

Asymptotic Eigenvalue Distribution of Random Matrices and Free Stochastic Analysis

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Abstract This is a survey on some recent work on free stochastic calculus and free Malliavin calculus. It is hoped that these theories will in the long run provide us with tools for qualitative descriptions of the asymptotic eigenvalue distribution of selfadjoint polynomials of independent Gaussian random matrices. The main concrete results center around the free Fourth Moment Theorem, which says that for a sequence of random variables which are constrained to live in a fixed free chaos, the convergence to the semicircle distribution can be controlled by the convergence of the second and the fourth moments.

1 Random Matrices

Our main interest lies in distributions of polynomials of free semicircular elements. Since those distributions arise as limits of eigenvalue distributions of quite natural random matrix models we will first look, to motivate the kind of questions, on those random matrices.

1.1 Wigner's Semicircle Law

Let $X_N = (X_N(i, j))_{i,j=1}^N$ be an $N \times N$ Gaussian random matrix. This means that

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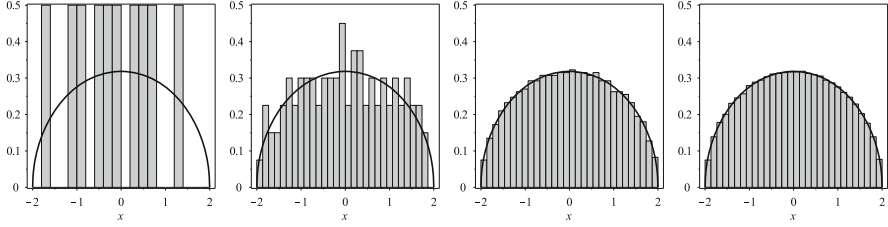


Fig. 1 Eigenvalue distribution of a $N \times N$ Gaussian random matrix, for $N = 10, 100, 1,000, 4,000$; the *solid curve* is the semicircular distribution

- X_N is selfadjoint, $X_N^* = X_N$
- $\{X_N(i, j) \mid 1 \leq i \leq j \leq N\}$ are i.i.d. random variables (say real-valued), with normal distribution of mean zero and variance $1/N$

Then, Wigner's famous semicircle law [1, 15] says that, for $N \rightarrow \infty$, we have convergence of the empirical eigenvalue distribution of X_N to the semicircle distribution. The latter lives on the interval $[-2, 2]$ and has there the density $1/(2\pi)\sqrt{4 - x^2}dx$.

Numerical simulations for this approximation of the semicircle are shown in Fig. 1.

One way to capture the essence of Wigner's semicircle law is the statement that the moments $\text{tr}[X_N^n]$ of the Gaussian random matrices converge, for $N \rightarrow \infty$, almost surely to the corresponding moments $\frac{1}{2\pi} \int_{-2}^{+2} x^n \sqrt{4 - x^2} dx$ of the semicircular distribution. With tr we denote the normalized trace on matrices,

$$\text{tr}((a_{ij})_{i,j=1}^N) := \frac{1}{N} \sum_{i=1}^N a_{ii}.$$

We write $X_N \rightarrow s$ to indicate the above kind of convergence and say that X_N *converges in distribution* to s . The limiting element s is a semicircular element living in an abstract non-commutative probability space $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a unital algebra and $\varphi : \mathcal{A} \rightarrow \mathbb{C}$ a unital linear functional; s is determined by the fact that its moments $\varphi(s^n)$ are the moments of the semicircular distribution. The notation $X_N \rightarrow s$ means here, by definition, the almost sure convergence

$$\lim_{N \rightarrow \infty} \text{tr}[X_N^n] = \varphi(s^n) \quad \forall n \in \mathbb{N}.$$

See, e.g., [9, 14].

