Chapter 9
Gauss–Kuzmin Statistics

It turns out that the frequency of a positive integer k in a continued fraction for almost all real numbers is equal to

$$\frac{1}{\ln 2}\ln\left(1+\frac{1}{k(k+2)}\right),$$

i.e., for a general real x we have 42 % of 1, 17 % of 2, 9 % of 3, etc. This distribution is traditionally called the *Gauss–Kuzmin* distribution. The statistics of the elements in continued fractions first appeared in the letters of K.F. Gauss to P.S. Laplace at the beginning of the nineteenth century (see [63]). The first proof with additional estimates was developed by R.O. Kuzmin in [121] in 1928 (see also [122]), and a little later, another proof with new estimates was given by P. Lévy in [131]. Further investigations in this directions were made by E. Wirsing in [209].

In this chapter we describe two strategies to study distributions of elements in continued fractions. A classical approach to the Gauss–Kuzmin distribution is based on the ergodicity of the Gauss map. The second approach is related to the geometry of continued fractions and its projective invariance. It is interesting to note that the frequencies of elements has an unexpected interpretation in terms of cross-ratios (see Remark 9.31). Unfortunately, the classical approach does not have a generalization to the case of multidimensional sails, since it is not clear what map is the multidimensional analogue of the Gauss map. We avoid this problem by using a geometric approach to define and investigate multidimensional statistical questions for multidimensional sails. We describe the multidimensional case in Chap. 19.

In the first five sections of this chapter we discuss the classical ergodic approach to the Gauss–Kuzmin distribution. In Sect. 9.1 we give some basic notions and definitions of ergodic theory. Further, in Sects. 9.2 and 9.3, we present a measure related to continued fractions and the Gauss map. We prove the pointwise Gauss–Kuzmin theorem and formulate the original Gauss–Kuzmin theorem in Sects. 9.4 and 9.5 respectively.

In the last five sections we study the statistic of edges of geometric one-dimensional continued fractions. After a brief discussion of cross-ratios (Sect. 9.6)

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we define a structure of a smooth manifold on the set of geometric continued fractions CF_1 in Sect. 9.7. Further, in Sect. 9.8, we define the Möbius measure on CF_1 which is invariant under the group $\mathrm{PGL}(\mathbb{R}, 2)$ acting on CF_1; we write the Möbius form explicitly in Sect. 9.9. Finally, in Sect. 9.10, we define related frequencies of edges of continued fractions and show that they coincide with the Gauss–Kuzmin statistics of the elements of continued fractions.

9.1 Some Information from Ergodic Theory

Let X be a set, Σ a σ-algebra on X, and μ a measure on the elements of Σ. The collection (X, Σ, μ) is called a *measure space*. If $\mu(X) = 1$, the measure space is called a *probability measure space*.

Given a transformation T of a set X to itself, for any μ-integrable function f on X one can define the *time average* for f at the point x to be

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f\bigl(T^k x\bigr).$$

The *space average* I_f is

$$I_f = \frac{1}{\mu(X)} \int f d\mu.$$

The space average always exists. The time average does not exist for all x. Nevertheless, in the case in which we are interested, it exists for almost all x. We formulate the related theorem after one important definition.

Definition 9.1 Let (X, Σ, μ) be a measure space. A transformation $T : X \to X$ is *measure-preserving* if it is measurable and

$$\mu\bigl(T^{-1}(A)\bigr) = \mu(A)$$

for every set A of Σ.

For measure-preserving transformations we have the following theorem.

Theorem 9.2 (Birkhoff's Pointwise Ergodic Theorem) *Consider an arbitrary measure space (X, Σ, μ) and a measure preserving transformation T on X. Let f be a μ-integrable function on X. Then the time average converges almost everywhere to an invariant function $\overline{f}$.*

Definition 9.3 Consider a probability measure space (X, Σ, μ). A measure-preserving transformation T on X is *ergodic* if for every $X' \in \Sigma$ satisfying $T^{-1}(X') = X'$, either $\mu(X') = 0$ or $\mu(X') = 1$.

Theorem 9.4 (Birkhoff–Khinchin's Ergodic Theorem) *Consider a probability measure space* (X, Σ, μ) *and a measure preserving transformation* T. *Suppose that* T *is ergodic. Then the values of the time average function are equal to the space average* (*i.e.*, $\overline{f}(x) = I_f$) *almost everywhere.*

9.2 The Measure Space Related to Continued Fractions

In this section we define a measure space that is closely related to distributions of the elements of continued fractions. For this measure we formulate a statement on the density of points for measurable subsets, which we use in the essential way in the proofs below.

