Chapter 2
Electromagnetic Wave Propagation Fundamentals

2.1 Maxwell's Equations

Here we give a brief summary of the features of the theory that are needed to understand the formation, emission and propagation of electromagnetic waves, in the CGS system. The quantities are the electric field intensity $\boldsymbol{E}$, the electric displacement $\boldsymbol{D}$, the magnetic field intensity $\boldsymbol{H}$, the magnetic induction $\boldsymbol{B}$, and the electric current density $\boldsymbol{J}$. The electric charge density is designated by ρ.

The relations of the five vector fields and one scalar field which are required to (properly) describe the electromagnetic phenomena are given by Maxwell's equations. These are conveniently divided into several groups. Some of the field components are related by the properties of the medium in which they exist. These are the so-called material equations

$$\boxed{\begin{aligned} \boldsymbol{J} &= \sigma\,\boldsymbol{E} && (2.1)\\ \boldsymbol{D} &= \varepsilon\,\boldsymbol{E} && (2.2)\\ \boldsymbol{B} &= \mu\,\boldsymbol{H} && (2.3)\end{aligned}} \quad .$$

σ, ε and μ are scalar functions that are almost constant in most materials. For the *Gaussian CGS system* the values of ε and μ are unity ($=1$) in vacuum, while (2.1) is the differential form of Ohm's law, where σ is the specific conductivity.

Maxwell's equations proper can now be further divided into two groups: The first group involves only the spatial structure of the fields

$$\boxed{\begin{aligned} \nabla\cdot\boldsymbol{D} &= 4\pi\rho && (2.4)\\ \nabla\cdot\boldsymbol{B} &= 0 && (2.5)\end{aligned}} \quad ,$$

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while the second group includes time derivatives

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\,\dot{\boldsymbol{B}} \tag{2.6}$$

$$\nabla \times \boldsymbol{H} = \frac{4\pi}{c}\,\boldsymbol{J} + \frac{1}{c}\,\dot{\boldsymbol{D}} \quad . \tag{2.7}$$

Taking the divergence of (2.7) the left side of the resulting equation is found to be equal to zero (see Appendix A). If we use (2.4), we obtain

$$\boxed{\nabla \cdot \boldsymbol{J} + \dot{\rho} = 0} \quad ; \tag{2.8}$$

that is, charge density and current obey a continuity equation. This leads to the conservation of energy including electromagnetic radiation under the *Poynting Vector*:

$$\boxed{\mid \boldsymbol{S} \mid = \frac{c}{4\pi} \mid \boldsymbol{E} \times \boldsymbol{H} \mid} \quad ; \tag{2.9}$$

2.2 Plane Waves in Nonconducting Media

Consider a homogeneous, nonconducting medium ($\sigma = 0$) that is free of currents and charges. In rectangular coordinates each vector component u of $\boldsymbol{E}$ and $\boldsymbol{H}$ obeys the homogeneous wave equation

$$\boxed{\nabla^2 u - \frac{1}{v^2}\ddot{u} = 0} \quad , \tag{2.10}$$

where

$$\boxed{v = \frac{c}{\sqrt{\varepsilon\mu}}} \tag{2.11}$$

is a constant with the dimension of velocity. For the vacuum this becomes

$$v = c\,. \tag{2.12}$$

When Kohlrausch and Weber in 1856 obtained this result experimentally, it became one of the basic facts used by Maxwell when he developed his electromagnetic theory predicting the existence of electromagnetic waves. Eventually this prediction was confirmed experimentally by Hertz.

Equation (2.10) is a homogeneous linear partial differential equation of second order. The complete family of solutions forms a wide and sometimes rather complicated group. No attempt will be made here to discuss general solutions, rather we will restrict our presentation to the properties of the harmonic waves.

$$u = u_0 \, \mathrm{e}^{\mathrm{i}\,(kx \pm \omega t)} \tag{2.13}$$

is a solution of (2.10) if the wave number k obeys the relation

$$\boxed{k^2 = \frac{\varepsilon \mu}{c^2} \, \omega^2} \tag{2.14}$$

This can be confirmed by the substitution of (2.13) into (2.10). If we set

$$\varphi = kx \pm \omega t \, , \tag{2.15}$$

where φ is the phase of the wave, we see that points of constant phase move with the phase velocity

$$\boxed{v = \frac{\omega}{k} = \frac{c}{\sqrt{\varepsilon\mu}}} \quad , \tag{2.16}$$