(Note that for the case of one random matrix the use of a non-commutative probability space is quite artificial; we can just take a classical random variable s with semicircle distribution and put $\mathcal{A}$ equal to the algebra generated by s , and take as φ the expectation. Since the semicircular distribution is determined by its moments, $X_N \rightarrow s$ is then the same as weak convergence of the empirical eigenvalue distribution of X_N to the semicircular distribution. However, in the multivariate case, where we consider several matrices, we will be leaving the commutative world, and the above non-commutative frame will then be unavoidable.)

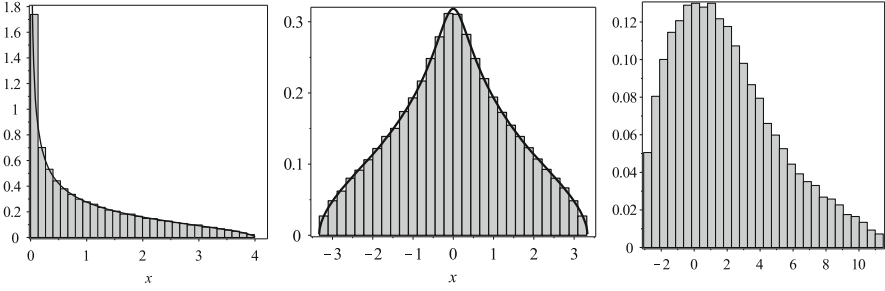


Fig. 2 Eigenvalue distributions of $p(x_1) = x_1^2$ (left), $p(x_1, x_2) = x_1x_2 + x_2x_1$ (middle) and $p(x_1, x_2, x_3) = x_1x_2^2x_1 + x_1x_3^2x_1 + 2x_2$ (right), where x_1, x_2 , and x_3 are independent $4,000 \times 4,000$ Gaussian random matrices; in the first two cases the asymptotic eigenvalue distribution can be calculated by free probability tools as the *solid curve*, for the third case no explicit solution exists

1.2 Polynomials in Independent Gaussian Random Matrices

Consider now m independent Gaussian random matrices $X_N^{(1)}, \dots, X_N^{(m)}$. Then, by Voiculescu's multivariate generalization [12] of Wigner's semicircular law, we know that those converge, for $N \rightarrow \infty$, to m free semicircular elements $s_1, \dots, s_m$, living in some non-commutative probability space $(\mathcal{A}, \varphi)$. We will define later the exact meaning of “freeness”; for the moment it suffices to say that this means that it allows in principle to calculate all mixed moments

$$\varphi(s_{i(1)} \cdots s_{i(n)}) = \lim_{N \rightarrow \infty} \text{tr}[X_N^{i(1)} \cdots X_N^{i(n)}].$$

In particular, for any selfadjoint polynomial p in m non-commuting variables we have that

$$p(X_N^{(1)}, \dots, X_N^{(m)}) \rightarrow p(s_1, \dots, s_m).$$

(A selfadjoint polynomial $p(x_1, \dots, x_m)$ in m non-commuting variables $x_1, \dots, x_m$ is one for which we have $p^* = p$, if we declare the variables to be selfadjoint, $x_i^* = x_i$ for all $i = 1, \dots, m$. This guarantees that the application of this polynomial to any m -tuple of selfadjoint operators yields a selfadjoint operator.)

So, in principle, we know everything about the limiting eigenvalue distribution of $p(X_N^{(1)}, \dots, X_N^{(m)})$. However, in practice, unless the polynomial is very simple (like the sum of one-variable polynomials), we cannot say anything concrete about this distribution. (For more information on the methods and results in the simple cases one might consult, e.g., [2, 9].)

In Fig. 2 we present histograms for eigenvalue distributions for various polynomials. One dimensional cases like $p(x) = x^2$ can be calculated directly by classical means, and some special multivariate cases like $p(x_1, x_2) = x_1x_2 + x_2x_1$ are simple enough to be calculated by free probability methods. However, for more generic

polynomials, like $p(x_1, x_2, x_3) = x_1 x_2^2 x_1 + x_1 x_3^2 x_1 + 2x_2$, there is no hope to derive any explicit formula for the limiting eigenvalue distribution.