9.2.1 Definition of the Measure Space Related to Continued Fractions

Consider the measure space of the segment $I = \{x \mid 0 \le x < 1\}$ with the Borel σ-algebra Σ and the measure $\hat{\mu}$ defined on a measurable set S as

$$\hat{\mu}(S) = \frac{1}{\ln 2} \int_S \frac{dx}{1+x}.$$

The coefficient $1/\ln 2$ is taken such that the measure of the segment I equals 1.

9.2.2 Theorems on Density Points of Measurable Subsets

We start with a classical theorem on Lebesgue measure space. Denote by $B(x, \varepsilon)$ the standard ball of radius ε centered at x.

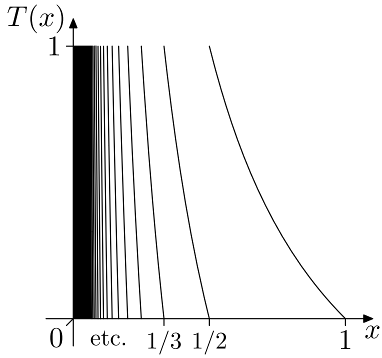
Theorem 9.5 (Lebesgue density) *Let* λ *be the n-dimensional Lebesgue measure on* $\mathbb{R}^n$. *If* $A \subset \mathbb{R}^n$ *is a Borel measurable set, then almost every point* $x \in A$ *is a Lebesgue density point*:

$$\lim_{\varepsilon \to 0} \frac{\lambda(A \cap B(x, \varepsilon))}{\lambda(B(x, \varepsilon))} = 1.$$

Here "almost every point" means "except for a subset of zero measure".

The measure $\hat{\mu}$ is equivalent to the one-dimensional Lebesgue measure λ on the segment [0, 1] (for more information on measure theory, see [137]). Hence we have a similar statement in the case of the measure space $(X, \Sigma, \hat{\mu})$.

Fig. 9.1 The Gauss map



Corollary 9.6 ($\hat{\mu}$-density) *Let $X = [0, 1]$ and let $\hat{\mu}$ be as above. If $A \subset X$ is a $\hat{\mu}$-measurable set with positive measure $\hat{\mu}(A)$, then almost every point in A satisfies*

$$\lim_{\varepsilon \to 0} \frac{\hat{\mu}(A \cap B(x, \varepsilon))}{\hat{\mu}(B(x, \varepsilon))} = 1.$$

9.3 On the Gauss Map

Let us introduce a transformation whose ergodic properties will form the basis for the proof of the Gauss–Kuzmin theorem.

9.3.1 The Gauss Map and Corresponding Invariant Measure

We consider the measure space $(X, \Sigma, \hat{\mu})$ defined in the previous section. Define the *Gauss map* T of a segment $[0, 1]$ to itself as follows:

$$T(x) = \{1/x\},$$

where $\{r\}$ denotes the fractional part $r - \lfloor r \rfloor$ (see Fig. 9.1).

Proposition 9.7 *The Gauss map T is measure-preserving for the measure space $(X, \Sigma, \hat{\mu})$.*

We start with the following lemma.

Lemma 9.8 *Let $x = [0; a_1 : a_2 : \cdots]$. Then*

$$T^{-1}(x) = \big\{[0; k : a_1 : a_2 : \cdots] | k \in \mathbb{Z}_+\big\} = \left\{ \frac{1}{x+k} \middle| k \in \mathbb{Z}_+ \right\}.$$

Proof The first equality follows directly from the fact that

$$T\big([0; b_1 : b_2 : \cdots]\big) = [0; b_2 : b_3 \cdots]$$

and the fact that every real number has a unique regular continued fraction expansion with the last element not equal to 1.

The second equality is straightforward. □

Proof of Proposition 9.7 Consider a measurable set S. From Lemma 9.8 it follows that

$$\begin{aligned}\hat{\mu}\big(T^{-1}(S)\big) &= \frac{1}{\ln 2}\int_{T^{-1}(S)} \frac{dx}{1+x} \\ &= \frac{1}{\ln 2}\sum_{k=1}^{\infty}\bigg(\int_{T^{-1}(S)\cap[1/(k+1),1/k]} \frac{dx}{1+x}\bigg).\end{aligned}$$