This gives a physical meaning to the constant v appearing in (2.10). Introducing the index of refraction n as the ratio of c to v this becomes

$$\boxed{n = \frac{c}{v} = \sqrt{\varepsilon\mu} = \frac{c}{\omega} k} \quad . \tag{2.17}$$

For plane electromagnetic waves, each component of $\boldsymbol{E}$ and $\boldsymbol{H}$ will have solutions (2.13) but with an amplitude, u_0, that generally is complex. The use of (2.13) permits us to introduce some important simplifications. For a traveling plane wave

$$\boldsymbol{A}(\boldsymbol{x}, t) = \boldsymbol{A}_0 \, \mathrm{e}^{\mathrm{i}\,(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \, , \qquad \boldsymbol{A}_0, \boldsymbol{k}, \omega = \text{const.}\, , \tag{2.18}$$

$$\dot{\boldsymbol{A}} = -\mathrm{i}\,\omega \boldsymbol{A} \, , \tag{2.19}$$

$$\ddot{\boldsymbol{A}} = -\omega^2 \boldsymbol{A} \, , \tag{2.20}$$

$$\nabla \cdot \boldsymbol{A} = \mathrm{i}\,\boldsymbol{k} \cdot \boldsymbol{A} \, , \tag{2.21}$$

$$\nabla^2 \boldsymbol{A} = -\boldsymbol{k}^2 \boldsymbol{A} \, . \tag{2.22}$$

The $\boldsymbol{E}$ and $\boldsymbol{H}$ fields of an electromagnetic wave are not only solutions of the wave equation (2.10), but these also must obey Maxwell's equations. Because of the decoupling of the two fields in the wave equation, this produces some additional constraints.

In order to investigate the properties of plane waves as simply as possible, we arrange the rectangular coordinate system such that the wave propagates in the positive z direction. A wave is considered to be plane if the surfaces of constant phase form planes $z =$ const. Thus all components of the $\boldsymbol{E}$ and the $\boldsymbol{H}$ field will be independent of x and y for fixed z; that is,

$$\begin{aligned}
&\frac{\partial E_x}{\partial x} = 0\,, \qquad \frac{\partial E_y}{\partial x} = 0\,, \qquad \frac{\partial E_z}{\partial x} = 0\,, \\
&\frac{\partial E_x}{\partial y} = 0\,, \qquad \frac{\partial E_y}{\partial y} = 0\,, \qquad \frac{\partial E_z}{\partial y} = 0\,,
\end{aligned} \tag{2.23}$$

and a similar set of equations for $\boldsymbol{H}$. But according to Maxwell's equations (2.4) and (2.5) with $\rho = 0$ and $\varepsilon =$ const.

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \qquad \text{and} \qquad \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0\,.$$

Because of (2.23) this results in

$$\boxed{\frac{\partial E_z}{\partial z} = 0 \qquad \text{and} \qquad \frac{\partial H_z}{\partial z} = 0}\,. \tag{2.24}$$

From the remaining Maxwell's equations (2.6) and (2.7) we similarly obtain

$$\boxed{\frac{\partial E_z}{\partial t} = 0 \qquad \text{and} \qquad \frac{\partial H_z}{\partial t} = 0}\,. \tag{2.25}$$

Therefore both the longitudinal components E_z and H_z must be constant both in space and time. Since such a constant field is of no significance here, we require that

$$\boxed{E_z \equiv 0\,,\ \ H_z \equiv 0} \tag{2.26}$$

that is, the plane electromagnetic wave in a nonconducting medium is *transverse* The remaining components have the form of traveling harmonic waves (as given by (2.13)). The only components of (2.6) and (2.7) which differ from zero are

$$\begin{aligned}
&\frac{\partial E_x}{\partial z} = -\frac{\mu}{c}\frac{\partial H_y}{\partial t}\,, \qquad && \frac{\partial H_x}{\partial z} = \frac{\varepsilon}{c}\frac{\partial E_y}{\partial t}\,, \\
&&\text{and}\qquad& \\
&\frac{\partial E_y}{\partial z} = \frac{\mu}{c}\frac{\partial H_x}{\partial t}\,, && \frac{\partial H_y}{\partial z} = -\frac{\varepsilon}{c}\frac{\partial E_x}{\partial t}\,.
\end{aligned} \tag{2.27}$$

