in this direction is that of Popescu who showed in [161, Theorem 4.3] that S/I satisfies Stanley's conjecture for a polynomial ring in at most five variables. Anwar and Popescu [11] also give an affirmative answer to Stanley's conjecture when I has at most three irreducible components:
([162, Theorems 5.6 and 5.9]) Stanley's conjecture holds true for $Q_1 \cap Q_2$ and for $S/(Q_1 \cap Q_2 \cap Q_3)$ where Q_1, Q_2, Q_3 are nonzero irreducible monomial ideals of S.

In [37, Theorem 2.3] Cimpoeaş showed that Stanley's conjecture holds for I and S/I when I is a monomial ideal generated by at most three elements.
For monomial ideals of low codimension the following is known by Soleyman Jahan, Yassemi and myself:
([98, Proposition 1.4 and Theorem 2.1]) Let I be a monomial ideal which is perfect of codimension 2 or Gorenstein of codimension 3. Then S/I satisfies Stanley's conjecture.

Another case of interest proved by Soleyman Jahan in the same paper is the following:
([178, Proposition 2.1]) Let $I \subset S = K[x_1, \dots, x_n]$ be a monomial ideal of height $\geq n-1$, then Stanley's conjecture holds for S/I.
In a very recent paper [14], Bandari, Divaani-Aazar and Soleyman Jahan showed that if $I \subset S = K[x_1, \dots, x_n]$ is a monomial ideal generated by monomials $u_1, u_2, \dots, u_t$, then S/I is pretty clean (and hence satisfies Stanley's conjecture) if either: $u_1, u_2, \dots, u_t$ is a filter-regular sequence, or d-sequence, or I is an almost complete intersection.
Finally we would like to mention the following result due to Pournaki, Fakhari and Yassemi.
([163, Corollary 2.8] Let I be the edge ideal of a graph. If G is a forest, then S/I^k satisfies Stanley's conjecture for all $k \gg 0$.

2. *Numerical bounds for the Stanley depth in special cases.*

The following results in this subsection are obtained by using Theorem 29 and extensions of the techniques of constructing partitions as developed by Biró, Howard, Keller, Trotter and Young in [18], and as outlined in Sect. 9.

We first would like to mention the following formula by Shen which is a natural extension of the result in [18] where it was shown that $\operatorname{sdepth}(\mathfrak{m}) = \lceil n/2 \rceil$ for the graded maximal ideal of $S = K[x_1, \dots, x_n]$.
([174, Theorem 2.4]) Let $I \subset K[x_1, \dots, x_n]$ be a complete intersection monomial ideal minimally generated by m elements. Then $\operatorname{sdepth}(I) = n - \lfloor m/2 \rfloor$.
This result has then been complemented by Okazaki who obtain the following nice lower bound for the Stanley depth of a monomial ideal
([154, Theorem 2.1]) Let $I \subset K[x_1, \dots, x_n]$ be a monomial ideal minimally generated by m monomials. Then $\operatorname{sdepth}(I) \geq n - \lfloor m/2 \rfloor$.
The squarefree version of this inequality by Okazaki was first proved by Keller and Young [120, Theorem 1.1].
Another extension of the result of Biró et al. concerns the so-called squarefree Veronese ideals $I_{n,d}$. The ideal $I_{n,d}$ is the ideal of all squarefree monomials in $K[x_1, \dots, x_n]$ of degree d. It has been conjectured by Cimpoeaş

[38, Conjecture 1.6] and by Keller et al. [119, Conjecture 2.4] that $\mathrm{sdepth}(I_{n,d}) = \lfloor \binom{n}{d+1}/\binom{n}{d} \rfloor + d$. Cimpoeaş [38] as well as Keller et al. [119] showed that the above conjectured Stanley depth is certainly an upper bound for $\mathrm{sdepth}(I_{n,d})$. The most far reaching result in this direction is the following:
([76, Theorem 1.2]) One has $\mathrm{sdepth}(I_{n,d}) = \lfloor \binom{n}{d+1}/\binom{n}{d} \rfloor + d$ for $1 \le d \le n \le (d+1)\lfloor (1+\sqrt{5+4d})/2 \rfloor + 2d$.