Notice that on each open (i.e., without the boundary points) segment $]1/k, 1/(k+1)[$ the operator T is in one-to-one correspondence with the open segment $]0, 1[$. Let us denote the inverse function to T on the segment $]1/k, 1/(k+1)[$ by $T_{(k)}^{-1}$. Therefore,

$$T\left(T^{-1}(S) \cap \left[\frac{1}{k+1}, \frac{1}{k}\right]\right) = T\big(T_{(k)}^{-1}(S)\big) = S,$$

and we can apply the rule of differentiation of a composite function. From Lemma 9.8 we know, that

$$T_{(k)}^{-1}(x) = \frac{1}{x+k}.$$

Then we have

$$\begin{aligned}\int_{T^{-1}(S)\cap[1/(k+1),1/k]} \frac{dx}{1+x} &= \int_{T_{(k)}^{-1}(S)} \frac{dx}{1+x} = \int_{T(T_{(k)}^{-1}(S))} \frac{dT_{(k)}^{-1}(x)}{1+T_{(k)}^{-1}(x)} \\ &= \int_S \frac{-d(1/(x+k))}{1+1/(x+k)} = \int_S \frac{dx}{(x+k)(x+k+1)}\end{aligned}$$

(the negative sign is taken, since the map $T_{(k)}^{-1} : x \to \frac{1}{x+k}$ changes the orientation). So we have

$$\hat{\mu}\big(T^{-1}(S)\big) = \frac{1}{\ln 2}\sum_{k=1}^{\infty}\int_S \frac{dx}{(x+k)(x+k+1)}.$$

Since the integrated functions are nonnegative, we can change the order of the summation and the integration operations. We get

$$\hat{\mu}\big(T^{-1}(S)\big) = \frac{1}{\ln 2}\int_S \left(\sum_{k=1}^{\infty} \frac{1}{(x+k)(x+k+1)}\right) dx$$
$$= \frac{1}{\ln 2}\int_S \left(\sum_{k=1}^{\infty} \left(\frac{1}{x+k} - \frac{1}{x+k+1}\right)\right) dx = \frac{1}{\ln 2}\int_S \frac{dx}{x+1} = \hat{\mu}(S).$$

So for every measurable set S we have

$$\hat{\mu}\big(T^{-1}(S)\big) = \hat{\mu}(S).$$

Therefore, the Gauss map T preserves the measure $\hat{\mu}$. □

Remark 9.9 (On the Euler–Mascheroni constant) By definition, the *Euler–Mascheroni constant* (traditionally denoted by γ) is the following infinite sum

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{1}{k} - \ln n\right).$$

It was first studied by L. Euler in 1734. It is not known whether γ is irrational. It turns out that the Euler–Mascheroni constant can be expressed as an integral of the Gauss map with respect to Lebesgue measure:

$$\gamma = 1 - \int_0^1 T(x)dx.$$

9.3.2 An Example of an Invariant Set for the Gauss Map

Let us consider one example of a measurable set that is invariant under the Gauss map.

Denote by Ψ the set of all irrational numbers in the segment [0, 1] whose continued fractions contain only finitely many 1's. It is clear that

$$T^{-1}(\Psi) = \Psi,$$

since the operation T^{-1} shifts elements of continued fractions by one and inserts the first element.

Proposition 9.10 *The set Ψ is measurable* (*i.e.*, $\Psi \in \Sigma$).

Proof Denote by Υ_n the set of all irrational numbers that contain the element 1 exactly at place n. Notice that

$$\Upsilon_1 = [1/2, 1],$$

and therefore, it is measurable. Hence for every n the set

$$\Upsilon_{n+1} = T^n(\Upsilon_1)$$

is measurable.

Denote by Ψ_0 the set of all irrational numbers that do not contain an element '1'. Since

$$\Psi_0 = X \setminus \bigcup_{n=1}^{\infty} \Upsilon_n,$$

the set Ψ_0 is also measurable. Then

$$T^{-n}(\Psi_0)$$

is measurable for any positive integer n. Hence

$$\Psi = \bigcup_{n=1}^{\infty} T^{-n}(\Psi_0)$$

is measurable. □

We will prove later that the Gauss map is ergodic, and therefore, Ψ is either of zero measure or full measure in X.

9.3.3 Ergodicity of the Gauss Map

In this subsection we prove the ergodicity of the Gauss map.

Proposition 9.11 *The Gauss map is ergodic.*

Before proving Proposition 9.11 we introduce some supplementary notation and prove two lemmas.