Applying the relations (2.19) and (2.21) for plane harmonic waves, we find

$$\begin{aligned}\frac{\partial E_x}{\partial z} &= \mathrm{i}\,k E_x = -\frac{\mu}{c}\dot{H}_y = \frac{\mathrm{i}\,\omega\mu}{c}H_y\,,\\ \frac{\partial E_y}{\partial z} &= \mathrm{i}\,k E_y = \frac{\mu}{c}\dot{H}_x = -\frac{\mathrm{i}\,\omega\mu}{c}H_x\,,\end{aligned} \tag{2.28}$$

resulting in

$$\boldsymbol{E}\cdot\boldsymbol{H} = E_x H_x + E_y H_y = -\frac{ck}{\omega\mu}E_x E_y + \frac{ck}{\omega\mu}E_y E_x = 0\,,$$

$$\boxed{\boldsymbol{E}\cdot\boldsymbol{H} = 0}\quad . \tag{2.29}$$

$\boldsymbol{E}$ and $\boldsymbol{H}$ are thus always perpendicular; together with the wave vector $\boldsymbol{k}$, these form an orthogonal system. For the ratio of their absolute values, (2.28) and (2.14) result in

$$\frac{|\,\boldsymbol{E}\,|}{|\,\boldsymbol{H}\,|} = \sqrt{\frac{\mu}{\varepsilon}}\,. \tag{2.30}$$

The unit of this *intrinsic impedance* of the medium in which the wave propagates is the Ohm (Ω). In a vacuum it has the value

$$Z_0 = 376.73\,\Omega\,. \tag{2.31}$$

Finally, the energy flux of the Poynting vector of this wave is of interest. We find

$$|\,\boldsymbol{S}\,| = \frac{c}{4\pi}\sqrt{\frac{\varepsilon}{\mu}}\,\boldsymbol{E}^2\,, \tag{2.32}$$

and $\boldsymbol{S}$ points in the direction of the propagation vector $\boldsymbol{k}$. The (time averaged) energy density, U, of the wave is then

$$U = \frac{1}{8\pi}\left(\varepsilon\,\boldsymbol{E}\cdot\boldsymbol{E}^* + \mu\,\boldsymbol{H}\cdot\boldsymbol{H}^*\right)\,. \tag{2.33}$$

In using (2.30) we find that (2.33) becomes

$$U = \frac{\varepsilon}{4\pi}\boldsymbol{E}^2\,. \tag{2.34}$$

The time averaged Poynting vector is often used as a measure of the intensity of the wave; its direction represents the direction of the wave propagation.

2.3 Wave Packets and the Group Velocity

A monochromatic plane wave

$$u(x,t) = A\ \mathrm{e}^{\mathrm{i}\,(kx-\omega t)} \tag{2.35}$$

propagates with the phase velocity

$$v = \frac{\omega}{k}\,. \tag{2.36}$$

If this velocity is the same for a whole range of frequencies, then a wave packet formed by the superposition of these waves will propagate with the same velocity. In general, however, the propagation velocity, v, will depend on the wave number k. Then such wave packets have some new and interesting properties. A wave with an arbitrary shape can be formed by superposing simple harmonic waves

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} A(k)\ \mathrm{e}^{\mathrm{i}\,(kx-\omega t)}\ \mathrm{d}k\,, \tag{2.37}$$

where $A(k)$ is the amplitude of the wave with the wave number k. The angular frequency of these waves will be different for different k; this distribution is

$$\omega = \omega(k) \tag{2.38}$$

and it will be referred to as the *dispersion equation* of the waves. If $A(k)$ is a fairly sharply peaked function around some k_0, only waves with wave numbers not too different from k_0 will contribute to (2.37), and quite often a linear approximation for (2.38)

$$\omega(k) = \omega_0 + \left.\frac{\mathrm{d}\omega}{\mathrm{d}k}\right|_0 (k-k_0) \tag{2.39}$$

will be sufficient. The symbol after the derivative indicates that it will be evaluated at $k = k_0$. Substituting this into (2.37) we can extract all factors that do not depend on k from the integral, obtaining

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \exp\left[\mathrm{i}\left(\left.\frac{\mathrm{d}\omega}{\mathrm{d}k}\right|_0 k_0 - \omega_0\right) t\right] \int\limits_{-\infty}^{\infty} A(k) \exp\left[\mathrm{i}\,k\left(x - \left.\frac{\mathrm{d}\omega}{\mathrm{d}k}\right|_0 t\right)\right] \mathrm{d}k\,. \tag{2.40}$$

According to (2.37), at the time $t = 0$ the wave packet has the shape

$$u(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\, \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}k\,.$$

