# Chapter 2
# Subgraphs, Paths, and Connected Graphs

## 2.1 Subgraphs and Spanning Subgraphs (Supergraphs)

*Subgraph*: Let $H$ be a graph with vertex set $V(H)$ and edge set $E(H)$, and similarly let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Then, we say that $H$ is a subgraph of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In such a case, we also say that $G$ is a supergraph of $H$.

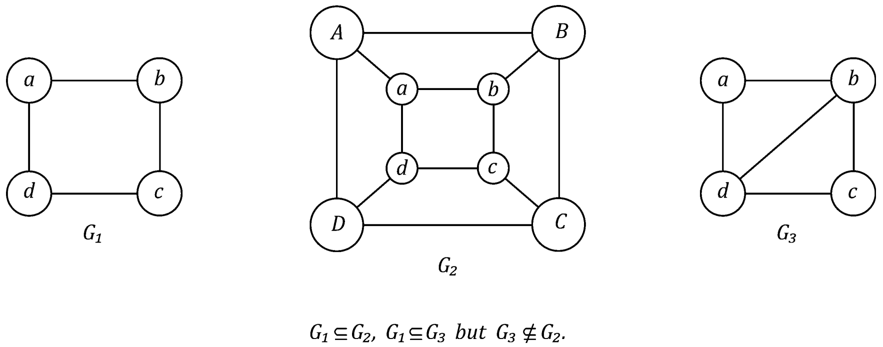


**Fig. 2.1** $G_1$ is a subgraph of $G_2$ and $G_3$

In Fig. 2.1, $G_1$ is a subgraph of both $G_2$ and $G_3$ but $G_3$ is not a subgraph of $G_2$.

Any graph isomorphic to a subgraph of $G$ is also referred to as a subgraph of $G$.

If $H$ is a subgraph of $G$ then we write $H \subseteq G$. When $H \subseteq G$ but $H \neq G$, i.e., $V(H) \neq V(G)$ or $E(H) \neq E(G)$, then $H$ is called a proper subgraph of $G$.

*Spanning subgraph* (*or Spanning supergraph*): A *spanning subgraph* (or *spanning supergraph*) of $G$ is a *subgraph* (or *supergraph*) $H$ with $V(H) = V(G)$, i.e. $H$ and $G$ have exactly the same vertex set.

It follows easily from the definitions that any simple graph on $n$ vertices is a subgraph of the complete graph $K_n$. In Fig. 2.1, $G_1$ is a proper spanning subgraph of $G_3$.


S. Saha Ray, *Graph Theory with Algorithms and its Applications*,
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## 2.2 Operations on Graphs

The *union* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is another graph $G_3 = (V_3, E_3)$ denoted by $G_3 = G_1 \cup G_2$, where vertex set $V_3 = V_1 \cup V_2$ and the edge set $E_3 = E_1 \cup E_2$.

The *intersection* of two graphs $G_1$ and $G_2$ denoted by $G_1 \cap G_2$ is a graph $G_4$ consisting only of those vertices and edges that are in both $G_1$ and $G_2$.

The *ring sum* of two graphs $G_1$ and $G_2$, denoted by $G_1 \oplus G_2$, is a graph consisting of the vertex set $V_1 \cup V_2$ and of edges that are either in $G_1$ or $G_2$, but not in both.

Figure 2.2 shows union, intersection, and ring sum on two graphs $G_1$ and $G_2$.

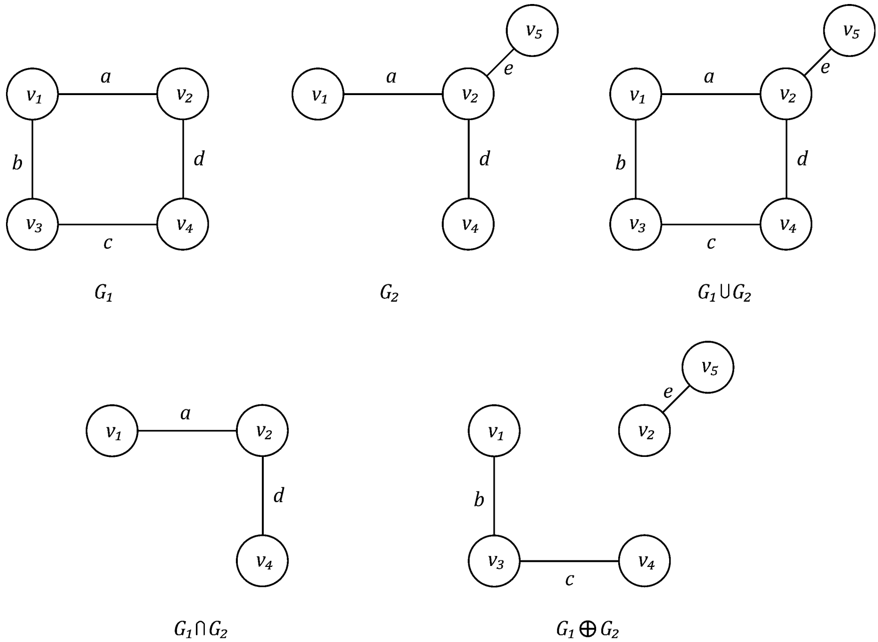



**Fig. 2.2** Union, intersection, and ring sum of two graphs

Three operations are *commutative*, i.e.,

$$G_1 \cup G_2 = G_2 \cup G_1, \quad G_1 \cap G_2 = G_2 \cap G_1, \quad G_1 \oplus G_2 = G_2 \oplus G_1$$