For a sequence of positive integers $(a_1, \ldots, a_n)$ denote by $I_{(a_1,\ldots,a_n)}$ the segment with endpoints $[0; a_1 : \cdots : a_{n-1} : a_n]$ and $[0; a_1 : \cdots : a_{n-1} : a_n + 1]$. It is clear that the map

$$T^n : I_{(a_1,\ldots,a_n)} \to [0, 1]$$

is one-to-one on the segment $I_{(a_1,\ldots,a_n)}$, and the inverse to T^n is

$$T^{-1}_{(a_1,\ldots,a_n)} : x \to [0; a_1 : \cdots : a_n : 1/x].$$

In terms of k-convergents $p_k/q_k = [0; a_1 : \cdots : a_k]$, the expression for $T^{-1}_{(a_1,\ldots,a_n)}(x)$ is as follows (see Proposition 1.13):

$$T^{-1}_{(a_1,\dots,a_n)}(x) = \frac{p_n/x + p_{n-1}}{q_n/x + q_{n-1}} = \frac{p_n + p_{n-1}x}{q_n + q_{n-1}x}.$$

Lemma 9.12 *The measure of a segment* $I_{(a_1,\dots,a_n)}$ *satisfies the following inequality*:

$$\hat{\mu}(I_{(a_1,\dots,a_n)}) < \frac{1}{\ln 2(q_n + q_{n-1})(p_n + q_n)}.$$

Proof We have

$$\begin{aligned}
\hat{\mu}(I_{(a_1,\dots,a_n)}) &= \frac{1}{\ln 2}\int_{I_{(a_1,\dots,a_n)}} \frac{dx}{1+x} = \frac{1}{\ln 2}\left|\int_{[0;a_1:\dots:a_n]}^{[0;a_1:\dots:a_n:1]} \frac{dx}{1+x}\right| \\
&= \frac{1}{\ln 2}\left|\ln\left(\left(1 + \frac{p_n + p_{n-1}}{q_n + q_{n-1}}\right)\Big/\left(1 + \frac{p_n}{q_n}\right)\right)\right| \\
&= \frac{1}{\ln 2}\left|\ln\left(1 + \frac{1}{(q_n + q_{n-1})(p_n + q_n)}\right)\right| \\
&< \frac{1}{\ln 2(q_n + q_{n-1})(p_n + q_n)}.
\end{aligned}$$

The last inequality follows from the concavity of the natural logarithm function. □

Lemma 9.13 *For any invariant set* S *of positive measure and any interval* $I_{(a_1,\dots,a_n)}$,

$$\hat{\mu}(S \cap I_{(a_1,\dots,a_n)}) \geq \frac{\ln 2}{2}\hat{\mu}(S)\hat{\mu}(I_{(a_1,\dots,a_n)}).$$

Proof Since the map T is surjective, we also have

$$T(S) = S.$$

Let $\hat{\mu}(S) = c > 0$. Let us prove that $c = 1$. We have

$$\begin{aligned}
\frac{1}{\ln 2}\int_{S\cap I_{(a_1,\dots,a_n)}} \frac{dx}{1+x} &= \frac{1}{\ln 2}\int_S d\left(\frac{p_n + p_{n-1}x}{q_n + q_{n-1}x}\right)\Big/\left(1 + \frac{p_n + p_{n-1}x}{q_n + q_{n-1}x}\right) \\
&= \frac{1}{\ln 2}\int_S \frac{dx}{(q_n + q_{n-1}x)(q_n + q_{n-1}x + p_n + p_{n-1}x)} \\
&\geq \frac{1}{\ln 2 \cdot q_n(q_n + q_{n-1} + p_n + p_{n-1})}\int_S \frac{dx}{1+x} \\
&= \frac{1}{q_n(q_n + q_{n-1} + p_n + p_{n-1})}\hat{\mu}(S) \\
&\geq \frac{1}{(q_n + q_{n-1})(2p_n + 2q_n)}\hat{\mu}(S) \geq \frac{\ln 2}{2}\hat{\mu}(S)\hat{\mu}(I_{(a_1,\dots,a_n)}).
\end{aligned}$$

The last inequality follows from Lemma 9.12. □

Proof of Proposition 9.11 Let S be a measurable subset of the unit interval such that $T^{-1}(S) = S$. Suppose also $\hat{\mu}(S) > 0$.

For any irrational number $y = [0; a_1 : a_2 : \cdots]$, from Lemma 9.13 we have

$$\frac{\hat{\mu}(X \setminus S) \cap B(y, \hat{\mu}(I_{(a_1,\dots,a_n)}))}{\hat{\mu}(B(y, \hat{\mu}(I_{(a_1,\dots,a_n)})))} \le 1 - \frac{\ln 2}{2} \frac{\hat{\mu}(S)\hat{\mu}(I_{(a_1,\dots,a_n)})}{\hat{\mu}(B(y, \hat{\mu}(I_{(a_1,\dots,a_n)})))} = 1 - \frac{\ln 2}{4}\hat{\mu}(S).$$