Therefore the integral in (2.40) is $u(x',0)$, where $x' = x - \frac{\mathrm{d}\omega}{\mathrm{d}k}\big|_0 t$. The entire expression is

$$u(x,t) = u\left(x - \frac{\mathrm{d}\omega}{\mathrm{d}k}\bigg|_0 t, 0\right) \exp\left[\mathrm{i}\left(k_0 \frac{\mathrm{d}\omega}{\mathrm{d}k}\bigg|_0 - \omega_0\right) t\right]. \tag{2.41}$$

The exponential in (2.41) has a purely imaginary argument and therefore is only a phase factor. Therefore, the wave packet travels undistorted in shape except for an overall phase factor with the group velocity

$$\boxed{v_\mathrm{g} = \frac{\mathrm{d}\omega}{\mathrm{d}k}}\,. \tag{2.42}$$

This is strictly true if the angular frequency is a linear function of k. If $\omega(k)$ is more general, the group velocity depends on wave number, and the form of the wave packet (made up of waves with a finite range of wave numbers) will be distorted in time. That is, the pulse will disperse.

Whether phase velocity (2.36) or group velocity (2.42), is larger depends on the properties of the medium in which the wave propagates. Writing (2.36) as

$$\omega = k\, v\,,$$

one finds

$$\frac{\mathrm{d}\omega}{\mathrm{d}k} = v_\mathrm{g} = v + k\,\frac{\mathrm{d}v}{\mathrm{d}k}\,. \tag{2.43}$$

Recalling the definition of the index of refraction (2.17)

$$n = \frac{c}{v}$$

and that the wavelength is given by

$$\lambda = \frac{2\pi}{k}\,, \tag{2.44}$$

we see that normal dispersion $\mathrm{d}n/\,\mathrm{d}\lambda < 0$ in the medium corresponds to $\mathrm{d}v/\,\mathrm{d}k < 0$. In a medium with normal dispersion therefore $v_\mathrm{g} < v$. Only for anomalous dispersion will we have $v_\mathrm{g} > v$.

Energy and information are usually propagated with the group velocity. The situation is, however, fairly complicated if propagation in dispersive media is considered. Details can be found in Sommerfeld (1959).

2.4 Plane Waves in Conducting Media

In Sect. 2.2 the propagation properties of plane harmonic waves in a *nonconducting* ($\sigma = 0$) medium have been investigated. Now this assumption will be dropped so that $\sigma \neq 0$, but we still restrict the investigation to strictly harmonic waves propagating in the direction of increasing x

$$\boldsymbol{E}(x,t) = \boldsymbol{E}_0 \, \mathrm{e}^{\mathrm{i}(kx-\omega t)} \,. \tag{2.45}$$

Both $\boldsymbol{E}_0$ and k are complex constants. Making use of (2.19)–(2.22), we have

$$\boxed{\left[k^2 - \left(\frac{\varepsilon\mu}{c^2}\omega^2 + \mathrm{i}\,\frac{4\pi\sigma\mu\,\omega}{c^2}\right)\right]\left\{\begin{matrix}\boldsymbol{E}\\ \boldsymbol{H}\end{matrix}\right\} = 0} \quad . \tag{2.46}$$

If these equations are to be valid for arbitrary $\boldsymbol{E}$ or $\boldsymbol{H}$ (of the form (2.45)) the square bracket must be zero, so that the *dispersion equation* becomes

$$\boxed{k^2 = \frac{\mu\varepsilon\,\omega^2}{c^2}\left(1 + \mathrm{i}\,\frac{4\pi\sigma}{\omega\varepsilon}\right)} \quad . \tag{2.47}$$

The wave number k thus is indeed a complex number. Writing

$$k = a + \mathrm{i}\, b \,, \tag{2.48}$$

we find

$$a = \sqrt{\varepsilon\mu}\,\frac{\omega}{c}\sqrt{\frac{1}{2}\left(\sqrt{1 + \left(\frac{4\pi\sigma}{\varepsilon\,\omega}\right)^2} + 1\right)} \tag{2.49}$$

$$b = \sqrt{\varepsilon\mu}\,\frac{\omega}{c}\sqrt{\frac{1}{2}\left(\sqrt{1 + \left(\frac{4\pi\sigma}{\varepsilon\,\omega}\right)^2} - 1\right)} \tag{2.50}$$

and the field therefore can be written

$$\boldsymbol{E}(x,t) = \boldsymbol{E}_0 \, \mathrm{e}^{-bx} \, \mathrm{e}^{\mathrm{i}(ax-\omega t)} \,. \tag{2.51}$$