3. *Further results on the Hilbert depth.*

In Sect. 12 we have seen that $\mathrm{hdepth}(\mathfrak{m}) = \mathrm{sdepth}(\mathfrak{m}) = \lceil n/2 \rceil$ where $\mathfrak{m}$ denotes the graded maximal ideal of $K[x_1,\ldots,x_n]$. Bruns, Krattenthaler and Uliczka generalized this result and showed
([32, Theorem 1.2]) For all n and k one has $\mathrm{hdepth}(\mathfrak{m}^k) = \lceil n/(k+1) \rceil$.
The proof of this result is surprisingly involved and needs a careful numerical analysis. It is conjectured by Cimpoeaş [36] that, similarly as for $\mathfrak{m}$ one has that $\mathrm{hdepth}(\mathfrak{m}^k) = \mathrm{sdepth}(\mathfrak{m}^k)$ for all k. Cimpoeaş actually showed in [36, Theorem 2.2] that $\mathrm{sdepth}(\mathfrak{m}^k) \le \lceil n/(k+1) \rceil$. Of course this result is also a consequence of the above quoted result of Bruns et al.

In [31] Bruns, Krattenthaler and Uliczka study the Stanley depth of the Koszul cycles, that is, of the syzygy modules of $S/\mathfrak{m}$. In their paper the kth syzygy module of $S/\mathfrak{m}$ is denoted $M(n,k)$. It is shown
([31, Theorem 3.5 and Proposition 2.6] $\mathrm{hdepth}(M(n,k)) = n-1$ for $n > k \ge \lfloor n/2 \rfloor$, and $\mathrm{hdepth}(M(n,k)) \le n - \lceil (n-k)/(k+1) \rceil$, if $k < \lfloor n/2 \rfloor$.
The precise Hilbert depth in the lower range for k is not known.

4. *Auxiliary results*

Let $I \subset S$ be a monomial ideal. We denote by $\sqrt{I}$ the radical of I. Is the any comparison between the Stanley depth of S/I and $S/\sqrt{I}$. The following results are known. Apel showed
([13, Theorem 1]) $\mathrm{sdepth}(S/\sqrt{I}) \ge \mathrm{sdepth}(S/I)$.
This result has been extended by Ishaq as follows:
([111, Theorem 2.1]) Let $J \subset I \subset S$ be monomial ideals:
then $\mathrm{sdepth}(\sqrt{I}/\sqrt{J}) \ge \mathrm{sdepth}(I/J)$.
Ishaq also proved the following interesting upper bound for the Stanley depth of a monomial ideal.
([112, Theorem 1.1]) Let $I \subset S$ be a monomial ideal with $\mathrm{Ass}(S/I) = \{P_1,\ldots,P_s\}$. Then $\mathrm{sdepth}(I) \le \min\{\mathrm{sdepth}(P_i)\colon i = 1,\ldots,s\}$.
A remarkable lower bound for the Stanley depth has been found by S.A. Seyed Fakhari:
([172, Corollary 3.4]) Let $I \subset K[x_1,\ldots,x_n]$ be a squarefree monomial ideal which is generated in a single degree. Then $\mathrm{sdepth}(I) \ge n - l(I) + 1$ and $\mathrm{sdepth}(S/I) \ge n - l(I)$, where $l(I)$ denotes the analytic spread of I.
One may ask how Stanley depth behaves with respect to monomial localization. Let $I \subset S = K[x_1,\ldots,x_n]$ be a monomial ideal. Monomial localization of I with respect to the variable x_n is the ideal $J \subset S' = K[x_1,\ldots,x_{n-1}]$ which is obtain

from I by the homomorphism which maps x_n to 1 and leaves the other variables unchanged. Note that $I_{x_n} = JS'[x_n, x_n^{-1}]$, where I_{x_n} denotes the usual localization of I with respect to x_n. Nasir showed that Stanley depth behaves as follows with respect to monomial localization.

([148, Corollary 3.2]) Let $I \subset S = K[x_1, \ldots, x_n]$ be a monomial ideal and $J \subset S' = K[x_1, \ldots, x_{n-1}]$ the monomial localization of I with respect to x_n. Then $\operatorname{sdepth}(S/I) \geq \operatorname{sdepth}(S'/J) - 1$.

Let $f = \prod_{j=1,\ldots,k} x_{i_j}$. Then $S_f = S[x_{i_1}^{-1}, \ldots, x_{i_k}^{-1}]$. In [149] Nasir and Rauf defined Stanley decomposition and Stanley depth for ideals in the ring S_f, and showed that the number o maximal Stanley spaces in any Stanley decomposition of S_f is equal to 2^k.