If $G_1$ and $G_2$ are edge disjoint, then $G_1 \cap G_2$ is a null graph, and $G_1 \oplus G_2 = G_1 \cup G_2$. If $G_1$ and $G_2$ are vertex disjoint, then $G_1 \cap G_2$ is empty.

For any graph $G$, $G \cap G = G \cup G = G$ and $G \oplus G =$ *a null graph*.

If $g$ is a subgraph of $G$, i.e., $g \subseteq G$, then $G \oplus g = G - g$, and is called a *complement* of $g$ in $G$.

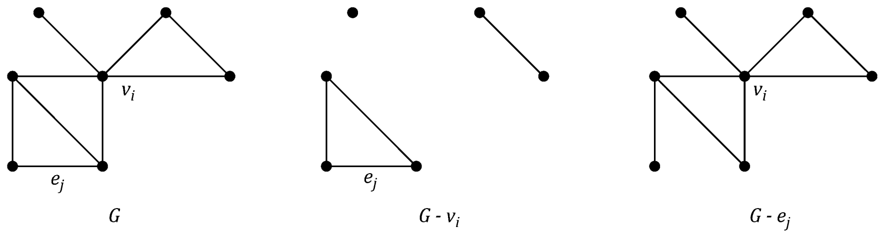



**Fig. 2.3** Vertex deletion and edge deletion from a graph $G$

*Decomposition*: A graph $G$ is said to be decomposed into two subgraphs $G_1$ and $G_2$, if $G_1 \cup G_2 = G$ and $G_1 \cap G_2$ is a *null graph*.

*Deletion*: If $v_i$ is a vertex in graph $G$, then $G - v_i$ denotes a subgraph of $G$ obtained by deleting $v_i$ from $G$. Deletion of a vertex always implies the deletion of all edges incident on that vertex. If $e_j$ is an edge in $G$, then $G - e_j$ is a subgraph of $G$ obtained by deleting $e_j$ from $G$. Deletion of an edge does not imply deletion of its end vertices. Therefore, $G - e_j = G \oplus e_j$ (Fig. 2.3).

*Fusion*: A pair of vertices $a, b$ in a graph $G$ are said to be *fused* if the two vertices are replaced by a single new vertex such that every edge, that was incident on either $a$ or $b$ or on both, is incident on the new vertex. Thus, fusion of two vertices does not alter the number of edges, but reduces the number of vertices by one (Fig. 2.4).

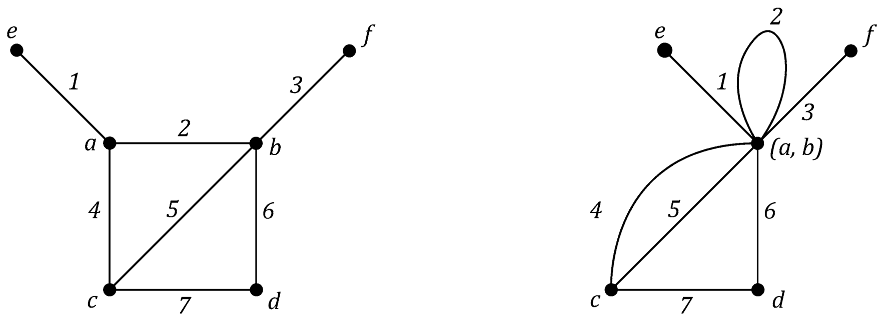



**Fig. 2.4** Fusion of two vertices $a$ and $b$

*Induced subgraph*: A subgraph $H \subseteq G$ is an induced subgraph, if $E_H = E_G \cap E(V_H)$. In this case, $H$ is induced by its set $V_H$ of vertices. In an induced subgraph $H \subseteq G$, the set $E_H$ of edges consists of all $e \in E_G$, such that $e \in E(V_H)$. To each nonempty subset $A \subseteq V_G$, there corresponds a unique induced subgraph $G[A] = (A, E_G \cap E(A))$ (Fig. 2.5).