Thus the real part of the conductivity gives rise to an exponential damping of the wave. If (2.51) is written using the index of refraction n and the absorption coefficient κ,

$$\boxed{\boldsymbol{E}(x,t) = \boldsymbol{E}_0 \exp\left(-\frac{\omega}{c}\, n\kappa x\right) \exp\left[\mathrm{i}\,\omega\left(\frac{n}{c}\, x - t\right)\right]} \quad , \tag{2.52}$$

we obtain

$$n\kappa = \sqrt{\varepsilon\mu}\,\sqrt{\frac{1}{2}\left(\sqrt{1+\left(\frac{4\pi\sigma}{\varepsilon\,\omega}\right)^2} - 1\right)} \tag{2.53}$$

$$n = \sqrt{\varepsilon\mu}\,\sqrt{\frac{1}{2}\left(\sqrt{1+\left(\frac{4\pi\sigma}{\varepsilon\,\omega}\right)^2} + 1\right)} \quad . \tag{2.54}$$

2.5 The Dispersion Measure of a Tenuous Plasma

The simplest model for a dissipative medium is that of a tenuous plasma where free electrons and ions are uniformly distributed so that the total space charge density is zero. This model was first given by Drude to explain the propagation of ultraviolet light in a transparent medium, but this model was later applied to the propagation of transverse electromagnetic radio waves in a tenuous plasma.

The free electrons are accelerated by the electric field intensity; their equation of motion is

$$m_\mathrm{e}\dot{\boldsymbol{V}} = m_\mathrm{e}\ddot{\boldsymbol{r}} = -e\,\boldsymbol{E}_0\,\mathrm{e}^{-\mathrm{i}\omega t} \tag{2.55}$$

with the solution

$$\boldsymbol{V} = \frac{e}{\mathrm{i}\,m_\mathrm{e}\,\omega}\,\boldsymbol{E}_0\,\mathrm{e}^{-\mathrm{i}\omega t} = -\mathrm{i}\,\frac{e}{m_\mathrm{e}\,\omega}\,\boldsymbol{E}\,. \tag{2.56}$$

Equation (2.56) describes the motion of the electrons. Moving electrons, however, carry a current, whose density is

$$\boldsymbol{J} = -\sum_\alpha e\boldsymbol{v}_\alpha = -Ne\boldsymbol{v} = \mathrm{i}\,\frac{Ne^2}{m_\mathrm{e}\,\omega}\,\boldsymbol{E} = \sigma\boldsymbol{E}\,. \tag{2.57}$$

This expression explains why the ions can be neglected in this investigation. Due to their large mass ($m_\mathrm{i} \simeq 2 \times 10^3\, m_\mathrm{e}$), the induced ion velocity (2.56) is smaller than that of the electrons by the same factor, and since the charge of the ions is the same as that of the electrons, the ion current (2.57) will be smaller than the electron current by the same factor.

According to (2.57) the conductivity of the plasma is purely imaginary:

$$\sigma = \mathrm{i}\,\frac{Ne^2}{m_\mathrm{e}\,\omega}\,. \tag{2.58}$$

Inserting this into (2.47) we obtain, for a thin medium with $\varepsilon \approx 1$ and $\mu \approx 1$

$$\boxed{k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_\mathrm{p}^2}{\omega^2}\right)}\quad, \tag{2.59}$$

where

$$\boxed{\omega_\mathrm{p}^2 = \frac{4\pi Ne^2}{m_\mathrm{e}}} \tag{2.60}$$

is the square of the *plasma frequency*. It gives a measure of the mobility of the electron gas. Inserting numerical values we obtain

$$\frac{\nu_\mathrm{p}}{\mathrm{kHz}} = 8.97\sqrt{\frac{N}{\mathrm{cm}^{-3}}} \tag{2.61}$$

if we convert (2.60) to frequencies by $\nu = \omega/2\pi$. For $\omega > \omega_\mathrm{p}$, k is real, and we obtain from (2.36)

$$\boxed{v = \frac{c}{\sqrt{1 - \dfrac{\omega_\mathrm{p}^2}{\omega^2}}}} \tag{2.62}$$

for the *phase velocity* v and so $v > c$ for $\omega > \omega_\mathrm{p}$. For the *group velocity* it follows from (2.42)

$$v_\mathrm{g} = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{1}{\mathrm{d}k/\,\mathrm{d}\omega}\,,$$

so that

$$\boxed{v_\mathrm{g} = c\sqrt{1 - \frac{\omega_\mathrm{p}^2}{\omega^2}}} \tag{2.63}$$

and $v_g < c$ for $\omega > \omega_p$. Both v and v_g thus depend on the frequency ω. For $\omega = \omega_p$, $v_g = 0$; thus for waves with a frequency lower than ω_p, no wave propagation in the plasma is possible. The frequency dependence of v and v_g are in the opposite sense; taking (2.62) and (2.63) together the relation

$$\boxed{v\, v_g = c^2} \tag{2.64}$$

is obtained.

