Preface

This proceedings volume contains a collection of referred contributions by participants of the international conference *Dynamical Systems: 100 years after Poincaré*, (held in Gijón, Spain, September 3–7, 2012) which commemorated H. Poincaré.

In the exceptional paradise of Asturias, with the sun shining brightly all week over the Cantabrian sea, more than 100 researchers from 20 countries joined together to discuss new trends in Dynamical Systems. In total, 71 talks, including 11 plenary lectures, completed more than 30 h of intensive scientific work as the waves beat against the impressive beach of San Lorenzo.

The name of Poincaré, often considered as the *last universalist*, can be claimed by the majority of mathematicians and a great part of contemporary scientists. His conjecture about the topological characterization of 3-dimensional spheres or his pioneering contributions to the Principles of Relativity interpreting Lorentz's theory are themselves reasons for such general acknowledgement in the worlds of Maths and Theoretical Physics. However, it is the field of Dynamical Systems in which his work reaches our time with the greatest influence and most promising future prospects.

Knowledgeable about the difficulties of solving differential equations, Poincaré understood the need for a new geometrical and qualitative approach. Solutions of differential equations, as particular functions, would no longer be the main interest. Instead, the behaviour of the solutions as a whole, that is, the phase portrait, would be the focus. A vector field on a differentiable manifold defines a partition of such manifold in orbits (trajectories or integral curves); this interpretation later led to the notion of foliation. Topological properties of these curves and their disposal inside the phase space gain a true dynamical significance and become the principal target in the study of differential equations. The functional approach gave way to a novel geometrical view. The analytical properties of the solutions were no longer so relevant and instead the study of invariant sets (their topological properties and their dynamical meaning) became the focus of attention.

But above all, H. Poincaré made it clear that each equation must be understood hereinafter as a whole and, from this new concept, presents the relationship between local and global behaviours, between singularities and closed orbits or other invariant sets. In any of these respects, all of H. Poincaré's contributions are characterized by an impressive creativity and significance.

Very early, in his doctoral thesis *Sur les propriétés des fonctions définies par les équations aux différences partielles*, he initiated the normal forms theory for singularities of vector fields. These techniques still remain as essential tools when dealing with local bifurcations. As inheritor of Darboux's concerns about the integrability of planar vector fields, Poincaré introduced the notion of remarkable value, bearing in mind the characterization of integrable planar polynomial vector fields. This attempt at relating the nature of a singularity to the global behaviour of a vector field also led him to the notion of index.

In order to study the flows associated with vector fields through the iterations of a diffeomorphism, H. Poincaré introduced what we now know as Poincaré maps, defined at first on transverse sections to a periodic orbit. The understanding of the dynamics around a periodic orbit became dependent on the analysis of the dynamics of a diffeomorphism in the vicinity of a fixed point. It was definitely an innovative idea that moved the world of continuous dynamical systems to the world of discrete dynamical systems. The different aspects of this genius are collected in four articles, published between 1881 and 1886, which completed his work *Sur les courbes définies par une équation différentielle*. This contribution marks a turning point in the way people looked at dynamics. However, an extremely significant boost was just ready to arrive with his work *Sur le probléme des trois corps et les équations de la dynamique*, with which he was awarded a prize granted by King Oscar II of Sweden in 1889.

Considering the equations that model the 3-body problem, he found hyperbolic periodic orbits whose two-dimensional invariant manifolds had non empty intersections and concluded, erroneously, that they should coincide. Hence these invariant manifolds should limit orbits that do not escape to infinity, which implied a false stability in the restricted 3-body problem. After reviewing this mistake, Poincaré introduced the term "homoclinic point" to refer to the points where manifolds intersect. The existence of one of these points implies the presence of many of them, where invariant manifolds intersect but do not coincide, describing a very complex tangle that he himself chose not to draw. Further understanding of the dynamics involved in these homoclinic configurations has led to intense and interesting research throughout the whole twentieth century until the present day, when many interesting open problems are still in need of answers. Thus, with this forcefulness, the prediction that Paul Apell made in 1925 is exceeded: "It is likely that, during the next half-century, this book will be the mine from which more modest researchers will extract their material", and that A. Chenciner reminds us in his contribution "A walk through the *New Methods of Celestial Mechanics*".