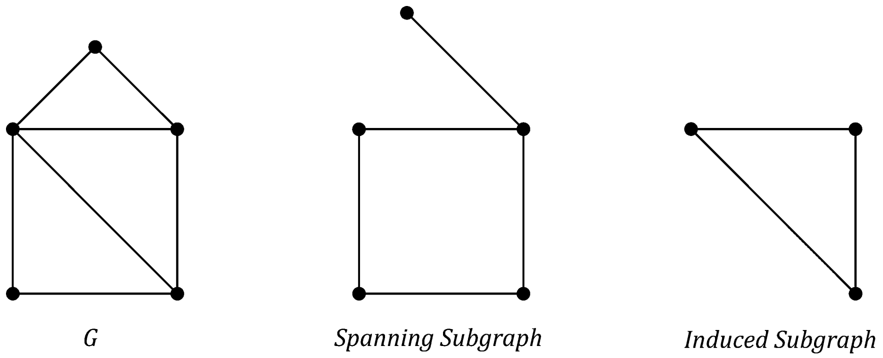



**Fig. 2.5** Spanning subgraph and induced subgraph of a graph $G$

*Trivial graph*: A graph $G = (V, E)$ is trivial, if it has only one vertex. Otherwise $G$ is nontrivial.

*Discrete graph*: A graph is called discrete graph if $E_G = \phi$.

*Stable*: A subset $X \subseteq V_G$ is stable, if $G[X]$ is a discrete graph.

## 2.3 Walks, Trails, and Paths

*Walk*: A *walk* in a graph $G$ is a finite sequence

$$W \equiv v_0 e_1 v_1 e_2 \cdots v_{k-1} e_k v_k$$

whose terms are alternately vertices and edges such that for $1 \le i \le k$, the edge $e_i$ has ends $v_{i-1}$ and $v_i$.

Thus, each edge $e_i$ is immediately preceded and succeeded by the two vertices with which it is incident. We say that $W$ is a $v_0 - v_k$ walk or a walk from $v_0$ to $v_k$.

*Origin and terminus*: The vertex $v_0$ is the *origin* of the walk $W$, while $v_k$ is called the *terminus* of $W$. $v_0$ and $v_k$ need not be distinct.

The vertices $v_1, v_2, \ldots, v_{k-1}$ in the above walk $W$ are called its *internal vertices*. The integer $k$, the number of edges in the walk, is called the *length of* $W$, denoted by $|W|$.

In a walk $W$, there may be repetition of vertices and edges.

*Trivial walk*: A *trivial walk* is one containing no edge. Thus for any vertex $v$ of $G$, $W \equiv v$ gives a trivial walk. It has length 0.

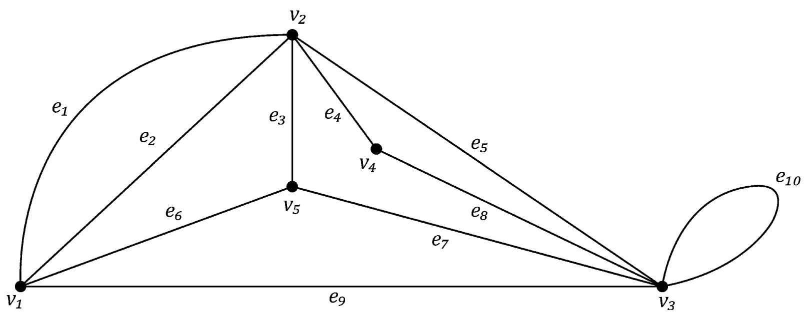



**Fig. 2.6** A graph with five vertices and ten edges

In Fig. 2.6, $W_1 \equiv v_1e_1v_2e_5v_3e_{10}v_3e_5v_2e_3v_5$ and $W_2 \equiv v_1e_1v_2e_1v_1e_1v_2$ are both walks of length 5 and 3, respectively, from $v_1$ to $v_5$ and from $v_1$ to $v_2$ , respectively.

Given two vertices $u$ and $v$ of a graph $G$, a $u$–$v$ walk is called *closed* or *open*, depending on whether $u = v$ or $u \neq v$.

Two walks $W_1$ and $W_2$ above are both open, while $W_3 \equiv v_1v_5v_2v_4v_3v_1$ is closed in Fig. 2.6.

*Trail*: If the edges $e_1, e_2, \ldots, e_k$ of the walk $W \equiv v_0e_1v_1e_2v_2 \cdots\cdots e_kv_k$ are distinct then $W$ is called a *trail*. In other words, a trail is a walk in which no edge is repeated. $W_1$ and $W_2$ are not trails, since for example $e_5$ is repeated in $W_1$, while $e_1$ is repeated in $W_2$. However, $W_3$ is a trail.

*Path*: If the vertices $v_0, v_1, \ldots, v_k$ of the walk $W \equiv v_0e_1v_1e_2v_2 \cdots e_kv_k$ are distinct then $W$ is called a *path*. Clearly, any two paths with the same number of vertices are isomorphic.

A path with $n$ vertices will sometimes be denoted by $P_n$.

Note that $P_n$ has length $n - 1$.