For some applications the *index of refraction* is a useful quantity. According to (2.17) and (2.59) it is

$$\boxed{n = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}} \quad . \tag{2.65}$$

Electromagnetic pulses propagate with the group velocity. This varies with frequency so that there is a dispersion in the pulse propagation in a plasma. This fact took on a fundamental importance when the radio pulsars were detected in 1967. The arrival time of pulsar pulses depends on the frequency: The lower the observing frequency, the later the pulse arrives. This behavior can easily be explained in terms of wave propagation in a tenuous plasma, as the following discussion shows.

The plasma frequency of the interstellar medium (ISM) is much lower than the observing frequency. For example, in the diffuse ISM, N is typically 10^{-3}–$10^{-1}\,\mathrm{cm}^{-3}$, so ν_p is in the range 2.85–0.285 kHz; however, the observing frequency must be $\nu > 10\,\mathrm{MHz}$ in order to propagate through the ionosphere of the earth. This is a reason for low frequency satellites or an antenna on the Moon. For v_g, we can use a series expansion of (2.63)

$$\frac{1}{v_g} = \frac{1}{c}\left(1 + \frac{1}{2}\frac{\nu_p^2}{\nu^2}\right) \tag{2.66}$$

with high precision. A pulse emitted by a pulsar at the distance L therefore will be received after a delay

$$\tau_D = \int_0^L \frac{\mathrm{d}l}{v_g} \cong \frac{1}{c}\int_0^L \left(1 + \frac{1}{2}\left(\frac{\nu_p}{\nu}\right)^2\right) \mathrm{d}l = \frac{1}{c}\int_0^L \left(1 + \frac{e^2}{2\pi\, m_e}\frac{1}{\nu^2}N(l)\right) \mathrm{d}l \,,$$

$$\tau_D = \frac{L}{c} + \frac{e^2}{2\pi c m_e}\frac{1}{\nu^2}\int_0^L N(l)\ \mathrm{d}l \,. \tag{2.67}$$

The difference between the pulse arrival times measured at two frequencies ν_1 and ν_2 therefore is given by

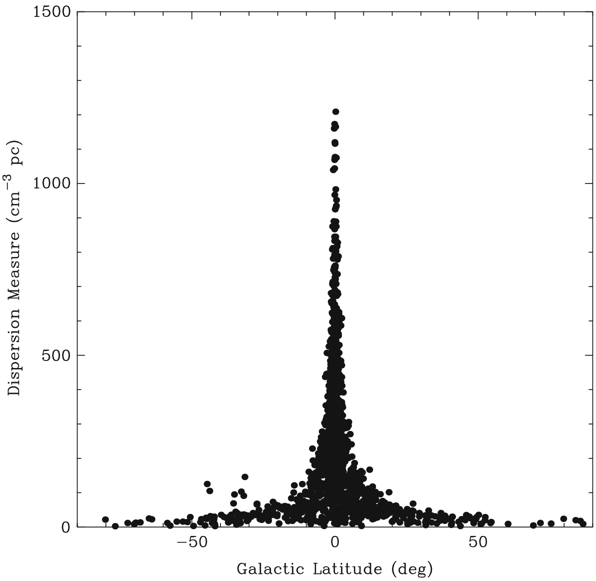

Fig. 2.1 Dispersion measure, DM, for pulsars at different galactic latitudes (adapted from B. Klein, unpublished)

$$\boxed{\Delta\tau_{\mathrm{D}} = \frac{e^2}{2\pi c m_{\mathrm{e}}}\left[\frac{1}{\nu_1^2} - \frac{1}{\nu_2^2}\right]\int\limits_0^L N(l)\,\mathrm{d}l} \quad . \tag{2.68}$$

