Chapter 2

Energy Preserving Boundary Conditions

As has been pointed out in the introduction the Navier-Stokes equations

$$(\partial_t \rho u) + \operatorname{div}(\rho u \otimes u - S) = \rho f, \qquad S = 2\mu D - p, \qquad \operatorname{div} u = 0 \qquad \text{in } J \times \Omega$$

form a system of partial differential equations and have to be complemented by suitable initial and boundary conditions. We employ the notation of Chapter 1, i. e. $J = (0, a)$ denotes the considered time interval and $\Omega \subseteq \mathbb{R}^3$ denotes the region in space that is occupied by the fluid. Moreover, $\rho > 0$ denotes the constant density of the fluid and $\mu > 0$ denotes its constant viscosity.

Unfortunately, in many situations the choice of reasonable boundary conditions is by no means obvious. To point out the main difficulties, which arise for the modelling of reasonable boundary conditions, it is convenient to distinguish the most relevant properties of the boundary under consideration.

Whenever a physical boundary is present, it separates the fluid from another phase, which may be a solid or another fluid. In both situations the boundary may be moving or non-moving. On the other hand, such a boundary may arise as an interface between two immiscible phases, e. g. between a fluid and a solid or between two immiscible fluids, or it may arise artificially, if the model describes only a part of a larger system. The last type of boundary, which is also called an artificial boundary, often appears in models for numerical simulations, since it is often convenient to save computation time by focusing on the interesting part of the flow while neglecting other parts of the system. Hence, such an artificial boundary "separates" a fluid from itself and may be freely passed in both directions by this fluid. As a consequence, no useful information is available about the exchange of mass, momentum and energy at such a boundary, which turns the modelling of reasonable artificial boundary conditions into a formidable task.

Now, the aim of this chapter is to give a tentative introduction to the modelling of boundary conditions for the above mentioned situations. Concerning artificial boundaries we will present a class of boundary conditions, which is derived based on suitable energy balances. The resulting *energy preserving boundary conditions* will then be the main topic of the following chapters, where we will show that these conditions are not only physically reasonable but also lead to mathematically well-posed models.

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2.1 Generic Transmission Conditions

2.1 As has been pointed out above, a boundary separates the fluid from another phase, which, however, may be the fluid itself as in the case of an artificial boundary. Nevertheless, the balance equations for extensive quantities, which encode their conservation, are always valid in the continuum mechanical setting. In Chapter 1 we used these balance equations for the extensive quantities mass and momentum to derive the Navier-Stokes equations in Ω. However, analogue arguments imply the balance equations

$$\llbracket \rho(u - u_\Gamma) \rrbracket_\Gamma \cdot \nu_\Gamma = 0, \quad \llbracket \rho u \otimes (u - u_\Gamma) - S \rrbracket_\Gamma \nu_\Gamma = f_\Gamma, \qquad t > 0,\ y \in \Gamma(t) \tag{2.1}$$

to be always valid. Here, $\llbracket \cdot \rrbracket_\Gamma$ denotes the jump of a quantity, if the boundary Γ is passed from the outside of Ω into the domain, i. e. in the direction opposite to the outer unit normal field ν_Γ. Moreover, u_Γ denotes the velocity of the boundary itself, e. g. $u_\Gamma = 0$ for a non-moving boundary or $u_\Gamma = u|_\Gamma$ for an interface separating two immiscible fluids, if the absence of phase transitions and interfacial slip is assumed. Hence, the first of the above *transmission conditions* encodes the conservation of mass, where we assume that there do not exist any sources or sinks for mass on the boundary. Analogously, the second transmission condition encodes the conservation of momentum, where we allow for the presence of sources or sinks given by their density f_Γ, which may e. g. arise due to surface tension in the case of a phase separating interface. Note that the density ρ and the viscosity μ, which enters the transmission conditions via the stress tensor S, will in general be discontinuous across the boundary and, thus, a jump of these quantities also contributes to the above balance equations.