In other words, *a path is a walk in which no vertex is repeated*. Thus, in a path no edge can be repeated either, so a every path is a trail. Not every trail is a path, though. For example, $W_3$ is not a path since $v_1$ is repeated. However, $W_4 \equiv v_2v_4v_3v_5v_1$ is a path in the graph $G$ as shown in Fig. 2.6.

## 2.4 Connected Graphs, Disconnected Graphs, and Components

*Connected vertices*: A vertex $u$ is said to be *connected* to a vertex $v$ in a graph $G$ if there is a path in $G$ from $u$ to $v$.

*Connected graph*: A graph $G$ is called *connected* if every two of its vertices are connected.

*Disconnected graph*: A graph that is not connected is called *disconnected*.

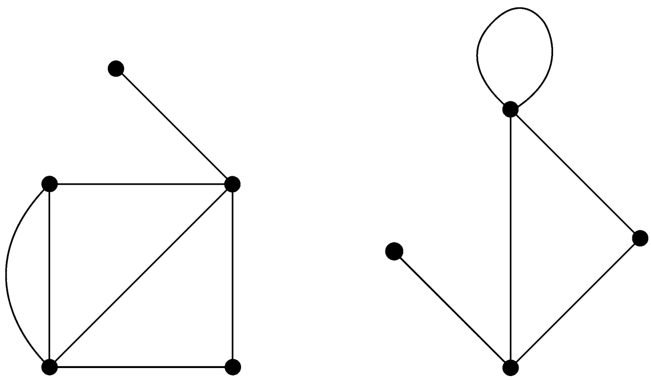


**Fig. 2.7** A disconnected graph with two components

It is easy to see that a disconnected graph consists of two or more *connected graphs*. Each of these connected subgraphs is called a component. Figure 2.7 shows a disconnected graph with two components.

**Theorem 2.1** *A graph G is disconnected iff its vertex set V can be partitioned into two non-empty, disjoint subsets* $V_1$ *and* $V_2$ *such that there exists no edge in G whose one end vertex is in subset* $V_1$ *and the other in subset* $V_2$.

*Proof* Suppose that such a partitioning exists. Consider two arbitrary vertices $a$ and $b$ of $G$, such that $a \in V_1$ and $b \in V_2$. No path can exist between vertices $a$ and $b$; otherwise there would be at least one edge whose one end vertex would be in $V_1$ and the other in $V_2$. Hence, if a partition exists, $G$ is not connected.

Conversely, let $G$ be a disconnected graph. Consider a vertex $a$ in $G$. Let $V_1$ be the set of all vertices that are connected by paths to $a$. Since $G$ is disconnected, $V_1$ does not include all vertices of $G$. The remaining vertices will form a (non-empty) set $V_2$. No vertex in $V_1$ is connected to any vertex in $V_2$ by path. Hence the partition exists. □

**Theorem 2.2** *If a graph* (*connected* or *disconnected*) *has exactly two vertices of odd degree, there must be a path joined by these two vertices.*

*Proof* Let $G$ be a graph with all even vertices except vertices $v_1$ and $v_2$, which are odd. From Handshaking lemma, which holds for every graph and therefore for every component of a disconnected graph, no graph can have an odd number of odd vertices. Therefore, in graph $G$, $v_1$ and $v_2$ must belong to the same component, and hence there must be a path between them. □

**Theorem 2.3** *A simple graph with n vertices and k components can have at most* $(n - k)(n - k + 1)/2$ *edges.*

*Proof* Let the number of vertices in each of the $k$ components of a graph $G$ be $n_1, n_2, \ldots, n_k$. Thus, we have

$$n_1 + n_2 + \cdots + n_k = n$$

where $n_i \geq 1 \quad for \quad i = 1, 2, \ldots, k$.

Now, $\sum_{i=1}^{k} (n_i - 1) = n - k$

$$\Rightarrow \left(\sum_{i=1}^{k}(n_i - 1)\right)^2 = n^2 + k^2 - 2nk$$
$$\Rightarrow [(n_1 - 1) + (n_2 - 1) + \cdots + (n_k - 1)]^2 = n^2 + k^2 - 2nk$$
$$\Rightarrow \sum_{i=1}^{k}(n_i - 1)^2 + 2\sum_{i,j=1,i\neq j}^{k}(n_i - 1)(n_j - 1) = n^2 + k^2 - 2nk$$
$$\Rightarrow \sum_{i=1}^{k}(n_i)^2 - 2\sum_{i=1}^{k} n_i + k + 2\sum_{i,j=1,i\neq j}^{k}(n_i - 1)(n_j - 1) = n^2 + k^2 - 2nk$$
$$\Rightarrow \sum_{i=1}^{k} n_i^2 - 2n + k + 2\sum_{i,j=1,i\neq j}^{k}(n_i - 1)(n_j - 1) = n^2 + k^2 - 2nk$$
$$\Rightarrow \sum_{i=1}^{n} n_i^2 + 2\sum_{i,j=1,i\neq j}^{k}(n_i - 1)(n_j - 1) = n^2 + k^2 - 2nk + 2n - k.$$