Where we have hope and what the following will be about, are qualitative features of the distributions which arise in this way.

2 Distribution of Non-commuting Variables

2.1 How Can We Deal with Multivariate Situations?

Let us start with some general remarks on understanding the distribution of non-commuting variables $X_1, \dots, X_m$? In particular, we want to point out that there is quite a difference between the commutative and the non-commutative situation in the multivariate case $m > 1$.

In the case of commuting variables, the distribution of one variable is a probability measure on $\mathbb{R}$, the distribution of m commuting variables is a probability measure on $\mathbb{R}^m$ – which is not so much of a difference.

In the case of non-commuting variables, the distribution of one variable is still a probability measure on $\mathbb{R}$, but the distribution of $m > 1$ variables is something quite different: analytically, it is a state on the algebra generated by the variables; or combinatorially, it is the collection of all their mixed moments, and there are much more mixed moments in non-commuting variables than in commuting ones.

At the moment we are still lacking the deeper analytic tools to deal directly with the distribution of several non-commuting variables. (Voiculescu's *free analysis* is aiming in this direction, see [13]; see also the Introduction of [5] for a discussion of the problem of multivariate non-commutative distributions.) What we will consider in the following is a kind of reduction to the one-dimensional classical case. Namely, in order to understand our non-commuting variables $X_1, \dots, X_m$, we will be satisfied with understanding all selfadjoint polynomials in $X_1, \dots, X_m$. The distribution of each such polynomial is just a probability measure on $\mathbb{R}$, so we might be able to use classical tools to deal with those. However, we should somehow understand all such polynomials.

2.2 Conjectures on Distributions of Several Non-commuting Variables?

There is no hope to have explicit formulas for the asymptotic eigenvalue distributions of general polynomials p in independent Gaussian random matrices or, equivalently, the distribution of polynomials p in free semicircular variables,

$$\lim_{N \rightarrow \infty} p(X_N^{(1)}, \dots, X_N^{(m)}) = p(s_1, \dots, s_m).$$

However, from the one-variable case and numerical simulations we expect that the distributions for such situations share some common qualitative features. In particular, we believe that for an arbitrary selfadjoint polynomial p the following should be true for the distribution of $p(s_1, \dots, s_m)$.

- It should have no atoms.
- It should have a density with respect to Lebesgue measure.
- This density should also have some nice regularity properties.

There is at least one statement which we can infer from general operator algebraic results. Namely, the facts that polynomials in semicircular elements can be realized in the C^* -algebra of the free group and that the latter has no non-trivial projections imply that the support of the distribution of $p(s_1, \dots, s_m)$ has to be an interval; see, e.g., [7]. The above mentioned regularity properties can then be stated a bit more precisely by conjecturing that the density is continuous or even analytic on the interior of this interval. (The example of the square of the semicircle, $p(s) = s^2$, shows that the density can go to infinity at the boundary of the support. see Fig. 2.)

2.3 A Side-Remark: The Linearization Trick

One possible simplification in this business is the following linearization trick. Instead of looking on all polynomials, it might be enough to consider just linear polynomials in $X_1, \dots, X_m$. In the commutative case this is actually enough; knowing the distribution of all linear combinations of the random variables $X_1, \dots, X_m$ is equivalent to the knowledge of the joint distribution of the random variables (and thus to the knowledge of the distribution of arbitrary polynomials in the variables). In the non-commutative case this is not true any more; what one needs there are all operator-valued linear polynomials, i.e., matrices have to be allowed as coefficients in the linear combinations. This is much more general than just scalar-valued linear polynomials, but on the other hand for many calculations the nature of the coefficients is not important, so much of the theory has still the flavour of the usual linear theory.