2.2 In order to exploit the transmission conditions (2.1) to derive boundary conditions for the flow in Ω it is convenient to rewrite them via

$$\llbracket \rho(u - u_\Gamma) \rrbracket_\Gamma \cdot \nu_\Gamma = \rho \llbracket u - u_\Gamma \rrbracket_\Gamma \cdot \nu_\Gamma + \llbracket \rho \rrbracket_\Gamma (\bar{u}|_\Gamma - u_\Gamma) \cdot \nu_\Gamma,$$

where $\llbracket \rho \rrbracket_\Gamma = \bar{\rho} - \rho$, if we assume the phase on the outside of Ω to have a constant density $\bar{\rho} > 0$, and where $\bar{u}$ denotes the velocity of the outer phase. Hence, we obtain

$$\rho \llbracket u - u_\Gamma \rrbracket_\Gamma \cdot \nu_\Gamma + (\bar{\rho} - \rho)(\bar{u} - u_\Gamma) \cdot \nu_\Gamma = 0, \qquad t > 0,\ y \in \Gamma(t). \tag{2.2}$$

Moreover, we may simplify the left hand side of the momentum transmission condition using

$$\begin{aligned} \llbracket \rho u \otimes (u - u_\Gamma) \rrbracket_\Gamma \nu_\Gamma &= (u|_\Gamma \otimes \llbracket \rho(u - u_\Gamma) \rrbracket_\Gamma)\nu_\Gamma \;+\; (\llbracket u \rrbracket_\Gamma \otimes \bar{\rho}(\bar{u}|_\Gamma - u_\Gamma))\nu_\Gamma \\ &= (\llbracket \rho(u - u_\Gamma) \rrbracket_\Gamma \cdot \nu_\Gamma)\, u|_\Gamma \;+\; (\llbracket u \rrbracket_\Gamma \otimes \bar{\rho}(\bar{u}|_\Gamma - u_\Gamma))\nu_\Gamma \\ &= (\llbracket u \rrbracket_\Gamma \otimes \bar{\rho}(\bar{u}|_\Gamma - u_\Gamma))\nu_\Gamma \end{aligned}$$

to obtain

$$(\llbracket u \rrbracket_\Gamma \otimes \bar{\rho}(\bar{u} - u_\Gamma))\nu_\Gamma - \llbracket S \rrbracket_\Gamma \nu_\Gamma = f_\Gamma, \qquad t > 0,\ y \in \Gamma(t). \tag{2.3}$$

For more details on the modelling of boundary conditions based on transmission conditions we refer to [BFG+12] and [BKP12].

2.2 Impermeable Walls

2.3 If the boundary Γ separates the fluid from a solid, i. e. Γ represents a solid wall, we necessarily have $u_\Gamma = \bar{u}|_\Gamma$, where $\bar{u}$ denotes the velocity field describing the motion of the solid. Now, the mass transmission condition (2.2) implies

$$u \cdot \nu_\Gamma = \bar{u} \cdot \nu_\Gamma, \qquad t > 0,\ y \in \Gamma(t).$$

Note that this condition encodes the fact, that the boundary is impermeable. In particular, for an impermeable, non-moving solid wall we have

$$u \cdot \nu_\Gamma = 0 \qquad \text{on } J \times \Gamma. \tag{2.4}$$

2.4 To obtain a complete set of boundary conditions for solid walls, we would need to exploit the momentum transmission condition (2.3), which in this case simplifies to

$$-[\![S]\!]_\Gamma \, \nu_\Gamma = f_\Gamma, \qquad t > 0,\ y \in \Gamma(t).$$

However, this would require to model the stress for the solid via constitutive equations, which would depend on the material properties of the solid, e. g. its rigidity or elasticity. Since this goes deeply beyond the scope of this thesis, we focus on rigid solids and assume the fluid to slip along the wall while being stressed in tangential directions due to friction. This leads to

$$P_\Gamma(u - \bar{u}) + \lambda P_\Gamma S \nu_\Gamma = 0, \qquad t > 0,\ y \in \Gamma(t), \tag{2.5}$$

with a parameter $\lambda > 0$, which is also called the *slip length*. For an impermeable, non-moving, rigid wall the combination of (2.4) and (2.5) reads

$$u \cdot \nu_\Gamma = 0, \qquad P_\Gamma u + \lambda P_\Gamma S \nu_\Gamma = 0 \qquad \text{on } J \times \Gamma, \tag{2.6}$$

which is known as the *Navier condition*, since it was first proposed by C. L. M. H. Navier in 1823 along with his derivation of the Navier-Stokes equations [Nav23].

