

Chapter 2

Image Reconstruction in MRI

WHEREAS the first chapter has described the basics of MRI signal formation, the present chapter deals with the reverse process: the problem of recovering the object information from the acquired signals. This is a difficult task because MRI measurements only indirectly represent the object, and there is only a finite amount of measurement data available for the vast amount of object information.

This inherently inverse problem has been solved for standard multi-coil acquisitions with a rigorous mathematical framework for linear image reconstruction [135]. In the preface, it has already been mentioned that conventional multi-coil imaging has more stringent imaging requirements than PatLoc because, conventionally, field linearity is required for gradient encoding; notwithstanding, the rigorous mathematical framework can easily be extended to be applicable also to PatLoc imaging (see chapter 4.2, page 140ff). Therefore this linear reconstruction framework is important for this thesis and it is reviewed at the beginning of the present chapter. The most important image reconstruction methods for single-coil imaging as well as multi-coil imaging are derived in this chapter using the same abstract framework.

Though this approach is more technical than standard descriptions of the different algorithms, it has the advantage of a unified portrayal of some of the most important reconstruction methods currently used in MRI. As a consequence, relations between individual methods can be elaborated (such as between SENSE and GRAPPA or between gridding reconstruction and the general matrix inversion approaches). The presentation tries to use a mathematical language which is as precise as possible. One consequence is for example the description of the SENSE reconstruction matrix with the help of the Kronecker product. New is also the explanation of the superresolution effect for certain PI reconstructions.

This chapter focuses on the essentials of MR image reconstruction. Not covered are dynamic imaging modalities like for example cardiac imaging, where several frames are recorded within the cardiac cycle (see e.g. [176]). A

good description of spatio-temporal reconstruction methods is found in [5]. Also not covered are nonlinear reconstruction methods. Such methods have proven advantageous in special situations like for example reconstruction from sparse data [28, 106] or reconstruction from subsampled radial imaging data [13, 83]. A problem with such nonlinear algorithms is that image properties are not easy to predict. This is different for linear reconstruction methods, where concrete results can be derived. Image properties like image resolution, aliasing and SNR can be calculated explicitly, and this is done in the present chapter for the general case and some of the discussed image reconstruction methods.

2.1 Basics of Linear Image Reconstruction

Linear image reconstruction is particularly beneficial because reconstruction can be described as a simple matrix-vector operation. Image reconstruction involves the inversion of the *encoding matrix*, which comprises the relevant information of the imaging process. With known gradient encoding scheme and RF-coil sensitivity data, the encoding matrix can easily be calculated. After presentation of the image reconstruction framework, a short section is devoted to the problem of obtaining reliable data from gradient and RF sensitivity encoding. After that, it is shown how basic image properties like image resolution, aliasing artifact and SNR can be calculated if reconstructed with the presented matrix approach.

2.1.1 Fundamental Reconstruction Algorithms

The basic principles of linear reconstruction theory for standard PI [135] are reviewed below before special imaging situations are discussed in later sections. The reconstruction theory is based on the signal equation presented in Eq. 1.31, which describes the general imaging process of PI. It is repeated here:

$$s_\alpha(\vec{k}_\kappa) = \int_V m(\vec{x}) c_\alpha(\vec{x}) e^{-i\vec{k}_\kappa \cdot \vec{x}} d\vec{x} \quad \text{for all } \alpha = 1, \dots, N_c. \quad (2.1)$$

The encoding basis for parallel imaging at measured k -space positions \vec{k}_κ ($\kappa = 1, \dots, N_\kappa$) is defined as:

$$\text{enc}_{\alpha, \kappa}(\vec{x}) = c_\alpha(\vec{x})e^{-i\vec{k}_\kappa \vec{x}}. \quad (2.2)$$

With this definition signal formation can be interpreted as the projection of the magnetization onto the encoding functions:

$$s_{\alpha, \kappa} := s_\alpha(\vec{k}_\kappa) = \int_V m(\vec{x}) \text{enc}_{\alpha, \kappa}(\vec{x}) d\vec{x}. \quad (2.3)$$

The linear relationship between signal and magnetization favors linear reconstruction methods, where the magnetization, collected in a vector \mathbf{m} of length N_ρ ¹ is reconstructed by adequately weighting the signal measurements $s_{\alpha, \kappa}$. The reconstruction is therefore described by a matrix \mathbf{F} , often termed *reconstruction matrix*:

$$\mathbf{m} = \mathbf{F}\mathbf{s}. \quad (2.4)$$

The reconstruction problem can then be formulated as finding a reconstruction \mathbf{F} , which produces a magnetization vector \mathbf{m} with elements m_ρ that approximate the magnetization at the corresponding position as closely as possible:

$$m_\rho = \sum_{\alpha, \kappa} F_{\rho, (\alpha, \kappa)} s_{\alpha, \kappa} \approx \int_V m(\vec{x}) i_\rho(\vec{x}) d\vec{x}. \quad (2.5)$$

