

Chapter 1

Introduction

It was in 1897 that J. J. Thomson made experiments which led to the discovery of *the electron*: the first elementary particle discovered in nature. After a century we now understand that all visible material in our Universe is composed of a few kinds of elementary particles, called quarks, leptons and gauge bosons. The Standard Model of particle physics is by now firmly established as *the* theory of elementary particles. It is a gauge theory based on the gauge group $U(1) \times SU(2) \times SU(3)$. While the $U(1) \times SU(2)$ sector takes care of the electromagnetism and the weak force, the $SU(3)$ sector, called *Quantum Chromodynamics (QCD)*, describes the strong interactions that act between quarks and gluons. The Lagrangian of QCD is given by¹

$$\mathcal{L}_{\text{QCD}} = \sum_{f=1}^{N_f} \bar{\psi}_f (i \gamma^\mu D_\mu - m_f) \psi_f - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} \quad (1.1)$$

$$D_\mu \equiv \partial_\mu + i g A_\mu^a T^a, \quad a = 1, 2, \dots, N_c^2 - 1 \quad (1.2)$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c, \quad (1.3)$$

where T^a and f^{abc} denote the generators and the structure constants of $SU(N_c)$, respectively. The integers N_f and N_c specify the number of flavors and colors; in reality $N_f = 6$ and $N_c = 3$. Usually we can neglect the contribution of heavier flavors (c , b , t) so that $N_f = 3$ or even 2 is a good approximation to the reality. Despite its simplicity Eq. (1.1) gives rise to incredibly diverse phenomena, such as the emergence of numerous bound states (pions, Kaons, vector mesons, baryons and atomic nuclei), chiral anomaly, asymptotic freedom, gluon condensate, spontaneous chiral symmetry breaking, and color confinement.

The asymptotic freedom [1, 2] is a property that the gauge coupling g , which depends on the energy scale under consideration as a result of renormalization, flows to zero at asymptotically high energy. This salient feature of QCD provided us with the opportunity to test QCD by making predictions on high-energy scattering

¹ Electromagnetic interactions are neglected.

processes using perturbation theory and comparing with experiments. On the other hand, g grows large at low energy where weak coupling methods are doomed to failure; phenomena such as bound state formation and symmetry breaking never occur within perturbation theory. Instead, we nowadays carry out massive computer simulations of QCD on a discrete lattice (called *lattice QCD*), and thereby succeeded in revealing various non-perturbative aspects of QCD from first principles. It has to be noted, however, that numerical simulations do not necessarily entail a physical understanding of the subject.

Chiral symmetry and its breaking are one of the central ingredients in low-energy QCD. When all quark masses are tuned off, the Lagrangian Eq. (1.1) is invariant under

$$\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_B \times \mathrm{U}(1)_A, \quad (1.4)$$

where $\mathrm{U}(1)_A$ is explicitly broken by chiral anomaly. In vacuum the chiral condensate $\langle \bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R \rangle \simeq - (240 \text{ MeV})^3$ develops, which breaks the symmetry as $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \rightarrow \mathrm{SU}(N_f)_V$. This *spontaneous breaking of chiral symmetry* is still poorly understood from a theoretical perspective owing to the strong coupling nature of QCD, but it has been confirmed in lattice simulations (see e.g., [3]). It gives rise to $N_f^2 - 1$ massless Nambu–Goldstone bosons (pions) and large dynamical mass for quarks, which is independent of the current quark masses m_f in Eq. (1.1). Those light pions dominate the infrared dynamics of QCD and particularly serve as a source of the attractive interaction between nucleons, which in turn leads to the formation of atomic nuclei. The chiral symmetry breaking therefore plays a major role in making our world appear as we see now.

It has been well known since the seminal work of Banks and Casher [4] that the magnitude of the chiral condensate $\langle \bar{\psi} \psi \rangle$ in the chiral limit is linked to the density of near-zero eigenvalues of the Euclidean Dirac operator $\gamma^\mu D_\mu$. That the spectral density indeed tends to a nonzero value near the origin has been explicitly demonstrated in a recent lattice simulation [5], as shown in Fig. 1.1. It is by now established that the way the thermodynamic limit is approached in the low end of the Dirac spectrum is described by *Chiral Random Matrix Theory* (ChRMT) [6]. ChRMT is one of rare examples in QCD where non-perturbative results can be obtained exactly. In ensuing chapters we will review this subject in much greater detail.