Since each $(n_i - 1) \geq 0$.

$$\sum_{i=1}^{n} n_i^2 \leq n^2 + k^2 - 2nk + 2n - k = n^2 + k(k - 2n) - (k - 2n)$$
$$= n^2 - (k - 1)(2n - k)$$

Now, the maximum number of edges in the $i$th component of $G$ is $n_i(n_i - 1)/2$. Since the maximum number of edges in a simple graph with $n$ vertices is $n(n-1)/2$ therefore, the maximum number of edges in $G$ is

$$\begin{aligned}
\frac{1}{2}\sum_{i=1}^{k} n_i(n_i - 1) &= \frac{1}{2}\sum_{i=1}^{n} n_i^2 - \frac{n}{2} \\
&\leq \frac{1}{2}\left[n^2 - (k - 1)(2n - k)\right] - \frac{n}{2} \\
&= \frac{1}{2}\left[n^2 - 2nk + 2n + k^2 - k - n\right] \\
&= \frac{1}{2}\left[(n - k)^2 + (n - k)\right] \\
&= \frac{1}{2}(n - k)(n - k + 1)
\end{aligned}$$

□

## 2.5 Cycles

*Cycle*: A nontrivial closed trail in a graph $G$ is called a cycle if its origin and internal vertices are distinct. In detail, the closed trail

$C \equiv v_1 v_2 \cdots v_n v_1$ is a cycle if

1. $C$ has at least one edge and
2. $v_1, v_2, \ldots, v_n$are $n$ distinct vertices.

*k-**Cycle***: A cycle of length $k$, , i.e., with $k$ edges, is called a $k$-cycle. A $k$-cycle is called odd or even depending on whether $k$ is odd or even.

Figure 2.8 cites $C_3, C_4, C_5$, and $C_6$. A 3-cycle is often called a triangle. Clearly, any two cycles of the same length are isomorphic.

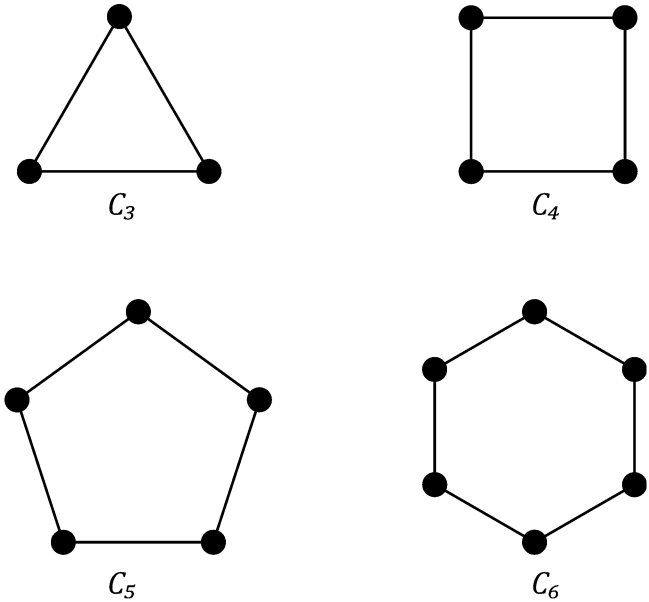
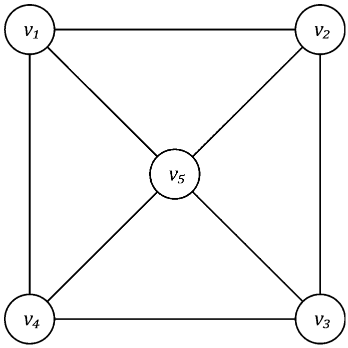



**Fig. 2.8** Cycles $C_3$, $C_4$, $C_5$ and $C_6$

An $n$-cycle, i.e., a cycle with $n$ vertices, will sometimes be denoted by $C_n$.

In Fig. 2.9, $C \equiv v_1 v_2 v_3 v_4 v_1$, is a 4-cycle and $T \equiv v_1 v_2 v_5 v_3 v_4 v_5 v_1$ is a non-trivial closed trail which is not a cycle (because $v_5$ occurs twice as an internal vertex) and $C' \equiv v_1 v_2 v_5 v_1$ is a triangle.

**Fig. 2.9** A graph containing 3-cycles and 4-cycles



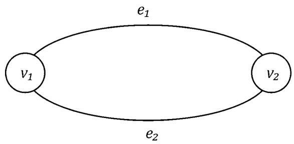



**Fig. 2.10** A 2-cycle

Note that, a loop is just a 1-cycle. Also, given parallel edges $e_1$ and $e_2$ in Fig. 2.10 with distinct end vertices $v_1$ and $v_2$, we can find the cycle $v_1e_1v_2e_2v_1$ of length 2. Conversely, the two edges of any cycle of length 2 are a pair of parallel edges.