Hence y is not a $\hat{\mu}$-density point of $X \setminus S$. Therefore, by Corollary 9.6 almost every point of $[0,1] \setminus \mathbb{Q}$ is not in $X \setminus S$, and therefore, it is in S. Hence

$$\hat{\mu}(S) \ge \hat{\mu}\big([0,1] \setminus \mathbb{Q}\big) = 1$$

Hence $\hat{\mu}(S) = 1$, concluding the proof of ergodicity of T. □

9.4 Pointwise Gauss–Kuzmin Theorem

Consider x in the segment $[0, 1]$. Let the regular continued fraction for x be $[0; a_1 : \cdots : a_n]$ (odd or infinite). For a positive integer k, set

$$\hat{P}_{n,k}(x) = \frac{\#(k,n)}{n},$$

where $\#(k, n)$ is the number of integer elements a_i equal to k for $i = 1, \dots, n$. Define

$$\hat{P}_k(x) = \lim_{n \to \infty} \hat{P}_{n,k}(x).$$

Theorem 9.14 *For every positive integer k and almost every x (i.e., in the complement of a set of zero measure) the following holds:*

$$\hat{P}_k(x) = \frac{1}{\ln 2} \ln\left(1 + \frac{1}{k(k+2)}\right).$$

We consider this theorem a *pointwise Gauss–Kuzmin theorem*. To prove this pointwise Gauss–Kuzmin theorem we use Birkhoff's ergodic theorems.

Proof of Theorem 9.14 Consider a subset $S \in I$. Let χ_S be the *characteristic function* of S, i.e.,

$$\chi_S(x) = \begin{cases} 1, & \text{if } x \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\hat{P}_{n,k}(x) = \frac{1}{n} \sum_{s=0}^{n-1} \chi_{]1/(k+1), 1/k]}\big(T^s x\big).$$

Hence, by Birkhoff's pointwise ergodic theorem, the limit $\hat{P}_k(x)$ exists almost everywhere. Since the transformation T is ergodic, we apply the Birkhoff–Khinchin ergodic theorem and get

$$\hat{P}_k(x) = \int_0^1 \chi_{]1/(k+1),1/k]} d\hat{\mu} = \frac{1}{\ln 2} \int_{1/(k+1)}^{1/k} \frac{dx}{1+x} = \frac{1}{\ln 2} \ln\left(1 + \frac{1}{k(k+2)}\right). \quad \square$$

9.5 Original Gauss–Kuzmin Theorem

Let α be some irrational number between zero and one, and let $[0; a_1 : a_2 : a_3 : \cdots]$ be its regular continued fraction.

Let $m_n(x)$ denote the measure of the set of real numbers α contained in the segment $[0, 1]$ such that $T^n(\alpha) < x$ (here T is the Gauss map). In his letters to P.S. Laplace C.F. Gauss formulated without proofs the following theorem.

Theorem 9.15 (Gauss–Kuzmin) *For* $0 \le x \le 1$ *the following holds*:

$$\lim_{n\to\infty} m_n(x) = \frac{\ln(1+x)}{\ln 2}.$$

This theorem is technically complicated. For the proof we refer to the original manuscripts of R.O. Kuzmin [121] and [122] (see also A.Ya. Khinchin [105]).

Denote by $P_n(k)$, for an arbitrary integer $k > 0$, the measure of the set of all real numbers α of the segment $[0, 1]$ such that each of them has the number k at the nth position. The limit $\lim_{n\to\infty} P_n(k)$ is called the *frequency of* k for regular continued fractions and is denoted by $P(k)$.

Corollary 9.16 *For every positive integer* k, *the following holds*:

$$P(k) = \frac{1}{\ln 2} \ln\left(1 + \frac{1}{k(k+2)}\right).$$

Proof Notice that $P_n(k) = m_n(\frac{1}{k}) - m_n(\frac{1}{k+1})$. Now the statement of the corollary follows from the Gauss–Kuzmin theorem. $\square$

9.6 Cross-Ratio in Projective Geometry

In this section we switch to the multidimensional case for a while in order to give some definitions that are similar to the one-dimensional case (we will use these later, in Chap. 19).

9.6.1 Projective Linear Group

The *projective linear group* (or the *group of projective transformations*) is the quotient group

$$\mathrm{PGL}(\mathbb{R}, n) = \mathrm{GL}(\mathbb{R}, n)/\mathrm{Z}(\mathbb{R}, n),$$

where $\mathrm{Z}(\mathbb{R}, n)$ is the one-dimensional subgroup of all nonzero scalar transformations of $\mathbb{R}^n$. The group $\mathrm{PGL}(\mathbb{R}, n)$ acts on the equivalence classes of vectors in $\mathbb{R}^n$ with respect to $\mathrm{Z}(\mathbb{R}, n)$. We have

$$\mathbb{R}^n/\mathrm{Z}(\mathbb{R}, n) = \mathbb{R}P^{n-1}.$$

Consider the affine part $\mathbb{R}^{n-1} \subset \mathbb{R}P^{n-1}$. The stabilizer for the affine part is exactly the group $\mathrm{Aff}(\mathbb{R}, n-1)$.