It took more than 30 years for a first result, shedding light on the dynamics near a homoclinic point, to arrive. In 1935, G. Birkhoff showed that in general, near a homoclinic point there exists an extremely intrincated set of periodic points, mostly with a very long period.

By the mid-1960s, S. Smale placed his geometrical device, the *Smale horseshoe*, in a neighbourhood of a transversal homoclinic orbit, thus explaining Birkhoff's result and arranged the complicated dynamics that takes place near such a orbit by means of a conjugation to the shift of Bernouilli. These horseshoe maps were, for the last decades, the most useful tools to classify dynamical systems, understand their transitions and explain chaotic behaviours.

Horseshoes and Anosov diffeomorphisms were the source of inspiration to introduce the notion of axiom-A diffeomorphisms or uniformly hyperbolic containing the open class of structurally stable diffeomorphisms. A mechanism of transition between this set and its complementary is the creation/destruction of horseshoes which takes place when a homoclinic tangency is unfolded. In this scenario, just for 2-dimensional diffeomorphisms, there appear non hyperbolic strange attractors (or repellers) and the Newhouse phenomenon, explaining the persistence of homoclinic tangencies.

For higher dimensional diffeomorphisms, the persistence of non uniformly hyperbolic dynamics is explained with the notion of *blender*, associated to heterodimensional cycles, as introduced by C. Bonatti and L. J. Díaz in 1996. Roughly speaking, a blender can be understood as a sufficiently thick hyperbolic set Γ such that the closure of an invariant manifold of dimension u of a saddle point in Γ contains an invariant manifold of dimension $u+1$. The most simple scenarios where blenders can be described are the skew products defined on a horseshoe map, where the dynamics on the fibers is given by an iterated system of functions, mostly contractions. In this volume, L. J. Díaz and K. Gelfert show a partially hyperbolic and topologically transitive which they call a porcupine-like horseshoe. Dynamics on this set is given by a skew product over a horseshoe, but the fiber dynamics is given by a one-dimensional genuinely non-contracting iterated function system.The authors explain how the properties of the iterated function system can be translated to topological and ergodic properties of the porcupines.

Ergodic Theory deals with measure preserving processes in a measure space and it appears after the Poincaré recurrence theorem. In particular, it tries to describe the average time spent by typical orbits in different regions of the phase space. According to Birkoff's Ergodic Theorem these times are well defined for almost all points, with respect to any invariant probability measure. However, the notion of typical orbit is usually understood in the sense of volume (Lebesgue measure), which is not always an invariant measure. It is a fundamental open problem to understand under which conditions the behavior of typical (positive Lebesgue measure) orbits is well defined from the statistical point of view. In chaotic dynamical systems this can be precisely formulated by means of Sinai-Ruelle-Bowen (SRB) measures, which were introduced by Sinai for Anosov diffeomorphisms and later extended by Ruelle and Bowen for Axiom A diffeomorphisms and flows. In trying to capture the persistence of the statistical properties of a dynamical system, J. F. Alves and M. Viana proposed the notion of statistical stability, which expresses the continuous variation of SRB measures as a function of the dynamical system. In the contribution by J. F. Alves and M. Soufi some results on the existence and continuous variation of physical measures for families of chaotic dynamical systems are given. In particular, quadratic maps and Lorenz flow are considered in more detail.

Chaotic dynamics, rigourously understood as the existence of strange attractors, is frequent outside the uniformly hyperbolic context. Persistence, in terms of positive probability of existence, of these attractors was proved, with great effort, for the Hénon family, for generic families of diffeomorphisms unfolding homoclinic tangencies and also for families of 3-dimensional vector fields unfolding Shil'nikov homoclinic orbits which are always accompanied by an infinity of horseshoes. In all cases the strange attractors are, as the unstable invariant manifold, of dimension one and therefore they have at most one positive Lyapunv exponent. Proofs are based in unfoldings of unimodal families, whose dynamics were studied along the second half of the past century. In order to prove the existence of persistent non hyperbolic strange attractors with more than one positive Lyapunov exponent it is necessary to consider 3-dimensional diffeomorphisms with homoclinic tangencies involving an unstable invariant manifold with dimension larger than two. There are numerical evidences of the existence of such attractors, but proofs require the study of certain non linear 2-dimensional maps (instead of unimodal maps). They will play the role of limit families that must be unfolded to conclude the existence of a strange attractor in the renormalization neighbourhood of a 3-dimensional homoclinic point.