The purpose of this thesis is to argue that a similar theoretical scheme can be developed for the Dirac operator in QCD with $N_c = 2$ (called *two-color QCD*) at large quark number density. To clarify how unexpected this extension is, let us review qualitative differences between QCD at zero density (vacuum) and QCD at high density:

1. At zero density the strong coupling effect is essential, while at high density the running coupling g reduces to zero owing to the asymptotic freedom, and it is the existence of the Fermi surface that plays a key role in various non-perturbative phenomena of dense QCD,
2. The patterns of spontaneous breaking of global symmetries are totally different in QCD vacuum and at high density, and

3. The eigenvalues of the Euclidean Dirac operator become complex in the presence of nonzero quark chemical potential μ . Consequently the statistical measure of the path integral is no longer positive definite, signaling the notorious sign problem.

Nonetheless it is possible to generalize various theoretical apparatuses originally developed for zero density QCD to the high density BCS regime. Let $D(\mu)$ denote the Euclidean Dirac operator at nonzero chemical potential μ . In this thesis, the following generalizations of seminal works in the literature are completed:

- Chiral Lagrangian at $\mu < \Lambda_{\text{QCD}}$ [7, 8]
 $\implies$ Chiral Lagrangian at $\mu \gg \Lambda_{\text{QCD}}$... Sect. 3.2.1
- Partition function in the ε -regime at $\mu \sim \frac{1}{\sqrt{V_4} F_\pi}$ [9, 10]
 $\implies$ Partition function in the new ε -regime at $\mu \gg \Lambda_{\text{QCD}}$... Sect. 3.2.2
- ChRMT for the eigenvalues $\{\lambda_n\}$ of $D(\mu)$ in the microscopic domain
 $\left(|\lambda_n| \sim \frac{1}{V_4 |\langle \bar{\psi} \psi \rangle|} \right)$ at $\mu \sim \frac{1}{\sqrt{V_4} F_\pi}$ [11–13]
 $\implies$ ChRMT for the eigenvalues λ_n of $D(\mu)$ in the new microscopic domain
 $\left(|\lambda_n| \sim \frac{1}{\sqrt{V_4 \Delta^2}} \right)$ at $\mu \gg \Lambda_{\text{QCD}}$... Sects. 3.3 and 3.4
- Analysis of the sign problem in the ε -regime at $\mu \sim \frac{1}{\sqrt{V_4} F_\pi}$ [14, 15]
 $\implies$ Analysis of the sign problem in the new ε -regime $\mu \gg \Lambda_{\text{QCD}}$... Sect. 3.5
- The “three-fold way” of ChRMT at $\mu \sim \frac{1}{\sqrt{V_4} F_\pi}$ [16]
 $\implies$ A new “three-fold way” of ChRMT at $\mu \gg \Lambda_{\text{QCD}}$... Chap. 4

It is impressive that in the ε -regime so many non-perturbative results can be obtained *exactly*. This tractability is due to the fact that the infinitely many degrees of freedom of quantum fields decouple dynamically in the ε -regime, leaving behind only a finite number of degrees of freedom that can be studied analytically.

Several comments are in order.

Why Two Colors Instead of Three Colors?

We would like to address a possible criticism: Why $N_c = 2$? Why not work on $N_c = 3$? The reasons will be spelled out in the following chapters and here we present a brief summary of them. First, contrary to $N_c = 3$ QCD which faces a severe sign problem at $\mu \neq 0$, two-color QCD is amenable to lattice simulations at any μ (as long as the masses are pairwise degenerate). Hence there is a hope that our

analytical results can be tested on the lattice directly. Such a comparison will serve as a useful check of simulation algorithms and will prompt further developments in this field. Secondly, two-color QCD and $N_c = 3$ QCD share the important physics of Cooper pairing near the Fermi surface at high density, even though $N_c = 2$ is theoretically much simpler than $N_c = 3$ (e.g., because the quark pair of the former does not break gauge symmetry). Thus one can expect that from a thorough analysis of dense two-color QCD we will learn at least certain universal aspects of dense $N_c = 3$ QCD. And last, but not least, our analysis of dense two-color QCD permits a straightforward generalization to other QCD-like theories, as is elucidated in Chap. 4. In this sense our current treatment is more general than it appears at first sight.

The extension to dense $N_c = 3$ QCD is a highly challenging attractive problem begging for a solution. This is left for future work.