2.5 The extremal case $\lambda = 0$ in (2.5) corresponds to the model assumption that the fluid is not allowed to slip along the wall and for an impermeable, non-moving, rigid wall the boundary condition becomes

$$u = 0 \qquad \text{on } J \times \Gamma. \tag{2.7}$$

This is called the *no-slip condition* and was already proposed by G. G. Stokes in 1845 along with his derivation of the Navier-Stokes equations [Sto45].

2.6 The extremal case $\lambda \to \infty$ in (2.5) corresponds to the model assumption that the fluid may freely slip along the wall without being stressed due to friction and for an impermeable, non-moving, rigid wall the boundary condition becomes

$$u \cdot \nu_\Gamma = 0, \qquad P_\Gamma S \nu_\Gamma = 0 \qquad \text{on } J \times \Gamma, \tag{2.8}$$

which is called the *free slip* or *perfect-slip condition*.

2.7 For more details on the family of boundary conditions (2.6), (2.7) and (2.8) we refer to [BLS07] and [LS03] and the references therein. A recent debate concerning the no-slip condition (2.7) may be found in [CD98].

2.3 Free Surfaces and Interfaces

2.8 If the boundary Γ separates two immiscible fluids, the situation is not as easy as for impermeable walls as described in 2.3. In fact, due to processes like phase transitions even the normal velocity might be discontinuous across the boundary and, moreover, the boundary velocity u_Γ may be totally different from the velocities of the fluids. However, if we exclude such phenomena and assume

$$[\![u]\!]_\Gamma \cdot \nu_\Gamma = 0, \qquad u_\Gamma \cdot \nu_\Gamma = u \cdot \nu_\Gamma, \qquad t > 0,\ y \in \Gamma(t),$$

the situation significantly simplifies and the boundary may be regarded as a *free interface* separating the two phases.

2.9 Now, if the interface carries no mass and no interfacial slip occurs, the tangential velocities may be neglected and we may assume

$$[\![u]\!]_\Gamma = 0, \qquad u_\Gamma = u, \qquad t > 0,\ y \in \Gamma(t).$$

Thus, the transmission conditions (2.2) and (2.3) simplify to

$$[\![u]\!]_\Gamma = 0, \qquad -[\![S]\!]_\Gamma\, \nu_\Gamma = f_\Gamma, \qquad t > 0,\ y \in \Gamma(t), \tag{2.9}$$

where the right hand side f_Γ has to be modelled by constitutive equations to take care of effects like *surface tension* or *surface viscosity*. This way we may model e. g. two-phase flows of immiscible, incompressible Newtonian fluids or an ocean, where the evolution of the atmosphere is included in the model, while the atmosphere itself is also regarded as an incompressible Newtonian fluid.

2.10 On the other hand, we may simplify the above model of an ocean by assuming the atmosphere to be an inviscid gas. This way it is reasonable to substitute the two transmission conditions (2.9) by the boundary condition

$$-S\nu_\Gamma = \bar{p}\nu_\Gamma + f_\Gamma, \qquad t > 0,\ y \in \Gamma(t), \tag{2.10}$$

where $\bar{p}$ denotes the pressure in the atmosphere and f_Γ has again to be modelled by constitutive equations to take care of effects like surface tension or surface viscosity. Hence, we may neglect the evolution of the atmosphere and exclude it from the model, which turns the boundary Γ into a *free surface*.