The contribution by J. C. Tatjer et al. initiates this program with the study of certain piecewise linear maps which generalize in dimension two the tent maps essential in the understanding of the unimodal dynamics. This idea of unfolding lower dimensional dynamics to understand the behaviour in higher dimensions and the use of Poincaré maps defined on crossings sections, sometimes including the hypothesis of invariance for certain foliations on the section, are two routes that justify the study of the iteration of maps on a manifold. The complexity of the dynamics in these maps has been settled down in terms of entropy, either topological or metric, and it has motivated a large number of papers. Two of the contributions featured in this volume are related to such questions. S. Aranzubía and R. Labarca study the continuity of the topological entropy in the Milnor-Thurston World. In the contribution by S. Cánovas the relation between both entropies is considered for the case of a non autonomous discrete system, that is, the ordered iteration $T_n \circ T_{n-1} \circ \cdots \circ T_1(x)$ of a sequence of maps $\{T_j\}_{j \in \mathbb{N}}$ defined on a topological or probabilistic space X.

Homoclinic orbits, or more generally, heteroclinic cycles play a crucial role to explain loss of stability or chaotic behaviour. Nevertheless, finding analytical proofs of the existence of chaos is not at all easy, particularly when the angle of intersection between the invariant manifolds is exponentially small or, for instance, when one wants to prove the existence of Shil'nikov homoclinic orbits. They appear for instance in the unfolding of Bykov cycles, a kind of heteroclinic cycle. In the contribution by I. Laboriau and A.P. Rodrigues, the authors consider an equivariant family of volume-contracting vector fields on the three-dimensional sphere. When part of the symmetry is broken, the vector fields exhibit Bykov cycles and close to the symmetry persistent suspended horseshoes accompanied by attracting periodic trajectories with long periods appear.

Existence of homoclinic and heteroclinic orbits has been commonly argued in the literature, particularly in Celestial Mechanics, taking small perturbations of hamiltonian vector fields. Regarding this approach one can see the paper by C. Simó and A. Vieiro where 2D diffeomorphisms with a homoclinic figure-eight to a dissipative saddle are perturbed by means of a periodic forcing.

In the more general case, when families of vector fields arise in given applications where the perturbation techniques cannot be applied, one can use numerical methods to prove, for instance, the existence of hypersurfaces of homoclinic bifurcations limiting stability domains in the parameter space or to compute Lyapunov exponents and conclude the existence of strange attractors when one finds any positive. Several contributions included in this volume are closely related to that approach.

The paper by M. Guardia et al. shows the existence of oscillatory motions for any value of the mass ratio in the restricted circular three body problem. The existence of these motions follows from the symbolic dynamics associated to the transversal intersection between the stable and unstable manifolds of infinity, which takes place for any value of the mass ratio and for big values of the Jacobi constant. Since the mass ratio is no longer small, this transversality cannot be checked by means of classical perturbation results.

C. Simó et al. consider again the restricted three-body problem and study the stability around the triangular libration points. The local stability follows from the well known KAM theory and Nekhorosev-like estimates. This paper examines what is the extent of the domains of practical stability. The answer requires the control of the intersection between the invariant manifolds associated to the different transient tori which leads to the arising of Arnold diffusion.