9.6.2 Cross-Ratio, Infinitesimal Cross-Ratio

Consider an arbitrary line in $\mathbb{R}^{n-1}$ with a Euclidean coordinate on it.

Definition 9.17 Consider a 4-tuple of points on a line with coordinates z_1, z_2, z_3, and z_4. The value

$$\frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}$$

is called the *cross-ratio* of the 4-tuple.

It is clear that the cross-ratio does not depend on the choice of the Euclidean coordinate on the line, and therefore, it a function on the space of ordered 4-tuples of distinct points in a line.

As we have already noted above, the space $\mathbb{R}^{n-1}$ can be considered an affine chart $\mathbb{R}^{n-1} \subset \mathbb{R}P^{n-1}$. Hence the action of $\mathrm{PGL}(\mathbb{R}, n)$ is well defined on the closure of $\mathbb{R}^{n-1}$ (which is actually $\mathbb{R}P^{n-1}$). The projective transformations take planes to planes and, in particular, lines to lines. So it is natural to ask what happens to the cross-ratios of four points on a line.

Proposition 9.18 *The cross-ratio of four points is an invariant of projective transformations of* $\mathbb{R}^n$.

We are interested in the *infinitesimal cross-ratio*, which is the following 2-form:

$$\frac{dxdy}{(x-y)^2}.$$

Notice that in the denominator we have

$$(x-y)^2 = \lim_{\varepsilon\to 0}\big(x-(y+\varepsilon dy)\big)\cdot\big((x+\varepsilon dx)-y\big).$$

Corollary 9.19 *The infinitesimal cross-ratio is an invariant of projective transformations of* $\mathbb{R}^{n-1}$.

Proof The density of the infinitesimal cross-ratio coincides with the asymptotic coefficient at ε^2 of the cross-ratios of 4-tuples of points:

$$x, y, x+\varepsilon dx, y+\varepsilon dy,$$

as ε tends to 0. Therefore, the infinitesimal cross-ratio is a projective invariant. □

9.7 Smooth Manifold of Geometric Continued Fractions

Denote the set of all geometric continued fractions by CF_1. Consider an arbitrary element of CF_1. It is a continued fraction defined by an (unordered) pair of nonparallel lines (ℓ_1, ℓ_2) passing through integer points.

Denote the sets of all ordered collections of two independent and dependent straight lines by FCF_1 and Δ_1 respectively. We say that FCF_1 is a space of *geometric framed continued fractions*. We have

$$FCF_1 = \big(\mathbb{R}P^1 \times \mathbb{R}P^1\big) \setminus \Delta_1 = T^2 \setminus \Delta_1 \quad \text{and} \quad CF_1 = FCF_1/(\mathbb{Z}/2\mathbb{Z}),$$

where $\mathbb{Z}/2\mathbb{Z}$ is the group transposing the lines in geometric continued fractions. Note that FCF_1 is a 2-fold covering of CF_1. We call the map of “forgetting” the order in the ordered collections the *natural projection* of the manifold FCF_1 to the manifold CF_1 and denote it by p (i.e., $p: FCF_1 \to CF_1$).

Notice that FCF_1 is homeomorphic to the annulus and CF_1 is homeomorphic to the Möbius band.

9.8 Möbius Measure on the Manifolds of Continued Fractions

The group $\mathrm{PGL}(2,\mathbb{R})$ of transformations of $\mathbb{R}P^1$ takes the set of all straight lines passing through the origin in the plane into itself. Hence, $\mathrm{PGL}(2,\mathbb{R})$ naturally acts on CF_1 and FCF_1. It is clear that the action of $\mathrm{PGL}(2,\mathbb{R})$ is transitive, i.e., it takes any (framed) continued fraction to any other. Notice that a stabilizer of any geometric continued fraction is one-dimensional.

Definition 9.20 A form on the manifold CF_1 (respectively FCF_1) is said to be a *Möbius form* if it is invariant under the action of $\mathrm{PGL}(2,\mathbb{R})$.

Remark 9.21 The name for the invariant forms comes from theory of energies of knots and graphs in low dimensional topology, where these forms are used as densities for Möbius energies that are invariant under the group of Möbius transformations in $\mathbb{R}^3$ (we refer the interested reader to the book by J. O'Hara [150]).