2.11 As may be expected, the mathematical treatment of problems including free surfaces and free interfaces is as difficult as their modelling. Nearly all approaches rely on a transformation of the problem to a fixed domain with a fixed boundary, which in case of the boundary condition (2.10) leads to model problems with an inhomogeneous *Neumann condition*

$$-S\nu_\Gamma = h \qquad \text{on } J \times \Gamma. \tag{2.11}$$

2.4 Artificial Boundaries

2.12 A third kind of boundaries occur, if the model describes only a part of a larger system. In this case *artificial boundaries* like the inflow and outflow boundaries in the figure on page 3 appear. Now, the transmission conditions (2.1) do not contain any useful information. Indeed, the density, the viscosity, the velocity and the pressure are continuous across the boundary Γ, since it "separates" the fluid from itself. Therefore, it is not at all obvious how *artificial boundary conditions* for such situations should be modelled. In fact, this is an active field of research in mathematics as well as in engineering science.

2.13 For an inflow boundary one may for example assume that the fluid enters the domain with a velocity profile, which is assumed to be known. This leads to an inhomogeneous *Dirichlet Condition*

$$u = h \qquad \text{on } J \times \Gamma, \tag{2.12}$$

where the right hand side h encodes the desired profile. Note that this condition is the inhomogeneous version of the no-slip condition (2.7).

2.14 In case of an outflow boundary as shown in the figure on page 3 it is completely unclear, how a reasonable boundary condition should be obtained. Ideally, such an outflow condition should, on one hand, lead to a well-posed problem and, on the other hand, ensure the unique solution to coincide with the solution to the model problem of an infinitely extended tube. Unfortunately, no such *transparent boundary condition* is available, if the model is restricted to the tube of finite length. This is one of the reasons, why we want to present a self-contained approach to derive a class of boundary conditions, which may especially be used at artificial boundaries. Their derivation is based on energy balances.

2.5 Energy Preserving Boundary Conditions

2.15 To derive the class of boundary conditions we have in mind we first observe that the *kinetic energy balance*

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \tfrac{1}{2}\rho|u|^2 \,\mathrm{d}\mathcal{H}^3 + 2\mu\int_\Omega |D|^2 \,\mathrm{d}\mathcal{H}^3 &= \int_\Omega \rho\partial_t \tfrac{1}{2}(u\cdot u)\,\mathrm{d}\mathcal{H}^3 + \int_\Omega \nabla u : 2\mu D\,\mathrm{d}\mathcal{H}^3 \\
&= \int_\Omega u\cdot\rho\partial_t u\,\mathrm{d}\mathcal{H}^3 + \int_\Omega \nabla u : S\,\mathrm{d}\mathcal{H}^3 \\
&= \int_\Omega u\cdot\big(\rho\partial_t u + \operatorname{div}(\rho u\otimes u - S)\big)\,\mathrm{d}\mathcal{H}^3 \\
&\qquad + \int_\Gamma (u\cdot\nu_\Gamma)\tfrac{1}{2}\rho|u|^2\,\mathrm{d}\mathcal{H}^2 + \int_\Gamma u\cdot S\nu_\Gamma\,\mathrm{d}\mathcal{H}^2 \\
&= \int_\Gamma (u\cdot\nu_\Gamma)\tfrac{1}{2}\rho|u|^2\,\mathrm{d}\mathcal{H}^2 + \int_\Gamma u\cdot S\nu_\Gamma\,\mathrm{d}\mathcal{H}^2
\end{aligned}$$

is valid for incompressible Newtonian flows, if we assume the absence of any driving forces, i. e. if we assume $f = 0$ in the Navier-Stokes equations. Note that we used

$$2\mu|D|^2 = 2\mu D : D = 2\mu(\tfrac{1}{2}(\nabla u + \nabla u^\mathsf{T}) : D) = 2\mu(\nabla u : D) = \nabla u : 2\mu D$$

based on the symmetry of D, which implies $\nabla u : D = \nabla u^\mathsf{T} : D$. Moreover, we used the incompressibility constraint, which implies $\nabla u : p = (\operatorname{div} u)\, p = 0$ as well as

$$\tfrac{1}{2}\operatorname{div}\big\{(\rho u \otimes u)u\big\} = \nabla u : (\rho u \otimes u).$$