The reconstructed values m_ρ might represent the total magnetization within the voxel of interest or the average density of the magnetization within the voxel. In the following, it is assumed that the goal of the reconstruction is the latter. The right hand side of Eq. 2.5 is the desired value for m_ρ . It depends on the chosen ideal voxel shape i_ρ . Often Dirac delta functions are chosen as ideal voxel shapes. This choice simplifies the involved calculations. Even though box functions are a better representation of image voxels it is usually acceptable to use delta functions because, typically, reconstruction grids are

¹The one-to-one procedure of mapping a matrix to a vector is often denoted as *vectorization*. The inverse mapping from the vector back to the matrix is termed in this thesis *de-vectorization*.

chosen not coarser than the encoded image resolution. Inserting Eq. 2.3 into the latter equation yields:

$$m_\rho = \int_V m(\vec{x}) f_\rho(\vec{x}) d\vec{x}. \quad (2.6)$$

where the voxel function f_ρ is given by:

$$f_\rho(\vec{x}) = \sum_{\alpha, \kappa} F_{\rho, (\alpha, \kappa)} \text{enc}_{\alpha, \kappa}(\vec{x}). \quad (2.7)$$

The voxel function f_ρ can also be interpreted as the *spatial response function*. The spatial response function and its relationship to the point spread function is explained in section 2.1.3, page 50ff. The reconstruction problem is thus reduced to finding good approximations of the voxel functions to the ideal voxel shapes:

$$f_\rho(\vec{x}) \approx i_\rho(\vec{x}).$$

Two different approaches are considered here. These approaches have been termed *weak* and *strong* reconstructions in [135].

a) Weak Matrix Approach

The weak approach only requires that ideal voxel shapes and voxel functions satisfy the orthogonality relation:²

$$\int_V i_\rho^*(\vec{x}) f_{\rho'}(\vec{x}) d\vec{x} = (\Delta V)^{-1} \delta_{\rho, \rho'}. \quad (2.8)$$

In chapter 4.2.2b, page 148ff, it is shown that the quantity ΔV represents the nominal voxel volume.³ For 2D imaging, it is given by $(\Delta x)^2$, where

²If the goal of the reconstruction is to find the total magnetization within the reconstructed voxels, $(\Delta V)^{-1}$ must be replaced by its inverse ΔV .

³This dependency on the voxel volume has not been observed in [135]. For standard rectilinear reconstruction grids, this dependency on the voxel volume can usually be ignored because the diagnostic value of MR images lies in *relative* intensity differences. Note however, that, without the introduction of the voxel volume in the latter equation, the physical units of that equation are no longer consistent. Under certain circumstances, the dependency on the voxel volume becomes important in PatLoc imaging, for example in chapter 5.1, page 155ff.

Δx is the discretization distance of the reconstruction grid.⁴ The entries of the encoding matrix are defined as:

$$E_{(\alpha,\kappa),\rho} := \int_V i_{\rho}^*(\vec{x}) \text{enc}_{\alpha,\kappa}(\vec{x}) d\vec{x} = \text{enc}_{\alpha,\kappa}(\vec{x}_{\rho}) = c_{\alpha}(\vec{x}_{\rho}) e^{-i\vec{k}_{\kappa}\vec{x}_{\rho}}. \quad (2.9)$$

In the latter equation the ideal voxel shapes were assumed to be delta functions ($i_{\rho}^*(\vec{x}) := \delta(\vec{x} - \vec{x}_{\rho})$). With the definitions of voxel functions and encoding matrix the orthogonality relation of Eq. 2.8 reduces to a matrix equation:

$$\mathbf{F}\mathbf{E} = (\Delta V)^{-1}\mathbf{1}. \quad (2.10)$$

Solving this matrix equation for the reconstruction matrix \mathbf{F} solves the reconstruction problem. Typically, the Moore-Penrose pseudo-inverse (MPPI) is taken as the solution:

$$\mathbf{F} = (\Delta V)^{-1}\mathbf{E}^+. \quad (2.11)$$

Note that the MPPI has different interpretations under different circumstances. Three different situations may occur:

1. Equation 2.10 has infinitely many solutions. This is generally the case when $N_c N_{\kappa} \geq N_{\rho}$, i.e., when a reconstruction grid is chosen which is not much finer than the corresponding grid of acquired data points.⁵ The MPPI then takes the solution with the smallest Euclidian norm. Problems like for example an underestimation of the spin density may therefore occur when the reconstruction grid is chosen too coarsely. The explicit solution then reads:

$$\mathbf{F} = (\Delta V)^{-1}(\mathbf{E}^H \mathbf{E})^{-1}\mathbf{E}^H. \quad (2.12)$$

⁴For 2D imaging it would be more precise to talk about *pixels* rather than *voxels*. In this thesis this imprecise terminology is accepted in favor of a unified presentation suited for 2D as well as 3D imaging. Also consider that, in reality, a 2D slice has a finite thickness. The discrepancy of calling quantities like $(\Delta x)^2$ “volumes” is therefore also resolved by simply multiplying such areas with the slice thickness.

⁵For $N_c N_{\kappa} \geq N_{\rho}$, the condition $\mathbf{F}\mathbf{E} = (\Delta V)^{-1}\mathbf{1}$ represents an *underdetermined* system of equations. Notwithstanding, this situation is usually considered as an *overdetermined* acquisition in the MRI literature because the amount of signal data exceeds the number of image voxels to be solved for. This terminology is also justified because the corresponding discretized signal equation $\mathbf{s} = \mathbf{E}\mathbf{m}$ represents an *overdetermined* system of equations, used for example for iterative CG reconstructions, cf. section 2.3.1f, page 89ff. In order to avoid confusion in this regard, only rare use of the terms *overdetermined/underdetermined* is made in this thesis.

2. No solution exists. This is the case when a very fine reconstruction grid is chosen. The MPPI is then the minimizer of a least-squares problem $\|[\Delta V \cdot \mathbf{F}\mathbf{E} - \mathbb{1}]\mathbf{W}\|_F^2$, where the diagonal matrix \mathbf{W} is a weighting function and the subscript F denotes the Frobenius matrix norm. Equal weighting for all image voxels is ensured if \mathbf{W} is chosen to be the unity matrix. Then the explicit solution is given by:

$$\mathbf{F} = (\Delta V)^{-1}\mathbf{E}^H(\mathbf{E}\mathbf{E}^H)^{-1}.$$

3. There is exactly one solution. This is a very special case. In this case, the two explicit solutions stated above are equivalent: $\mathbf{F} = (\Delta V)^{-1}(\mathbf{E}^H\mathbf{E})^{-1}\mathbf{E}^H = (\Delta V)^{-1}\mathbf{E}^H(\mathbf{E}\mathbf{E}^H)^{-1}$.

The solutions to the three cases rely on the property that either the matrix $\mathbf{E}^H\mathbf{E}$ or $\mathbf{E}\mathbf{E}^H$ is invertible. Invertibility is ensured only if \mathbf{E} has full rank. This can, however, not always be guaranteed. One example would be an image acquisition, where some k -space locations are sampled several times and subsequent reconstruction is performed onto a dense grid. It is therefore necessary to use a reconstruction, which can cope with this potential problem. And the MPPI can! To show this, consider the singular value decomposition (SVD) of a matrix \mathbf{A} : $\mathbf{A} = \mathbf{P}\mathbf{\Sigma}\mathbf{Q}^H$, where \mathbf{P} and \mathbf{Q} are unitary and $\mathbf{\Sigma}$ is a potentially non-square matrix with entries only along the main diagonal. These entries form the singular values of \mathbf{A} . Then, the MPPI of \mathbf{A} is given by: $\mathbf{A}^+ = \mathbf{Q}\mathbf{\Sigma}^+\mathbf{P}^H$. The MPPI solution $\mathbf{\Sigma}^+$ is simply defined as the transpose of $\mathbf{\Sigma}$ with inverted singular values. In [170] it is recalled that the MPPI of a matrix \mathbf{A} has two different kinds of expansions, which are mathematically equivalent to the MPPI⁶:

$$\mathbf{A}^+ = (\mathbf{A}^H\mathbf{A})^+\mathbf{A}^H = \mathbf{A}^H(\mathbf{A}\mathbf{A}^H)^+.$$

The first expansion reduces to the first case above if $\mathbf{E}^H\mathbf{E}$ is invertible and the second expansion reduces to the second case above if $\mathbf{E}\mathbf{E}^H$ is invertible. These equivalences show that all discussed cases are covered by the MPPI.

