Preface

Surfaces of constant mean curvature (*cmc surfaces*) play an important role in classical differential geometry because they are the local solutions of one of the oldest problems in geometry, namely, the isoperimetric problem: find those surfaces of least area enclosing a prescribed volume. These surfaces are of great interest not only for mathematicians, but also for physicists and engineers since they provide mathematical models in contexts where a physical system seeks a state of least energy. A number of texts are available which deal with this type of surface in more or less depth. The classical book of H. Hopf [Hop83] contains interesting material that can be used as an introduction. We also refer to Nitsche and Osserman for minimal surfaces [Nit89, Oss86] and the recent book of K. Kenmotsu [Ken03]. See also [Boy59, EJ07, Law80, Opr00].

This book introduces the reader to that part of the subject concerning 'surfaces with boundary'. Our aim is to present as much as possible of the theory of compact constant mean curvature surfaces spanning a given boundary curve, selecting the results that represent the core of this theory. The initial motivating problem is the simplest case of boundary, say a circle, and we ask what type of symmetries of the circle are inherited by the surface that spans it. We describe various methods, such as, the tangency principle, the Alexandrov method, the flux formula and the Dirichlet problem and we shall emphasize the geometric aspects of these techniques in the context of the theory of surfaces with constant mean curvature. Most of the results are given in Euclidean space and in particular in the three-dimensional case, but we can consider other ambient spaces and dimensions. In this monograph we only extend the discussion to hyperbolic space and we dedicate the last chapter to the study of these surfaces in Lorentz-Minkowski space. The references that appear in this book are only a small part of the bibliography in this theory.

One of the main goals of this book is to help graduate students to get started on research in the theory of surfaces in Euclidean space with constant mean curvature. We have tried to write a reasonably self-contained text so that the reader can learn the techniques of the subject in such a way that they will begin with basic methods and results and end with recent research topics. We do assume that the reader knows the elementary theory of surfaces, as presented for example in do Carmo [Car76] and Montiel and Ros [MR09] and the basic theory of integration and submanifolds in Riemannian manifolds as described in selected parts of [Car92, Chv94].

The text is organized as follows. In Chap. 1 we present an historical review of surfaces with constant mean curvature motivating their study by the classical isoperimetric problem. Historically there are two parallel problems depending on whether we consider only one constraints (the volume) or two constraints (the volume and the boundary). In the first case, we recall the different characterizations of the sphere within the family of closed surfaces with constant mean curvature. The absence of new examples brought as a consequence a great effort to find new surfaces (and techniques), finally succeeding in the discovery by H.C. Wente of a torus immersed in Euclidean space with constant mean curvature. This opened new avenues of research in the theory. For surfaces with boundary, Douglas and Radó considered the Plateau problem for minimal surfaces and later, in the 1950s, Heinz and others considered the general case of mean curvature.

In Chap. 2, anticipating the main discussions of the book, we focus on the basic properties of cmc surfaces. We derive the first variation of the area and we give the variational characterization of a cmc surface. We give the notion of stability of a surface when we compute the second variation of the area. Next we prove the Hopf theorem, introducing complex coordinates on a cmc surface. Following this, we compute the Laplacian of the position vector and the Gauss map. With these techniques we obtain height estimates for a cmc graph and finally we prove the classical results of Jellet and Barbosa-do Carmo that characterize the sphere within the family of closed cmc surfaces under the corresponding hypothesis that the surface is starshaped and stable, respectively.

Chapter 3 introduces the comparison principle and the tangency principle, two of the main tools employed in this book. These are obtained first by writing a surface locally as the graph of a function $z = u(x, y)$. Then the constancy of the mean curvature implies that u satisfies a second order partial differential equation of elliptic type. The ellipticity of the equation allows the use of the classical Hopf maximum principle. We will obtain an initial set of results on compact cmc surfaces with planar boundary giving conditions which ensure that the surface lies on one side of the boundary plane. Comparing the surface with spheres and cylinders, we obtain characterizations of a sphere if the surface is included in the closure of a Euclidean ball or a cylinder whose radii are related to the value of the mean curvature of the surface.

In Chap. 4 we study embedded cmc surfaces using the reflection technique of Alexandrov. With this method we will give conditions determining whether the symmetries of the boundary are inherited by the surface, in particular, when the boundary is a plane curve. We employ the method of reflection by means of planes orthogonal to the plane containing the boundary, proving that, under some conditions, if the boundary is a circle the surface is rotational. In this context, we also use the reflection method with planes parallel to the boundary plane, obtaining results that prove that the surface is a graph. Finally, we use the Alexandrov method for embedded cmc surfaces whose boundary lies in a sphere and we give conditions that ensure that the surface lies on one side of the sphere.

In Chap. 5 we study how the geometry of a given closed curve imposes restrictions on the existence of surfaces with constant mean curvature that span such a curve. We shall obtain a flux formula that will play an important role, not only in this chapter but throughout the rest of the book. With the aid of the flux formula, and jointly the tangency principle, we will derive some geometric configurations where the surface is contained in a Euclidean ball or a cylinder whose size is related with the value of mean curvature of the surface. In the last section, we will prove that if an embedded surface with convex boundary is transverse to the boundary plane then it lies on one side of this plane.

Chapter 6 addresses the study of the area and the volume of a compact cmc surface with boundary. The control of the area or the volume will give information on the geometry of the surface. The chapter begins by obtaining a monotonicity formula for the area in terms of the height of the surface and we will show that the solutions of the isoperimetric problem for convex planar domains are graphs if the volume is sufficiently small. In the last section, we will consider embedded surfaces with large volume and we will prove that, under some conditions, the surface converges to a large spherical cap if the volume is sufficiently big.