The remaining equalities may then be obtained by partial integration as

$$\begin{aligned}
\int_\Omega u \cdot \operatorname{div}(\rho u \otimes u)\, \mathrm{d}\mathcal{H}^3 &= \int_\Gamma (u \cdot \nu_\Gamma)\rho|u|^2\, \mathrm{d}\mathcal{H}^2 - \int_\Omega \nabla u : (\rho u \otimes u)\, \mathrm{d}\mathcal{H}^3 \\
&= \int_\Gamma (u \cdot \nu_\Gamma)\rho|u|^2\, \mathrm{d}\mathcal{H}^2 - \tfrac{1}{2}\int_\Omega \operatorname{div}\big\{(\rho u \otimes u)u\big\}\, \mathrm{d}\mathcal{H}^3 \\
&= \int_\Gamma (u \cdot \nu_\Gamma)\tfrac{1}{2}\rho|u|^2\, \mathrm{d}\mathcal{H}^2
\end{aligned}$$

and

$$\int_\Omega \nabla u : S\, \mathrm{d}\mathcal{H}^3 = \int_\Gamma u \cdot S\nu_\Gamma\, \mathrm{d}\mathcal{H}^2 - \int_\Omega u \cdot \operatorname{div} S\, \mathrm{d}\mathcal{H}^3.$$

Of course, we assume the velocity field, the pressure and the domain to be sufficiently regular to allow for the above manipulations of the equations.

2.16 Now, according to the kinetic energy balance derived in 2.15, the rate of change of total kinetic energy plus the loss of kinetic energy due to internal friction are given as the power

$$\pi_{\mathrm{NF}} := \int_\Gamma \Big\{u \cdot S\nu_\Gamma - (u \cdot \nu_\Gamma)\, \tfrac{1}{2}\rho|u|^2\Big\}\, \mathrm{d}\mathcal{H}^2,$$

which changes the total amount of kinetic energy of the system via the boundary. A boundary condition that implies this contribution via the boundary to vanish may therefore be considered as an *energy preserving boundary condition* for incompressible Newtonian flows. In the case of a local boundary condition given by a linear operator $\mathcal{B}$ this contribution surely vanishes, if

$$\begin{aligned}
&\mathcal{B}(u,\, p) = 0 \quad \text{on } J \times \Gamma \\
&\qquad \Rightarrow\; u \cdot \nu_\Gamma = 0,\; u \cdot S\nu_\Gamma = 0 \quad \text{on } J \times \Gamma.
\end{aligned} \tag{2.13a}$$

2.17 The energy preserving boundary conditions that are subject to the constraint (2.13a) always ensure the total kinetic energy to be monotonically decreasing as they imply the rate of its change to equal its loss due to internal friction. However, due to the incompressibility condition we have

$$\operatorname{div} D = \operatorname{div} R = \tfrac{1}{2}\Delta u, \qquad \operatorname{div} S = \operatorname{div} V = \mu\Delta u - \nabla p,$$

where $R = \frac{1}{2}(\nabla u - \nabla u^\mathsf{T})$ denotes the rate of rotation tensor and $V := 2\mu R - p$ denotes the antisymmetric counterpart of the stress tensor S. Hence, the *alternative form*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \tfrac{1}{2}\rho|u|^2\,\mathrm{d}\mathcal{H}^3 + 2\mu\int_\Omega |R|^2\,\mathrm{d}\mathcal{H}^3 = \int_\Gamma (u\cdot\nu_\Gamma)\tfrac{1}{2}\rho|u|^2\,\mathrm{d}\mathcal{H}^2 + \int_\Gamma u\cdot V\nu_\Gamma\,\mathrm{d}\mathcal{H}^2$$

of the kinetic energy balance is also available. Note that this alternative form may be obtained as in 2.15. Here we use

$$2\mu|R|^2 = 2\mu(\,R:R\,) = 2\mu(\tfrac{1}{2}(\nabla u - \nabla u^\mathsf{T}):R) = 2\mu(\nabla u:R) = \nabla u : 2\mu R$$

based on the antisymmetry of R, which implies $R:\nabla u = -R:\nabla u^\mathsf{T}$.

