

Preface

The extraordinary increase of computer capacity now encourages physicists, biologists, economists, and engineers to model and simulate numerically tremendously complex phenomena, in order to answer scientific questions, industrial needs, and societal requirements for risk evaluation and control. Therefore, experts in various fields aim to solve incredibly complex high dimensional systems of partial differential equations (PDE), couplings of local stochastic dynamics and deterministic macroscopic equations, etc.

Stochastic approaches appear to be useful, and sometimes mandatory, in the following two contexts. First, one cannot expect that very complex phenomena lead to perfectly calibrated mathematical models, or even to perfect mathematical models, so that uncertainties or stochastic components are involved in the equations. Second, stochastic numerical methods allow one to solve deterministic problems, of which the high dimension or singularities render classical deterministic methods of resolution intractable or inaccurate, provided that the solutions can be represented in terms of probability distributions of random variables or stochastic processes.

The combination of stochastic analysis and PDE theory are necessary to:

- obtain stochastic representations of solutions of deterministic PDE,
- construct effective stochastic numerical methods,
- obtain precise error estimates in terms of the numerical parameters of these methods, under the constraint that they take into account the critical situations where the stochastic numerical methods are used and the objectives of their users.

This monograph aims to introduce the reader to these difficult issues in a self-contained way. We particularly emphasize the essential role played by martingale theory in all the theoretical and numerical aspects of these issues. We also have devoted a substantial part of the book to the construction of simulation algorithms: the readers who are mainly interested in numerical issues may skip the mathematical proofs and concentrate on the algorithms and the convergence rate estimates.

Compared to other textbooks, this monograph presents the specificity of developing the mathematical tools which are necessary to construct effective stochastic

simulation methods and obtain accurate error estimates, and of reuniting in a non-classical way several advanced theoretical topics, some of which we now list.

We focus on non-asymptotic error estimates for Monte Carlo methods, on the backward martingale technique to prove the Strong Law of Large Numbers, and on elementary notions on logarithmic Sobolev inequalities to prove basic concentration inequalities. We place great emphasis on the pathwise construction and simulation of Poisson processes, discrete space Markov processes, and solutions of Itô stochastic differential equations.

We use the notions of infinitesimal generators and stochastic flows to establish stochastic representations of parabolic partial differential equations or integro-differential equations. We intensively use these stochastic representations of evolution equations to prove optimal error estimates for stochastic simulation methods. As explained in Chap. 1, stochastic numerical methods are used to compute quantities expressed in terms of the probability distribution of stochastic processes; we therefore essentially consider numerical errors in the weak sense rather than in an L^p -norm or in a pathwise sense which only provide crude information on the accuracy of practical simulations.

We present variance reduction techniques in nontrivial situations, which leads us to use optimized Girsanov transformations and to introduce the reader to stochastic optimization procedures.

For further information on the contents of the first two chapters, the reader is advised to consult the huge literature which concerns Central Limit Theorems, Edgeworth expansions, Large Deviations Principles, concentration inequalities, and simulation algorithms for finite-dimensional random variables. See, e.g., Devroye [10], Feller [15, 16], Petrov [43], Shiriyayev [44], and references therein.

The first book on the discretization of stochastic differential equations is due to Milstein [37]. Several other books have been published on this topic with a rather different point of view to ours: most of them focus on particular applications or various discretization methods whereas, as already emphasized, we concentrate on the mathematical methodologies which allow one to get sharp convergence rates. Therefore we encourage the reader to consult the selected references below, references therein, and other useful references, to get further algorithmic or applied information on stochastic simulations.

For time dependent models, Monte Carlo methods are derived from the simulation of Markov processes, possibly discretized. In this context, e.g., Asmussen and Glynn [4] treat the mathematics of queueing theory and some related areas, with an emphasis on stationary regimes. Glasserman [20] focuses on numerical methods for financial models. Kloeden and Platen [28] present an extended catalog of variants of the Euler and Milstein discretization schemes for stochastic differential equations. Lapeyre, Pardoux and Sentis [32] present an overview of applications of Monte Carlo simulations of stochastic processes. Milstein and Tretyakov [38] notably study discretization methods for stochastic differential systems with symplectic structure, Hamiltonian systems, and small noise systems, layer simulation methods and random walk simulations for stochastic systems with boundary conditions. The CIME volume [21] developed the weak convergence of discretized pro-

cesses, with a strong emphasis on stochastic interacting particle systems related to non-linear partial differential equations.

We here have not tackled such important topics as Malliavin calculus techniques to get optimal convergence rates and develop variance reduction methods, long time simulations of ergodic Markov processes and the stochastic approximation of their invariant probability distributions, approximation methods for reflected and stopped diffusion processes, simulation of Lévy processes and discretization of Lévy driven stochastic differential equations, quantization simulation techniques, or exact simulation methods. These subjects need mathematical tools which are beyond the objectives of this first volume, and we will address them in a forthcoming volume.

Our other volumes will concern the stochastic simulation methods for Partial Differential Equations with non-smooth coefficients and the stochastic particle methods for the analysis and the numerical resolution of non-linear Partial Differential Equations.

We hope that Master or Ph.D. students with notions on stochastic calculus and researchers interested in the mathematical or numerical aspects of stochastic simulations will find this series of monographs useful, in order to be introduced into some advanced topics in Probability theory, to improve the accuracy and the confidence intervals of their simulations, or to acquire some fundamental knowledge before reading advanced research papers on numerical probability.

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