Matrix Models as a Phenomenological Model of QCD

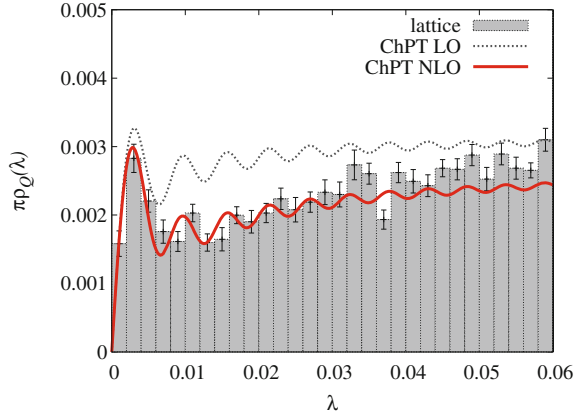
While this thesis is mainly concerned with ChRMT as an *exact* theory of the QCD Dirac spectrum in the microscopic domain, the ChRMT has been also used as a phenomenological mean-field model of QCD at finite temperature and density. In this context, the universal correspondence of ChRMT to the ε -regime QCD is lost, but we can learn about qualitative features of the QCD phase diagram. The reader interested in this direction is referred to a recent review [17].

Subsequent Progress

As we will discuss in the main text, the high-density regime and the low-density regime of dense two-color QCD are described by distinct low-energy effective theories, because the patterns of symmetry breaking are different. This observation triggers a question about the physics at intermediate densities. In [18] it was shown, on the basis of the putative BEC-BCS crossover scenario, that the three regimes of dense two-color QCD with high, intermediate, and low density, respectively, are governed by three distinct effective theories, with overlapping domains of validity. The relation between those theories was analyzed in detail and found to be totally consistent with each other. Eventually it was shown that the scheme of the present thesis can be successfully extended to *all densities*.

Furthermore, we also succeeded in extending the universal correspondence between ChRMT and the Dirac spectrum in QCD to a novel correspondence between ChRMT and the *singular value spectrum* of the Dirac operator [18]. The key observation in this work is that the singular value spectrum can be probed through the addition of the *diquark sources* in place of usual quark masses. Important results such as the Banks–Casher relation, the Smilga–Stern relation and the Leutwyler–Smilga sum rules, for the Dirac eigenvalues are now all generalized to the case of the Dirac singular values.

Fig. 1.1 Spectral density of the Dirac operator measured in a lattice QCD simulation. (Reprinted from [5]. Copyright(2012) with permission from Elsevier)



Since these developments substantially widen and deepen the framework proposed in this thesis, one may view this thesis and [18] as comprising a single integrated research. The reader interested in this thesis is therefore highly recommended to take a look at [18] as well.

This thesis is organized as follows. In Sect. 2.1 we will explain the motivation to look into QCD at nonzero μ and review past studies on the phase diagram of QCD. In Sects. 2.2 and 2.3 we will review important features of two-color QCD and ChRMT. In Chap. 3 a thorough investigation is given into various properties of the Dirac spectrum in dense two-color QCD: we will construct a low energy effective theory of Nambu–Goldstone bosons at high density, and use it to derive spectral sum rules for the complex eigenvalues of the Euclidean Dirac operator. In Sect. 3.3 we introduce a non-Hermitian random matrix ensemble and argue that it belongs to the same universality class as the Dirac operator in dense two-color QCD. In Sect. 3.4 the new ChRMT is solved exactly and microscopic spectral densities are obtained in the limit where the matrix size tends to infinity. It is discovered that the microscopic spectrum is quite sensitive to the detuning of mass parameters. In Sect. 3.5 we evaluate the severity of the sign problem in the ε -regime of dense two-color QCD on the basis of the universal correspondence to the new ChRMT. It is observed that, as we detune the masses, the expectation value of the sign of the fermion determinant drops to zero, signaling a severe sign problem. Chapter 4 discusses possible extensions of our analysis on two-color QCD ($\beta = 1$) to other QCD-like theories, namely QCD with isospin chemical potential ($\beta = 2$) and QCD with fermions in a *real* representation of the gauge group ($\beta = 4$) where β is the so-called Dyson index. Chapter 5 is devoted to summary and conclusions. In appendices we clarify mathematical conventions of this thesis and present technical details of the calculations that have been omitted in the main text.

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Kanazawa, T.

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