2.18 Therefore, the rate of change of total kinetic energy together with the loss of kinetic energy due to rotation are given as the power

$$\bar{\pi}_{\mathrm{NF}} := \int_\Gamma \Big\{u\cdot V\nu_\Gamma - (u\cdot\nu_\Gamma)\,\tfrac{1}{2}\rho|u|^2\Big\}\,\mathrm{d}\mathcal{H}^2,$$

which changes the total amount of kinetic energy of the system via the boundary. Thus, the vanishing of this contribution also implies the total kinetic energy to be monotonically decreasing and we want to consider any boundary condition that ensures this contribution to vanish as an energy preserving boundary condition for incompressible Newtonian flows, too. In the case of a local boundary condition given by a linear operator $\mathcal{B}$ the contribution above surely vanishes, if

$$\text{(2.13b)}\qquad \begin{aligned} &\mathcal{B}(u,\,p) = 0 \quad \text{on } J\times\Gamma \\ &\qquad\Rightarrow\; u\cdot\nu_\Gamma = 0,\; u\cdot V\nu_\Gamma = 0 \quad \text{on } J\times\Gamma. \end{aligned}$$

As will be revealed below there are several boundary conditions satisfying the constraint (2.13b), which lead to physically reasonable and mathematically interesting models, and, which have been treated recently in an analytically rigorous way, see [BKP12].

2.19 To develop a first impression of how local, linear boundary conditions satisfying (2.13a) or (2.13b) may be constructed we decompose the inner product into a tangential and a normal part according to

$$u\cdot S\nu_\Gamma = P_\Gamma u\cdot P_\Gamma S\nu_\Gamma + (u\cdot\nu_\Gamma)(S\nu_\Gamma\cdot\nu_\Gamma),$$

An analogous decomposition is valid for $u\cdot V\nu_\Gamma$. Due to

$$P_\Gamma S\nu_\Gamma = 2\mu P_\Gamma D\nu_\Gamma \qquad \text{and} \qquad P_\Gamma V\nu_\Gamma = 2\mu P_\Gamma R\nu_\Gamma = 2\mu R\nu_\Gamma$$

there are three directly arising energy preserving boundary conditions satisfying one of the constraints (2.13a) or (2.13b), which read

$$(\mathrm{B}\,|\,a,\,\Omega)_0^{0,0}\qquad \begin{aligned} P_\Gamma u &= 0 \quad \text{on } J\times\partial\Omega, \\ u\cdot\nu_\Gamma &= 0 \quad \text{on } J\times\partial\Omega, \end{aligned}$$

which equals the no-slip condition (2.7),

$$(\mathrm{B}\,|\,a,\,\Omega)_0^{+1,0} \qquad \begin{aligned} 2\mu P_\Gamma D\nu_\Gamma &= 0 \quad \text{on } J\times\partial\Omega, \\ u\cdot\nu_\Gamma &= 0 \quad \text{on } J\times\partial\Omega, \end{aligned}$$

which equals the free slip condition (2.8), and, finally,

$$(\mathrm{B}\,|\,a,\,\Omega)_0^{-1,0} \qquad \begin{aligned} 2\mu R\nu_\Gamma &= 0 \quad \text{on } J\times\partial\Omega, \\ u\cdot\nu_\Gamma &= 0 \quad \text{on } J\times\partial\Omega, \end{aligned}$$

which corresponds to the prescription of the vorticity and the normal velocity. Note that we use tags with two upper indices, since the following chapters will refer to these conditions as $(\mathrm{B}\,|\,\cdot,\,\cdot)_{\cdot}^{\alpha,\beta}$ with suitable parameters $\alpha,\,\beta\in\{-1,\,0,\,+1\}$. The lower index will be used to prescribe a suitable right hand side.

2.20 In the case of a Stokes flow, which is governed by the Stokes equations

$$\rho\partial_t u - \operatorname{div} S = \rho f, \qquad S = 2\mu D - p, \qquad \operatorname{div} u = 0 \qquad \text{in } J\times\Omega,$$

analogous considerations as in 2.15 and 2.17 result in the kinetic energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \tfrac{1}{2}\rho|u|^2\,\mathrm{d}\mathcal{H}^3 + 2\mu\int_\Omega |D|^2\,\mathrm{d}\mathcal{H}^3 = \int_\Gamma u\cdot S\nu_\Gamma\,\mathrm{d}\mathcal{H}^2,$$