In [136] MacLagan and Smith describe an algorithm to obtain a Stanley decomposition of S/I. A special case of this algorithm occurs implicitly already in the proof of [186, Lemma 2.4].

([136, Algorithm 3.4]) If I is (monomial) prime ideal, then $S/I = K[Z]$ where Z is the set of variables not belonging to I. Now assume I is not a prime ideal, and choose a variable x_i that is a proper divisor of a minimal generator of I. One obtains the exact sequence

$$0 \longrightarrow (S/(I : x_i))(-\epsilon_i) \xrightarrow{x_i} S/I \longrightarrow S/(I + x_i) \longrightarrow 0.$$

Here ϵ_i is the canonical ith unit vector in $\mathbb{Z}^n$.

By Noetherian induction we may assume that we have a Stanley decomposition $\mathscr{D}_1 \colon (S/(I : x_i)) = \bigoplus_{j=1}^r u_j K[Z_j]$, and a Stanley decomposition $\mathscr{D}_2 \colon S/(I + x_i) = \bigoplus_{k=1}^s v_k K[W_k]$. Then

$$\mathscr{D} \colon\ S/I = \bigoplus_{j=1}^{r} x_i u_j K[Z_j] \oplus \bigoplus_{k=1}^{s} v_k K[W_k]$$

is a Stanley decomposition of S/I.

5. *Some open problems.*

Let $S = K[x_1, \ldots, x_n]$ be the polynomial over K in n-indeterminates. In the previous subsection we mentioned the conjecture by Cimpoeaş according to which one should have $\operatorname{sdepth}(\mathfrak{m}^k) = \lceil n/(k+1) \rceil$ for all n and k where $\mathfrak{m}$ is the graded maximal ideal of S. The squarefree version of this conjecture is the following: $\operatorname{sdepth}(I_{n,d}) = \lfloor \binom{n}{d+1} / \binom{n}{d} \rfloor + d$. It would be a challenge to settle these conjectures by finding suitable partitions of the attached characteristic posets.

Assuming Cimpoeaş's conjecture one has $\operatorname{sdepth}(\mathfrak{m}^k) \geq \operatorname{sdepth}(\mathfrak{m}^{k+1})$ for all k.

Question 58. Let $I \subset S$ be a monomial ideal. Is it true that $\operatorname{sdepth}(I^k) \geq \operatorname{sdepth}(I^{k+1})$ for all k? In general such an inequality is not true for the ordinary depth. There exist examples with $\operatorname{depth}(I^k) < \operatorname{depth}(I^{k+1})$, see [89, 145]. In general this inequality is also not true for the Stanley depth. Consider for example the monomial ideal $I = (x_1^4, x_1^3x_2, x_1x_2^3, x_2^4, x_1^2x_2^2x_3)$. Then $\operatorname{depth}(S/I) = 0$ and $\operatorname{depth}(S/I^2) = 1$. It follows from Theorem 27 that $\operatorname{sdepth}(S/I) = 0$ and $\operatorname{sdepth}(S/I^2) > 0$.

On the other hand, S.A. Seyed Fakhari [171] showed that for each monomial ideal I and integer $k \geq 1$ the inequalities $\operatorname{sdepth}(S/\overline{I}) \geq \operatorname{sdepth}(S/\overline{I^k})$ and $\operatorname{sdepth}(\overline{I}) \geq \operatorname{sdepth}(\overline{I^k})$ hold. Here $\overline{J}$ denotes the integral closure of the ideal J. Thus Question 58 may have a positive answer for normal monomial ideals, that is, for monomial ideals for which all of its powers are integrally closed.

It is known by Brodmann [28] that for any graded ideal $I \subset S$ there exists an integer such that $\operatorname{depth}(I^k) = \operatorname{depth}(I^{k+1})$ for all $k \geq k_0$.

Conjecture 59. Let $I \subset S$ be a monomial ideal. Then there exists an integer k_0 such that $\operatorname{sdepth}(I^k) = \operatorname{sdepth}(I^{k+1})$ for all $k \geq k_0$. A similar statement holds for the Hilbert depth.

For the maximal ideal $\mathfrak{m}$ we have $\operatorname{sdepth}(\mathfrak{m}^k) = 1$ for all $k \geq n$, by [36, Theorem 2.2] or [32, Theorem 1.2].