In the paper by L. Benet and A. Jorba, a simple model for the confinement of Saturn's F ring is considered and some preliminary numerical results are discussed. The classical Hindmarsh-Rose neuron model is studied numerically in the paper by M. A. Martínez et al. for certain parameter values where chaotic dynamics exists. The contribution by R. Barrio et al. is also of a computational nature. The authors examine spiral structures in 2-parametric diagrams of dissipative systems with strange attractors. Existence of chaotic dynamics is concluded in the last two mentioned papers by finding attractors with positive Lyapunov exponents. In the paper by R. Barrio et al. a new computational technique is proposed for explorations of parametric chaos in Lorenz like attractor. Numerical analysis of bifurcation diagrams, Lyapunov exponents and stability regions can be found also in the contribution by C. Simó and A. Vieiro, where they study the dynamics of a parametrically driven dissipative pendulum with a magnetic kick force acting on it. P. Benítez et al. study the LiNC/LiCN triatomic molecule vibrational dynamics including all three degrees of freedom by using frequency maps and small alignment index. With these tools they obtain numerical representations of global chaotic dynamics.

From Poincaré on, a vector field was understood as the qualitative picture of all orbits in its phase portrait. First, efforts were addressed to the study of all possible configurations, but bearing in mind vector fields possessing some particular interest. Nowadays, in contrast to such a approach, research is focused on whole sets $\mathscr{X}$ of dynamical systems, either consisting on vector fields or diffeomorphisms, defined on a manifold M. Given a topology and a dynamical equivalence relation, usually the topological equivalence, defined on $\mathscr{X}$, the first focus of attention is the subset $\Sigma \subset \mathscr{X}$ given by the structurally stable systems, which is obtained by taking the union of the interiors of all equivalence classes. It coincides with the set consisting of all hyperbolic vector fields or diffeomorphisms satisfying the transversality condition. The complement $\mathscr{B}$ of Σ is called the bifurcation set and there one finds those systems whose invariant manifolds are tangent: with non transversal homoclinic points. Homoclinic tangencies are persistent generically on hypersurfaces $\mathscr{H} \subset \mathscr{X}$ of codimension one and just as the existence of a homoclinic point implies the existence of an infinite number of such points, the Newhouse phenomenon, for example, shows that the existence of a hypersurface $\mathscr{H}$ of vector fields or diffeomorphisms with homoclinic tangencies implies the existence of many others arbitrarily close. Hypersurface $\mathscr{H}$ can contain others of higher codimension matching with other dynamical transitions and in between these hypersurfaces there can arise very complicated dynamics which are persistent but not structurally stable (Hénon like strange attractors for instance). All of this shows that $\mathscr{B}$ has a very complex stratification. In order to unravel its structure we can pay attention, emulating Poincaré again, to the simplest elements in $\mathscr{B}$ playing a role in its organization. These elements are those systems with a non hyperbolic equilibrium (or fixed) point and we refer to them as singularities, either thinking in the point or in the system where it appears. The number of strata in $\mathscr{B}$ which are adjacent to a given singularity will depend on its codimension. This approach recalls again Poincaré. On one hand, to understand what happens close to a singularity, we study the singularity itself. In the same way, from the study of the equilibrium points of a vector field (topological index, remarkable values, ...) one can obtain properties of the behaviour in a neighbourhood of the equilibrium point. One of the essential tools when dealing with singularities is the reduction to normal form. This technique was introduced by Poincaré in his thesis and it is based in the use of change of coordinates to obtain simplified expressions of the singularity, where most of the terms in the Taylor expansion are appropriately removed. Since the structure of the bifurcation set is extremely complicated, in order to deal with a given singularity X we consider families of dynamical systems X_μ contained in $\mathscr{X}$ and satisfying $X_0 = X$. These families are called unfoldings of the singularity. The knowledge of the different dynamics present in the families X_μ allows for an understanding of the structure of $\mathscr{B}$ in the vicinity of X. Some examples of unfoldings are considered in the contribution by H. Broer and G. Vegter, which considers the unfolding of some singularities of diffeomorphisms, mainly focusing on the phenomenon of resonance, which is the main concern of this paper. The concept of resonance is discussed and illustrated with a collection of clever examples.

The great variety of feasible dynamics for vector fields in dimension higher that three is not possible in the planar case. The Jordan Curve Lemma permits to prove the Poincaré Bendixson Theorem which characterize the limit sets of orbits in the plane. Thanks to the restrictions stated in that theorem many questions regarding planar flows dynamics have been solved. Nevertheless many other problems remain unsolved, including the integrability, the center-focus problem or the 16th Hilbert problem.