A striking example of how far this linearization trick can be pushed is the following result of Haagerup and Thorbjørnsen [6] (see also [7]) about the largest eigenvalue of any selfadjoint polynomial in independent Gaussian random matrices:

$$\lambda_{\max}(p(X_N^{(1)}, \dots, X_N^{(m)})) \rightarrow \|p(s_1, \dots, s_m)\| \quad \text{almost surely,}$$

where $\|p(s_1, \dots, s_m)\|$ denotes the operator norm of the operator $p(s_1, \dots, s_m)$ (for example, in a concrete realization of $s_1, \dots, s_m$ as operators on some Hilbert space).

We will, however, not follow this direction, but will try to understand the distribution of general polynomials $p(s_1, \dots, s_m)$ in free semicircular elements by invoking free stochastic calculus.

3 Free Stochastic Calculus

Our approach to the question of qualitative features of distributions of polynomials in free semicircular elements will rely on stochastic calculus, the rough idea being to build a polynomial $p(s_1, \dots, s_m)$ in a differential way and control the differential changes along the way by free stochastic calculus and free Malliavin calculus. So, in the following we will first recall a bit of free stochastic calculus before we come back to our original problem.

As a general reference to basic free probability notions we refer to [9, 14]. Free stochastic calculus was introduced by Biane and Speicher in [3], a good summary can also be found in [8].

3.1 Freeness

Definition 1. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space; i.e., $\mathcal{A}$ is a unital algebra and $\varphi : \mathcal{A} \rightarrow \mathbb{C}$ is a unital linear functional. Unital subalgebras $(\mathcal{A}_i)_{i \in I}$ (for some index set I) are *free* if $\varphi(a_1 \cdots a_n) = 0$ whenever for each $k = 1, \dots, n$ we have $a_k \in \mathcal{A}_{i(k)}$ for some $i(k) \in I$ such that $i(k) \neq i(k+1)$ for all $k = 1, \dots, n-1$ and such that $\varphi(a_k) = 0$ for all $k = 1, \dots, n$.

Random variables $x_i \in \mathcal{A}$ ($i \in I$) are free, if their generated unital subalgebras are free.

3.2 Free Brownian Motion

Definition 2. A *free Brownian motion* is given by a family $(S(t))_{t \geq 0}$ of random variables $S(t) \in \mathcal{A}$ ($t \geq 0$), living in a non-commutative probability space $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a von Neumann algebra and φ a faithful trace, such that

- $S(0) = 0$
- Each increment $S(t) - S(s)$ ($0 \leq s < t$) is semicircular with mean 0 and variance $t - s$, i.e.,

$$d\mu_{S(t)-S(s)}(x) = \frac{1}{2\pi(t-s)} \sqrt{4(t-s) - x^2} dx$$

- Disjoint increments are free: for $0 < t_1 < t_2 < \dots < t_n$, the increments $S(t_1)$, $S(t_2) - S(t_1)$, $\dots$, $S(t_n) - S(t_{n-1})$ are free.

A free Brownian motion can also be realized concretely: by the sum of creation and annihilation operators on the full Fock space; and as the limit of matrix-valued (Dyson) Brownian motions. This latter realization is as follows.

Let $(X_N(t))_{t \geq 0}$ be a symmetric $N \times N$ -matrix-valued Brownian motion, i.e.,

$$X_N(t) = \begin{pmatrix} B_{11}(t) & \dots & B_{1N}(t) \\ \vdots & \ddots & \vdots \\ B_{N1}(t) & \dots & B_{NN}(t) \end{pmatrix}$$

where

- B_{ij} are, for $i \geq j$, independent classical Brownian motions
- $B_{ij}(t) = B_{ji}(t)$.