**Theorem 2.4** *Given any two vertices u and v of a graph G, every u–v walk contains a u–v path.*

*Proof* We prove the statement by induction on the length $l$ of a $u$–$v$ walk $W$.

*Basic step*: $l = 0$, having no edge, $W$ consists of a single vertex ($u = v$). This vertex is a $u$–$v$ path of length 0.

*Induction step*: $l \geq 1$. We suppose that the claim holds for walks of length less than $l$. If $W$ has no repeated vertex, then its vertices and edges form a $u$–$v$ path. If $W$ has a repeated vertex $w$, then deleting the edges and vertices between appearances of $w$ (leaving one copy of $w$) yields a shorter $u$-$v$ walk $W'$ contained in $W$. By the induction hypothesis, $W'$ contains a $u$–$v$ path $P$, and this path $P$ is contained in $W$ (Fig. 2.11). This proves the theorem. □

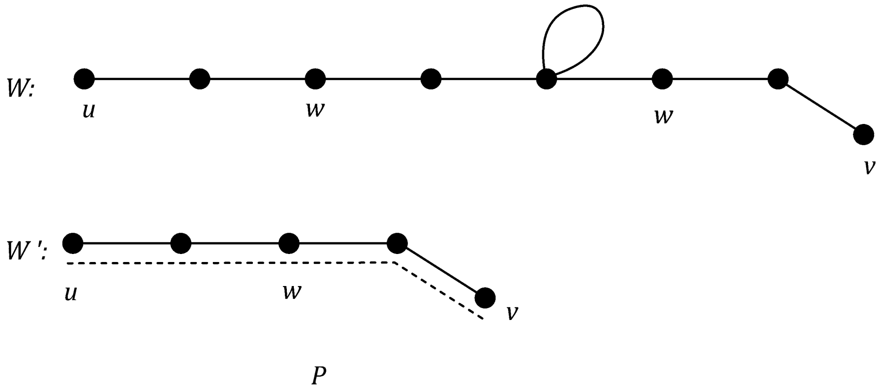



**Fig. 2.11** A walk $W$ and a shorter walk $W'$ of $W$ containing a path $P$

**Theorem 2.5** *The minimum number of edges in a connected graph with n vertices is* $n - 1$.

*Proof* Let $m$ be the number of edges of such a graph. We have to show $m \geq n - 1$. We prove this by method of induction on $m$. If $m = 0$ then obviously $n = 1$ (otherwise $G$ will be disconnected). Clearly, then $m \geq n - 1$. Let the result be true for $m = 0, 1, 2, 3, \ldots, k$. We shall show that the result is true for $m = k + 1$. Let $G$ be a graph with $k + 1$ edges. Let $e$ be an edge of $G$. Then the subgraph $G - e$ has

$k$ edges and $n$ number of vertices. If $G - e$ is also connected then by our hypothesis $k \geq n - 1$, i.e., $k + 1 \geq n > n - 1$.

If $G - e$ is disconnected then it would have two connected components. Let the two components have $k_1, k_2$ number of edges and $n_1, n_2$ number of vertices, respectively. So, by our hypothesis, $k_1 \geq n_1 - 1$ and $k_2 \geq n_2 - 1$. These two imply that $k_1 + k_2 \geq n_1 + n_2 - 2$, i.e., $k \geq n - 2$ (since, $k_1 + k_2 = k,\ n_1 + n_2 = n$), i.e., $k + 1 \geq n - 1$.

Thus, the result is true for $m = k + 1$. □

**Theorem 2.6** *A graph G is bipartite if and only if it has no odd cycles.*

*Proof* Necessary condition:

Let $G$ be a bipartite graph with bipartition $(X, Y)$, i.e., $V = X \cup Y$.

For any cycle $C : v_1 \to v_2 \cdots \to v_{k+1}(= v_1)$ of length $k, v_1 \in X \Rightarrow v_2 \in Y, v_3 \in X \Rightarrow v_4 \in Y \cdots \Rightarrow v_{2m} \in Y \Rightarrow v_{2m+1} \in X$. Consequently, $k + 1 = 2m + 1$ is odd and $k = |C|$ is even. Hence, $G$ has no odd cycle.

Sufficient condition:

Suppose that, all the cycles in $G$ are even, i.e., $G$ be a graph with no odd cycle.

To show: $G$ is a bipartite graph. It is sufficient to prove this theorem for the connected graph only.

Let us assume that $G$ is connected. Let $v \in G$ be an arbitrary chosen vertex.

Now, we define,

$$X = \{x | d_G(v, x) \text{ is even}\},$$

i.e., $X$ is the set of all vertices $x$ of $G$ with the property that any shortest $v - x$ path of $G$ has even length and $Y = \{y | d_G(v, y)\ \textit{is odd}\}$, i.e., $Y$ is the set of all vertices $y$ of $G$ with the property that any shortest $v - y$ path of $G$ has odd length.