Proposition 9.22 *All Möbius forms of the manifolds CF_1 and FCF_1 are proportional.*

Proof Transitivity of the action of $\mathrm{PGL}(2, \mathbb{R})$ implies that all Möbius forms of the manifolds CF_1 and FCF_1 are proportional. □

Let ω be some volume form of the manifold M. Denote by μ_ω a measure of the manifold M that for every open measurable set S contained in the same piecewise connected component of M is defined by the equality

$$\mu_\omega(S) = \left| \int_S \omega \right|.$$

Definition 9.23 A measure μ of the manifold CF_1 (FCF_1) is said to be a *Möbius measure* if there exists a Möbius form ω of CF_1 (FCF_1) such that $\mu = \mu_\omega$.

From Proposition 9.22 we have the following.

Corollary 9.24 *Any two Möbius measures are proportional.*

Remark 9.25 The projection p takes the Möbius measures of the manifold FCF_1 to the Möbius measures of the manifold CF_1. This establishes an isomorphism between the spaces of Möbius measures for CF_1 and FCF_1. Since the manifold of framed continued fractions possesses simpler chart systems, all formulas of the work are given for the case of the framed continued fraction manifold. To calculate a measure of some set F of the unframed continued fraction manifold, one should: take $p^{-1}(F)$; calculate the Möbius measure of the obtained set of the manifold of framed continued fractions, and divide the result by 2.

9.9 Explicit Formulas for the Möbius Form

Let us write down Möbius forms of the framed one-dimensional continued fraction manifold FCF_1 explicitly in special charts.

Consider a vector space $\mathbb{R}^2$ equipped with standard metrics on it. Letting l be an arbitrary straight line in $\mathbb{R}^2$ that does not pass through the origin, choose some Euclidean coordinates $O_l X_l$ on it. Denote by $FCF_{1,l}$ a chart of the manifold FCF_1 that consists of all ordered pairs of straight lines both intersecting l. Let us associate to any point of $FCF_{1,l}$ (i.e. to a collection of two straight lines) coordinates (x_l, y_l), where x_l and y_l are the coordinates on l for the intersections of l with the first and the second straight lines of the collection respectively. Denote by $|\overline{v}|_l$ the Euclidean length of a vector $\overline{v}$ in the coordinates $O_l X_l Y_l$ of the chart $FCF_{1,l}$. Note that the chart $FCF_{1,l}$ is the space $\mathbb{R} \times \mathbb{R}$ minus its diagonal.

Consider the following form in the chart $FCF_{1,l}$:

$$\omega_l(x_l, y_l) = \frac{dx_l \wedge dy_l}{|x_l - y_l|_l^2}.$$

Proposition 9.26 *The measure μ_{ω_l} coincides with the restriction of some Möbius measures to $FCF_{1,l}$.*

Proof Notice that the form $\omega_l(x_l, y_l)$ coincides with the infinitesimal cross-ratio on the line l. Hence it is invariant under projective transformations of l (on an everywhere dense subset) in the chart $FCF_{1,l}$. Therefore, the measure μ_{ω_l} coincides with the restriction of some Möbius measures to $FCF_{1,l}$. □

Corollary 9.27 *A restriction of an arbitrary Möbius measure to the chart $FCF_{1,l}$ is proportional to μ_{ω_l}.*

Proof The statement follows from the proportionality of any two Möbius measures. □

Consider now the manifold FCF_1 as a set of ordered pairs of distinct points on a circle $\mathbb{R}/\pi\mathbb{Z}$ (this circle is a one-dimensional projective space obtained from the unit circle by identifying antipodal points). The doubled angular coordinate φ of the circle $\mathbb{R}/\pi\mathbb{Z}$ induced by the coordinate x of the straight line $\mathbb{R}$ naturally defines the coordinates (φ_1, φ_2) of the manifold FCF_1.

Proposition 9.28 *The form $\omega_l(x_l, y_l)$ is extendible to some form ω_1 of FCF_1. In coordinates (φ_1, φ_2), the form ω_1 can be written as follows*:

$$\omega_1 = \frac{1}{4}\cot^2\left(\frac{\varphi_1 - \varphi_2}{2}\right) d\varphi_1 \wedge d\varphi_2.$$

The proof of Proposition 9.28 is left as an exercise for the reader.

9.10 Relative Frequencies of Edges of One-Dimensional Continued Fractions

Without loss of generality, in this section we consider only the Möbius form ω_1 of Proposition 9.28. Denote the natural projection of the form μ_{ω_1} to the manifold of one-dimensional continued fractions CF_1 by μ_1.