The quantity $\int_0^L N(l)\,\mathrm{d}l$ is the column-density of the electrons in the intervening space between pulsar and observer. Since distances in astronomy are measured in parsecs (1 pc $= 3.085677 \times 10^{18}$ cm), it has become customary to measure $N(l)$ in cm^{-3} but dl in pc. The integral is referred to as the *dispersion measure* (Fig. 2.1)

$$\boxed{\mathrm{DM} = \int\limits_0^L \left(\frac{N}{\mathrm{cm}^{-3}}\right)\mathrm{d}\left(\frac{l}{\mathrm{pc}}\right)} \tag{2.69}$$

and therefore we find

$$\frac{\Delta\tau_{\mathrm{D}}}{\mu\mathrm{s}} = 1.34 \times 10^{-9}\left[\frac{\mathrm{DM}}{\mathrm{cm}^{-2}}\right]\left[\frac{1}{\left(\frac{\nu_1}{\mathrm{MHz}}\right)^2} - \frac{1}{\left(\frac{\nu_2}{\mathrm{MHz}}\right)^2}\right] \tag{2.70}$$

or

$$\boxed{\frac{\Delta\tau_{\mathrm{D}}}{\mu\mathrm{s}} = 4.148 \times 10^9 \left[\frac{\mathrm{DM}}{\mathrm{cm}^{-3}\,\mathrm{pc}}\right] \left[\frac{1}{\left(\frac{\nu_1}{\mathrm{MHz}}\right)^2} - \frac{1}{\left(\frac{\nu_2}{\mathrm{MHz}}\right)^2}\right]} \quad . \tag{2.71}$$

Since both the time delay $\Delta\tau_{\mathrm{D}}$ and the observing frequencies ν_1 and ν_2 can be measured with high precision, a very accurate value of DM for a given pulsar can be determined from

$$\boxed{\frac{\mathrm{DM}}{\mathrm{cm}^{-3}\,\mathrm{pc}} = 2.410 \times 10^{-4} \left(\frac{\Delta\tau_{\mathrm{D}}}{\mathrm{s}}\right) \left[\frac{1}{\left(\frac{\nu_1}{\mathrm{MHz}}\right)^2} - \frac{1}{\left(\frac{\nu_2}{\mathrm{MHz}}\right)^2}\right]^{-1}} . \tag{2.72}$$

Provided the distance L to the pulsar is known, this gives a good estimate of the average electron density between observer and pulsar. However since L is usually known only very approximately, only approximate values for N can be obtained in this way. Quite often the opposite procedure is used: From reasonable guesses for N, a measured DM provides information on the unknown distance L to the pulsar.

Dispersion in the ISM, combined with a finite pulse width, sets a limit to the fine structure one can resolve in a pulse. The frequency dependence of the pulse arrival time is τ_{D} from (2.67). This gives a condition for the bandwidth b needed to resolve a feature in a time τ.

$$\left(\frac{b}{\mathrm{MHz}}\right) = 1.205 \times 10^{-4} \frac{1}{\left[\frac{\mathrm{DM}}{\mathrm{cm}^{-3}\,\mathrm{pc}}\right]} \left[\frac{\nu}{\mathrm{MHz}}\right]^3 \left[\frac{\tau}{\mathrm{s}}\right] . \tag{2.73}$$

Since the pulses will have a finite width in both time and frequency, a differential form of (2.73) will give a limit to the maximum bandwidth that can be used at a given frequency and DM if a time resolution τ is wanted. This will be rediscussed in the context of pulsar back ends.

Problems

1. There is a proposal to transmit messages to mobile telephones in large U.S. cities from a transmitter hanging below a balloon at an altitude of 40 km. Suppose the city in question has a diameter of 40 km. What is the solid angle to be illuminated? Suppose mobile telephones require an electric field strength, E, of 200 µV per meter. If one uses $S = E^2/R$ with $R = 50, \Omega$, what is the E field at the transmitter? How much power must be transmitted? At what distance from the transmitter would the microwave radiation reach the danger level, 10 mW cm^{-2}?