Chapter 7 treats the case where the given boundary is a circle. When the topology of the surface is a disk, we shall characterize a spherical cap under hypotheses on the area. In this chapter, it will be shown that the planar disk and the spherical cap are the only stable disks with constant mean curvature spanning a circle.

Chapters 8 and 9 are devoted to the Dirichlet problem of the constant mean curvature equation. First we consider the case where the domain is bounded, whereas in Chap. 9 the domain is unbounded. We will describe the techniques used to solve the Dirichlet problem, namely, the continuity method and the Perron method. We shall give a set of existence results when the domain is convex under hypotheses on the curvature and length of the boundary curve or the area of the domain. If the domain is not convex, we assume a uniform circle exterior condition. In order to get the desired estimates, we shall employ pieces of rotational cmc surfaces as barriers. In addition, we will give estimates of the height of a cmc graph. If the boundary values are bounded, the estimates depend only on the mean curvature and the boundary data. When the domain is bounded, we also obtain estimates of the graph in terms of the area of the domain.

Chapter 10 concerns the study of compact cmc surfaces in hyperbolic space $\mathbb{H}^3$ and we ask questions similar to those posed in Euclidean space. A basic difference in this ambient space is the variety of umbilical surfaces because, besides totally geodesic surfaces and spheres, there are equidistant surfaces and horospheres. We shall see that the problems differ depending on the value of the mean curvature H. For example, if $|H| > 1$, the surface exhibits behaviour similar to that in $\mathbb{R}^3$, if $|H| = 1$, the surface has some similarities with a minimal surface of $\mathbb{R}^3$ and finally, if $|H| < 1$, the nature of the problem has no equivalent in Euclidean space.

Chapter 11 deals with the Dirichlet problem for geodesic graphs in the hyperbolic setting. We shall consider the solvability of the problem when the domain is included in a geodesic plane, an equidistant surface and a horosphere. Here we restrict to the case that the domain is bounded and strictly convex.

Finally, in Chap. 12 we consider cmc surfaces with boundary in Lorentz-Minkowski space $\mathbb{L}^3$. Here we study spacelike surfaces because the induced metric on the surface is Riemannian. We shall characterize the umbilical surfaces as the only spacelike compact surfaces of constant mean curvature spanning a circle. We also treat techniques such as the tangency principle and the flux formula, and finally, we shall consider the Dirichlet problem for the constant mean curvature equation.

There is an appendix, where we derive the first and second variation of the area, and we finish with an open problems section.

We point out some topics on cmc surfaces that are beyond the scope of this book.

1. Complete surfaces with constant mean curvature. After the discovery by H.C. Wente of an immersed torus with constant mean curvature in Euclidean space [Wen86], a great deal of work has been focused on the description of complete and closed cmc surfaces with arbitrary topology. This effort was initiated by N. Kapouleas, who constructed surfaces with high genus. See [Kap90, Kap91, Kap92].
2. Existence of parametric constant mean curvature surfaces. The techniques are based on functional analysis. The pioneering results were established by, among others, E. Heinz and S. Hildebrandt. The two texts of Struwe are good introductions to the topic [Str88, Str00].
3. Existence of non-parametric surfaces whose mean curvature is not constant. The constant mean curvature equation is just one of a whole family of equations of divergence type, which have been widely studied in the theory of partial differential equations. The text of Gilbarg and Trudinger [GT01] is a suitable starting point for the techniques and the first results.
4. The DPW method. For a non-minimal cmc surface, J. Dorfmeister, F. Pedit and H. Wu developed a generalized Weierstrass representation in terms of holomorphic functions by combining the Sym-Bobenko formula with integrable systems methods [DPW98]. Using loop groups, this technique can be used to construct cmc surfaces. We refer to [FKR05, Hel01].
5. Capillarity. The condition that the boundary curve is prescribed is replaced by the fact that the angle between the surface with a given support is constant. We refer to [Fin86, Lan02].
6. Minimal surfaces, that is, surfaces with zero mean curvature. See [BaCo86, CM11, DHKW92, Nit89, Oss86].
7. Generalizations in other ambient spaces, such as homogeneous spaces. In the last decade, there has been an intensive effort to develop the theory of cmc surfaces, including minimal surfaces, in Thurston's eight models for 3-dimensional geometries [Thu97]. This interest is due after the recent work of U. Abresch and H. Rosenberg concerning the extension of the Hopf theorem [AR04]. See the surveys [DHM09, MP12]. Some works that consider the topics of this book in these spaces are [Bar13, CS12, Daj06, ET08, ET11, EFR10, FM11, FR12, HRS09, Lop12, NEST08, Pin09, Sen11, Spr07].

This monograph is an outgrowth of a series of seminars entitled "Surfaces with constant mean curvature in Euclidean space" that I gave in 2010 at the University of Granada. The origin of some of the material can also be found in the author's articles [Lop96, Lop05, Lop06a, Lop10a].

Some of the results of this book which relate to my own research grew out of discussions with colleagues and collaborators. Firstly, I am grateful to Sebastián Montiel, the advisor of my Ph. Doctoral Thesis [Lop96], who first introduced me to the field of differential geometry and, in particular, to the topic of surfaces with constant mean curvature. Also, I am indebted to Antonio Ros who has always stimulated my work and with whom I have enjoyed many fruitful discussions. Finally, I wish to thank Luis J. Alías, Bennett Palmer, Joaquín Pérez and Miguel Sánchez for valuable conversations.

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