which implies the rate of change of total kinetic energy together with its loss due to internal friction to equal the power

$$\pi_{\mathrm{SF}} := \int_\Gamma u\cdot S\nu_\Gamma\,\mathrm{d}\mathcal{H}^2,$$

and the alternative form of the kinetic energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \tfrac{1}{2}\rho|u|^2\,\mathrm{d}\mathcal{H}^3 + 2\mu\int_\Omega |R|^2\,\mathrm{d}\mathcal{H}^3 = \int_\Gamma u\cdot V\nu_\Gamma\,\mathrm{d}\mathcal{H}^2,$$

which implies the rate of change of total kinetic energy together with its loss due to rotation to equal the power

$$\bar\pi_{\mathrm{SF}} := \int_\Gamma u\cdot V\nu_\Gamma\,\mathrm{d}\mathcal{H}^2.$$

2.21 In analogy to the discussion in 2.16 and 2.18 we will consider a boundary condition that ensures one of the contributions π_{SF} or $\bar\pi_{\mathrm{SF}}$ to vanish as an energy preserving boundary condition for incompressible Newtonian Stokes flows, since these boundary conditions imply the total kinetic energy to be monotonically decreasing. In the case of a local boundary condition given by a linear operator $\mathcal{B}$ we therefore require

$$\text{(2.13c)} \qquad \mathcal{B}(u,\,p) = 0 \quad \text{on } J\times\Gamma \qquad \Rightarrow \qquad \begin{aligned} u\cdot S\nu_\Gamma = 0 \text{ on } J\times\Gamma \\ \text{or } u\cdot V\nu_\Gamma = 0 \text{ on } J\times\Gamma, \end{aligned}$$

which is a weaker condition than (2.13a) respectively (2.13b).

2.22 To develop a first impression of how local, linear boundary conditions satisfying (2.13c) may be constructed we again decompose the inner product into a tangential and a normal part. Due to

$$\begin{aligned} S\nu_\Gamma \cdot \nu_\Gamma &= 2\mu D\nu_\Gamma \cdot \nu_\Gamma - p = 2\mu\,\partial_\nu u \cdot \nu_\Gamma - p \\ \text{and} \quad V\nu_\Gamma \cdot \nu_\Gamma &= 2\mu R\nu_\Gamma \cdot \nu_\Gamma - p = -p \end{aligned}$$

there are, in addition to the energy preserving boundary conditions derived in 2.19, four directly arising energy preserving boundary conditions for Stokes flows satisfying the constraint (2.13c). These read

$$(\mathrm{B}\,|\,a,\,\Omega)_0^{0,+1} \qquad \begin{aligned} P_\Gamma u &= 0 \quad \text{on } J \times \partial\Omega, \\ 2\mu\,\partial_\nu u \cdot \nu_\Gamma - p &= 0 \quad \text{on } J \times \partial\Omega, \end{aligned}$$

which corresponds to the prescription of tangential velocities and the normal component of normal stress,

$$(\mathrm{B}\,|\,a,\,\Omega)_0^{+1,+1} \qquad \begin{aligned} 2\mu P_\Gamma D\nu_\Gamma &= 0 \quad \text{on } J \times \partial\Omega, \\ 2\mu\,\partial_\nu u \cdot \nu_\Gamma - p &= 0 \quad \text{on } J \times \partial\Omega, \end{aligned}$$

which equals the Neumann condition (2.11),

$$(\mathrm{B}\,|\,a,\,\Omega)_0^{0,-1} \qquad \begin{aligned} P_\Gamma u &= 0 \quad \text{on } J \times \partial\Omega, \\ -\,p &= 0 \quad \text{on } J \times \partial\Omega, \end{aligned}$$

which corresponds to the prescription of tangential velocities and the external pressure, and, finally,

$$(\mathrm{B}\,|\,a,\,\Omega)_0^{-1,-1} \qquad \begin{aligned} 2\mu R\nu_\Gamma &= 0 \quad \text{on } J \times \partial\Omega, \\ -\,p &= 0 \quad \text{on } J \times \partial\Omega, \end{aligned}$$

which corresponds to the prescription of the vorticity and the external pressure.