Consider an arbitrary segment F with vertices at integer points. Denote by $CF_1(F)$ the set of continued fractions that contain the segment F as an edge.

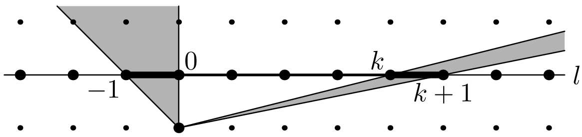


Fig. 9.2 Rays defining a continued fraction should lie in the domain *shaded in gray*

Definition 9.29 The quantity $\mu_1(CF_1(F))$ is called the *relative frequency* of the edge F.

Note that the relative frequencies of edges of the same integer-linear type are equivalent. Every edge of a one-dimensional continued fraction is at unit integer distance from the origin. Thus, the integer-linear type of a segment is defined by its integer length (the number of inner integer points plus unity). Denote the relative frequency of the edge of integer length k by $\mu_1(''k'')$.

Proposition 9.30 *For every positive integer k, the following holds*:

$$\mu_1(''k'') = \ln\left(1 + \frac{1}{k(k+2)}\right).$$

Proof Consider a particular representative of an integer-linear type of a length-k segment: the segment with vertices $(0, 1)$ and $(k, 1)$. The one-dimensional continued fraction contains the segment as an edge if and only if one of the straight lines defining the fraction intersects the interval with vertices $(-1, 1)$ and $(0, 1)$ while the other straight line intersects the interval with vertices $(k, 1)$ and $(k+1, 1)$ (see Fig. 9.2).

For the straight line l defined by the equation $y = 1$, we calculate the Möbius measure of the Cartesian product of the described pair of intervals. By the last section it follows that this quantity coincides with the relative frequency $\mu_1(''k'')$. So,

$$\mu_1(''k'') = \int_{-1}^{0}\int_{k}^{k+1} \frac{dx_l dy_l}{(x_l - y_l)^2} = \int_{k}^{k+1}\left(\frac{1}{y_l} - \frac{1}{y_l+1}\right)dy_l$$
$$= \ln\left(\frac{(k+1)(k+1)}{k(k+2)}\right) = \ln\left(1 + \frac{1}{k(k+2)}\right).$$

This proves the proposition. □

Remark 9.31 Note that the argument of the logarithm $\frac{(k+1)(k+1)}{k(k+2)}$ is the cross-ratio of points $(-1, 1)$, $(0, 1)$, $(k, 1)$, and $(k+1, 1)$.

Corollary 9.32 *The relative frequency $\mu_1(''k'')$, up to the factor*

$$\ln 2 = \int_{-1}^{0}\int_{1}^{+\infty} \frac{dx_l dy_l}{(x_l - y_l)^2},$$

coincides with the Gauss–Kuzmin frequency $P(k)$.

9.11 Exercises

Exercise 9.1

(a) Prove that the measure

$$\mu(S) = \frac{1}{\ln 2} \int_S \frac{dx}{1+x}$$

is a probability measure on the segment [0, 1], i.e., $\mu([0, 1]) = 1$.

(b) Find $\mu([a, b])$ for $0 \leq a < b \leq 1$, where μ is as above.

Exercise 9.2 *Ergodicity of the doubling map.* Consider the space (S^1, Σ, λ), where X is the unit circle, Σ is the Borel σ-algebra, and λ is the Lebesgue measure. Consider the doubling map $T : S^1 \to S^1$ such that

$$T(\varphi) = 2\varphi.$$

Prove that T is measure-preserving and ergodic.

Exercise 9.3 Define the frequencies of subsequences in continued fractions. What is the frequency of the sequence $(1, 2, 3)$?

Exercise 9.4 Prove the $\hat{\mu}$-density theorem from the Lebesgue density theorem.

Exercise 9.5 Recall that Ψ_0 is the subset irrational numbers in [0, 1] whose continued fractions do not contain 1 as an element. Prove by elementary means (without using ergodic theorems) that

$$\hat{\mu}(\Psi_0) = 0.$$

Exercise 9.6 Prove the projective invariance of the cross-ratio.

Exercise 9.7 Prove that any two triples of points on a line are projectively equivalent. Find a criterion for two 4-tuples of points on a line to be projectively equivalent.

Exercise 9.8 Prove that

(a) $\mathbb{R}P^1$ is homeomorphic to a circle;
(b) FCF_1 is homeomorphic to an annulus;
(c) CF_1 is homeomorphic to the Möbius band.

Exercise 9.9 Prove Proposition 9.28.

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