Remark: In the MRI literature, one often encounters reconstructions based on the discretization of the forward model using a Riemann sum approximation. The resulting equation $\mathbf{s} = \mathbf{E}\mathbf{m}$ is then solved using the MPPI with solution $\mathbf{m} = \mathbf{E}^+\mathbf{s}$. Such a “forward” approach is problematic because

⁶That result can be proved easily with the help of the SVD of \mathbf{A} .

reconstruction solves an inverse problem; however, the discussion above shows that the forward discretization is justified because the more sophisticated inverse approach has the same solution under the conditions of (a) weak reconstruction (b) delta functions as ideal voxel shapes (c) Cartesian reconstruction grids.⁷

b) Strong Matrix Approach

In the strong reconstruction approach voxel shapes are chosen to represent the least-squares approximation to the ideal voxel shapes. In Appendix B in [135] it is shown that this approach results, with $\Delta V := 1$, in the solution:

$$\mathbf{F} = \mathbf{E}^H \mathbf{B}^+, \quad (2.13)$$

where \mathbf{B} is the correlation matrix of the encoding functions:

$$B_{(\alpha, \kappa), (\alpha', \kappa')} = \int_V \text{enc}_{\alpha, \kappa}(\vec{x}) \text{enc}_{\alpha', \kappa'}^*(\vec{x}) d\vec{x}. \quad (2.14)$$

and V is the volume over which the least-squares approximation of the voxel function to the ideal voxel shape is calculated. For comparison, the concepts of weak and strong reconstruction are illustrated in Fig. 2.1.

Remark: If additional factors are added to the encoding functions, defined in Eq. 2.2, this formalism can incorporate effects such as relaxation, field inhomogeneities due to non-uniform B_0 -field, susceptibility differences, or, in a slightly modified form, chemical shift imaging. Interesting in the context of this thesis is also that RF pulses can be designed to influence the phase of the magnetization. This effect can also be considered with an additional factor in the encoding functions. In chapter 4 it is shown that this formalism can also be used in PatLoc imaging with a modified encoding matrix.

c) Relationship Between Weak and Strong Reconstruction

Both reconstruction methods have in common that the solution is determined from a comparison with an ideal reconstruction. The strong reconstruction aims at approximating the ideal voxel shapes in a least-squares

⁷This approach of discretizing the forward model prior to inversion is also justified for non-Cartesian reconstruction grids. The discretization (see Eq. 4.25) results in Eq. 4.20.

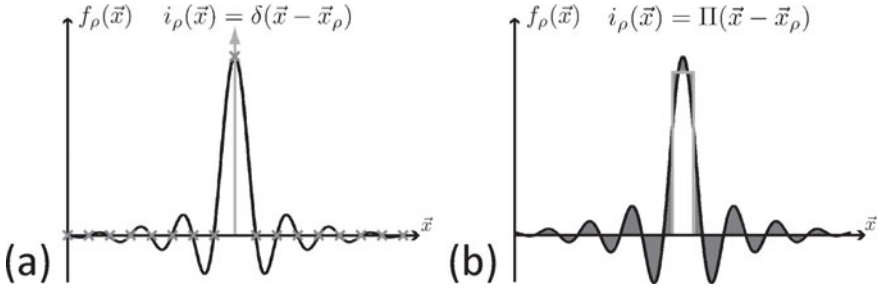


Figure 2.1: The concepts of weak and strong reconstruction. Illustrated are voxel functions (black) and ideal voxel shapes (gray); in the shown example, a delta function ($\delta(\cdot)$) has been chosen as ideal voxel shape for the weak approach, and a box function ($\Pi(\cdot)$) for the strong approach. With both approaches a voxel function is sought that resembles the ideal voxel shape as closely as possible. (a) The weak reconstruction requires that, for a delta function as ideal voxel shape, the voxel function is unity at the voxel center and zero at the centers of the neighboring voxels. No conditions are imposed on the behavior in-between. (b) The strong reconstruction aims at minimizing the least-squares deviation from the ideal voxel shape; thus, a reconstruction is chosen which minimizes the (square of the) gray-shaped area. Close inspection shows that, in the depicted example, both approaches lead to voxel functions which differ only very slightly from one another.

sense, whereas the weak reconstruction only requires that the voxel function is defined at a finite number of grid points.⁸ For the weak reconstruction, no condition is stated for what happens in between the grid points. The strong approach is therefore more convincing and should in general provide higher reliability than the weak approach.

But also the weak approach normally leads to reliable reconstructions: The reconstruction grid is typically chosen dense enough to avoid loss of acquired image information. Correspondingly, the encoding functions vary smoothly on a voxel scale and high amplitude variations of the voxel function are unlikely between the undefined grid points. This is particularly true when $\mathbf{F}\mathbf{E} = \mathbf{1}$ can be fulfilled. If not (i.e., for dense reconstruction grids) the minimum-norm solution serves a similar purpose.