2.6 Vorticity and Pressure Boundary Conditions

2.23 The boundary conditions introduced in 2.19 and 2.22 are constructed based on the constraint that there should be no contribution to the kinetic energy of the system via the boundary. On the other hand, all those conditions have a similar structure, since they are all decomposed into a tangential part and a normal part based on the velocity u, the stress tensor S and its antisymmetric counterpart V. However, the two boundary conditions

$$(\mathrm{B}\,|\,a,\,\Omega)_0^{+1,-1} \qquad \begin{aligned} 2\mu P_\Gamma D\nu_\Gamma &= 0 \quad \text{on } J \times \partial\Omega, \\ -\,p &= 0 \quad \text{on } J \times \partial\Omega, \end{aligned}$$

which corresponds to the prescription of the tangential part of the normal deformation rate and the external pressure, and

$$(\mathrm{B}\,|\,a,\,\Omega)_0^{-1,+1} \qquad \begin{aligned} 2\mu R\nu_\Gamma &= 0 \quad \text{on } J \times \partial\Omega, \\ 2\mu\,\partial_\nu u \cdot \nu_\Gamma - p &= 0 \quad \text{on } J \times \partial\Omega, \end{aligned}$$

which corresponds to the prescription of the vorticity and the normal part of the normal stress, are not included in Section 2.5, because they do not directly arise from an energy balance.

2.24 As a consequence, their influence on the kinetic energy of the system may not be predicted. Both conditions may cause a contribution to or a consumption of kinetic energy via the boundary and may therefore not be considered as energy preserving boundary conditions, even in the case of a Stokes flow. However, we want to keep the well-posedness and maximal regularity results obtained in the next chapters as general as possible, since all of the conditions $(\mathrm{B}\,|\,\cdot,\,\cdot)_\cdot^{\alpha,\beta}$ with $\alpha,\,\beta \in \{-1,\,0,\,+1\}$, may prove useful in several situations, e. g. as model problems or as artificial boundary conditions. Therefore, all of these conditions are covered by our analysis.

Remarks

2.25 The tags of the boundary conditions derived in 2.19, 2.22 and 2.23 are explained as follows: A boundary condition $(\mathrm{B}\,|\,\cdot,\,\cdot)_\cdot^{0,\cdot}$ always includes a tangential part of order zero, which prescribes the tangential velocity. A boundary condition $(\mathrm{B}\,|\,\cdot,\,\cdot)_\cdot^{\pm 1,\cdot}$ always includes a tangential part of order one, which either prescribes the tangential part of the normal deformation rate or the vorticity via the prescription of

$$\mu P_\Gamma(\nabla u \pm \nabla u^\mathsf{T})\nu_\Gamma.$$

Analogously, a boundary condition $(\mathrm{B}\,|\,\cdot,\,\cdot)_\cdot^{\cdot,0}$ always includes a normal part of order zero, which prescribes the normal velocity. A boundary condition $(\mathrm{B}\,|\,\cdot,\,\cdot)_\cdot^{\cdot,\pm 1}$ always includes a normal part of order one, which either prescribes the normal part of the normal stress or the pressure via the prescription of

$$\mu(\nabla u \pm \nabla u^\mathsf{T})\nu_\Gamma \cdot \nu_\Gamma - p.$$

Note that

$$\mu(\nabla u + \nabla u^\mathsf{T})\nu_\Gamma \cdot \nu_\Gamma - p = 2\mu\,\partial_\nu u \cdot \nu_\Gamma - p$$

and

$$\mu(\nabla u - \nabla u^\mathsf{T})\nu_\Gamma \cdot \nu_\Gamma - p = -p.$$

2.26 The no-slip condition (2.7), the perfect-slip condition (2.8), the Neumann condition (2.11) as well as their inhomogeneous versions like e. g. the Dirichlet condition (2.12) are included in the family of (energy preserving) boundary conditions $(\mathrm{B}\,|\,a,\,\Omega)_\cdot^{\alpha,\beta}$ via suitable choices of $\alpha,\,\beta \in \{-1,\,0,\,+1\}$.

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