Then, $(X_N(t))_{t \geq 0} \rightarrow (S(t))_{t \geq 0}$, in the sense that almost surely

$$\lim_{N \rightarrow \infty} \text{tr}(X_N(t_1) \cdots X_N(t_n)) = \varphi(S(t_1) \cdots S(t_n)) \quad \forall 0 \leq t_1, t_2, \dots, t_n$$

3.3 Stochastic Analysis on “Wigner” Space

3.3.1 Definition of Free Stochastic Integrals

Starting from a free Brownian motion $(S(t))_{t \geq 0}$ we define multiple *Wigner integrals*

$$I(f) = \int \cdots \int f(t_1, \dots, t_n) dS(t_1) \cdots dS(t_n)$$

for scalar-valued functions $f \in L^2(\mathbb{R}_+^n)$, by avoiding the diagonals, i.e. we understand in the above that all times t_i are different. The definition is made rigorous, by first defining $I(f)$ for step functions f which have no support on the diagonals; for those one establishes an Ito isometry

$$\|I(f)\|_2 = \|f\|_{L^2(\mathbb{R}_+^n)},$$

and then uses this to extend the definition to all $f \in L^2(\mathbb{R}_+^n)$. The integral $I(f)$ is then an element in the L^2 -space of the free Brownian motion. The 2-norm on $\mathcal{A}$ is as usual given by $\|A\|_2^2 := \varphi(AA^*)$, and the L^2 -space is then the completion of $\mathcal{A}$ with respect to $\|\cdot\|_2$.

3.3.2 A Side Remark: Free Stochastic Integrals are Usually Bounded Operators

We want to remark that free stochastic integrals are not just elements in the L^2 -space, but usually are bounded operators and thus their distribution has compact support. Note that the fact that all our $S(t)$ and their increments are bounded does

not imply the same property for the integrals (as they are limits of sums), but one needs the freeness of the increments as the essential input to prove this. More precisely, we have the following estimate (which is the analogue of a Haagerup inequality) [3, 4]:

$$\left\| \int \cdots \int f(t_1, \dots, t_n) dS(t_1) \dots dS(t_n) \right\| \leq (n+1) \|f\|_{L^2(\mathbb{R}_+^n)}.$$

The operator norm in the above can be recovered from the moments of the variables even without having the free Brownian motion concretely realized as operators on a Hilbert space in the following way:

$$\|A\| = \lim_{m \rightarrow \infty} \sqrt[2m]{\varphi((AA^*)^m)}.$$

3.3.3 Ito Formula and Multiplication of Multiple Wigner Integrals

The Ito formula shows up in the multiplication of two multiple Wigner integrals.

As first example, consider the product of two first order Wigner integrals.

$$\begin{aligned} \int f(t_1) dS(t_1) \cdot \int g(t_2) dS(t_2) &= \iint f(t_1) g(t_2) dS(t_1) dS(t_2) + \int f(t) g(t) \underbrace{dS(t) dS(t)}_{dt} \\ &= \iint f(t_1) g(t_2) dS(t_1) dS(t_2) + \int f(t) g(t) dt. \end{aligned}$$

We have invoked here the free Ito formula $dS(t)dS(t) = dt$. This is the same as the Ito formula for classical Brownian motion $dB(t)dB(t) = dt$. In order to see a difference between the classical and the free situation, we consider a more complicated case, the product of a first order with a second order integral.

$$\begin{aligned} \iint f(t_1, t_2) dS(t_1) dS(t_2) \cdot \int g(t_3) dS(t_3) &= \iiint f(t_1, t_2) g(t_3) dS(t_1) dS(t_2) dS(t_3) \\ &\quad + \iint f(t_1, t) g(t) dS(t_1) \underbrace{dS(t) dS(t)}_{dt} + \iint f(t, t_2) g(t) \underbrace{dS(t) dS(t_2) dS(t)}_{dt \varphi[dS(t_2)] = 0}. \end{aligned}$$

In the classical case, the last term would also contribute as $dB(t)dB(t_2)dB(t) = dB(t_2)dB(t)dB(t) = dB(t_2)dt$, whereas in the free case this term vanishes according to the general version of the free Ito formula

$$dS(t)AdS(t) = \varphi(A)dt \quad \text{for } A \text{ adapted.}$$

(Adapted means here as in the classical case that A is only a function of the free Brownian motion up to time t .)