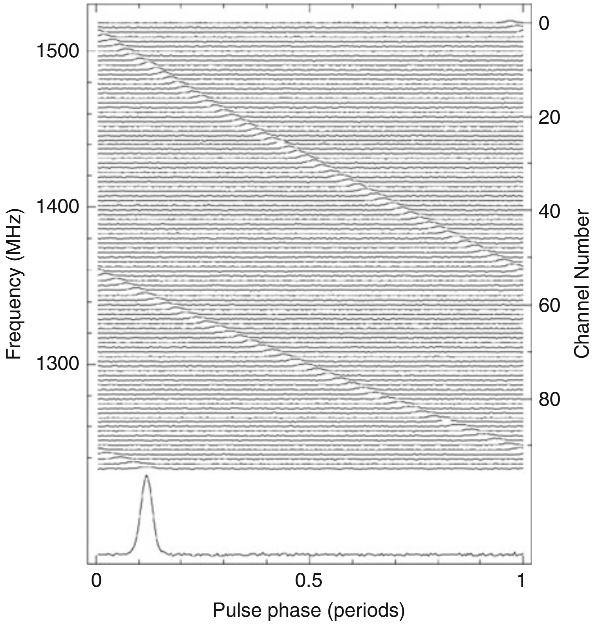

Fig. 2.2 Pulse arrival times at frequencies from 1.24 to 1.5 GHz. The data were taken with the CSIRO Parkes 64-m antenna by Andrew Lyne. The source is the pulsar B1356-60. The coherently summed pulse shown at the *bottom* has a period of 128 ms. The dispersion measure, DM, is 295 cm^{-3} pc

2. Radiation from an astronomical source at a distance of 1.88 kpc, ($= 7.1 \times 10^{21}$ cm) has a flux density of 10^3 Jy over a frequency band of 600 Hz. If it is isotropic, what is the power radiated? Suppose the source size is 1 mas (see (1.34)). What is the value of T_B?

3. Suppose that $v_{\text{phase}} = \frac{c}{\sqrt{1-(\lambda_0/\lambda_c)^2}}$. What is v_{group}? Evaluate both of these quantities for $\lambda_0 = \frac{1}{2}\,\lambda_c$.

4. There is a 1 D wave packet. At time $t = 0$, the amplitudes are distributed as $a(k) = a_0 \exp(-k^2/(\Delta k)^2)$, where a_0 and Δk are constant. From the use of Fourier transform relations in Appendix A, determine the product of the width of the wave packet, Δk, and the width in time, Δt.

5. Repeat problem 7 with $a(k) = a_0 \exp(-(k - k_0)^2/(\Delta k)^2)$. Repeat for $a(k) = a_0$ for $k_1 < k < k_2$, otherwise $a(k) = 0$.

6. Assume that pulsars emit narrow periodic pulses at all frequencies simultaneously. Use (2.67) to show that a narrow pulse (width of order $\sim 10^{-6}$ s) will traverse the radio spectrum at a rate, in MHz s^{-1}, of $\dot{\nu} = 1.2 \times 10^{-4}\,(\mathrm{DM})^{-1}\,\nu\,[\mathrm{MHz}]^3$.

7. **(a)** Show that using a receiver bandwidth B will lead to the smearing of a very narrow pulse, which passes through the ISM with dispersion measure DM, to a width $\Delta t = 8.3 \times 10^3$ DM $[\nu\ [(\mathrm{MHz})]^{-3}\ B$ s.

(b) Show that the ionosphere (electron column density $10^{12}\,\mathrm{cm}^{-2}$, referred to as 1 TEC) has little influence on the pulse shape at 100 MHz.

8. **(a)** Show that the smearing Δt, in milli seconds, of a short pulse is $(202/\nu_{\mathrm{MHz}})^3$ DM ms per MHz of receiver bandwidth.

(b) If a pulsar is at a distance of 5 kpc, and the average electron density is $0.05\,\mathrm{cm}^{-3}$, find the smearing at 400 MHz. Repeat for 800 MHz.

9. Suppose you would like to detect a pulsar located at the center of our Galaxy. The pulsar may be behind a cloud of ionized gas of size 10 pc, and electron density $10^3\,\mathrm{cm}^{-3}$. Calculate the dispersion measure, DM. What is the bandwidth limit if the observing frequency is 1 GHz, and the pulsar frequency is 30 Hz?

10. A typical value for DM is $30\,\mathrm{cm}^{-3}$ pc, which is equivalent to an electron column density of $10^{20}\,\mathrm{cm}^{-2}$. For frequencies of 400 and 1,000 MHz, use (2.71) to predict how much a pulse will be delayed relative to a pulse at an infinitely high frequency.

11. To resolve a pulse feature with a width of 0.1 μs at a received frequency of 1,000 MHz and DM $= 30\,\mathrm{cm}^{-3}$ pc, what is the maximum receiver bandwidth?

12. In Fig. 2.2, about 10 % of the pulse width is 10 ms. What bandwidth is needed to resolve this pulse for a DM $= 295\,\mathrm{cm}^{-3}$ pc?

References

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