Conjecture 60. Let $I \subset S$ be a monomial ideal. Then there exists an integer k_1 such that $\operatorname{sdepth}(\mathfrak{m}^k I) = \operatorname{hdepth}(\mathfrak{m}^k I) = 1$ for all $k \geq k_1$.

For the proof of this conjecture it suffices to show that $\operatorname{hdepth}(\mathfrak{m}^k I) = 1$ for all $k \geq k_1$, because $1 \leq \operatorname{sdepth}(\mathfrak{m}^k I) \leq \operatorname{hdepth}(\mathfrak{m}^k I)$ for all k.

It is known that for any graded ideal $I \subset S$, the regularity $\operatorname{reg}(I^k)$ is a linear function of k for $k \gg 0$.

Conjecture 61. Let $I \subset S$ be a monomial ideal. Then there exists integers k_2, a and b such that $\operatorname{sreg}(I^k) = ak + b$ for $k \geq k_2$.

Let $I \subset S$ be a monomial ideal. In [178, Theorem 3.10] Soleyman Jahan showed that S/I is pretty clean if and only if S/I^p is clean. Here I^p denotes the polarization of I, see [30, Lemma 4.2.16]. Consequently, if S/I is pretty clean, then not only S/I but also S/I^p satisfy Stanley's conjecture.

Conjecture 62. Let $I \subset S$ be a monomial ideal. Then

$$\operatorname{sdepth}(S/I) - \operatorname{depth}(S/I) = \operatorname{sdepth}(S^p/I^p) - \operatorname{depth}(S^p/I^p).$$

Here S^p is the polynomial ring where I^p is defined.

The conjecture in combination with Corollary 37 implies that Stanley's conjecture holds for all K-algebras with monomial relations if and only if any Cohen–Macaulay simplicial complex is partionable.

For the proof of this conjecture it suffices to consider a 1-step polarization. The complete polarization is obtained by a sequence of 1-step polarizations. A 1-step polarization is obtained as follows: Let $I = (u_1, \ldots, u_m)$ be the minimal set of monomial generators of I. Fix a number $i \in [n]$. We define the 1-step polarization of I with respect to i to be the monomial ideal $J = (v_1, \dots, v_m) \subset S[y]$, where y is an indeterminate over S and

$$v_j = \begin{cases} y(u_j/x_i), & \text{if } x_i^2 \text{ divides } u_j, \\ u_j, & \text{otherwise.} \end{cases}$$

It is known that $\operatorname{depth}(S[y]/J) = \operatorname{depth}(S/I) + 1$. Thus for the proof of Conjecture 62 one has to show that $\operatorname{sdepth}(S[y]/J) = \operatorname{sdepth}(S/I) + 1$.

Let M be a finitely generated $\mathbb{Z}^n$-graded S-module with syzygy module Z_k for $k = 1, 2 \dots$. We have seen in Theorem 28 that $\operatorname{sdepth}(Z_k) \geq k$.

Question 63. Is it true that $\operatorname{sdepth}(Z_{k+1}) \geq \operatorname{sdepth}(Z_k)$?

In [31, Lemma 3.2] Bruns et al. showed that this question has a positive answer for the syzygies of the graded maximal ideal of S. On the other hand, even the following conjecture (which is a very special case of the preceding question) is widely open.

Conjecture 64. Let $I \subset S$ be a monomial ideal.
Then $\operatorname{sdepth}(I) \geq \operatorname{sdepth}(S/I)$.

In all known cases one even has $\operatorname{sdepth}(I) > \operatorname{sdepth}(S/I)$.

Question 65. Does there exist an algorithm to compute the Stanley depth of finitely generated $\mathbb{Z}^n$-graded S-modules?

The following problem has a good chance to be solved.

Problem 66. Find an algorithm to compute the Stanley depth for finitely generated graded S-modules M with $\dim_K M_a \leq 1$ for all a.

However if we drop the assumption that $\dim_K M_a \leq 1$ for all a, an answer to Question 65 seems to be hard. The next problem demonstrates how little is known.

Problem 67. Let M and N be finitely generated graded S-modules. Then

$$\operatorname{sdepth}(M \oplus N) \geq \min\{\operatorname{sdepth}(M), \operatorname{sdepth}(N)\}.$$

Do we have equality?

Let $I \subset S$ be a monomial ideal. If equality holds in Problem 67, then in particular one has $\operatorname{sdepth}(I \oplus S) = \operatorname{sdepth}(I)$. To the best of my knowledge this is not known.

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