3.3.4 Contractions

For a systematic description of the product of two Wigner integrals it is useful to introduce the notion of contraction of functions. This is defined as follows. Consider $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$. Then we define, for $0 \leq p \leq \min(n, m)$, the p -th contraction of f and g , $f \stackrel{p}{\frown} g \in L^2(\mathbb{R}_+^{n+m-2p})$, by

$$f \stackrel{p}{\frown} g(t_1, \dots, t_{m+n-2p}) := \int f(t_1, \dots, t_{n-p}, s_1, \dots, s_p) g(s_p, \dots, s_1, t_{n-p+1}, \dots, t_{n+m-2p}) ds_1 \cdots ds_p$$

With this definition we have the following general formula for the product of two Wigner integrals.

$$I(f) \cdot I(g) = \sum_{p=0}^{\min(n,m)} I(f \stackrel{p}{\frown} g) \quad (f \in L^2(\mathbb{R}_+^n), g \in L^2(\mathbb{R}_+^m))$$

3.3.5 Free Chaos Decomposition

As for classical Brownian motion one has now the canonical isomorphism

$$L^2(\{S(t) \mid t \geq 0\}) \hat{=} \bigoplus_{n=1}^{\infty} L^2(\mathbb{R}_+^n), \quad f \hat{=} \bigoplus_{n=0}^{\infty} f_n,$$

via

$$f = \sum_{n=0}^{\infty} I(f_n) = \sum_{n=0}^{\infty} \int \cdots \int f_n(t_1, \dots, t_n) dS(t_1) \cdots dS(t_n).$$

This decomposition of the L^2 -space of free Brownian motion as a direct sum of Wigner integrals is called *free chaos decomposition* and the set $\{I(f_n) \mid f_n \in L^2(\mathbb{R}_+^n)\}$ is called the n -th *free chaos*.

4 Distributions in Chaoses

4.1 Conjectures on Distributions in Fixed or Finite Chaoses

Since free semicircular elements $s_1, \dots, s_m$ can in this frame be realized as elements in the first chaos (namely, as stochastic integrals over characteristic functions

of disjoint intervals of length 1), polynomials in our free semicircular elements, $p(s_1, \dots, s_m)$, are then, by the Ito formula, realized as elements from finite chaos

$$\{I(f) \mid f \in \bigoplus_{\text{finite}} L^2(\mathbb{R}_+^n)\}$$

Thus our conjectures on qualitative features of the distribution of $p(s_1, \dots, s_m)$ can now also be rephrased as follows.

The distribution of selfadjoint elements from a finite chaos with no constant ($n = 0$) contribution should have no atoms and they should have a density with respect to Lebesgue measure.

Motivated by the corresponding situation from classical chaos decomposition we conjecture also that selfadjoint variables $I(f)$ and $I(g)$ from different chaoses cannot have the same distribution.

It might not be clear to the reader why our original problem (on densities of polynomials in free semicircular elements) should be more accessible if reformulated in the language of stochastic integrals. As an answer to this we can only offer the observation that stochastic analysis offers some additional tools to attack this question. In particular, it is our hope that free Malliavin calculus, when sufficiently developed, should be able to answer our questions. It should be remarked that classical Malliavin calculus is indeed answering the classical analogue of our question – i.e., it allows to prove that selfadjoint elements from finite chaos of classical Brownian motion (like polynomials in independent normal variables) have a smooth density.

4.2 The Free Fourth Moment Theorem

Most of the above conjectures are still open, but we have at least some positive results which show that looking on elements from a fixed chaos can give some quite unexpected constraints. Namely, we have the following free analogue of a striking classical result of Nualart and Peccati [11]. There, f^* , for $f \in L^2(\mathbb{R}_+^n)$, is defined by

$$f^*(t_1, \dots, t_n) := \overline{f(t_n, \dots, t_1)}.$$

The condition $f^* = f$ just ensures that $I(f)$ is a selfadjoint operator.