Here,

$$\begin{aligned} d_G(u, v) &= \text{shortest distance from the vertex } u \text{ to the vertex } v \\ &= \min\left\{k : u \xrightarrow{k} v\right\} \end{aligned}$$

[If the graph $G$ is connected then this shortest distance should be finite, i.e., $d_G(u, v) < \infty$ for $\forall u, v \in G$. Otherwise, $G$ is disconnected]

Then clearly, since the graph $G$ is connected $V = X \cup Y$ and also by definition of distance $X \cap Y = \emptyset$.

Now, we show that $V = X \cup Y$ is a bipartition of $G$ by showing that any edge of $G$ must have one end vertex in $X$ and another in $Y$.

Suppose that $u, w \in V(G)$ are both either in $X$ or in $Y$ and they are adjacent.

Let $P : v \xrightarrow{*} u$ and $Q : v \xrightarrow{*} w$ be the two shortest paths from $v$ to $u$ and $v$ to $w$, respectively.

Let $x$ be the last common vertex of the two shortest paths $P$ and $Q$ such that $P = P_1 P_2$ and $Q = Q_1 Q_2$ where $P_2 : x \xrightarrow{*} u$ and $Q_2 : x \xrightarrow{*} w$ are independent (Fig. 2.12).

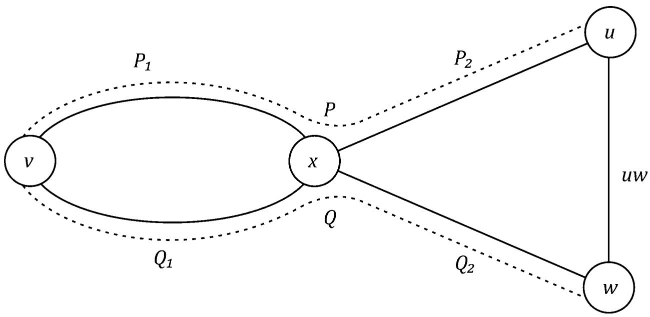



**Fig. 2.12** Two shortest paths $P$ and $Q$

Since $P$ and $Q$ are shortest paths, therefore, $P_1 : v \xrightarrow{*} x$ and $Q_1 : v \xrightarrow{*} x$ are shortest paths from $v$ to $x$.

Consequently, $|P_1| = |Q_1|$

Now consider the following two cases.

Case 1: $u, w \in X$, then $|P|$ is even and $|Q|$ is even (Also, $|P_1| = |Q_1|$)

Case 2: $u, w \in Y$, then $|P|$ is odd and $|Q|$ is odd (Also, $|P_1| = |Q_1|$)

Therefore, in either case, $|P_2| + |Q_2|$ must be even and so $uw \notin E(G)$. Otherwise, $x \xrightarrow{*} u \longrightarrow w \xrightarrow{*} x$ would be an odd cycle, which is a contradiction.

Therefore, $X$ and $Y$ are stable subsets of $V$. This implies $(X, Y)$ is a bipartition of $G$. Therefore, $G[X]$ and $G[Y]$ are discrete induced subgraphs of $G$.

Hence, $G$ is a bipartite graph.

If $G$ is disconnected then each cycle of $G$ will belong to any one of the connected components of $G$ say $G_1, G_2, \ldots, G_p$.

If $G_i$ is bipartite with bipartition $(X_i, Y_i)$, then $\left(X_1 \cup X_2 \cup X_3 \cup \cdots\cdots \cup X_p, Y_1 \cup Y_2 \cup \cdots\cdots \cup Y_p\right)$ is a bipartition of $G$.