The contribution by A. Gasull and H. Giacomini is related to the 16th Hilbert problem. They use an extension of the Bendixson-Dulac Theorem to control the number and disposal of the limit cycles. The importance of the use of the Bendixson-Dulac results is that in many cases they translate the problem of knowing the number of periodic solutions of a planar polynomial differential equation to a problem of semi-algebraic nature: the control of the sign of a polynomial in a suitable domain.

Questions regarding integrability, limit cycles and center problem can be found in the paper by A. Buică et al., but posed for 3-dimensional vector fields or 2-dimensional non autonomous systems. The principal purpose of this paper is two fold: first to prove the existence and smoothness of inverse Jacobi multiplier V in the region of interest in the phase space and second to show that the invariant set under the flow given by the zero-set of an inverse Jacobi multiplier contains under some assumptions orbits which are relevant in its phase portrait such as periodic orbits, limit cycles, stable, unstable and center manifolds and so on. In the non-autonomous T-periodic case, A. Buică et al. show some relationships between T-periodic orbits and T-periodic inverse Jacobi multipliers.

In the paper by C. Alonso-González et al. an infinitesimal version of the Poincaré-Bendixson problem in dimension three is given. They describe the sets of accumulation of secants for orbits of real analytic vector fields in dimension three with the origin as only ω-limit point. These sets have structure of cyclic graph when the singularities are isolated under one blow-up. If the reduction of singularities is hyperbolic, under conditions of Morse-Smale type, they prove that the accumulation set is a single point or homeomorphic to S^1.

Recently there exist many papers devoted to the study of piecewise linear vector fields. These systems are easily designable using electric circuits and this can motivate the growing interest in their study. From the theoretical point of view researchers are interested in developing a theory which resembles that already known for dynamical systems with higher regularity hypothesis. It was expected that this should be an easy task and also that the main dynamics and transitions already observed in the regular case could be obtained for piecewise linear systems. None of these expectations have been met, but it is true that many of the possible dynamics in regular case can be observed also in this new context.

E. Ponce et al. provide two contributions to this volume. In the first, they consider planar discontinuous piecewise linear systems with two linearity zones, one of them being of focus type. By using an adequate canonical form under certain hypotheses, they characterize the bifurcation of a limit cycle when the focus changes its stability after becoming a linear center. The studied bifurcation appears in real world applications, as shown by the analysis of an electronic Wien bridge oscillator without symmetry. In their second paper, E. Ponce et al. study a possible degeneration of the Hopf-zero bifurcation for a two- parameters family of symmetric three dimensional piecewise linear differential systems with three zones of linearity. Then they show that around the critical point in such parameter plane the unfolding is very similar to the one appearing in the generalized Hopf bifurcation of differentiable dynamics.

E. Freire et al. consider a family of planar piecewise linear systems with a discontinuity line where the crossing set is maximal and it has two dynamics of focus type. Then they give some new results and a survey of known bifurcations for this family.

This volume is completed with four more papers, which also deal with topical issues in dynamical systems. D. Peralta-Salas presents a series of realization problems regarding Foliation Theory which are addressed using the theory of integrable embeddings. This theory is a very rich framework where many classical tools from differential and algebraic topology play a prominent role: Gromov h-principle, Hirch-Smale theory of immersions, complete intersections and obstruction theory. The paper by F. A. Carnicero and F. Sanz deals with real meromorphic linear systems of two ordinary differential equations and studies the asymptotic behaviour of solutions defined either to the left or to the right of 0. In another paper, S. Gefter and T. Stulova present the differential-difference equation $w'(z) = Aw(z-h) + f(z)$, where f is an entire function of zero exponential type and A is a closed linear operator on a complex Banach E space having a bounded inverse operator and whose domain $D(A)$ is not necessarily dense in E. Then they find the explicit formula for zero exponential type entire solutions. Finally, G. de la Vega and S. López de Medrano consider the generalization of the May-Leonard system to the case of a larger number of species.

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