Theorem 1 ([8]). *Consider, for fixed n , a sequence $f_1, f_2, \dots \in L^2(\mathbb{R}_+^n)$ with $f_k^* = f_k$ and $\|f_k\|_2 = 1$ for all $k \in \mathbb{N}$. Then the following statements are equivalent.*

1. We have $\lim_{k \rightarrow \infty} \varphi[I(f_k)^4] = 2$.
2. We have for all $p = 1, 2, \dots, n-1$ that

$$\lim_{k \rightarrow \infty} f_k \stackrel{p}{\sim} f_k = 0 \quad \text{in } L^2(\mathbb{R}_+^{2n-2p}).$$

3. The selfadjoint variable $I(f_k)$ converges in distribution to a semicircular variable of variance 1.

The striking direction of these equivalences is that (1) implies (3); i.e., if we constrain our variables to live in one fixed chaos then, under fixed second moment, the convergence of the fourth moment implies the convergence of all other moments. The second condition is the key for the proof of this. Convergence of the fourth moments implies the vanishing of the non-trivial contractions; but all moments can be calculated in terms of contractions and the vanishing of the non-trivial ones implies then that this calculations reproduces just the Catalan numbers, which are the moments of a semicircular variable.

As a consequence of the above theorem we get that we can at least distinguish elements from the first chaos (all of which are semicircular variables) from all the other chaoses.

Corollary 1 ([8]). *For $n \geq 2$ and $f \in L^2(\mathbb{R}_+^n)$, the law of $I(f)$ is not semicircular*

4.3 Quantitative Estimates

In the classical case one can refine the qualitative statement of convergence to a normal variable to quantitative estimates of the distance to a normal variable in terms of differences of the fourth moments. Such estimates exist for various distances (like Kolmogorov or Wasserstein distance). In the free case, those “classical” distances do not fit too well with the non-commutative nature of the problem (i.e., even though in the end one wants to compare just two single variable distributions, i.e., two probability distributions, the constraint that they live in some fixed free chaos puts them into a quite non-commutative context). The following distance seems to be more appropriate for our free setting.

4.3.1 Distance Between Operators

Given two self-adjoint random variables X, Y , we define their distance

$$d_{\mathcal{C}_2}(X, Y) := \sup\{|\varphi[h(X)] - \varphi[h(Y)]| : \mathcal{I}_2(h) \leq 1\};$$

where $\mathcal{I}_2(h)$ is formally the norm $\|\partial h'\|$ of the non-commutative derivative of the classical derivative of h ; the non-commutative derivative ∂ is defined by

$$\partial X^n = \sum_{k=0}^{n-1} X^k \otimes X^{n-1-k}.$$

One way to make this rigorous is as follows. If h is the Fourier transform of a complex measure ν on $\mathbb{R}$,

$$h(x) = \hat{\nu}(x) = \int_{\mathbb{R}} e^{ix\xi} \nu(d\xi)$$

then we define

$$\mathcal{J}_2(h) = \int_{\mathbb{R}} \xi^2 |\nu|(d\xi).$$

The above definitions are motivated by the fact that $\partial h'$ is essentially the Ito correction term showing up in the free Ito formula applied to $dh(S_t)$.

4.3.2 The Free Gradient Operator

As in the classical setting we have a free version of a Malliavin calculus. The basic definitions and properties were given in [3]; however, the more advanced features of this theory still have to be developed.

Here is the definition of the free Malliavin gradient operator. Since it is related with the non-commutative derivative, it will, like ∂ , also map into the tensor product.