Hence, the disconnected graph $G$ is bipartite. □

***Exercises:***

1. Show that the following two graphs are isomorphic (Fig. 2.13).

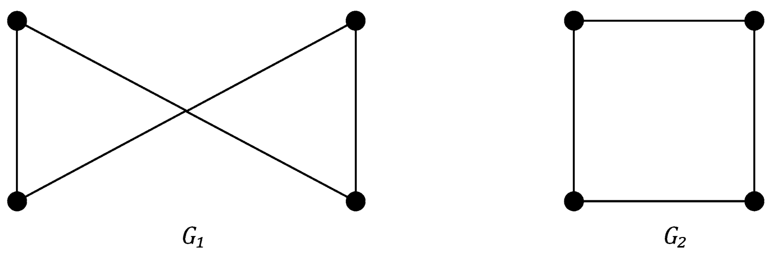



**Fig. 2.13**

2. Check whether the following two graphs are isomorphic or not (Fig. 2.14).

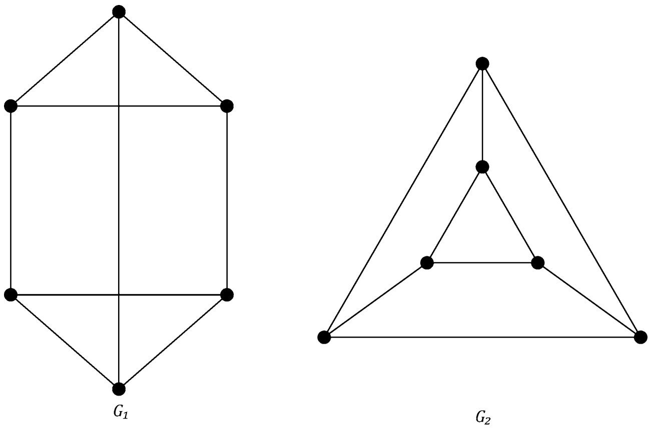



**Fig. 2.14**

3. Show that the following graphs are isomorphic and each graph has the same bipartition (Fig. 2.15).

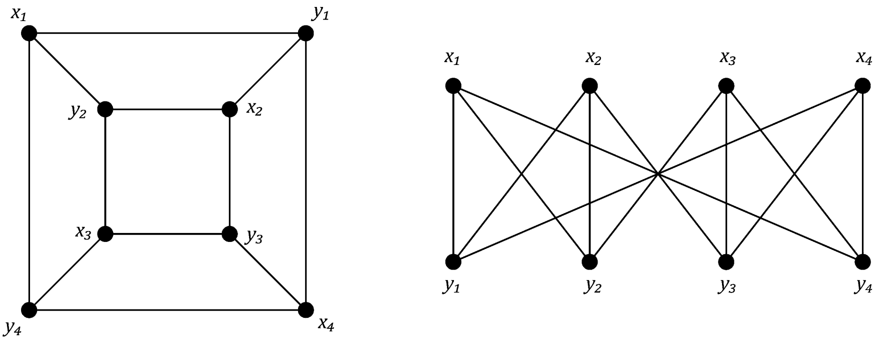



**Fig. 2.15**

4. What is the difference between a closed trail and a cycle?
5. Are the following graphs isomorphic? (Fig. 2.16).

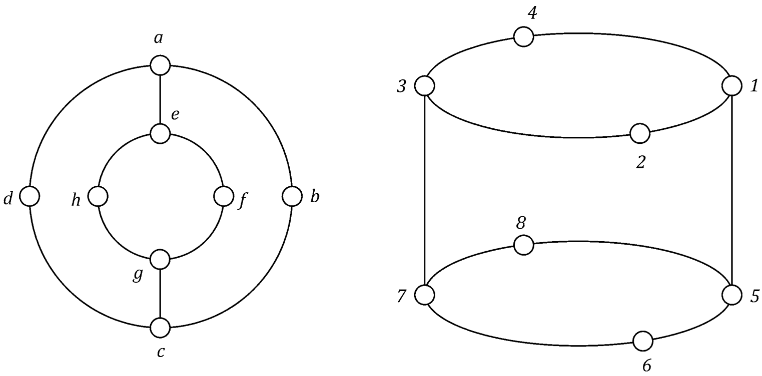



**Fig. 2.16**

6. Prove that a simple graph having $n$ number of vertices must be connected if it has more than $(n-1)(n-2)/2$ edges.
7. Check whether the following two given graphs $G_1$ and $G_2$ are isomorphic or not (Fig. 2.17).

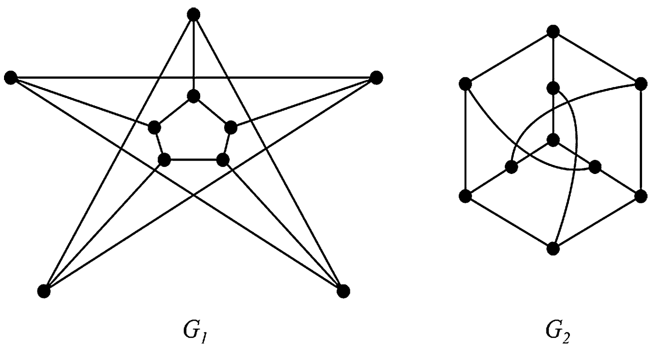



**Fig. 2.17**

8. Prove that the number of edges in a bipartite graph with $n$ vertices is at most$(n^2/2)$.
9. Prove that there exists no simple graph with five vertices having degree sequence $4, 4, 4, 2, 2$.
10. Find, if possible, a simple graph with five vertices having degree sequence $2, 3, 3, 3, 3$.
11. If a simple regular graph has $n$ vertices and 24 edges, find all possible values of $n$.

12. If $\delta(G)$ and $\Delta(G)$ be the minimum and maximum degrees of the vertices of a graph $G$ with $n$ vertices and $e$ edges, show that

$$\delta(G) \leq \frac{2e}{n} \leq \Delta(G)$$

13. Show that the minimum number of edges in a simple graph with $n$ vertices is $n - k$, where $k$ is the number of connected components of the graph.
14. Find the maximum number of edges in
    (a) a simple graph with $n$ vertices
    (b) a bipartite graph with bipartition $(X, Y)$ where $|X| = m$ and $|Y| = n$, respectively.

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