Both approaches are in fact closely related to each other. In the limit of infinitely dense reconstruction grids both methods are equivalent because in this case $\mathbf{B} = \mathbf{E}\mathbf{E}^H$ and therefore the strong approach has the recon-

⁸At least when delta functions are chosen as ideal voxel shapes.

struction $\mathbf{F} = \mathbf{E}^H(\mathbf{E}\mathbf{E}^H)^+$, which corresponds to the MPPI solution of the weak reconstruction. This perfect congruence of the weak and the strong approach is technically broken, once realistic reconstruction grids of finite density are considered. However, note that in most cases the correlation matrix \mathbf{B} cannot be calculated analytically, but has to be determined by means of numerical integration. If a Riemann-sum is chosen with a step-size corresponding to the voxel size of the (sufficiently dense) reconstruction grid, the correlation matrix is again given by $\mathbf{B} = \mathbf{E}\mathbf{E}^H$, and both, weak and strong reconstruction, yield exactly the same results once again. Closely related to this discussion is the minimum-norm reconstruction presented in [170].

d) Conclusion

With sufficiently dense reconstruction grids, it is appropriate to consider only delta functions as ideal voxel shapes in favor of minimizing reconstruction time. Owing to the fact that in this case, weak and strong approach have similar solutions, the following presentation is oriented toward the solution of the weak approach, unless stated otherwise:

$$\mathbf{F} = \mathbf{E}^+, \quad \text{with} \quad E_{(\alpha, \kappa), \rho} = \text{enc}_{\alpha, \kappa}(\vec{x}_\rho) = c_\alpha(\vec{x}_\rho) e^{-i\vec{k}_\kappa \vec{x}_\rho}, \quad (2.15)$$

where the \mathbf{E} must be replaced by $\Delta V \cdot \mathbf{E}$ if the volume information is of interest. This reconstruction is denoted in this thesis as the *MPPI reconstruction* or *MPPI solution*.

In theory, the MPPI approach is straightforward. Once the encoding matrix is determined, images can be reconstructed by simply inverting this matrix. In practice, however, two issues must be addressed: The first has already been mentioned above and concerns the problem that the large dimensions of the encoding matrix complicate direct inversion. One focus of this chapter is to tackle this inversion problem for different situations. Depending on the structure of the encoding matrix, direct inversion can be an option. If not, iterative methods often lead to acceptable results. The second issue relates to the fact that image reconstruction is based on accurate knowledge of the encoding matrix. This issue is the topic of the following section.

2.1.2 Determination of the Encoding Matrix

The correct determination of the encoding matrix is crucial for image reconstruction. According to Eq. 2.15, this matrix consists of two factors: the gradient encoding factor $e^{-i\vec{k}_\kappa \vec{x}_\rho}$ and the RF-sensitivity encoding factor $c_\alpha(\vec{x}_\rho)$. The determination of these two factors is treated separately here.

a) Gradient Encoding

The gradient encoding part $e^{-i\vec{k}_\kappa \vec{x}_\rho}$ of the encoding matrix is based on two assumptions: First, it is assumed that the trajectory \vec{k}_κ matches the true trajectory. Second, it is assumed that the gradients generate exactly linear encoding fields (cf. Eq. 1.23).

The hardware of state-of-the-art MRI systems is particularly well optimized for accurate control of the k -space trajectories. For many standard imaging sequences and applications it can therefore be assumed that the desired k -space trajectory is accurate. If a certain application requires a higher accuracy, established calibration methods (see e.g. chapter 2.2.4 of [211]) or promising new methods like for example magnetic field monitoring [4] could be considered to improve the reliability of gradient encoding.

Typically less demanding are the requirements on gradient linearity. Linearity is very accurate only at the isocenter of the MRI bore. Outside of the center, non-linearities occur. The non-linearities are often accepted in order to permit improvements in other performance measures such as power consumption or especially switching speed of the gradients. Typical artifacts resulting from gradient non-linearities are image distortions. The distortions can be corrected if the spatial distributions of the magnetic gradient fields are known. The spatial distributions can be measured indirectly with the help of calibration phantoms [149] or directly by acquiring field maps of the gradient fields (cf. chapter 5.1.2c, page 178ff, and 6.2.1b, page 220f; further references may be found in [29]). These methods are particularly important in the context of PatLoc imaging, where strong deviations from gradient linearity are generated intentionally.

b) Sensitivity Encoding

Reconstruction also requires the determination of the complex-valued (cf. Eq. 1.16) spatial distributions $\mathbf{c}(\cdot) := (c_1(\cdot), \dots, c_{N_c}(\cdot))^T$ of the RF-coil

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