Definition 3. For $f \in L^2(\mathbb{R}_+^n)$ we define the *free (Malliavin) gradient* $\nabla I(f)$ of $I(f)$ by

$$\begin{aligned} \nabla_t \left(\int f(t_1, \dots, t_n) dS_{t_1} \cdots dS_{t_n} \right) := \\ \sum_{k=1}^n \int f(t_1, \dots, t_{k-1}, t, t_{k+1}, \dots, t_n) dS_{t_1} \cdots dS_{t_{k-1}} \otimes dS_{t_{k+1}} \cdots dS_{t_n}. \end{aligned}$$

$\nabla_t I(f)$ is to be considered as a function in t , taking on values in the tensor product $L^2 \otimes L^2$.

The adjoint of ∇ is the *free Skorohod integral*, denoted by δ . It maps bi-processes, i.e., functions which take values in the tensor product into the L^2 -space of our free Brownian motion.

4.3.3 Estimate of the Distance Against the Gradient

Let us denote by N the number operator, which acts just by multiplication by n in the n -th chaos, i.e.,

$$NI(f) = nI(f), \quad \text{for } f \in L^2(\mathbb{R}_+^n).$$

Since one has, as in the classical case, $\delta \nabla = N$, one can derive the following estimate.

Theorem 2 ([8]). *Let $F = F^*$ be an element from a finite chaos with mean 0 and variance 1. Then we have the following estimate for the distance between F and a semicircular variable S of mean 0 and variance 1:*

$$d_{\mathcal{E}_2}(F, S) \leq \frac{1}{2} \varphi \otimes \varphi \left(\left| \int \nabla_s (N^{-1} F) \sharp (\nabla_s F)^* ds - 1 \otimes 1 \right| \right).$$

The symbol $\sharp$ denotes here the following product in the tensor product:

$$(a_1 \otimes b_1) \sharp (a_2 \otimes b_2) := a_1 a_2 \otimes b_2 b_1.$$

In the classical case one can estimate the corresponding expression of the above gradient, for F living in some n -th chaos, in terms of the fourth moment of the considered variable, thus giving a quantitative estimate for the distance between the considered variable (from a fixed chaos) and a normal variable in terms of the difference between their fourth moments. In the free case such a general estimate does not seem to exist; at the moment we are only able to do this for elements F from the second chaos.

Corollary 2 ([8]). *Let $F = I(f) = I(f)^*$ ($f \in L^2(\mathbb{R}_+^2)$) be an element from the second chaos with variance 1, i.e., $\|f\|_2 = 1$, and let S be a semicircular variable with mean 0 and variance 1. Then we have*

$$d_{\mathcal{E}_2}(F, S) \leq \frac{1}{2} \sqrt{\frac{3}{2}} \sqrt{\varphi(F^4) - 2}.$$

4.3.4 A Multivariate Fourth Moment Theorem

In the long run, we are looking for multivariate versions of the above results. On the qualitative level, this extension is quite straightforward and does not require new ideas. Here we have the following result.

Theorem 3 ([10]). *Let, for each $i = 1, \dots, d$, $(f_k^{(i)})_{k \in \mathbb{N}}$ be a sequence in $L^2(\mathbb{R}_+^{n_i})$, such that for all i, j*

$$\lim_{k \rightarrow \infty} \varphi[I(f_k^{(i)}) \cdot I(f_k^{(j)})] = \delta_{ij}.$$

The following statements are equivalent.

1. *$((I(f_k^{(1)}), \dots, I(f_k^{(d)})))$ converges in distribution to a free semicircular family $(s_1, \dots, s_d)$ as $k \rightarrow \infty$.*
2. *For each $i = 1, \dots, d$, $I(f_k^{(i)})$ converges to s_i as $k \rightarrow \infty$.*

3. For each $i = 1, \dots, d$, the fourth moments of $I(f_k^{(i)})$ converge to 2:

$$\lim_{k \rightarrow \infty} \varphi[I(f_k^{(i)})^4] = 2.$$

However, when it comes to quantitative results, nothing is known at the moment for the multivariate situation. In particular, we still don't know what a reasonable multivariate version for the distance $d_{\mathcal{C}_2}$ should be.

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